MATHEMATICS FOR ECONOMISTS

SETS, NUMBERS, AND FUNCTIONS

Sets

Definition 1 *A* **set** *is a collection of objects thought of as a whole.*

- Describe a set by enumeration: list all the **elements** of the set e.g. $S = \{2, 4, 6, 8, 10\} = \{4, 10, 6, 2, 8\}$
- Describe a set by property: state the property shared by all the elements in the set, e.g.

$$S = \{x : x \text{ is an even number between 1 and 11}\}$$

- $x \in S$: x is in the set S.
- $x \notin S$: x is not in the set S.

Examples

$\{x : x \text{ is a firm producing computers}\}$ ("computer industry")

$\{x : x \text{ is a bundle of goods a consumer can afford}\}$

("budget set")

Definition 2 *If all the elements of a set X are also elements of a set Y, then X is a* **subset** *of Y: $X \subseteq Y$*

$\{4, 8, 10\} \subseteq \{2, 4, 6, 8, 10\}$

$\{2, 4, 6, 8, 10\} \subseteq \{2, 4, 6, 8, 10\}$

Computer industry $\subseteq$ IT industry.

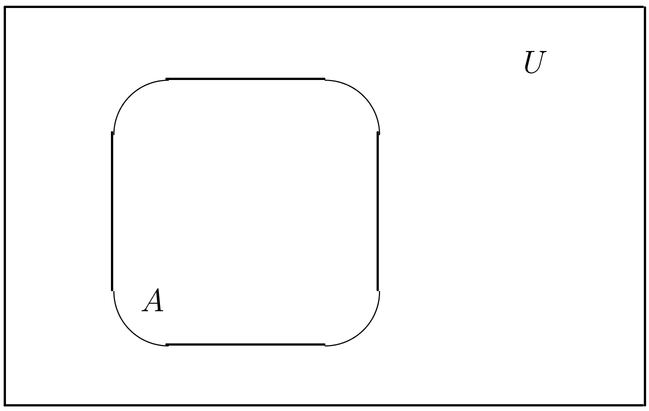
Definition 3 *If all the elements of a set X are also elements of a set Y, but not all the elements of Y are in X, then X is a* **proper subset** *of Y: $X \subset Y$*

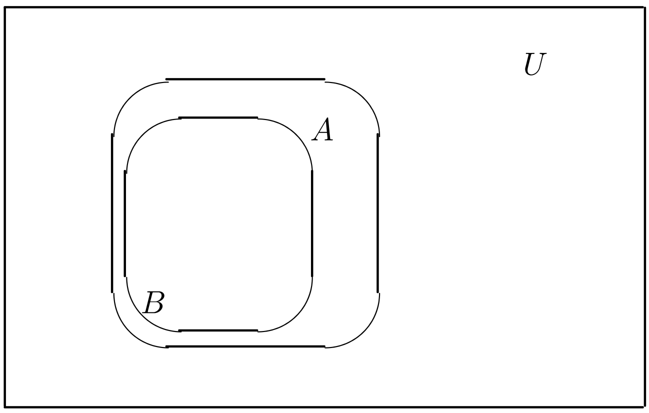
Definition 4 *Two sets X and Y are equal if they contain exactly the same elements: $X = Y$.*

$X \subseteq Y$ and $Y \subseteq X$ implies and is implies by $X = Y$.

Venn Diagram

- **Universal Set**: the set that contains all possible objects under consideration

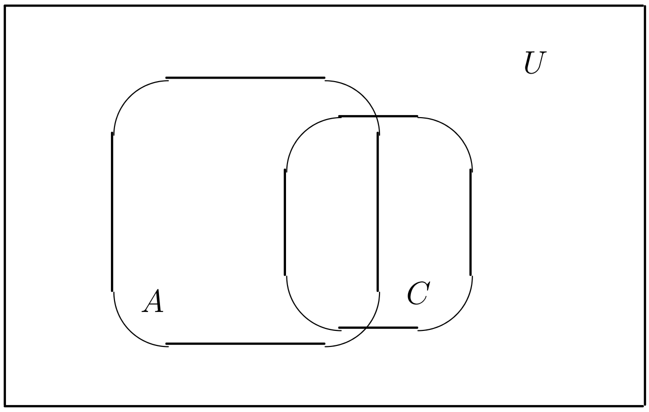




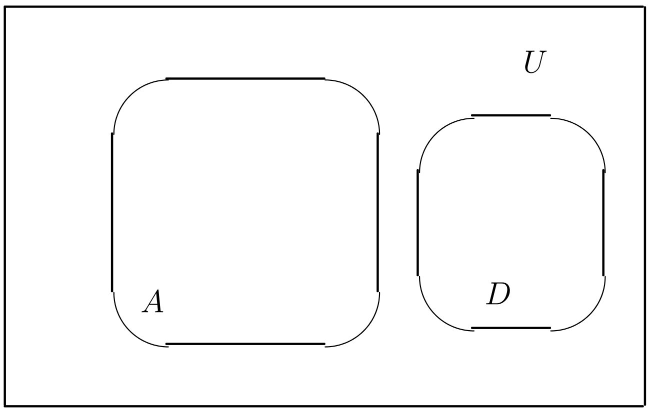
$B \subset A$

Definition 5 *The* **intersection** *of the two sets X and Y is the set of elements that are in both X and Y:*

$$X \cap Y = \{x : x \in X \textit{ and } x \in Y\}$$



If $A = \{2, 4, 6, 8\}, B = \{3, 4, 6, 7\},\ A \cap B = \{4, 6\}$



Definition 6 *The* **empty set** *is the set with no elements: $\emptyset$.*

A set with only one element is a **singleton**

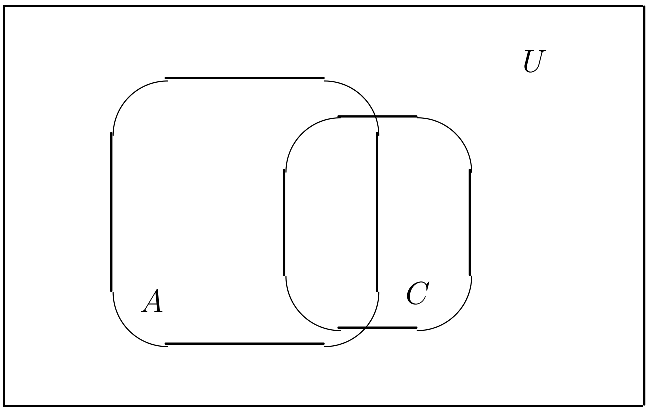
$A \cap D = \emptyset$

$\{x : 2x + 3 = 1\}$

Definition 7 *The* **union** *of two set X and Y is the set of elements in one or the other of the sets:*

$$W = X \cup Y = \{x : x \in X \ \ or \ Y\}$$

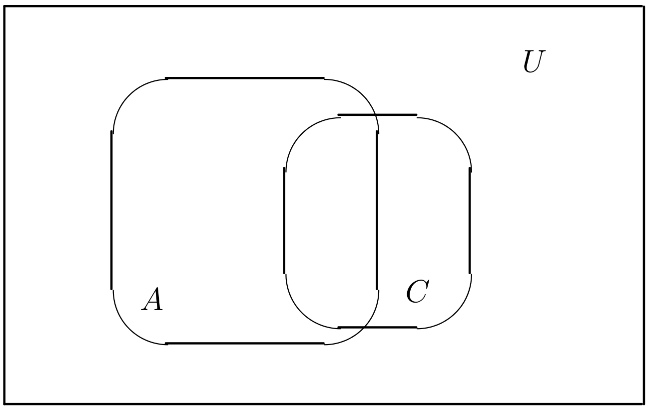
If $A = \{2, 4, 6, 8\}, B = \{3, 4, 6, 7\}$, $A \cup B = \{2, 3, 4, 6, 7, 8\}$

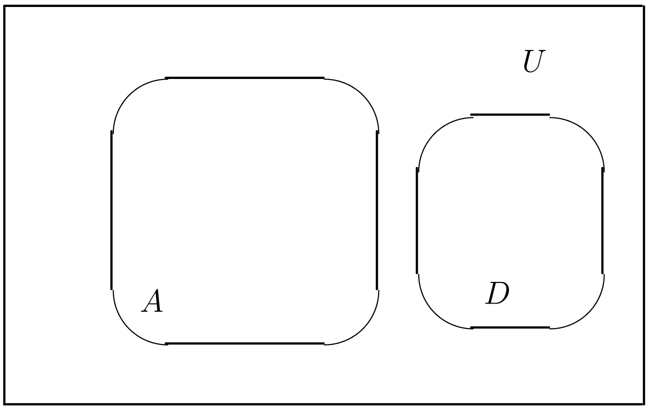


Definition 8 *The* **relative difference** *of X and Y, $X - Y$, is the set of elements of X that are not also in Y.*

$$X - Y = \{x : x \in X \ \ and \ x \notin Y\}$$

If $A = \{2, 4, 6, 8\}, B = \{3, 4, 6, 7\}$, $A - B = \{2, 8\}$

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Necessary and sufficient conditions.

"If the GDP of Germany is twice as large as that of England, then the GDP of England is less than that of Germany."

$A =$ "the GDP of Germany is twice as large as that of England."

$B =$ "the GDP of England is less than that of Germany."

- $A \Rightarrow B$

  If A, then B.

  A implies B.

  (Whenever A is true, B is true.)

- A is a **sufficient condition** for B.

  (The truth of A guarantees the truth of B.)

- A only if B

  B is a **neccessary condition** for A.

  (If B is not true, then A is not true.)

- x: GDP of England, y: GDP of Germany.

  A: $2x = y$, B: $x < y$

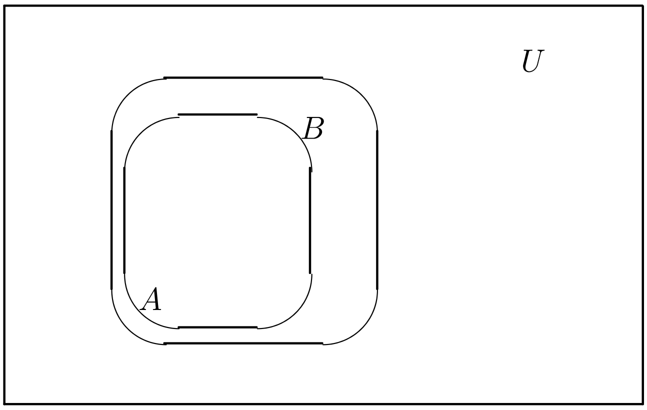
  $A = \{x, y : 2x = y\}$

  (the set of all the objects that satisfy the condition A)

  $B = \{x, y : x < y\}$

  (the set of all the objects that satisfy the condition B)

  $A \subseteq B$



Suppose $A \Rightarrow C$ and $C \Rightarrow A$.

($C =$ "The economy of England is half that of Germany.")

$(x = 1/2y))$

- $A \Longleftrightarrow C$
- A if and only if C. (A iff C.)
- A is equivalent to C
- A is a neccessary and sufficient condition for C
- A implies and is implied by C
- $C = \{x, y : x = 1/2y\}, A \subseteq C$ and $C \subseteq A$, $A = C$.

Example

$E =$ Demark is in the Euro Zone.

$F =$ Demark's interest rate is set by the European Central Bank.

Numbers

Natural numbers: $N = \{1, 2, 3, \ldots\}$

(arise naturally from counting objects).

- Closed under addition and multiplication:

$$x, y \in N \Rightarrow x + y \in N, xy \in N$$

- Not closed under substraction and division:

$$x, y \in N, x \leq y \Rightarrow x - y \leq 0 \notin N$$

$$2, 3 \in N, \frac{2}{3} \notin N$$

Integers: $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

- Closed under addition, substraction, multiplication,

  but not division.

Rational numbers:

$$Q = \{\frac{a}{b} : a \in I, b \in I - \{0\}\}$$

- $I \subset Q$: choose $b = 1$.

- Infinitely many rational numbers between any two integers, e.g., 1 and 2:

$$1 + \frac{1}{c}, c \in N$$

Irrational numbers: numbers that cannot be expressed as ratios of integers. e.g. $\sqrt{2}$ (between 1 and 2, not rational).

Real numbers (R): Union of rational and irrational numbers.

- extending along a line to infinity in both directions with no breaks or gaps: the **real line**.

Intervals: subsets of R,

$a, b \in R, a < b$

- Closed interval: $[a, b] = \{x \in R : a \leq x \leq b\}$
- Half-open intervals:

  $(a, b] = \{x \in R : a < x \leq b\}$

  $[a, b) = \{x \in R : a \leq x < b\}$
- Open interval: $(a, b) = \{x \in R : a < x < b\}$

The **Cartesian product** of two sets X and Y, $X \times Y$, is the set of **ordered pairs** formed by taking in turn each element in X and associating with it each element in Y.

$X \times Y = \{(x, y) : x \in X, y \in Y\}$

$X = \{1, 2, 3\}, Y = \{a, b\},$

$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

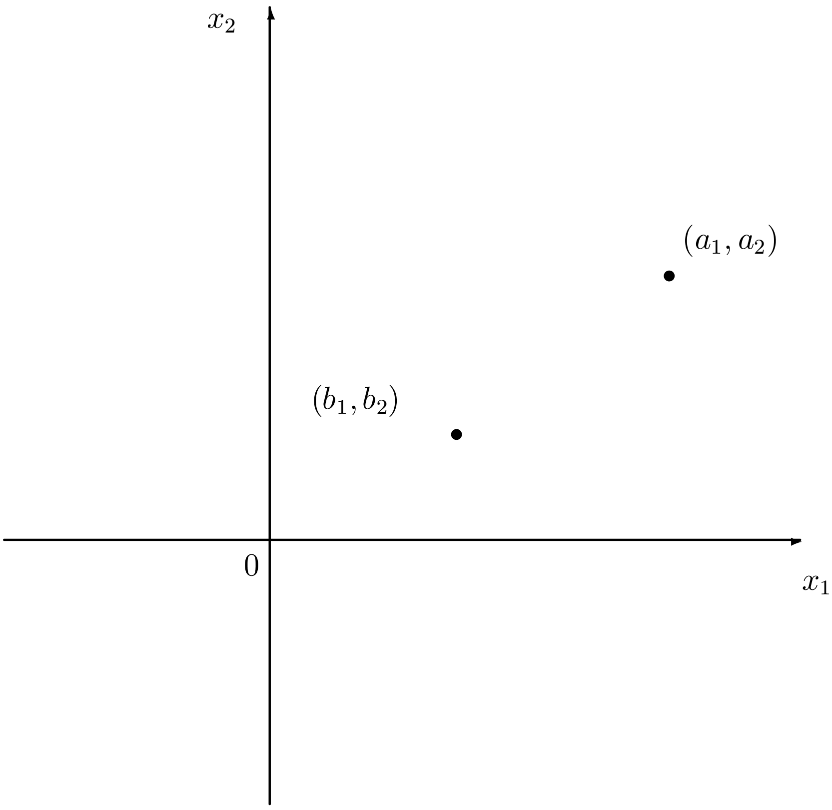
$(1, \sqrt{2}) \in Q \times N?$

$(1, \sqrt{2}) \in N \times Q?$

The Cartesian product of R with itself.

$$R \times R = \{(x_1, x_1) : x_1 \in R, x_2 \in R\} = R^2$$

- All points in R^2:



- $x = (x_1, x_2) = 0$: $x_1 = 0$ and $x_2 = 0$

  $x = (x_1, x_2) \neq 0$: $x_1 \neq 0$ or $x_2 \neq 0$

- Graph of $[2, 3] \times [1, 2]$?

- Distance between $a = (a_1, a_2)$ and $b = (b_1, b_2)$

$$d(a, b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \text{ (Pythagorean Theorem)}$$

Definition 9 *An ϵ-**neighborhood** of a point $a \in R^2$ is the set*

$$N_\epsilon(a) = \{x \in R^2 : d(a, x) < \epsilon\}$$

$N_\epsilon[(2,3)] = \{(x_1, x_2) \in R^2 : \sqrt{(x_1 - 2)^2 + (x_2 - 3)^2} < \epsilon\}$.

Graph?

Definition 10 *Given two points $x = (x_1, x_2), x' = (x_1', x_2') \in R^2$, a **convex combination** of x and x' is*

$$\lambda x + (1 - \lambda)x' = (\lambda x_1 + (1 - \lambda)x_1', \lambda x_2 + (1 - \lambda)x_2')$$

$$= (x_1' + \lambda(x_1 - x_1'), x_2' + \lambda(x_2 - x_2'),$$

for some $\lambda \in [0, 1]$. (point on line segment between x and x'.)

Definition 11 *A set $X \subset R^2$ is **convex** if for any two points $x, x' \in X$, $\lambda x + (1-\lambda)x' \in X$ for all $\lambda \in [0, 1]$.*

(A set is convex if any convex combination of any two points in the set is in the set.)

Functions

Definition 12 *Given two sets X and Y, a* **function (mapping)** *f from X to Y, $f : X \to Y$, is a rule that associates each element of X with one and only one element of Y.*

("element of X you pick *determines* the element of Y you get.")

("Consumption is a function of income.")

- X: **domain**

- For each $x \in X$, $y = f(x) \in Y$: the **image** of x

  (value of f at x).

- $f(X) = \{y \in Y : y = f(x), x \in X\}$: the **range**

  Examples

  1. X: set of countries, $Y \subset R$, f: "the GDP of"

     $f(UK) = 21,000$

  2. $X = R, Y = R$, $y = f(x) = 2x + 3$

  3. $y^2 = 2x + 3$: association of $x \in X$ and $y \in Y$,

     but y is not a function of x

- Different $x \in X$ may have the same image

  e.g., $y = f(x) = x^2$

- If each x has a different image: the function is **one-to-one**

  can be **inverted**: $x = f^{-1}(y)$

  $f^{-1}(y)$: **inverse function**

  (the rule that associates the image of x with x)

e.g.,

$$y = f(x) = 2x + 3 \Rightarrow x = f^{-1}(y) = \frac{y-3}{2}$$

Definition 13 *The* **Composite function** *of two functions* $f : X \to Y$ *and* $g : Y \to Z$ *is*

$$g \circ f : X \to Z$$

or

$$z = g(f(x))$$

- The range of f must be a subset of the domain of g.

  Examples

  1. "Consumption is a function of income."

     "income is a function of age."

     "Consumption is a function of age."

  2. $y = f(x) = 2x + 3, \quad z = g(y) = y^2$

     $z = g(f(x)) = g \circ f(x) = (2x + 3)^2$

Types of $R \to R$ functions

$f : R \to R \subset R \times R$

Graph of f: $\{(x, f(x)) : x \in R, f(x) \in R\} \subset R^2$.

- **Identity function**: $f(x) = x$
- **Constant functions**: $f(x) = a$
- **Linear functions**: $f(x) = ax + b$
- **Quadratic functions**: $f(x) = ax^2 + bx + c$
- **Power functions**: $f(x) = ax^b$
- **Exponential functions**: $f(x) = b^x$, b: **base**

  If $b = e \approx 2.718$ (Napier's constant) $f(x) = e^x = \exp(x)$.

  If $y = b^x$, x: the **logarithm** of y to base b: $x = \log_b y$
- **Logarithmic functions**: $f(x) = \log_b x$

  If $b = e$, $f(x) = \ln x$: natural logarithm
- **Absolute Value**: $f(x) = |x| = x$ if $x \geq 0$

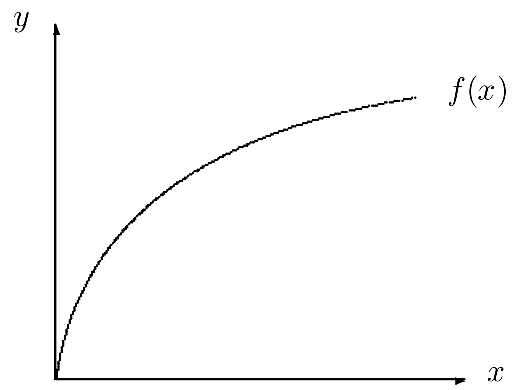
$$= -x \text{ if } x < 0$$

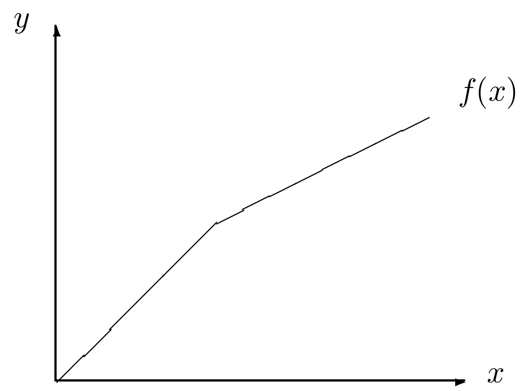
Definition 14 *$X \subseteq R$ or R^2 and convex, $Y \subseteq R$*

The function $f : X \to Y$ is **concave** *if for any $x, x' \in X$, $\lambda \in [0, 1]$*

$$f(\lambda x + (1 - \lambda)x') \geq \lambda f(x) + (1 - \lambda)f(x')$$

It is **strictly concave** *if the inequality holds when $\lambda \in (0, 1)$.*





Definition 15 *The function $f : X \to Y$ is* **convex** *if for any $x, x' \in x$, $\lambda \in [0, 1]$*

$$f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda)f(x')$$

It is **strictly convex** *if the inequality holds when $\lambda \in (0, 1)$.*

Definition 16 $y = f(x) : R \to R$ *is*

increasing *if* $\bar{x} > \hat{x} \Rightarrow f(\bar{x}) \geq f(\hat{x})$.

strictly increasing *if* $\bar{x} > \hat{x} \Rightarrow f(\bar{x}) > f(\hat{x})$.

decreasing *if* $\bar{x} > \hat{x} \Rightarrow f(\bar{x}) \leq f(\hat{x})$.

strictly decreasing *if* $\bar{x} > \hat{x} \Rightarrow f(\bar{x}) < f(\hat{x})$.

monotonic *if it is strictly increasing or strictly decreasing.*

GDP is growing: GDP$= f(\text{time})$ is increasing.

$y = f(x) = 2x + 3, \quad y = f(x) = x^2 - 2x$

Definition 17 $y = f(x_1, x_2) : R^2 \to R$ *is* **increasing**

in x_1 *if* $\bar{x}_1 > \hat{x}_1 \Rightarrow f(\bar{x}_1, \bar{x}_1) \geq f(\hat{x}_1, \bar{x}_2)$

in x_2 *if* $\bar{x}_2 > \hat{x}_2 \Rightarrow f(\bar{x}_1, \bar{x}_1) \geq f(\bar{x}_1, \hat{x}_2)$

$$y = f(x_1, x_2) = 2x_1 + \frac{1}{x_2}$$

Implicit Functions

$y = f(x) = 3x^2$: an *explicit function.*

$y - 3x^2 = 0$: its equivalent *implicit function*

General form : $F(x, y) = 0$

e.g., utility function: $U(x, y)$

$\Rightarrow U(x, y) = c$ (constant): an indifference curve

Not all equations of the form $F(x, y) = 0$ are implicit functions.

$F(x, y) = x^2 + y^2 = 9$

Problem Set I

1. Define the relationships ($\subseteq$, $\subset$, $=$), if any, among the following sets:

$$A = \{x : 0 \leq x \leq 1\}$$
$$B = \{x : 0 < x < 1\}$$
$$C = \{x : 0 \leq x < 1\}$$
$$D = \{x : 0 \leq x^2 \leq 1\}$$
$$E = \{x : 0 \leq x < 1/2 \text{ and } 1/2 \leq x \leq 1\}$$

   Is $x \in C$ a necessary condition for $x \in D$? Is it sufficient?

   Is $x \in E$ a sufficient condition for $x \in D$?

2. Let

$$X = \{x \in N : x \leq 20 \text{ and } x/2 \in N\}$$
$$Y = \{x \in N : 10 \leq x \leq 24 \text{ and } x/2 \in N\}$$

   What are $X \cap Y$, $X \cup Y$, $X - Y$, $Y - X$, $(X \cup Y) - (X \cap Y)$, and $(X \cap Y) - (X \cup Y)$?

3. The overall effect of a change in the price of a good on the demand for it is the sum of two separate effects: the *substitution effect* (demand for the good will increase when price falls because it becomes cheaper relative to is substitues); and the *income effect* (a fall in the price of a good increases the consumer's real income, leading to an increase in demand if the good is a *normal good* and a fall in demand if the good is an *inferior good*).

   In a Venn diagram, illustrate the relationship among the following four sets

   (i) the set of goods for which demand increases when prices fall.

   (ii) the set of goods for which demand falls when prices fall.

(iii) the set of normal goods.

(iv) the set of inferior goods.

4. Describe $X \times Y$ algebraically and graphically for

   (i) $X = [-1, 1], Y = (-1, 1)$.

   (ii) $X = \{x \in R : x > 3\}$, $Y = \{x \in R : x < -1\}$.

   (iii) $X = \{x : x \leq 4, x/2 \in N\}$, $Y = \{x : x < 9, x/3 \in N\}$.

5. A consumer's budget set is

$$B = \{(x, y) \in R^2 : p_1 x + p_2 y \leq m, x \geq 0, y \geq 0\}$$

   where $p_1, p_2 > 0$ are prices and $m > 0$ is income. Is the set convex? What about the set

$$\{(x, y) \in R^2 : 2x + y \leq 4, x \geq 0, y \geq 0\} \cup \{(x, y) \in R^2 : x + 2y \leq 4, x \geq 0, y \geq 0\}$$

6. For $\epsilon = 0.1$, describe the ϵ-neighborhood $N_\epsilon[(-1, 1)]$.

7. What is the range of the function $y = f(x) = x^2 + 4$ if the domain is

   (i) $[1, 4]$, (ii) R.

8. Suppose quantity demanded q as a function of price p is given by

$$q = 8 - 2p$$

   Is total revenue as a function of price concave or convex? Can the demand function and the revenue function be inverted?

9. Let X be the set of countries and Y the set of positive real numbers. Define the funtion $f : X \to Y$ to be "y is the GDP of x". Is f one-to-one?

10. Which of the following implicitly defines y as a function of x:

$$F(x, y) = e^x + 1/y = 0$$

$$F(x, y) = 2|x + 2y| = 0$$

$$F(x, y) = |x| - 2|y| = 0$$

11. Is $g \circ f$ (a) concave or convex (b) increasing or decreasing if

$$\text{(i) } f : R^2 \to R : f(x_1, x_2) = 2x_1 - 3x_2$$

$$g : R \to R : g(y) = \frac{y}{\sqrt{2}} \ \ ?$$

$$\text{(ii) } f : R \to R : f(x) = \frac{x^4}{4}$$

$$g : R \to R : g(y) = 4 - 2\sqrt{y} \ \ ?$$

$$\text{(iii) } f : R \to R : f(x) = \frac{x^4}{9} - 3$$

$$g : R \to R : g(y) = 3\sqrt{y + 3} \ \ ?$$

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$f : R \longrightarrow R : y = f(x)$

Continuity

(The idea: will a small change in x cause a drastic change in y?)

Definition 18 *Suppose f is well-defined to the left of the point $x = a$. The* **left-hand limit** *of a function $f(x)$ at the point $x = a$ exists and is equal to L^L*

$$\lim_{x \to a^-} f(x) = L^L$$

if for any $\epsilon > 0$, however small, there exists some $\delta > 0$, such that $|f(x) - L^L| < \epsilon$ for all $x \in (a - \delta, a)$.

- $f(x) = \ln x$: not defined to the left of $x = 0$.
- Left-hand limit exists at $x = 2$ for

$$f(x) = \begin{cases} x & x < 2 \\ 2x & x \geq 2 \end{cases} \qquad ?$$

Definition 19 *Suppose f is well-defined to the right of the point $x = a$. The* **right-hand limit** *of a function $f(x)$ at the point $x = a$ exists and is equal to L^R*

$$\lim_{x \to a^+} f(x) = L^R$$

if for any $\epsilon > 0$, however small, there exists some $\delta > 0$, such that $|f(x) - L^R| < \epsilon$ for all $x \in (a, a + \delta)$.

Definition 20 *Suppose $f(x)$ is well-defined on an open interval containing the point $x = a$. The* **limit** *of $f(x)$ at the point $x = a$ exists and is equal to L*

$$\lim_{x \to a} f(x) = L$$

if for any $\epsilon > 0$, however small, there exists some $\delta > 0$, such that $|f(x) - L| < \epsilon$ for all $x \in (a - \delta, a + \delta)$ except possibly a.

- $$\lim_{x \to a} f(x) \text{ exists} \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Definition 21 *Suppose $f(x)$ is well-defined on an open interval containing the point $x = a$. $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists, and $\lim_{x \to a} f(x) = f(a)$*

Definition 22 *Suppose $f(x)$ is well-defined on an open interval containing the point $x = a$. $f(x)$ is continuous at $x = a$ if for any $\epsilon > 0$ however small, there exists some $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.*

Examples

$$f(x) = 2x$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \begin{cases} +1, & x \leq 0 \\ -1, & x > 0 \end{cases}$$

$$f(x) = |x|$$

Bertrand competition Two firms, firms 1 and 2, produce an identical product and compete in prices. Each firm sets a price and then meet whatever demand exists for its product at that price. If one firm charges a lower price, then all consumers will purchase from that firm. If the two firms charge the same price, consumers' purchases will be split evenly btween the two firms. The demand function is given by $y = 20 - 2p$ and the cost function for both firms is $C(y) = 4y$. Suppose firm 2 charges a price $p_2 = 7$. Derive firm 1's profit as a function of its price.

Derivatives and Differentials

$y = f(x)$

How y changes in response to a (small) change in x?

Rate of change? e.g., "marginal cost", "marginal tax rate".

Definition 23 *Given two points $P = (x_1, f(x_1))$ and $Q = (x_2, f(x_2))$ on the graph of a function $f(x)$ where $x_2 = x_1 + \Delta x$, the* **secant line** *is the straight line joining the two points. The slope of the secant line is*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

Definition 24 *$f(x)$ is well-defined on an open interval containing $x = x_1$. The* **derivative** *of $f(x)$ at the point x_1 is*

$$\frac{dy}{dx} = f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

f is **differentiable** *at $x = x_1$ and $f'(x_1)$ the slope of the graph of $f(x)$ at the point $(x_1, f(x_1))$ if the derivative exists at $x = x_1$, i.e.,*

$$\lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

("rate of change" when the change is "very small".)

Examples

$f(x) = x^2$

$f(x) = |x|$

Income tax function

$$T(x) = \begin{cases} 0 & 0 \le x < 5000 \\ 0.15(x - 5000) & 5000 \le x < 15,000 \\ 0.15 \cdot 10,000 + 0.25(x - 15,000) & x \ge 15,000 \end{cases}$$

For f to be differentiable at x_1, $\lim_{\Delta x \to 0} f(x_1 + \Delta x)$ must exist and $\lim_{\Delta x \to 0} f(x_1 + \Delta x) = f(x)$

Theroem 1 *If $f(x)$ is differentiable, then $f(x)$ must be continuous at $x = x_1$.*

Rules of Differentiation

- $f(x) = c$, a constant, $\Rightarrow f'(x) = 0$
- $f(x) = ax + b \Rightarrow f'(x) = a$
- $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
- $f(x) = cg(x) \Rightarrow f'(x) = cg'(x)$
- $f(x) = h(x) + g(x) \Rightarrow f'(x) = h'(x) + g'(x)$

  $f(x) = h(x) - g(x) \Rightarrow f'(x) = h'(x) - g'(x)$
- $f(x) = g(x)h(x) \Rightarrow f'(x) = g'(x)h(x) + g(x)h'(x)$
- $y = f(u), u = g(x), y = f(g(x)) = h(x)$

  $$\Rightarrow h'(x) = f'(u)g'(x) \text{ or } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
- $$f(x) = \frac{g(x)}{h(x)} \Rightarrow f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$
- $f(x) = e^x \Rightarrow f'(x) = e^x$
- $$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

  Examples

$$f(x) = 3x^5 + 6x^3 + 2$$

$$f(x) = \sqrt{\frac{x^2}{2(x+1)}}$$

$$f(x) = \ln(x^3 + \frac{1}{x})$$

$$f(x) = e^{\sqrt{x}}$$

- The derivative of a function $y = f(x)$,

$$dy/dx = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is also a function of x. Can take its derivative

$$\frac{d(dy/dx)}{dx} = \frac{df'(x)}{dx} = \frac{d^2 y}{dx^2} = f''(x)$$

- $f'(x)$: **first derivative**, $f''(x)$: **second derivative**.

Definition 25 *If the first two derivatives of a function exist, then the function is* **twice differentiable**.

Theroem 2 *A twice differentiable function is* **convex** *iff, at all points on its domain,* $f''(x) \geq 0$

(slope of the graph is increasing.)

Theroem 3 *A twice differentiable function is* **strictly convex** *iff* $f''(x) > 0$ *except possibly at a single point.*

e.g. $f(x) = x^4$.

Theroem 4 *A twice differentiable function is* **concave** *iff, at all points on its domain,* $f''(x) \leq 0$

Theroem 5 *A twice differentiable function is* **strictly concave** *iff* $f''(x) < 0$ *except possibly at a single point.*

Examples: Graphs and convexity/concavity of the following.

$f(x) = 2x^2 - 4x + 3$

$f(x) = x^3$

$f(x) = -x^4$

$f(x) = \ln x$

$f(x) = e^x$

$f(x) = \frac{1}{x}$

Marginal Revenue of a monopolist The market demand function is given by $q(p)$, which is decreasing in the market price p. Show that the marginal revenue of a monopolist in this market is lower than the market price at every level of output. If $q(p) = 20 - 1/2p$, graph the total revenue function and derive the monopolist's marginal revenue function.

Definition 26 *If $f'(x_1)$ is the derivative of $y = f(x)$ at x_1, then the* **total differential** *at x_1 is*

$$dy = df(x_1, dx) = f'(x_1)dx$$

(a function of both x and dx).

- $f'(x_1)$: rate of change.
- $dy = f'(x_1)dx$: "magnitude" of change

  $\Delta y = f'(x_1)\Delta x$: good approximation if Δx small.

  (dy, dx short-hand for $\Delta y, \Delta x$ very small.)

Unconstrained Optimization

- Given $y = f(x)$, we **optimize** it by finding a value of x at which it takes on a maximum or minimum value (**extreme values**)

  Optimizing: rational economic behavior

  (rational agents should not consistently choose suboptimal options.)

- **Unconstrained optimization**: can choose any $x \in R$.

- **Constrained optimization**: can choose $x \in$ subset of R:

Definition 27 *x^* is a* **global maximum** *if $f(x^*) \geq f(x)$, for all x. $\hat{x}$ is a* **local maximum** *if there exists $\epsilon > 0$, however small, such that $f(\hat{x}) \geq f(x)$, for all $x \in [\hat{x} - \epsilon, \hat{x} + \epsilon]$.*

Definition 28 *x^* is a* **global minimum** *if $f(x^*) \leq f(x)$, for all x. $\hat{x}$ is a* **local minimum** *if there exists $\epsilon > 0$, however small, such that $f(\hat{x}) \leq f(x)$, for all $x \in [\hat{x} - \epsilon, \hat{x} + \epsilon]$.*

- x^* a global max (min) $\Rightarrow$ x^* a local max (min).

Theroem 6 *If the differentiable function f takes a local extreme value (maximum or minimum) at a point x^*, then $f'(x^*) = 0$.*

- $f'(x^*) = 0$: **first-order condition**

- **Necessary**: $dy = f'(x^*)dx$.

  If $f'(x^*) > 0$, choose $dx > 0 \Rightarrow dy > 0$: y increases.

  $f(x^*)$ not maximum.

  If $f'(x^*) < 0$, choose $dx < 0 \Rightarrow dy > 0$: y increases.

  $f(x^*)$ not maximum.

- **Not Sufficient**: e.g., $f(x) = x^3$, $f'(x) = 3x^2$, $f'(0) = 0$

  $f(0)$ not max or min: $f(0) < f(x)$ for $x > 0$, $f(0) > f(x)$ for $x < 0$.

  $f(x)$ is "**stationary**" at $x = 0$.

Definition 29 *$\bar{x}$ is a* **stationary point** *of a differentiable $f(x)$ if $f'(\bar{x}) = 0$.*

$$\{\text{extreme values}\} \subset \{\text{stationary points}\}$$

- Need **second-order conditions** to distinguish different stationary points.

Theroem 7

(i) If $f'(x^*) = 0$ *and* $f''(x^*) < 0$*, then* f *has a local maximum at* x^**.*

(ii) If $f'(x^*) = 0$ *and* $f''(x^*) > 0$*, then* f *has a local minimum at* x^**.*

- $f'(x^*) = 0$ and $f''(x^*) < 0$ sufficient but not necessary

  e.g., $f(x) = -x^4 \leq 0$ for all x, max at $x = 0$

  $f'(x) = -4x^3, f''(x) = -12x^2 \Rightarrow f'(0) = 0, f''(0) = 0.$

- $f'(x^*) = 0$ and $f''(x^*) \leq 0$ not sufficient,

  e.g., $f(x) = x^3$ at $x = 0$

Examples.

$$f(x) = 2x^3 - \frac{1}{2}x^2 + 2$$

$$f(x) = \frac{1}{2}x^4 - 3x^3 + 2x^2$$

$$f(x) = x \ln x - x, \text{ for } x > 0$$

Theroem 8

(i) Suppose $f(x)$ is concave. Then x^* is a global maximum of $f(x)$ **iff** *x^* is a stationary point, i.e., $f'(x^*) = 0$.*

(ii) Suppose $f(x)$ is convex. Then x^* is a global minimum of $f(x)$ **iff** *x^* is a stationary point, i.e., $f'(x^*) = 0$.*

Examples

$$f(x) = e^{3x^2-6x}$$

$$f(x) = x + e^{-x}$$

$$f(x) = \sqrt{x} - 2x \quad (x \geq 0)$$

$$f(x) = \ln(x+4) \quad (x > -4)$$

Profit-maximizing monopolist. For a monopolist who faces a market demand function $q = 25 - 1/2p$ and has a cost function $c(q) = 20 + 2q + 0.5q^2$, what are the profit-maximizing price and output?

Constrained Optimization

Upper bounds

$$\max_x f(x) \;\; s.t. \;\; x \leq a$$

(firm chooses production level to max profit subject to capacity.)

3 possibilities:

- no max.
- the constraint is binding and solution $x^* = a$.
- the constaint is slack and the solution is a stationary point $f'(x^*) = 0$.

Theroem 9 *If x^* maximizes $f(x)$ subject to $x \leq a$, then*

(i) $f'(x^*) \geq 0$

(ii) $x^* \leq a$

(iii) either $f'(x^*) = 0$ or $x^* = a$.

If $f(x)$ is concave, then (i)-(iii) are necessary and sufficient.

Examples

$$\max_x f(x) = -x^2 + 2x \;\; s.t. \;\; x \leq 2$$

$$\max_x f(x) = -x^2 + 6x \;\; s.t. \;\; x \leq 2$$

Lower bounds

$$\max_x f(x) \;\; s.t. \;\; x \geq b$$

(e.g. minimum production level or non-negativity constraint.)

3 possibilities:

- no max.
- the constraint is binding and solution $x^* = b$.
- the constaint is slack and the solution is a stationary point $f'(x^*) = 0$.

Theroem 10 *If x^* maximizes $f(x)$ subject to $x \geq b$, then*

(i) $f'(x^*) \leq 0$

(ii) $x^* \geq b$

(iii) either $f'(x^*) = 0$ or $x^* = b$.

If $f(x)$ is concave, then (i)-(iii) are necessary and sufficient.

Examples

$$\max_x f(x) = -x^2 + 2x \;\; s.t. \;\; x \geq 2$$

$$\max_x f(x) = -x^2 + 6x \;\; s.t. \;\; x \geq 2$$

Lagrangian Method

$$\max_x f(x) \;\; s.t. \;\; g(x) \geq 0$$

- $L(x) = f(x) + \lambda g(x)$
- λ: **Lagrange multiplier**

Theroem 11 *If x^* maximizes $f(x)$ subject to $g(x) \geq 0$, then*

(i) $L'(x^*) = 0$ (stationarity)

(ii) $g(x^*) \geq 0$ (constraint)

(iii) $\lambda \geq 0$ (non-negativity)

(iv) either $\lambda = 0$ or $g(x^*) = 0$ (complementary slackness).

(either the constraint is not binding ($\lambda = 0$) or it is $g(x^*) = 0$) If both $f(x)$ and $g(x)$ is concave, then (i)-(iv) are necessary and sufficient.

Examples.

$$\max_x \;\; [-x^2 + 2x] \;\; s.t. \;\; x^2 \leq 4$$

$$\max_x \;\; [6 \ln x - 2x] \;\; s.t. \;\; x^2 - 3x + 2 \leq 0$$

Constrained minimization

$$\max_x f(x) \;\; s.t. \;\; g(x) \geq 0 \Longleftrightarrow \min_x [-f(x)] \;\; s.t. \;\; g(x) \geq 0$$

Example

$$\min_x [e^{2x-4} - 2x] \;\; s.t. \;\; x + 3 \leq 0$$

Price-regulated monopolist A monopolist faces a demand function $q = 20 - 1/2p$ and has a cost function $c(q) = 8q$. The monopolist however is not free to set the price as the industry is regulated and the regulator stipulates that the price cannot be higher than £20 or lower than £15. What are the monopolist's profit-maximizing price and output?

Problem Set II

1. Continuity is a necesary but not sufficient condition for differentiability. True or False?

2. For each of the following functions, (a) determine whether it is continuous, (b) determine whether it is differentiable, (c) sketch its graph.

$$\text{(i) } f(x) = \frac{1}{x-2}$$

$$\text{(ii) } f(x) = |3x - 9|$$

$$\text{(iii) } f(x) = \begin{cases} 2, & x \leq 3 \\ 1, & x > 3 \end{cases}$$

$$\text{(iv) } f(x) = \begin{cases} x, & x \leq 3 \\ 3 - x, & x > 3 \end{cases}$$

$$\text{(v) } f(x) = \begin{cases} 3x, & x < 2 \\ 8 - x, & x \geq 2 \end{cases}$$

3. For each of the following functions, (a) find the first and second derivatives where they exist, (b) determine whether it is convex or concave, (c) sketch its graph.

$$\text{(i) } f(x) = 2x^3 + 3x^2 + 2$$

$$\text{(ii) } f(x) = \frac{1}{x+1} \text{ (for } x \neq -1)$$

$$\text{(iii) } f(x) = \ln\left(\frac{2}{2x+1}\right) \text{ (for } x > -\frac{1}{2})$$

$$\text{(iv) } f(x) = e^{\sqrt{x}} \text{ (for } x \geq 0)$$

4. Consider the following income tax structure:

   The first £5,000 of income is not subject to any tax.

   The next £15,000 is subject to a tax rate of 25%.

   The next £30,000 is subject to a tax rate of 40%.

   Any additional income is subject to a tax rate of 50%.

   (i) Find and graph the tax function $T(y)$, defined on $y \geq 0$.

   (ii) Determine the points of nondifferentiability.

   (iii) Graph the marginal tax function.

5. For each of the functions below, identify (a) stationary points, (b) any local maxima or minima, (c) any global maxima or minima.

$$\text{(i) } f(x) = -4x^3$$

$$\text{(ii) } f(x) = 4x^3 - \frac{1}{3}x^2 + 9$$

$$\text{(iii) } f(x) = x^4 - 8x^2$$

$$\text{(iv) } f(x) = x^2 + x^{-2} \text{ where } x \neq 0$$

$$\text{(v) } f(x) = \ln(1 + x^2)$$

$$\text{(vi) } f(x) = e^{-x^2}$$

6. Consider a firm in a competitive industry which takes the market price $p = \pounds 36$ as given and has a cost function $c(q) = 3q^2 + 1$. Find the profit maximizing output.

7. Solve the following problems.

$$\text{(i) } \max_x f(x) = -x^2 + 8x \;\; s.t. \;\; x \leq 2$$

(ii) $\max_x f(x) = -x^2 + 8x \;\; s.t. \;\; x \leq 6$

(iii) $\max_x f(x) = -\frac{1}{x} - 4x \;\; s.t. \;\; x \geq 2$

(iv) $\max_x f(x) = -\frac{1}{x} - \frac{x}{9} \;\; s.t. \;\; x \geq 2$

(v) $\min_x \; [x^2 - 3x + 4] \;\; s.t. \;\; x^2 \leq 4$

(vi) $\max_x \; [15 \ln x - 3x] \;\; s.t. \;\; x^2 - 8x + 7 \leq 0$

(vii) $\min_x [e^{3x-9} - 3x] \;\; s.t. \;\; x - 2 \leq 0$

8. A monopolist faces a demand function $q = 12 - 1/3p$ and has a cost function $c(q) = 6q$. The monopolist however is not free to set the price as the industry is regulated and the regulator stipulates that the price cannot be higher than £18 or lower than £12. What are the monopolist's profit-maximizing price and output?

LINEAR ALGEBRA

Linear function (equation): $y = ax + b$

Example: $q^D = -2p + 10$

q^D: quantity demanded for milk, p: market price of milk

Graph: all values of (q^D, p) satisfying the equation: straight line

(simple, could be a good approximation of reality)

$q^S = 4p - 8$, q^S: quantity supplied of milk.

Equilibrium $q^D = q^S = q$

$$q = -2p + 10$$
$$q = 4p - 8$$

Solution: Values of (q, p) satisfying both equations,

Graphically, the intersection.

q, p: unknowns

$$q + 2p = 10$$
$$q - 4p = 8$$

A system of 2 simultaneous equations in 2 unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Graph of $a_{11}x_1 + a_{12}x_2 = b_1$
- There may be one, none, or infinitely many solutions (intersections).

$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$
$$\Rightarrow \text{No solution.}$$

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$
$$\Rightarrow \text{Infinitely many solutions.}$$

- General principles on finding solutions?
- Systems of n simultaneous equations in n unknowns?

Matrix Algebra

Definition 30 *A* **matrix** *is a rectangular array of numbers enclosed in parentheses, conventionally denoted by a capital letter. The number of rows (say m) and the number of columns (say n) determine the* **order** *of the matrix ($m \times n$).*

$$P = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 3 \\ 4 & 3 \\ 1 & 5 \end{bmatrix}$$

A general 2×2 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Definition 31 *An array that consists of only one row or column is known as a* **vector***.*

$$\begin{bmatrix} 2 & 3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Definition 32 *A matrix that has the same number of rows and columns is a* **square matrix***.*

Definition 33 *A square matrix that has only nonzero entries on the main diagonal and zero everywhere else is a* **diagonal matrix***.*

A diagonal matrix whose diagonal elements are one is the **identity matrix***, denoted by* I_n *where* n *is the order of the matrix.*

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 34 *A square matrix with all its entries being zero if the* **null matrix***.*

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Definition 35 *The* **sum** *of two matrices is a matrix, the elements of which are the sums of the corresponding elements of the matrices. Two matrices are* **conformable** *for addition or substraction if they are of the same order.*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{33} \end{bmatrix}$$

Car production A car manufacturer who produces 3 different models in 3 different plants A, B, and C in the first half and second half of the year as follows

First Half

	Model 1	Model 2	Model 3
Plant A	27	44	51
Plant B	35	39	62
Plant C	33	50	47

Second Half

	Model 1	Model 2	Model 3
Plant A	25	42	48
Plant B	33	40	66
Plant C	35	48	50

Summarize the total production for the whole year.

Definition 36 Scalar multiplication *is carried out by multiplying each element of the matrix by the scalar.*

$$P = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \Rightarrow 3P = \begin{bmatrix} 6 & 9 & 12 \\ 9 & 3 & 15 \end{bmatrix}$$

$$3P = P + P + P$$

Definition 37

(i) Two matrices A and B of dimensions $m \times n$ and $n \times l$ respectively are **conformable** *to form the product matrix $C = AB$, since the number of columns of A is equal to the number of rows of B. (ii) The product matrix AB is of dimension $m \times l$ and its ijth element, c_{ij} is obtained by multiplying the elements of the ith row of A by the corresponding elements of the jth column of B and adding the resulting products.*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

The product of any matrix A and a conformable identity matrix I is equal to the original matrix A.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Example Determine the revenue of a car park on a given Monday, Tuesday, and Wednesday based on the following data.

	Number of cars	Number of buses
Monday	30	5
Tuesday	25	5
Wednesday	35	15

The parking charge is £4 per car and £8 per bus.

Definition 38 *The* **transpose matrix**, A^T *is the original matrix* A *with its rows and columns interchanged.*

$$P = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}, \quad P^T = \begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 4 & 5 \end{bmatrix}$$

Definition 39 *A matrix* A *that is equal to its transpose* A^T *is a* **symmetric matrix**.

$$\begin{bmatrix} 1 & 5 & 6 \\ 5 & 2 & 0 \\ 6 & 0 & 4 \end{bmatrix}$$

Definition 40 *The* **inverse matrix** A^{-1} *of a square matrix* A *of order* n *is the matrix such that*

$$AA^{-1} = A^{-1}A = I_n$$

Definition 41

Any matrix A for which A^{-1} does not exist is **singular***.*

A matrix A for which A^{-1} exists is **non-singular***.*

Let A be $n \times n$, x and b be $n \times 1$

$$Ax = b$$

is a **(linear) matrix equation** and

defines a system of n simultaneous equations in n unknown, x.

If A is nonsingular,

$$A^{-1}Ax = A^{-1}b \;\; \Rightarrow \;\; I_n x = A^{-1}b \;\; \Rightarrow \;\; x = A^{-1}b$$

Definition 42 *The determinant of a 2×2 matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by $(a_{11}a_{22} - a_{21}a_{12})$ and is denoted by $|A|$ or $detA$.

The inverse of a 2×2 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Theroem 12 *A is singular* **iff** $|A| = 0$.

System of 2 simultaneous equations in 2 unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow Ax = b \Rightarrow x = A^{-1}b$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Example

$$\begin{cases} 2x + 3y = 5 \\ 4x - 7y = 8 \end{cases}$$

$$\begin{cases} 2x + 3y = 1 \\ 10x + 15y = 12 \end{cases}$$

Linear Production Technology A firm produces two outputs, y_1 and y_2, with two inputs, x_1 and x_2. a_{ij} denote the amount of input i required to produce 1 unit of output j.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$

Find the quantities of x_1 and x_2 needed to produce 20 units of y_1 and 15 units of y_2. Suppose we are given the quantities of the inputs: $x_1 = 10, x_2 = 20$. Find the quantities of y_1 and y_2 that can be produced.

Quadratic Forms

Definition 43 *Given a 2×2 matrix A and a 2×1 vector x,*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

the **quadratic form** *is*

$$q(x) = x^T A x = x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= a_{11}x_1^2 + (a_{12} + a_{21})x_1 x_2 + a_{22}x_2^2$$

Definition 44

(i) $q(x)$ and A are **positive definite** *if $q(x) = x^T A x > 0$, for all $x \neq 0$.*

(ii) $q(x)$ and A are **positive semi-definite** *if $q(x) = x^T A x \geq 0$, for all $x \neq 0$.*

(iii) $q(x)$ and A is **negative definite** *if $q(x) = x^T A x < 0$, for all $x \neq 0$.*

(iv) $q(x)$ and A **negative semi-definite** *if $q(x) = x^T A x \leq 0$, for all $x \neq 0$.*

Theroem 13 *Suppose A is symmetric.*

A is **positive definite** *iff $a_{11} > 0$, $a_{22} > 0$ and $|A| > 0$.*

A is **negative definite** *iff $a_{11} < 0$, $a_{22} < 0$ and $|A| > 0$.*

A is **positive semi-definite** *iff $a_{11} \geq 0$, $a_{22} \geq 0$, and $|A| \geq 0$.*

A is **negative semi-definite** *iff $a_{11} \leq 0$, $a_{22} \leq 0$, and $|A| \geq 0$.*

A symmetric $\Rightarrow a_{12} = a_{21}$

$$q(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

$$= a_{11}(x_1^2 + \frac{2a_{12}}{a_{11}}x_1x_2 + \frac{a_{12}^2}{a_{11}^2}x_2^2) - \frac{a_{12}^2}{a_{11}}x_2^2 + a_{22}x_2^2$$

$$= a_{11}(x_1 + \frac{a_{12}}{a_{11}}x_2)^2 + \frac{a_{11}a_{22} - a_{12}^2}{a_{11}}x_2^2$$

Examples

The quadratic forms and "definiteness" of

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Problem Set III

1. Find the value of x, y and z.

$$\begin{bmatrix} 3 & 2 \\ x+y & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2y \\ 2 & y-z \end{bmatrix}$$

$$3\begin{bmatrix} 3 & y \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 4 & 2 \\ x & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & y \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 0 & y & 1 \\ 0 & z & 10 \end{bmatrix}\begin{bmatrix} x \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

2. Solve the matrix equation $Ax = b$ for the following pairs of matrix A and column vector b.

(i) $A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. A firm produces two outputs, y_1 and y_2, with two inputs, x_1 and x_2. a_{ij} denote the amount of input i required to produce 1 unit of output j.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

   Find the quantities of x_1 and x_2 needed to produce 20 units of y_1 and 15 units of y_2. Suppose we are given the quantities of the inputs: $x_1 = 20, x_2 = 12$. Find the quantities of y_1 and y_2 that can be produced.

4. Fine the quadratic forms and "definiteness" of the following.

$$\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

MULTIVARIATE CALCULUS AND OPTIMIZATION

$f : R^2 \to R : y = f(x_1, x_2)$

Partial Derivatives

Definition 45 *The* **partial derivatives** *of* $y = f(x_1, x_2)$ *with respect to* x_1 *and* x_2 *are*

$$f_1(x_1, x_2) = \frac{\partial y}{\partial x_1} = \frac{\partial f(x_1, x_2)}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1}$$

$$f_2(x_1, x_2) = \frac{\partial y}{\partial x_2} = \frac{\partial f(x_1, x_2)}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2) - f(x_1, x_2)}{\Delta x_2}$$

Examples

$U(x_1, x_2)$: utility function, $U_1(x_1, x_2)$: marginal utility.

$F(K, L)$: production function, $F_L(K, L)$: marginal product of labor.

- $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ are functions of x_1, x_2.

- Taking partial derivatives of the partial derivatives:

$$f_{11}(x_1, x_2) = \frac{\partial f_1(x_1, x_2)}{\partial x_1}, \quad f_{12}(x_1, x_2) = \frac{\partial f_1(x_1, x_2)}{\partial x_2}$$

$$f_{21}(x_1, x_2) = \frac{\partial f_2(x_1, x_2)}{\partial x_1}, \quad f_{22}(x_1, x_2) = \frac{\partial f_2(x_1, x_2)}{\partial x_2}$$

- $f_{11}(x_1, x_2), f_{22}(x_1, x_2)$: **second partial derivatives**

  $f_{12}(x_1, x_2), f_{21}(x_1, x_2)$: **cross partial derivatives**

Hessian Matrix

$$H(x_1, x_2) = \begin{bmatrix} f_{11}(x_1, x_2) & f_{12}(x_1, x_2) \\ f_{21}(x_1, x_2) & f_{22}(x_1, x_2) \end{bmatrix}$$

Theroem 14 *(***Young's Theorem***) If $f(x_1, x_2)$ has continuous first and second partial derivatives, then $f_{12}(x_1, x_2) = f_{21}(x_1, x_2)$*

The Hessian matrix is symmetric.

Examples. Hessian matrices of

$$f(x_1, x_2) = 5x_1^2 x_2^4$$

$$f(x_1, x_2) = \ln x_1 x_2$$

$$f(x_1, x_2) = e^{5x_1 + x_2^4}$$

Definition 46

The **first-order total differential** *for* $y = f(x_1, x_2)$ *is*

$$dy = f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2$$

Allowing both x_1, x_2 to change (i.e., change in all "directions")

Implicit function: $F(x, y) = 0$

$$F_x dx + F_y dy = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Utility function $U(x, y)$. Slope of indifference curve (MRS)?

Indifference curve: $U(x, y) = c$

$$U_x dx + U_y dy = 0$$

$$\frac{dy}{dx} = -\frac{U_x}{U_y}$$

Indifference curves downward-sloping if marginal utilities positive.

Examples: Total differentials and slopes of indifference curves of

$$u = U(x_1, x_2) = 5x_1^{2/3}x_2^{1/3}$$

$$u = U(x_1, x_2) = \ln(2x_1 + 3x_2)^2$$

Second-order total differential

$$dy = f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2$$

(function of x_1, x_2, dx_1, dx_2)

$$d[dy] = \frac{\partial[dy]}{\partial x_1}dx_1 + \frac{\partial[dy]}{\partial x_2}dx_2$$

$$= \frac{\partial[f_1dx_1 + f_2dx_2]}{\partial x_1}dx_1 + \frac{\partial[f_1dx_1 + f_2dx_2]}{\partial x_2}dx_2$$

$$= [f_{11}dx_1 + f_{21}dx_2]dx_1 + [f_{12}dx_1 + f_{22}dx_2]dx_2$$

$$= f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

Theroem 15 *$y = f(x_1, x_2)$ is twice differentiable.*

It is strictly convex **if** *for all $(x_1, x_2) \in R^2$ and $(dx_1, dx_2) \neq 0$*

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2 > 0$$

It is strictly concave **if** *for all $(x_1, x_2) \in R^2$ and $(dx_1, dx_2) \neq 0$*

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2 < 0$$

Theroem 16 *$y = f(x_1, x_2)$ is twice differentiable.*

It is convex **iff** *for all $(x_1, x_2) \in R^2$,*

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2 \geq 0$$

It is concave **iff** *for all $(x_1, x_2) \in R^2$,*

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2 \leq 0$$

Let
$$dx = \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \Rightarrow dx^T = [dx_1, dx_2]$$

$$d^2y = dx^T H(x_1, x_2) dx$$

Theroem 17

$y = f(x_1, x_2)$ is twice differentiable with Hessian $H(x_1, x_2)$.

(i) $f(x_1, x_2)$ is strictly convex **if** *$H(x_1, x_2)$ is positive definite for all $(x_1, x_2) \in R^2$.*

(ii) $f(x_1, x_2)$ is strictly concave **if** *$H(x_1, x_2)$ is negative definite for all $(x_1, x_2) \in R^2$.*

(iii) $f(x_1, x_2)$ is convex **iff** *$H(x_1, x_2)$ is positive semi-definite for all $(x_1, x_2) \in R^2$.*

(iv) $f(x_1, x_2)$ is concave **iff** *$H(x_1, x_2)$ is negative semi-definite for all $(x_1, x_2) \in R^2$.*

Theroem 18

$y = f(x_1, x_2)$ is twice differentiable with Hessian $H(x_1, x_2)$.

(i) $f(x_1, x_2)$ is strictly convex **if** *$f_{11} > 0$, $f_{22} > 0$ and $|H| > 0$ for all $(x_1, x_2) \in R^2$.*

(ii) $f(x_1, x_2)$ is strictly concave **if** *$f_{11} < 0$, $f_{22} < 0$ and $|H| > 0$ for all $(x_1, x_2) \in R^2$.*

(iii) $f(x_1, x_2)$ is convex **iff** *$f_{11} \geq 0$, $f_{22} \geq 0$, and $|H| \geq 0$ for all $(x_1, x_2) \in R^2$.*

(iv) $f(x_1, x_2)$ is concave **iff** *$f_{11} \leq 0$, $f_{22} \leq 0$, and $|H| \geq 0$ for all $(x_1, x_2) \in R^2$.*

Examples

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$f(x_1, x_2) = x_1^2 + x_2^2 - 5x_1x_2$$

$$f(x_1, x_2) = x_1^4 + x_2^4$$

$$f(x_1, x_2) = \sqrt{x_1 + x_2}$$

$$f(x_1, x_2) = 5 - (x_1 + x_2)^2$$

$$f(x_1, x_2) = 3x_1 + x_2^2$$

Functions with Economic Applications

Definition 47

A function $f : R^2 \to R$ *is* **homogeneous of degree** k *if*

$$f(\alpha x_1, \alpha x_2) = \alpha^k f(x_1, x_2)$$

Examples

$f(x_1, x_2) = x_1 x_2^2$: homogeneous of degree 3.

$f(x_1, x_2) = x_1^{1/4} x_2^{1/2}$: homogeneous of degree 3/4.

$f(x_1, x_2) = x_1^a x_2^{1-a}$ (**Cobb-Douglas**): homogeneous of degree 1.

$f(x_1, x_2) = x_1^2 + x_2^2$: homogeneous of degree 2.

$f(x_1, x_2) = x_1^{1/4} x_2^{1/2} + x_1$: not homogeneous.

- A production function $F(K, L)$ is homogeneous of degree k.

  Then the production function exhibits

  increasing returns to scale if $k > 1$

  decreasing returns to scale if $k < 1$

  constant returns to scale if $k = 1$.

Theroem 19 *If f is homogeneous of degree k, then its first partial derivatives are homogeneous of degree $(k-1)$.*

$$f(\alpha x_1, \alpha x_2) = \alpha^k f(x_1, x_2)$$

$$\frac{\partial f(\alpha x_1, \alpha x_2)}{\partial x_1} = \alpha f_1(\alpha x_1, \alpha x_2), \quad \frac{\partial \alpha^k f(x_1, x_2)}{\partial x_1} = \alpha^k f_1(x_1, x_2)$$

Example: $f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$

Theroem 20 *If $y = f(x_1, x_2)$ is a production (utility) function which is homogeneous of degree k and has continuous first partial derivatives, then along any ray from the origin the slope of all isoquants (indifference curves) is equal.*

The ratio x_1/x_2 constant along any ray from the origin.

To show

$$\text{slope of isoqauant } = \frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

depends only on the ratio x_1/x_2.

We know

$$f_1(\alpha x_1, \alpha x_2) = \alpha^{k-1} f_1(x_1, x_2), \quad f_2(\alpha x_1, \alpha x_2) = \alpha^{k-1} f_2(x_1, x_2)$$

Choose $\alpha = 1/x_2$

$$\frac{f_1(x_1/x_2, 1)}{f_2(x_1/x_2, 1)} = \frac{(1/x_2)^{k-1} f_1(x_1, x_2)}{(1/x_2)^{k-1} f_2(x_1, x_2)} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

Example:

$f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$

Definition 48 *A function is homothetic if it is a monotonic transformation of some homogeneous function.*

$f(x_1, x_2)$ homothetic if $f(x_1, x_2) = Q[h(x_1, x_2)]$

$h(x_1, x_2)$ homogeneous, $Q(z)$ monotonic.

Example:

$f(x_1, x_2) = 1 + x_1^{1/3} x_2^{2/3}$

Theroem 21 *$y = f(x_1, x_2)$ is a homothetic production (utility) function* **iff** *along any ray from the origin the slope of all isoquants (indifference curves) is equal.*

$f(x_1, x_2)$ homothetic:

$$f(x_1, x_2) = Q[h(x_1, x_2)]$$

$h(x_1, x_2)$ homogeneous, $Q(z)$ monotonic.

$$f_1(x_1, x_2) = Q'[h(x_1, x_2)]h_1(x_1, x_2)$$

$$f_2(x_1, x_2) = Q'[h(x_1, x_2)]h_2(x_1, x_2)$$

$$\frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{Q'[h(x_1, x_2)]h_1(x_1, x_2)}{Q'[h(x_1, x_2)]h_2(x_1, x_2)} = \frac{h_1(x_1, x_2)}{h_2(x_1, x_2)} = \frac{h_1(x_1/x_2, 1)}{h_2(x_1/x_2, 1)}$$

Unconstrained Optimization

- Given $y = f(x_1, x_2)$, find values of x_1, x_2 at which f takes on an *extreme value*.
- **Unconstrained optimization**: can choose any $x \in R^2$.
- **constrained optimization**: can choose $x \in$ subset of R^2.

Definition 49 *(i) At (x_1^*, x_2^*) we have a local maximum of $f(x_1, x_2)$ if there exists ϵ, however small, such that $f(x_1^*, x_2^*) \geq f(x_1, x_2)$ for all $(x_1, x_2) \in N_\epsilon(x_1^*, x_2^*)$.*
(ii) At (x_1^*, x_2^*) we have a local minimum of $f(x_1, x_2)$ if there exists ϵ, however small, such that $f(x_1^*, x_2^*) \leq f(x_1, x_2)$ for all $(x_1, x_2) \in N_\epsilon(x_1^*, x_2^*)$.

Definition 50 *(i) At (x_1^*, x_2^*) we have a global maximum of $f(x_1, x_2)$ if $f(x_1^*, x_2^*) \geq f(x_1, x_2)$ for all $(x_1, x_2) \in R^2$.*
(ii) At (x_1^*, x_2^*) we have a global minimum of $f(x_1, x_2)$ if $f(x_1^*, x_2^*) \leq f(x_1, x_2)$ for all $(x_1, x_2) \in R^2$.

Global max (min) $\Rightarrow$ local max (min)

Definition 51 $(\bar{x}_1, \bar{x}_2)$ *is a* **stationary point** *of* $f(x_1, x_2)$ *if*

$f_1(x_1, x_2) = f_2(x_1, x_2) = 0.$

Theroem 22 *If at* (x_1^*, x_2^*) *we have a local maximum or minimum of* $f(x_1, x_2)$*, then* $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0.$

- $dy = f_1 dx_1 + f_2 dx_2$: can make $dy > 0$ if $f_1 \neq 0$ or $f_2 \neq 0$.

Theroem 23 $y = f(x_1, x_2)$ *is twice differentiable with Hessian* $H(x_1, x_2)$*.*

(i) If $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$ *and* $H(x_1^*, x_2^*)$ *is negative definite, then* f *has a local maximum at* (x_1^*, x_2^*)*.*

(ii) If $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$ *and* $H(x_1^*, x_2^*)$ *is positive definite, then* f *has a local minimum at* (x_1^*, x_2^*)*.*

Example.

$f(x_1, x_2) = -x_1^3 + 6x_1 - x_2^2$, local max at $(\sqrt{2}, 0)$.

$f(x_1, x_2) = -x_1^4 - x_2^4$ (condition sufficient, not necessary).

Theroem 24

(i) If $y = f(x_1, x_2)$ *is concave for all* $(x_1, x_2) \in R^2$, *then* f *has a global maximum at* (x_1^*, x_2^*) **iff** $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$.

(ii) If $y = f(x_1, x_2)$ *is convex for all* $(x_1, x_2) \in R^2$, *then* f *has a global minimum at* (x_1^*, x_2^*). **iff** $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$.

Examples.

$f(x_1, x_2) = -x_1^2 - x_2^2$,

$f(x_1, x_2) = -x_1^2 + 4x_1x_2 - x_2^2$

Profit-maximizing input choice A competitive firm produces output y using two inputs, labor L and capital K. The production function is given by $F(K, L) = K^{0.6}L^{0.2}$. The firm takes the input and output prices as given and they are: output price $p = 100$, price of labor $w = 10$, and price of capital $r = 20$. What are the profit-maximizing input levels?

Constrained Optimization

Standard Consumer Problem

$$\max_{x_1,x_2} U(x_1,x_2)$$

$$s.t.\;\; p_1x_1 + p_2x_2 \le y$$

$$(p_1x_1 + p_2x_2 \le y \text{ defines a subset of } R^2)$$

Constrained Maximization

$$\max_{x_1,x_2} f(x_1,x_2) \;\; s.t. \;\; g(x_1,x_2) \ge 0$$

Lagrangian Method

$$L(x_1,x_2) = f(x_1,x_2) + \lambda g(x_1,x_2)$$

λ: **Lagrange multiplier**

Theroem 25

If (x_1^*, x_2^*) *maximizes* $f(x_1, x_2)$ *subject to* $g(x_1, x_2) \geq 0$*, then*

(i) $L_1(x_1^*, x_2^*) = L_2(x_1^*, x_2^*) = 0$ *(stationarity)*

(ii) $g(x_1^*, x_2^*) \geq 0$ *(constraint)*

(iii) $\lambda \geq 0$ *(non-negativity)*

(iv) either $\lambda = 0$ *or* $g(x_1^*, x_2^*) = 0$ *(complementary slackness).*

(either constraint is not binding ($\lambda = 0$*) or it is (*$g(x_1^*, x_2^*) = 0$*).*

If both $f(x_1, x_2)$ *and* $g(x_1, x_2)$ *are concave, then (i)-(iv) are necessary and sufficient.*

Examples

$$\max_{x_1, x_2} [\ln x_1 + \ln x_2] \;\; s.t. \;\; 2x_1 + 3x_2 \leq 12$$

(utility maximization subject to budget constraint.)

$$\max_{x_1, x_2} [-(x_1 - 1)^2 - (x_2 - 2)^2] \;\; s.t. \;\; x_1 + x_2 \leq 4$$

$$\max_{x_1, x_2} [\ln x_1 + \ln x_2] \;\; s.t. \;\; 2x_1 + 3x_2 \geq 12$$

Constrained Minimization

$$\max_{x_1,x_2} f(x_1,x_2) \;\; s.t. \;\; g(x_1,x_2) \geq 0 \Longleftrightarrow$$

$$\min_{x_1,x_2}[-f(x_1,x_2)] \;\; s.t. \;\; g(x_1,x_2) \geq 0$$

Examples.

$$\min_{x_1,x_2} \,[2x_1 + 3x_2] \;\; s.t. \;\; 2\sqrt{x_1 x_2} \geq 1$$

(cost minimization subject to an output level.)

Problem Set IV

1. Hessian matrices of

$$f(x_1, x_2) = 2x_1^2 - 3x_2^4$$

$$f(x_1, x_2) = x_1^3 - 6x_1x_2$$

$$f(x_1, x_2) = 7x_1^3x_2^4$$

$$f(x_1, x_2) = 3\ln x_1x_2 + x_1^2$$

$$f(x_1, x_2) = 8e^{5x_1+x_2^3}$$

2. Total differentials and slopes of indifference curves of

$$u = U(x_1, x_2) = 4\ln x_1 + 3\ln x_2$$

$$u = U(x_1, x_2) = 5x_1^{4/3}x_2^{2/3} + 3$$

$$u = U(x_1, x_2) = 4\ln(7x_1 + 3x_2)$$

3. Determine the convexity/concavity of the following functions.

   $f(x_1, x_2) = 3x_1^2 + 5x_2^2$

   $f(x_1, x_2) = 3x_1^2 + 3x_2^2 - 5x_1x_2$

   $f(x_1, x_2) = -x_1^4 - 2x_2^4$

   $f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$

   $f(x_1, x_2) = (x_1 + x_2)^2 - 100x_1 - 50x_2$

   $f(x_1, x_2) = -x_1^3 + 6x_2$

4. If $y = 4^{3/2} x^a y^{2/3}$ is homogeneous of degree $7/6$, find the value of a.

5. Which of the following are homothetic but not homogeneous

   (i) $f(x_1, x_2) = e^{x_1^2 x_2}$.

   (ii) $f(x_1, x_2) = -\ln(x_1 + x_2)$.

   (iii) $f(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2$.

   (iv) $f(x_1, x_2) = -4x_1^3 x_2 + 7x_1^2 x_2^2$.

   (v) $f(x_1, x_2) = \sqrt{x_1 x_2^2 + 2}$.

6. For each of the functions below, identify (a) stationary points, (b) any local maxima or minima, (c) any global maxima or minima.

   $f(x_1, x_2) = -3x_1 + 4x_2$.

   $f(x_1, x_2) = -3x_1^2 - 4x_2^2$.

   $f(x_1, x_2) = -3x_1^2 + 4x_2^2$.

   $f(x_1, x_2) = x_1^3 - 3x_1 + x_2^2$.

   $f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2$.

   $f(x_1, x_2) = x_1^3 + x_1 x_2 - x_2^2$.

7. A competitive firm produces output y using two inputs, labor L and capital K. The production function is given by $F(K, L) = K^{0.4} L^{0.4}$. The firm takes the input and output prices as given and they are: output price $p = 10$, price of labor $w = 2$, and price of capital $r = 2$. What are the profit-maximizing input levels?

8. Solve the following problems.

$$\max_{x_1, x_2} [\ln x_1 + 2 \ln x_2] \;\; s.t. \;\; 3x_1 + 5x_2 \leq 18$$

$$\max_{x_1, x_2} [2x_1 + 4x_2] \;\; s.t. \;\; \ln x_1 + \ln x_2 \geq \ln 8$$

$$\max_{x_1, x_2} [30 - x_1^2 - x_2^2] \;\; s.t. \;\; x_1 + x_2 \geq 10$$

$$\min_{x_1, x_2} [(x_1 - 3)^2 + (x_2 - 5)^2] \;\; s.t. \;\; x_1 + x_2 \leq 10$$

Constrained Optimization (With Equality Constraints)

$$\max_{x_1,x_2} f(x_1, x_2) \;\; s.t. \;\; ax_1 + bx_2 = c$$

$$(ax_1 + bx_2 = c \text{ defines a subset of } R^2)$$

Standard Consumer Problem

$$\max_{x_1,x_2} U(x_1, x_2)$$

$$s.t. \;\; p_1x_1 + p_2x_2 = y$$

Example:

$$\max_{x_1,x_2} (\ln x_1 + \ln x_2)$$

$$s.t. \;\; 3x_1 + x_2 = 180$$

Method I (by substitution)

$$ax_1 + bx_2 = c \Rightarrow x_2 = \frac{c - ax_1}{b}$$

An equivalent unconstrained univariate optimization problem

$$\max_{x_1} f(x_1, \frac{c - ax_1}{b})$$

Method II (Lagrangian Method)

$$L(x_1, x_2) = f(x_1, x_2) + \lambda(c - ax_1 - bx)$$

λ: **Lagrange multiplier**

Theroem 26

If (x_1^*, x_2^*) *maximizes* $f(x_1, x_2)$ *subject to* $ax_1 + bx_2 = c$*, then*

(i) $L_1(x_1^*, x_2^*) = L_2(x_1^*, x_2^*) = 0$ *(stationarity)*

(ii) $ax_1^* + bx_2^* = c$ *(constraint)*

If $f(x_1, x_2)$ *is concave, then (i)-(ii) are necessary and sufficient.*

Constrained Minimization

$$\max_{x_1,x_2} f(x_1, x_2) \;\; s.t. \;\; ax_1 + bx_2 = c \Longleftrightarrow$$

$$\min_{x_1,x_2}[-f(x_1, x_2)] \;\; s.t. \;\; ax_1 + bx_2 = c$$

Solve the following problems.

$$\max_{x_1,x_2} [\ln x_1 + 2\ln x_2] \;\; s.t. \;\; 3x_1 + 5x_2 = 18$$

$$\max_{x_1,x_2} [30 - x_1^2 - x_2^2] \;\; s.t. \;\; x_1 + x_2 = 10$$

$$\min_{x_1,x_2} [(x_1 - 3)^2 + (x_2 - 5)^2] \;\; s.t. \;\; 2x_1 + x_2 = 10$$

INTEGRATION

- "Expected Utility"

  Lottery: $[(x_1, p_1), (x_2, p_2)]$

  $Eu(x) = p_1 u(x_1) + p_2 u(x_2)$

  Continuous probability distribution $F(x)$ over $[a, b]$, density $f(x)$

  $Eu(x) = \int_a^b u(x) f(x) dx$

- $u(t)$ instantaneous utility at $t \in [a, b]$

  Life-time utility $= \int_a^b u(t) dt$

- Solving "differential equations".

- "Consumer Surplus".

Indefinite Integrals

Definition 52 *A function $F(x)$ is an* **indefinite integral** *(or* **antiderivative***) of the function $f(x)$ if $F'(x) = f(x)$ for all x. We usually write $F(x) = \int f(x)dx$. $f(x)$ is called the* **integrand***.*

Rules of Integration

- **Power Rule**

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

- **Linear Rule**

$$\int [\alpha f(x) \pm \beta g(x)] dx = \alpha \int f(x)dx \pm \beta \int g(x)dx$$

- **Exponential Rule**

$$\int e^x dx = e^x + C$$

- **Logarithmic Rule**

$$\int \frac{1}{x} dx = \ln x + C$$

$$\int (\ln x) dx = x \ln x - x + C$$

- **Integration by Substitution**

$$f(x) = g(h(x))h'(x)$$

$$\Rightarrow \quad \int f(x)dx = \int g(h(x))h'(x)dx = \int g(h(x))dh(x) = \int g(h)dh$$

$$(h'(x) = \frac{dh(x)}{dx})$$

General log rule

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$$

- **Integration by Parts**

$$f(x) = g(x)h'(x)$$

$$\Rightarrow \quad \int f(x)dx = \int g(x)h'(x)dx = \int g(x)dh(x)$$

$$= g(x)h(x) - \int h(x)dg(x) = g(x)h(x) - \int h(x)g'(x)dx$$

Examples

$$\int (x^2 + \frac{2}{x} + 3e^x)dx = \frac{1}{3}x^3 + 2\ln x + 3e^x + C$$

$$\int \frac{2x}{x^2+1} dx$$

$$\int xe^x dx$$

Definite Integrals

Definition 53 *A* **partition** *of a closed interval* $[a, b]$ *is any decomposition of* $[a, b]$ *into subintervals of the form*

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

where

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$$

For $i = 1, 2, \ldots, n$ *the length of* $[x_{i-1}, x_i]$ *is denoted by* $\Delta_i = x_i - x_{i-1}$.

Definition 54 *Let* $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ *be a partition of* $[a, b]$ *and* $w_i \in [x_{i-1}, x_i]$ *for* $i = 1, 2, \ldots, n$. *Then*

$$\sum_{i=1}^{n} f(w_i)\Delta_i$$

is a **Riemann sum** *of* $f(x)$.

Definition 55 *$f(x)$ is* **integrable** *on the interval $[a,b]$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|\sum_{i=1}^{n} f(w_i)\Delta_i - L| < \epsilon$$

for any partition of $[a,b]$ such that $\max \Delta_i < \delta$ and for any $w_i \in [x_{i-1}, x_i]$. This value is called the **definite integral** *of $f(x)$ from a to b and denoted*

$$\int_a^b f(x)dx$$

$$(= \lim_{\max \Delta_i \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i)$$

- **Area under the graph**

Theroem 27 (Fundamental Theorem of Calculus)

Suppose $f(x)$ is continuous on a closed intervale $[a, b]$

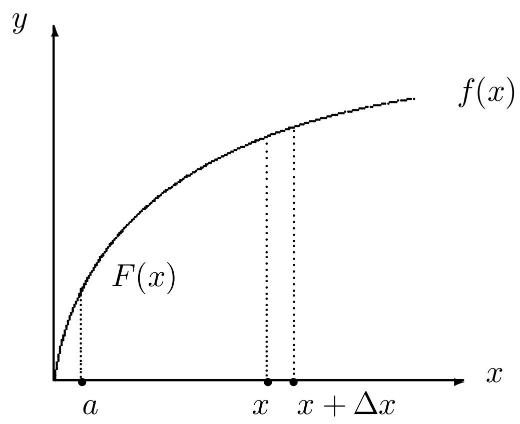
(i) If we define

$$F(x) = \int_a^x f(t)dt$$

for all $x \in [a, b]$, then $F(x)$ is an antiderivative of $f(x)$, i.e., $F'(x) = f(x)$.

(ii) If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$



Given $f(x)$, function $F(x)$ = area under graph of f from a to x

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x)$$

Properties of Integrals

- If $a < b < c$
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

- $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Examples

$$\int_0^1 2xdx$$

$$\int_0^1 (x^3 + e^x)(3x^2 + e^x)dx$$

- Suppose a firm begins at time $t = 0$ with a capital stock $K(0) = 500,000$ and, in addition to replacing any depreciated capital, is planning to invest in new capital at the rate $I(t) = 600t^2$ over the next ten years. What would be the capital stock at the end of the ten years if the plan is carried out?

- The demand function for a product is $q = 15 - 3p^{1/2}$. Compute the consumer surplus at price $p = 9$.

Problem Set V

1. Find the indefinite integrals (antiderivatives) of the following.

$$f(x) = (x^2 + \frac{2}{x} + 3e^x)$$

$$f(x) = \frac{2x}{x^2+1}$$

$$f(x) = xe^x$$

$$f(x) = 2e^{2x} + \frac{14x}{7x^2+5}$$

$$f(x) = 6x^2(x^3+2)^{99}$$

2. Evaluate the following definite integrals

$$\int_5^1 3x^2 dx$$

$$\int_{-1}^1 2e^x dx$$

$$\int_1^2 6xe^{x^2} dx$$

$$\int_0^{64} (x^{1/2} + 5x^{-2/3}) dx$$

$$\int_1^5 \frac{2x^3+1}{x^4+2x} dx$$

$$\int_0^4 (\frac{1}{x+1} + 2x) dx$$

3. Suppose an economy's net investment flow is $I(t) = 10t^{1/2}$. Letting $K(0) = 5,000,000$ be the current stock of capital, find the level of capital five years from now.

4. The demand function for a product is $q = 10 - 2p^{1/2}$. Compute the consumer surplus at prices $p = 4$ and $p = 1$.