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Separating Discounting from Changing Utility

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Separating Discounting from Changing Utility.

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Abstract

Discounting of delayed utilities is the guiding principle for evaluating policies with future implications. Many public institutions commit to using a fixed schedule of discount rates. But the decision makers in these institutions change over time. What normative principle justifies using a fixed schedule of discount rates when the tastes of the decision makers are not fixed over time? To address this question, this paper introduces a framework of discounted *current* utility that separates discounting from changing utility. An axiom of *time tradeoff invariance* is shown to be the key characterising condition. This is the normative principle that justifies using a given schedule of discount rates, irrespective of whether utility changes over time. We also compare constant discount rates, adopted in US public institutions, with the declining rates adopted by various European institutions. What normative principle separates these approaches? We identify a condition called *time tradeoff consistency* that characterises discounting at a constant rate. Time tradeoffs are used to derive *subjective delay midpoints*, which provide a powerful tool for identifying and comparing decreasing impatience across decision makers, who have different utilities, even if both of their utilities change over time.

Keywords: Discounted Utility, Dynamic Preferences, Time Tradeoffs.

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1 Introduction

When evaluating prospects with future implications, a first consideration is how to rank the various outcomes, the relative costs and benefits, that will result from these decisions. Such a ranking might be summarised by a utility function. It is typically assumed that utilities are stable over time. But decision makers change. For example, different people, with different political ideologies, will occupy the same policy decision-making role over time. Therefore, as time passes, one might expect a decision maker's ranking of outcomes to change and so the associated utility values must also change. Even for individual preferences over monetary outcomes, presumably always ranked the same way (more is always preferred), there is growing evidence that utility curvature is not stable over time (Meier and Sprenger, 2015; Schildberg-Hörisch, 2018).

The second consideration in the evaluation of intertemporal prospects is how to discount utilities when they are delayed. For policy evaluation, such decisions will typically make use of a *fixed* schedule of discount rates. As decision makers change, the same discount rates continue to be used. The specific values of these (social) discount rates are idiosyncratic, often set by a government or other authority at a national level. In the United States, it is the US Office of Management and Budget (OMB) that sets the schedule of discount rates for evaluating public projects, which currently suggests a constant annual discount rate of 2% (OMB, 2023). Figure 1 of Groom et al. (2022) summarises the rates of various other countries. Social discount rates are crucial for analysing public projects with implications in the future, such as mitigating climate change, improving infrastructure, long-term health projects, and so on (Gollier, 2012). Setting the social discount rate used to evaluate public projects is a difficult but important task. A social discount rate that is set incorrectly in any direction, too high or too low, will inevitably result in misallocation of resources and welfare losses.

Beyond setting the discount rate, the nature of discounting is also a contentious issue. Cropper et al. (2014) and others (e.g. Arrow et al., 2014), for example, argued for changing the US approach such that the schedule of discounts exhibits decreasing impatience. Declining discount rates have been incorporated into the appraisal of policies by several countries. The UK, for example, applies a discount rate of 3.5% for delays up to 30 years, 3% for delays between 31 and 75 years, and 2.5% for delays of 76 years or longer (HM Treasury, 2026). The adoption of a schedule of declining discount rates, it is suggested, better reflects the findings of a substantial literature on measuring discounting in experiments (e.g. Andersen et al., 2008; Andreoni and Sprenger, 2012; Attema et al., 2010; Attema et al., 2016; Harrison et al., 2002) where *decreasing impatience* is the prevailing finding. Most studies assume either that utility does not change over time or consider only preferences at a fixed decision time.

This paper considers a question that precedes any discussion of the appropriate level and the appropriate nature of discounting used in the evaluation of public policy. We ask, what normative principles of decision making justify using any fixed schedule of discounts when utility changes over time? That is, under what conditions can discounting and changing utility be separated? Uncovering this normative principle is important because so many institutions have already adopted fixed schedules of discounts and so, to some extent, have already taken whatever this principle may be to be acceptable. We also ask, what does it mean for such a schedule of discounts to exhibit constant or decreasing impatience when utility is changing? When does one decision maker's schedule of discounts exhibit *greater* decreasing impatience than another decision maker's schedule of discounts, when both decision makers have utilities that change over time?

To address these questions, it is necessary to extend the standard framework, preferences over consumption streams, to *dynamic preferences*. A dynamic preference is a collection of current preferences, one for each decision time, and these current preferences are used to evaluate consumption streams at that decision time. A dynamic preference might correspond to the various decision makers who occupy the same decision-making role within an institution, such as a government, over time. Alternatively, it might correspond to one decision maker who has multiple selves over time. We introduce a framework of *discounted current utility* that separates discounting from changing utility. To disentangle and test the key normative properties of exponential discounting, but with unchanging utility, Halevy (2015) assumed a dynamic preference structure over *timed outcomes*. Timed outcomes, one payment at one time, are simple objects that are ideally suited to experimental testing. In this paper, we consider a much broader framework of consumption streams, and a much broader class of dynamic preferences.

In Section 2 we introduce the framework of discounted current utility. In Section 3 we explore a basic implication for decision making when one has adopted a given schedule of discount rates. Section 4 introduces *time tradeoffs*. Decisions that represent time tradeoffs constitute a more general class of decisions. Nevertheless, we show that decisions that involve time tradeoffs are determined entirely by the adopted discount factors, independent of current utility. Section 5 introduces the key normative principle of *time tradeoff invariance*. When combined with basic dynamic preference axioms, time tradeoff invariance characterises discounted current utility. Therefore, time tradeoff invariance is the precise normative principle that justifies using a fixed schedule of discount rates when utility is not stable over time.

The schedule of discount factors adopted by many institutions, for example the OMB in the US, correspond to a single, constant discount rate. Discounting at a constant rate is characteristic of *exponential* discounting, which is the central model used in economics. In Section 6 we identify the condition that reduces the general form of discounted current utility to exponential discounting of current utility. This provides a more robust dynamic foundation of exponential discounting that does not require utility to be stable over time. It is well-known that *time consistency of preferences* is a characterising condition of the standard version of exponential discounting (Halevy, 2015). Here, we show that *time consistency of time tradeoffs* is the characterising dynamic condition of exponential discounting of a changing utility. This principle separates, for example, the US approach and the UK approach to social discounting. The US applies a constant rate and so conforms to time tradeoff consistency, whereas the declining discount rates of the UK and others necessarily violate time tradeoff consistency. In Section 7, we use time tradeoffs to identify and compare decreasing impatience across decision makers with different utilities, even if their utilities both change over time. Finally, in Section 8 we identify the dynamic preference condition such that utility does not change over time, precisely identifying the property that distinguishes discounted current utility from the standard discounted utility model. The Appendix contains a summary of the basic axioms used, further details on discounted current utility, and all proofs.

2 Discounted Current Utility.

For the theory developed here to be broadly applicable, we will use continuous time and an arbitrary set of outcomes. The theory will apply if outcomes are consumption outcomes (divisible or not) or outcomes such as environmental or health conditions. Let time be $T = [0, \infty)$ and let X be an arbitrary set of outcomes. Consumption streams, or simply streams, are step functions $\mathbf{c} : T \rightarrow X$, and the set of consumption streams is $\mathcal{C}$. Outcomes are flow variables and streams specify the outcome at each point in time. At any time a, a decision maker must compare and rank consumption streams. This decision maker considers consumption streams with times before a truncated, the set of which is $\mathcal{C}_a$, so has current preferences $\succcurlyeq_a$ over $\mathcal{C}_a$. A collection of current preferences $\{\succcurlyeq_a\}_{a \in T}$, one preference relation for each decision time, is called a *dynamic preference*.

The decision maker's dynamic preferences will be represented by a combination of two things: a discount factor and a set of *current* utility functions. Current utility represents how the decision maker rates outcomes at the current decision time. We are interested in changing tastes and so allow there to be a different current utilities. The decision maker chooses the stream that maximises the *discounted current utility* as follows:

Definition 1 (Discounted Current Utility)**.** There is a continuous and strictly decreasing discount factor $D : T \rightarrow \mathbb{R}$, with $D(0) = 1$ and $\lim_{t \rightarrow \infty} D(t) = 0$, and a set of real-valued current utility functions $\{u_a\}_{a \in T}$ for outcomes such that the decision maker at time a maximises:

$$\int_a^\infty u_a(\mathbf{c}(t))\mathrm{d}D(t-a). \tag{1}$$

The function D is called a *discount factor*. To capture impatience, discount factors are strictly decreasing in time and approach zero in the limit. Pivato (2021) referred to these kind of discount factors as *generalised discount factors*. Discount factors are the dual of what Attema et al. (2016, p.1477) referred to as *cumulative (discount) weights*. In what follows, it will be convenient to use the following notation:

$$\begin{bmatrix} c_1 & [t_0, t_1) \\ \vdots & \vdots \\ c_n & [t_{n-1}, t_n) \end{bmatrix}$$

which indicates a consumption stream that gives outcome c_i on interval $[t_{i-1}, t_i)$ for all $i = 1, \dots, n$, where $a = t_0 < \cdots < t_n = \infty$. Under discounted current utility, the utility of this stream at time a is given by:[1]

$$\sum_{i=1}^{n-1} u_a(c_i)\left(D(t_{i-1}-a) - D(t_i - a)\right) + u_a(c_n)D(t_{n-1}-a).$$

When evaluating a consumption stream under discounted current utility, the utility of consumption c_i in interval $[t_{i-1}, t_i)$ is weighted by the difference in discount factors:[2]

$$D(t_{i-1}-a) - D(t_i - a).$$

What distinguishes discounted current utility from the standard discounted utility is that, as the name suggests, it is the *current* utility u_a being discounted to inform decisions at any given time. Current utility depends on what the current date happens to be and so can change as the decision time changes. The decision maker is not anticipating changes in tastes, however, but simply applying their current preferences. Like a policy maker, enjoying their time in the seat of power, as if it will last forever. Discounted

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[1] See the Appendix for some basic details on this Stieltjes-Riemann integral.

[2] The utility of consumption at the end of the stream, c_n, is weighted only by the discount factor $D(t_{n-1} - a)$ because discount factors vanish in the limit.

utility, in the standard sense, refers to the case where $u_a \equiv u$. Addressing the central questions of this paper requires that we do not impose this restriction. The main restriction of the discounted current utility model is that the same discount factors for delays are used at every decision time. This is precisely the condition that interests us here – when decision makers commit to using a given schedule of discount factors, even as their utility functions change over time.

3 Basic Implications of Discounting Current Utility.

When decision makers, be they individuals or institutions, commit to using a given schedule of discount factors, there are implications. In this section, we look at perhaps the most basic implication. Suppose that, starting from a constant stream, the outcome can be improved on a specific time interval before returning to the baseline level. There are various time intervals that can be chosen for this temporary boost. How would the decision maker rank these objects? We shall see that such rankings are determined only by the given discount factors.

To formalise this, we start with a constant stream that delivers a baseline level of consumption b. If the stream is improved to a consumption level g during the interval $[s, t)$, we can represent the resulting stream as:

$$\begin{bmatrix} g & [s,t) \\ b & \text{otherwise} \end{bmatrix}$$

where g and b represent good and baseline outcomes ($g \succ_a b$), respectively, *as ranked by current preferences*.[3] The decision maker compares two streams that differ only in the time intervals over which the good outcome is available:

$$\begin{bmatrix} g & [s,t) \\ b & \text{otherwise} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} g & [s',t') \\ b & \text{otherwise} \end{bmatrix}.$$

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[3] Current preferences for outcomes are identified with current preferences for *constant* streams .

Taking the constant stream b as a point of reference, the question is simply whether an improvement in consumption from b to g is better on the interval $[s, t)$ or on the interval $[s', t')$. How might a strict preference for the stream with good outcome on $[s, t)$ over the stream with good outcome on $[s', t')$ be interpreted? If the decision maker strictly prefers the good outcome on $[s, t)$, then it is reasonable to conclude that an improvement in consumption on the interval $[s, t)$ is "weighted more" than on the interval $[s', t')$. This weight, however, has been decided in advance. The decision maker, committed to using a given schedule of discount factors, applies discounted current utility and it is immediate that:

$$\begin{bmatrix} g & [s,t) \\ b & \text{otherwise.} \end{bmatrix} \succ_a \begin{bmatrix} g & [s',t') \\ b & \text{otherwise.} \end{bmatrix}$$

if and only if the following holds:

$$D(t-a) - D(s-a) > D(t'-a) - D(s'-a),$$

Weak preferences and indifference can be similarly interpreted, with the strict inequality replaced by weak or equality. For experiments, the above choices represent a simple way to elicit the discount function. For our question, the basic implication of committing to given set of discount factors is now clear. If policy makers must use a given schedule of discounts, then all such decision makers who rank the outcomes the same way will also rank these types of binary streams the same way. Except for the initial ranking of good versus baseline outcomes, their ranking of these binary streams is entirely determined by the given discount factors.

Now let us consider a later decision time, $a + \tau$ where $\tau > 0$. Apply the same delay to the intervals being compared. If it were the case that $g \succ_{a+\tau} b$, so preferences have not changed concerning which outcome is good and which outcome is bad, then the ranking of these binary streams is preserved. Indeed, one need only ensure that the outcome considered good, compared to the baseline, reflects the *current* preference. For the purposes of eliciting discount factors, the outcomes can be replaced at each decision time to guarantee this.[4] If preferences are represented by discounted current utility, then for all decision times $a < a + \tau$, all outcomes $g \succ_a b$ and $g' \succ_{a+\tau} b'$, and all times s, t, s', t' we have

$$\begin{bmatrix} g & [s,t) \\ b & \text{otherwise} \end{bmatrix} \succcurlyeq_a \begin{bmatrix} g & [s',t') \\ b & \text{otherwise} \end{bmatrix}$$

if and only if

$$\begin{bmatrix} g' & [s+\tau,t+\tau) \\ b' & \text{otherwise} \end{bmatrix} \succcurlyeq_{a+\tau} \begin{bmatrix} g' & [s'+\tau,t'+\tau) \\ b' & \text{otherwise} \end{bmatrix}.$$

This relationship between different current preferences above is a weak form of Halevy's (2015) *time invariance* that applies only to binary streams, with outcomes also adjusted to reflect current preferences, and must be satisfied by discounted current utility.[5] In terms of our question, when an institution commits to applying a given set of discount factors, they are ensuring that all future decision makers in that position abide with the invariance condition. What other restrictions are being made? We next introduce *time tradeoffs* and this will allow us, amongst other things, to precisely identify the invariance properties, the normative content, that characterises discounted current utility.

4 Further Implications: Time Tradeoffs.

We have seen in the previous section some of the basic implications of decision makers, with utilities changing over time, committing to a given schedule of discount factors. In this section, to expand further on this, we introduce a more involved type of comparison called a *time tradeoff.* The comparisons considered in the previous section considered improvements only to constant streams. That is, we took a constant stream b as a benchmark, improved b to g on each of the intervals of interest, and asked which improvement

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[4] Analogous to how Savage (1954) compared subjective likelihoods of uncertain events and the P4 axiom.

[5] Discounted current utility preferences do not satisfy the full version of time invariance, as it would naturally be defined for steams in continuous time. See Section **??**.

is preferred. Time tradeoffs generalise this by considering improvements to (currently) indifferent streams that are not necessarily constant. We will see that time tradeoffs are characterised by the adopted discount factors, independent of utility and whether changes over time. In this sense, committing to a given schedule of discount factors means that all decisions that involve time tradeoffs are already decided.

Let us start by taking an outcome b and intervals $[s,t)$ and $[s',t')$. We now find *reference streams* $\mathbf{c}$ and $\mathbf{d}$ to establish an indifference:

$$\begin{bmatrix} b & [s,t) \\ \mathbf{c} & \text{otherwise} \end{bmatrix} \sim_a \begin{bmatrix} b & [s',t') \\ \mathbf{d} & \text{otherwise} \end{bmatrix}. \tag{2}$$

Having established an indifference as a benchmark, a time tradeoff is revealed by improving each stream only on the intervals being considered. That is, for an outcome $g \succ_a b$, let us improve each of the streams above by replacing b with g. These replacements yield two new streams:

$$\begin{bmatrix} g & [s,t) \\ \mathbf{c} & \text{otherwise} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} g & [s',t') \\ \mathbf{d} & \text{otherwise} \end{bmatrix},$$

and we ask whether the decision maker prefers the first stream to the second. If the decision maker has preference,

$$\begin{bmatrix} g & [s,t) \\ \mathbf{c} & \text{otherwise} \end{bmatrix} \succcurlyeq_a \begin{bmatrix} g & [s',t') \\ \mathbf{d} & \text{otherwise} \end{bmatrix}, \tag{3}$$

then the claim is that this reveals the same "$[s,t)$ is weighted no less than interval $[s',t')$" conclusion as in the basic approach. Of course, if $\mathbf{c}$ and $\mathbf{d}$ are constant streams, then time tradeoffs are equivalent to the basic approach. Time tradeoffs allow for more general streams. These general streams are not, however, arbitrary. They must be such that indifference (2) holds. This step allows the reference streams to be factored out, allowing us to focus on the time intervals being compared. We will adopt the following notation for time tradeoffs:

Definition 2 (Time Tradeoffs)**.** For all pairs of times (s, t) and (s', t') we write:

$$(s, t) \succcurlyeq_a^* (s', t')$$

if there are outcomes $g \succ_a b$ and streams $\mathbf{c}, \mathbf{d}$ such that indifference (2) and preference (3) hold. The symmetric and strict parts, $\sim_a^*$ and $\succ_a^*$ respectively, correspond to the latter preference being an indifference or a strict preference.

The following proposition characterises the time tradeoff ordering $\succcurlyeq_a^*$ under discounted current utility:

Proposition 1. *Suppose discounted current utility $(D, \{u_a\}_{a \in T})$ holds. Then the ranking of time tradeoffs is characterised entirely by the discount factor D, independent of the current utility function. Specifically, we have $(s, t) \succcurlyeq_a^* (s', t')$ if and only if:*

$$D(s-a) - D(t-a) \geqslant D(s'-a) - D(t'-a). \tag{4}$$

That $(s, t) \succcurlyeq_a^* (s', t')$ implies (4) follows from routine substitution of discounted current utility. For the converse, beginning with (4) and showing the existence of the required time tradeoff streams, it is sufficient to consider revelations the basic sense, with $\mathbf{y} = \mathbf{z} = w$. We give details in the Appendix.

If time tradeoff indifference $\sim^*$ holds for two intervals, then the discount factor D equally spaces the difference in discount factors for those intervals. Working with this type of information, one can elicit the entire discount function. Hence, time tradeoffs can be used as an experimental method to elicit and analyse individual discount factors. Attema et al. (2010) used a version of time tradeoffs in discrete time to do this. The approach here is conceptually similar, but using a continuous time framework, formally equivalent to the likelihood method of Abdellaoui and Wakker (2005). Our focus in this paper has not been elicitation but, instead, the question of how prescribed discount rates inform decisions. Proposition 1 confirms, if a decision maker is required to apply a given schedule of discounts when evaluating policies with intertemporal implications, that the ranking of those policies that represent time tradeoffs is determined entirely by the adopted discount factors.

5 Separating Discounting from Changing Utility.

The previous section introduced time tradeoffs at a given decision time a. Some caution must be taken when considering time tradeoffs at a later decision time $a + \tau$ because outcomes being compared may be ranked differently. It was noted in Section 3 that the ranking of simple, binary streams under discounted current utility is invariant to delays, provided that the outcomes and are adjusted to reflect current preferences. In this section we show that time tradeoffs are also invariant to delays, provided that both the outcomes and the reference streams are adjusted to reflect current preferences. We call this *time tradeoff invariance*. Time tradeoff invariance can be considered the characterising property of discounted current utility. It is the normative principle that we sought. Beyond some basic conditions, any decision-making institution that commits to using the same discount factors over time, even as utility for outcomes changes, is committing to adhering to the principle of time tradeoff invariance and that principle alone.

The concept of time tradeoff invariance is simple. If a time tradeoff $(s, t) \succcurlyeq_a^* (s', t')$ holds at decision time a, then the same time tradeoff must hold when the decision time and all streams are delayed by a common τ, provided that the outcomes and reference streams are based only on current preferences. The adjustment is necessary because, as noted above, the outcomes being compared may be ranked differently at a later decision time. Proposition 1 shows that the time tradeoff ordering $\succcurlyeq_a^*$ is equivalent to a difference in discount factors. Applying Proposition 1 twice yields:

$$\begin{aligned} & (s,t) \succcurlyeq_a^* (s',t') \\ \Leftrightarrow \quad & D(s-a) - D(t-a) \geqslant D(s'-a) - D(t'-a) \\ \Leftrightarrow \quad & (s+\tau, t+\tau) \succcurlyeq_{a+\tau}^* (s'+\tau, t'+\tau). \end{aligned}$$

If discount factors depend only on the delay from decision time, as assumed in discounted current utility, and time tradeoffs are characterised only by differences in discount factors, then the time tradeoff ordering must be invariant to delays. That is, under discounted current utility, we can never have a time tradeoff $(s,t) \sim_a^* (s',t')$ at decision time a and a different time tradeoff $(s+\tau, t+\tau) \succ_{a+\tau}^* (s'+\tau, t'+\tau)$ at decision time $a+\tau$. We formulate this as an axiom:

Axiom (Time Tradeoff Invariance)**.** *For no a, s, t, s', t' and τ do we ever have both*

$$(s,t) \sim_a^* (s',t') \qquad \text{and} \qquad (s+\tau, t+\tau) \succ_{a+\tau}^* (s'+\tau, t'+\tau).$$

The double application of Proposition 1 above shows that time tradeoff invariance is a necessary condition for discounted current utility. The following theorem, the paper's main theorem, shows that time tradeoff invariance is also sufficient for discounted current utility, when combined with basic dynamic preference axioms. The following theorem, the paper's main theorem, characterises discounted current utility:

Theorem 1. *The following are equivalent:*

1. *$\{\succcurlyeq_a\}_{a \in T}$ satisfies weak order, monotonicity, strong monotone continuity, and time tradeoff invariance.*
2. *$\{\succcurlyeq_a\}_{a \in T}$ can be represented by discounted current utility $(D, \{u_a\}_{a \in T})$.*

In statement 2, the discount factor D is unique and each of the utilities in $\{u_a\}_{a \in T}$ is unique up to positive affine transformation.

The property of time tradeoff invariance, when combined with basic dynamic preference axioms detailed in Appendix A.1, characterises discounted current utility. In this sense, time tradeoff invariance is the precise normative principle that justifies using a fixed schedule of discount rates when utility is not stable over time. If an institution commits to using a fixed schedule of discount factors over time, irrespective of whether different decision makers with different current preferences are in charge, then this institution is committing to uphold the time tradeoff invariance axiom.

6 Constant Discount Rates with Changing Utility.

Discounted current utility captures decision makers / institutions that make use of a fixed schedule of discount factors when utility is not stable over time. Often, institutions will adopt a schedule discount factors that corresponds to a fixed *discount rate*. This is the case of the US, as noted in the introduction, which adopts discount factors corresponding to a constant annual rate of 2%. The UK, however, adopts a schedule of discount factors corresponding to declining discount rates. What normative principle separates these decisions? It is well-known that *exponential discounting* captures discounting with a constant rate. Exponential discounting is the central model of intertemporal choice in economics. In this section, we consider exponential discounting of *current* utility as follows:

Definition 3 (Exponentially Discounted Current Utility)**.** There is a constant discount factor $\delta \in (0, 1)$ and a set of real-valued current utility functions $\{u_a\}_{a \in T}$ for outcomes such that the decision maker at time a maximises:

$$\int_a^\infty \delta^{t-a} u_a(\mathbf{c}(t))\mathrm{d}t. \tag{5}$$

In discounted current utility, the discount factor D is essentially a (decumulative) distribution function. Exponentially discounted current utility can therefore be seen as the special case such that the discount factor D has density of the form δ^t for some

$\delta \in (0, 1)$. The US adoption of a 2% discount rate corresponds to a discount factor of $\delta = \frac{1}{1+0.02} = 0.98$. Exponentially discounted current utility must, as a special case of discounted current utility, satisfy the time tradeoff invariance axiom. What is the *additional* normative content that guarantees the decision maker applies a fixed discount rate when utility is changing over time? The answer is an axiom that we call *time tradeoff consistency*.

If we assume that exponentially discounted current utility holds, then the time tradeoff ordering $\succcurlyeq_a^*$ is characterised by the following difference in discount factors:

$$(s, t) \succcurlyeq_a^* (s', t') \quad \Leftrightarrow \quad \delta^s - \delta^t \geqslant \delta^{s'} - \delta^{t'},$$

It is apparent that the time tradeoff ordering $\succcurlyeq_a^*$ does not depend on the decision time a. Therefore, if two intervals are revealed today by time tradeoffs to be equally-weighted, then the same will be revealed at later decision times. For the time tradeoff invariance axiom, the intervals were subject to the same change as the decision time. In the following axiom, the intervals are not at all changed. Only the decision time changes, and this cannot reverse revealed time tradeoffs:

Axiom (Time Tradeoff Consistency)**.** *For no a, s, t, s', t', and τ do we ever have both*

$$(s, t) \sim_a^* (s', t') \qquad \textit{and} \qquad (s, t) \succ_{a+\tau}^* (s', t').$$

To see that time tradeoff consistency is necessary for exponentially discounted current utility, the first time tradeoff $(s, t) \sim_a^* (s', t')$ is equivalent to $\delta^t - \delta^s = \delta^{t'} - \delta^{s'}$. If it were the case that $(s, t) \succ_{a+\tau}^* (s', t')$ also held, this would require $\delta^t - \delta^s > \delta^{t'} - \delta^{s'}$, a contradiction. Hence, the consistent time tradeoffs axiom is necessary for $\{\succcurlyeq_a\}_{a \in T}$ to be represented by $(\delta, \{u_a\}_{a \in T})$. Combined with the basic dynamic preference conditions and time tradeoff invariance, it is also sufficient, which we state as a theorem below. Before stating the theorem, we note that time tradeoff invariance and time tradeoff consistency can be stated as a single, powerful condition:

Axiom (Time Tradeoff Persistency). *For no a, s, t, s', t', τ and τ' do we ever have both*

$$(s, t) \sim_a^* (s', t') \qquad \textit{and} \qquad (s + \tau', t + \tau') \succ_{a+\tau}^* (s' + \tau', t' + \tau').$$

In the time tradeoff persistency axiom, the decision time is delayed by τ and the intervals are delayed by τ'. If $\tau = \tau'$, a common delay, then the axiom reduces to time tradeoff invariance. Taking $\tau' = 0$, the axiom reduces to time tradeoff consistency.

Theorem 2. *The following are equivalent:*

1. $\{\succcurlyeq_a\}_{a \in T}$ *satisfies weak order, monotonicity, strong monotone continuity, time tradeoff invariance, and time tradeoff consistency.*
2. $\{\succcurlyeq_a\}_{a \in T}$ *satisfies weak order, monotonicity, strong monotone continuity, and time tradeoff persistency.*
3. $\{\succcurlyeq_a\}_{a \in T}$ *can be represented by exponentially-discounted current utility* $(\delta, \{u_a\}_{a \in T})$.

In statement 2, each of the utilities in $\{u_a\}_{a \in T}$ *is unique up to positive affine transformation and the discount factor* $\delta \in (0, 1)$ *is unique.*

For decision making institutions that adopt a given schedule of discount factors, hence adopt discounted current utility in some form, time tradeoff consistency is additional condition that characterises a constant rate of discounting. It seems, therefore, that UK and various other European public institutions do not subscribe to this principle, adopting instead a schedule of declining discount rates. Public institutions in the US, by adopting a constant discount rate, do commit to upholding the principle of time tradeoff consistency. We hope that identifying the principle that distinguishes to the two approaches will inform the ongoing discussion on this matter.

7 Time Tradeoffs and Comparing Attitudes to Delay.

In this section, as another application of time tradeoffs, we consider the problem of observing and comparing decreasing impatience when utility changes over time. Although one cannot directly observe a decision maker's changing level of patience, one can observe preferences that would seem best explained by such a change. We will consider two approaches to decreasing impatience, static and dynamic. The static approach is based on a fixed decision time, while the dynamic approach considers a preference reversal between an earlier and a later decision time.

Definition 4 (Static Decreasing Impatience)**.** For all $a, s, s', t, t' \in T$, with $a \leqslant s \leqslant s'$, $g \succ_a b$, and $\tau > 0$, we have:

$$\begin{bmatrix} g & [s,t) \\ b & \text{otherwise} \end{bmatrix} \sim_a \begin{bmatrix} g & [s',t') \\ b & \text{otherwise} \end{bmatrix}$$

only if:

$$\begin{bmatrix} g & [s+\tau, t+\tau) \\ b & \text{otherwise} \end{bmatrix} \preccurlyeq_a \begin{bmatrix} g & [s'+\tau, t'+\tau) \\ b & \text{otherwise} \end{bmatrix}.$$

This definition of decreasing impatience is static in the sense that decision time is fixed. The first indifference suggests that increasing the delay for the good outcome g from $s - a$ to $s' - a$ is only just acceptable, provided that this outcome is enjoyed for a longer period, until t' rather than t. When a common delay of τ is added to the best outcome in both streams, the increase in delay is more than acceptable. Hence the decision maker seems to be more patient or, in more common terms, their impatience has decreased.

In the dynamic framework, decreasing impatience can be revealed by a preference reversal between an earlier and a later decision time.

Definition 5 (Dynamic Decreasing Impatience)**.** For all $a, s, s', t, t' \in T$, with $a + \tau \leqslant$

$s \leqslant s'$, $g \succ_a b$, and $g' \succ_{a+\tau} b'$, we have:

$$\begin{bmatrix} g & [s,t) \\ b & \text{otherwise} \end{bmatrix} \sim_a \begin{bmatrix} g & [s',t') \\ b & \text{otherwise} \end{bmatrix}$$

only if:

$$\begin{bmatrix} g' & [s,t) \\ b' & \text{otherwise} \end{bmatrix} \succcurlyeq_{a+\tau} \begin{bmatrix} g' & [s',t') \\ b' & \text{otherwise} \end{bmatrix}.$$

In the case of dynamic decreasing impatience, the streams are fixed. Pinned to the calendar. Initially, at decision time a, the decision maker is indifferent to the streams such that the intervals $[s,t)$ and $[s',t)$ seem to be equally weighted. That is, at time decision time a, it is judged that increasing the delay for the good outcome g, from $s - a$ to $s' - a$, is acceptable, provided it can be enjoyed for a presumably longer interval $[s',t')$. At a later the decision time $a + \tau$, the delays $s - (a + \tau)$ and $s' - (a + \tau)$ are shorter, and a strict preference for the new good outcome g' sooner suggests impatience has increased. Of course, impatience increasing with shorter delays means impatience decreasing with longer delays. By ensuring that outcomes g and b, and g' and b', are appropriately ranked at the relevant decision time, we see that this reversal is a consequence of changing impatience and not changing tastes.

It is straightforward to show that, under discounted current utility, static and dynamic decreasing impatience are both characterised by the difference $D(t - a) - D(s - a)$ decreasing with a. Because D is continuous, this is equivalent to *convexity*. A third approach, based on time tradeoffs, is now given. We will use the following notion of *subjective delay midpoints*, which are defined using the time tradeoff method:

Definition 6 (Subjective Delay Midpoint)**.** Let $a \in T$ and $s < t$. A time $\mu(s,t,a)$ is a *subjective delay midpoint* of s and t, as measured at decision time a, if:

$$(s, \mu(s,t,a)) \sim_a^* (\mu(s,t,a), t).$$

According to Proposition 1, the $\mu(s,t,a)$ is a subjective delay midpoint of s and t, as measured at decision time a, if and only if:

$$D(\mu(s,t,a)-a)=\frac{1}{2}D(s-a)+\frac{1}{2}D(t-a),$$

It is for this reason, that we call such $\mu(s,t,a)$ a *subjective delay midpoint* of s and t at decision time a. Such a $\mu(s,t,a)$ need not be an *objective delay midpoint*, that is, it may not equal $\frac{1}{2}(s-a)+\frac{1}{2}(t-a)$. Comparing subjective and objective delay midpoints gives a direct method to analyse the shape of discount factors. Convexity of D can be characterised by the simple condition that *subjective delay midpoints do not exceed objective delay midpoints.*

Theorem 3. *Let $\succcurlyeq_a$ be represented by discounted current utility $(D,\{u_a\}_{a\in T})$. Then, the following statements are equivalent:*

1. *$\{\succcurlyeq_a\}_{a\in T}$ exhibits static decreasing impatience.*
2. *$\{\succcurlyeq_a\}_{a\in T}$ exhibits dynamic decreasing impatience.*
3. *Subjective delay midpoints do not exceed objective delay midpoints. That is, given a, we have*

$$\mu(s,t,a)\leqslant\frac{1}{2}s+\frac{1}{2}t \qquad \textit{for all } s,t\in T.$$

4. *The discount factor D is convex.*

Theorem 3 fully characterises the conditions such that a schedule of discounts exhibit decreasing impatience when utility is changing. Decreasing impatience is separated from changing utility and characterised entirely by convexity of the discount factor. We omit details of the proof. The equivalence of the first two statements with convexity has been established in the text. The equivalence of the third, subjective delay midpoints not exceeding objective delay midpoints, is similar to Theorem 2.2 of Baillon, Driesen, and Wakker (2012, p.483). Their proof, which characterises concavity of a strictly increasing utility function using midpoints, can be adapted in a straightforward manner to show convexity of a strictly decreasing discount function.

If one is only concerned with identifying decreasing impatience, the first two concepts in Theorem 3 are surely simpler than the tradeoff-based subjective delay midpoint technique. The benefit of the latter approach is revealed, however, when considering *interpersonal* comparisons of decreasing impatience. Indeed, subjective delay midpoints obtained using time tradeoffs can be used directly to compare the decreasing impatience of different decision makers, each with utilities that change over time. If we fix two times, say $s < t$, and two decision makers, A and B, we can elicit two indifferences each to reveal their subjective delay midpoints of s and t. These can be compared to see which decision maker is more decreasingly impatient.

Definition 7 (Relative Decreasing Impatience)**.** Individual A is *more decreasingly impatient* than individual B if, given decision time a, we have

$$\mu_A(s,t,a) \leqslant \mu_B(s,t,a) \qquad \text{for all } s,t \in T.$$

That is, A's subjective delay midpoints do not exceed B's subjective delay midpoints. As one might expect, greater decreasing impatience is characterised by a more convex discount function. The following theorem establishes the relationship between relative convexity of discount functions and relative decreasing impatience:

Theorem 4. *Let decision makers A and B have dynamic preferences represented by $(D_A, \{u_a^A\}_{a\in T})$ and $(D_B, \{u_a^B\}_{a\in T})$, respectively. Then, the following are equivalent:*

1. *A is more decreasingly impatient than B.*
2. *There exists a strictly increasing and convex function ψ such that $D_A = \psi \circ D_B$.*

The proof of Theorem 4 is in the Appendix. Theorem 4 shows that the relative convexity of discount functions is equivalent to the relative decreasing impatience of decision makers. This means that we can compare the decreasing impatience of decision makers, with possibly different utilities, regardless of whether those utilities change over time, by comparing their subjective delay midpoints.

8 Time Invariance and Discounted Utility.

The standard discounted utility model is the special case of discounted current utility such that $u_a = u_{a'} = u$ for all $a, a' \in T$. In this case, both the discount function for delays and the utility function for outcomes are the same at all decision times. We write a discounted utility representation as (D, u). Let us briefly show the condition under which discounted current utility reduces discounted utility. The condition, *time invariance*, is well-known. The following notation is used. Given a stream $\mathbf{x} \in \mathcal{C}_a$ and $0 \leq b \leq a$, denote by $\mathbf{x}_b$ the stream $\mathbf{x}_b \in \mathcal{C}_{a-b}$ such that $\mathbf{x}_b(t) = \mathbf{x}(t-b)$ for all $t \geq b$.

Definition 8. *Time invariance* holds if, for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}$, $\mathbf{x} \succcurlyeq_0 \mathbf{y}$ if and only if $\mathbf{x}_a \succcurlyeq_a \mathbf{y}_a$.

Although time tradeoff invariance suffices for establishing a common discount function of delays at every decision time in Theorem 1, it allows utility to depend on decision time. To obtain the standard discounted utility model, in which utility is taken to be unchanging with decision time, time invariance of preferences is required:

Proposition 2. *Suppose that $\{\succcurlyeq_a\}_{a \in T}$ is represented by $(D, \{u_a\}_{a \in T})$. Then $\{\succcurlyeq_a\}_{a \in T}$ can be represented by discounted utility (D, u) if and only if $\{\succcurlyeq_a\}_{a \in T}$ satisfies time invariance.*

As a corollary, exponentially-discounted current utility reduces to exponential discounting if and only if those dynamic preferences satisfy time invariance. Time invariance, therefore, seems to provide a simple test of whether a decision maker's utility is changing with decision time or not. Of course, this conclusion holds only in the context of discounted current utility, that is, when discount factors can be separated from current utility and depend only on delay from decision time. Halevy (2015) tested time invariance in the context of timed outcomes and, except for low-stakes decisions, the null hypothesis of time invariance was rejected.

A Appendix A: Technical Details.

A.1 Dynamic Preferences: Basic Axioms.

A dynamic preference $\{\succcurlyeq_a\}_{a\in T}$ is complete and transitive if, for all $a \in T$, $\succcurlyeq_a$ is a complete and transitive order on $C_a = \{\mathbf{c}|_{[a,\infty)} : \mathbf{c} \in C\}$.

Axiom (Weak Order)**.** $\{\succcurlyeq_a\}_{a\in T}$ *is complete and transitive.*

A stream $\mathbf{x}$ is constant if there is $x \in X$ such that $\mathbf{x} \equiv x$. We will often identify outcomes and constant streams in notation. Preferences over outcomes are derived from preferences over constant streams, and so for $x, y \in X$ we write $x \succcurlyeq_a y$ to mean that the stream that gives x at all points in time is preferred to the stream that gives y at all points in time.

Axiom (Monotonicity)**.** *For all $a \in T$ we have $\mathbf{x}(t) \succcurlyeq_a \mathbf{y}(t)$ for all $t \in T$ implies $\mathbf{x} \succcurlyeq_a \mathbf{y}$. If, in addition, there is $b < c$ such that $\mathbf{x}(t) \succ_a \mathbf{y}(t)$ for all $t \in [b, c)$, then $\mathbf{x} \succ \mathbf{y}$.*

In the text, we used the notation:

$$\begin{bmatrix} x & t \in [s,t) \\ \mathbf{y} & t \notin [s,t) \end{bmatrix}$$

to denote the stream with outcome x for all times in $[s,t)$ and $\mathbf{y}$ otherwise. In the Appendix, we will use the more efficient notation $x_{[s,t)}\mathbf{y}$.

Axiom (Strong Monotone Continuity)**.** *For all $a \in T$, all $\mathbf{x}, \mathbf{y} \in C_a$, all $y \in X$, and all $[b_1, c_1) \supset [b_2, c_2) \supset \cdots$ with $\bigcap_{i=1}^{\infty}[b_i, c_i)$ either empty or a singleton, we have:*

1. *If $\mathbf{x} \succcurlyeq_a z_{[b_i,c_i)}\mathbf{y}$ for all i, then $\mathbf{x} \succcurlyeq_a \mathbf{y}$.*
2. *If $z_{[b_i,c_i)}\mathbf{x} \succcurlyeq_a \mathbf{y}$ for all i, then $\mathbf{x} \succcurlyeq_a \mathbf{y}$.*

The three axioms above are the basic axioms that will be assumed throughout the paper.

A.2 Discounted Utility and Subjective Expected Utility.

Although interpreted differently, discounted current utility representations are formally equivalent to subjective expected utility representations in the following sense. Suppose that, for each a, there is a probability measure p_a over half-open intervals and, further, that for all a and all t we have $p_a([t,\infty)) = p_0([t-a,\infty))$. We can write that $\mathbf{x} \in \mathcal{C}_a$ gives outcome x_i on interval $[t_{i-1},t_i)$ for all $i = 1,\dots,n$, where $a = t_0 < \cdots < t_n = \infty$. Then, we have:

$$\int_T u_a(\mathbf{x})\mathrm{d}p_a = \sum_{i=1}^{n} u_a(x_i)p_a([t_{i-1},t_i))$$

$$= \sum_{i=1}^{n-1} u_a(x_i)\big(p_a([t_{i-1},\infty) - p_a([t_i,\infty))\big) + u_a(x_n)p_a([t_{n-1},\infty))$$

The discounted current utility evaluation of $\mathbf{x}$ at decision time a is given by:

$$\int_a^{\infty} u_a(\mathbf{x}(t))\mathrm{d}D(t-a) = \lim_{b\to\infty}\int_a^{b} u_a(\mathbf{x}(t))\mathrm{d}D(t-a)$$

$$= \sum_{i=1}^{n-1} u_a(x_i)\big(D(t_{i-1}-a) - D(t_i-a)\big) + u_a(x_n)\big(D(t_{n-1}-a) - \lim_{b\to\infty} D(b-a)\big) \tag{6}$$

If D is such that $D(t-a) = p_a([t,\infty)]) = p_0([t-a,\infty))$, then discounted utility coincides with subjective expected utility. Hence, in the proof of Proposition 1, we will derive subjective expected utility representations (p_a, u_a) for each $\succcurlyeq_a$, show that the probability measure p_a associated with each decision time a must satisfy $p_a([t,\infty)]) = p_0([t-a,\infty))$, and use this to *define* a discount factor.

B Appendix B: Proofs.

B.1 Proof of Proposition 1.

Suppose that $\{\succcurlyeq_a\}_{a\in T}$ is represented by discounted current utility $(D,\{u_a\}_{a\in T})$. Suppose that $[s,t) \succcurlyeq_a^* [s',t')$. This implies the existence of $g \succ_a b$ and $\mathbf{y},\mathbf{z}$ such that

$b_{[s,t)}\mathbf{y} \sim_a b_{[s',t')}\mathbf{z}$ and $g_{[s,t)}\mathbf{y} \succcurlyeq_a g_{[s',t')}\mathbf{z}$. Subsitution of discounted current utility for each of these yields:

$$\begin{aligned}
&\big(D(s-a)-D(t-a)\big)u_a(b)+\int_a^s u_a(\mathbf{y}(t))\mathrm{d}D(t-a)+\int_t^\infty u_a(\mathbf{y}(t))\mathrm{d}D(t-a)\\
&=\big(D(s'-a)-D(t'-a)\big)u_a(b)+\int_a^s u_a(\mathbf{z}(t))\mathrm{d}D(t-a)+\int_t^\infty u_a(\mathbf{z}(t))\mathrm{d}D(t-a),
\end{aligned}$$

and:

$$\begin{aligned}
&\big(D(s-a)-D(t-a)\big)u_a(g)+\int_a^s u_a(\mathbf{y}(t))\mathrm{d}D(t-a)+\int_t^\infty u_a(\mathbf{y}(t))\mathrm{d}D(t-a)\\
&\geqslant\big(D(s'-a)-D(t'-a)\big)u_a(g)+\int_a^s u_a(\mathbf{z}(t))\mathrm{d}D(t-a)+\int_t^\infty u_a(\mathbf{z}(t))\mathrm{d}D(t-a).
\end{aligned}$$

Deducting the former equation from the above inequality, cancelling common terms, yields the required $D(s-a)-D(t-a) \geqslant D(s'-a)-D(t'-a)$. For the converse, suppose that discounted utility holds and there are s, s', t, t' and a such that $D(s-a)-D(t-a) \geqslant D(s'-a)-D(t'-a)$. Choose any $g \succ_a b$. As $\sim_a$ is reflexive, $b_{[s,t)}b \sim_a b_{[s',t')}b$. We must also have $g_{[s,t)}w \succcurlyeq_a g_{[s',t')}w$, or else substitution of discounted current utility will reveal a contradictory inequality. Hence, the result follows from taking $\mathbf{y} = \mathbf{z} = b$. ■

B.2 Proof of Theorem 1.

The main step in the proof is obtained by applying, at each decision time, a subjective expected utility theorem due to Abdellaoui and Wakker (2005). However, the structural requirements to apply this theorem are not met and so the theorem cannot be directly applied. Our time tradeoff invariance axiom is formulated only for half-open time intervals, which is neither an algebra nor a mosaic (Kopylov, 2007). Under an axiom of strong monotone continuity, we show that (Lemma 3) the time tradeoff invariance condition does indeed hold on the algebra generated by half-open time intervals, and so the theorem of Abdellaoui and Wakker (2005) can be applied. This is similar to Corollary 4 of Kopylov (2010) who showed, under the same conditions, that Savage's P2 and P4 on half-open time intervals extends to the algebra generated by those intervals.

Let $\mathcal{I} = \{[a, b) : a, b \in T, a \leq b\}$ denote the set of half-open time intervals. For $A = [a, b)$ and τ we write $A + \tau$ for $[a + \tau, b + \tau)$. The time tradeoff invariance (TTI) axiom can be written as: For all $a, a + \tau \geqslant 0$, all outcomes $g \succ_a b$, $g' \succ_{a+\tau} b'$, all $\mathbf{y}, \mathbf{z} \in \mathcal{C}_a$, all $\mathbf{y}', \mathbf{z}' \in \mathcal{C}_{a+\tau}$, and all $A, B \in \mathcal{I}$, we have $b_A\mathbf{y} \sim_a b_B\mathbf{z}$, $g_A\mathbf{y} \sim_a g_B\mathbf{z}$, and $b'_{A+\tau}\mathbf{y}' \sim_{a+\tau} b'_{B+\tau}\mathbf{z}'$, only if $g'_{A+\tau}\mathbf{y}' \sim_{a+\tau} g'_{B+\tau}\mathbf{z}'$.

TTI holds for $A, B \in \mathcal{I}$ that are intervals of the form $A = [a, b)$, $B = [a', b')$. In this section it will be shown that, given axioms 1-3, the TTI condition holds for the case where A and B are elements of the algebra generated by $\mathcal{I}$. We let $\alpha(\mathcal{I})$ denote the algebra generated by $\mathcal{I}$.

Axiom (TTI*)**.** *For all $a, a + \tau \geqslant 0$, all outcomes $g \succ_a b$, $g' \succ_{a+\tau} b'$, all $\mathbf{y}, \mathbf{z} \in \mathcal{C}_a$, all $\mathbf{y}', \mathbf{z}' \in \mathcal{C}_{a+\tau}$ and all $A, B \in \alpha(\mathcal{I})$, we have $b_A\mathbf{y} \sim_a b_B\mathbf{z}$, $g_A\mathbf{y} \sim_a g_B\mathbf{z}$, and $b'_{A+\tau}\mathbf{y}' \sim_{a+\tau} b'_{B+\tau}\mathbf{z}'$, only if $g'_{A+\tau}\mathbf{y}' \sim_{a+\tau} g'_{B+\tau}\mathbf{z}'$.*

Some preparatory lemmas:

Lemma 1. *For all $g' \succ_a b$ and $g' \succ_{a+\tau} b'$ and $A, B \in \mathcal{I}$, we have $g_A b \sim_a g_B b$ only if $g'_{A+\tau} b' \sim_{a+\tau} g'_{B+\tau} b'$.*

Proof. In the TTI condition, take $\mathbf{y} = \mathbf{z} \equiv b$ and $\mathbf{y}' = \mathbf{z}' \equiv b'$. Note, if $\tau = 0$, this is Savage's (1954) P4 condition for indifferences. ☐

Lemma 2. *For all $g \succ_a b$ and $A, B \in \mathcal{I}$, we have $g_A b \sim_a g_B b$ and $b_A\mathbf{x} \sim_a b_B\mathbf{y}$ only if $g_A\mathbf{x} \sim_a g_B\mathbf{y}$.*

Proof. It is trivial that $b_A b \sim_a b_B b$. Hence, if $g_A b \sim_a g_B b$ and $b_A\mathbf{x} \sim_a b_B\mathbf{y}$, then $g_A\mathbf{x} \sim_a g_B\mathbf{y}$ follows from TTI with $\tau = 0$. ☐

We will show that, under the basic axioms, time tradeoff invariance implies the same condition extended to $\alpha(\mathcal{I})$. It is similar to Corollary 4 of Kopylov (2010), which showed the same for Savage's (1954) axioms:

Lemma 3. *If $\{\succcurlyeq_a\}_{a\in T}$ satisfies weak order, monotonicity, strong monotone continuity and TTI, then $\{\succcurlyeq_a\}_{a\in T}$ satisfies TTI*.*

Proof. Assume the three prerequisite indifferences hold. If $A, B \in \alpha(\mathcal{I})$ then there are $0 \leq s_1^A \leq t_1^A \leq \cdots \leq s_n^A \leq t_n^A \leq +\infty$, and $0 \leq s_1^B \leq t_1^B \leq \cdots \leq s_m^B \leq t_m^B \leq +\infty$, with $0 < m, n < \infty$, such that $A = \bigcup_{i=1}^n [s_i^A, t_i^A)$ and $B = \bigcup_{i=1}^m [s_i^B, t_i^B)$. We proceed by induction on $m+n$. If $m = n = 1$, then $A, B \in \mathcal{I}$, and TTI* is equivalent to TTI. Next suppose that TTI* holds for $m+n-1$ and consider the $m+n$ case.

Suppose that there are $0 \leq j \leq n$ and $0 \leq k \leq m$ such that $g_{[s_j^A, t_j^A)} b \succcurlyeq_a g_{[s_k^B, s_k^B)} b$. If not, it is a simple matter to relabel. Monotonicity implies that $b \preccurlyeq_a g_{[s_k^B, t_k^B)} b$. Let $c_j = \sup_{c \in [s_j^A, t_j^A)} g_{[t_j^A, c)} b \preccurlyeq_a g_{[s_k^B, t_k^B)} w$. Strong monotone continuity implies that (see Kopylov, 2010, p.875):

$$g_{[s_j^A, c_j)} b \sim_a g_{[s_k^B, t_k^B)} b, \tag{7}$$

and, by Lemma 1, we have:

$$g'_{[s_j^A+\tau, c_j+\tau)} b' \sim_{a+\tau} g'_{[s_k^B+\tau, t_k^B+\tau)} b'. \tag{8}$$

Let $\tilde{A} = \bigcup_{i \neq j} [s_i^A, t_i^A) \cup [c_j, t_j^A)$ and let $\tilde{B} = \bigcup_{i \neq k} [s_i^B, t_i^B)$. Note that $\tilde{A}$ is a union of n intervals, and $\tilde{B}$ is a union of $m-1$ intervals, and so a total of $m+n-1$. We can rewrite the assumed indifferences as:

$$b_A \mathbf{y} = b_{[s_j^A, c_j)} b_{\tilde{A}} \mathbf{y} \sim_a b_B \mathbf{z} = b_{[s_k^B, t_k^B)} b_{\tilde{B}} \mathbf{z}, \tag{9}$$

$$g_A \mathbf{y} = g_{[s_j^A, c_j)} g_{\tilde{A}} \mathbf{y} \sim_a g_B \mathbf{z} = g_{[s_k^B, t_k^B)} g_{\tilde{B}} \mathbf{z}, \tag{10}$$

$$b'_{A+\tau} \mathbf{y}' = b'_{[s_j^A+\tau, c_j+\tau)} b'_{\tilde{A}+\tau} \mathbf{y}' \sim_{a+\tau} b'_{B+\tau} \mathbf{z}' = b'_{[s_k^B+\tau, t_k^B+\tau)} x_{\tilde{B}+\tau} \mathbf{z}'. \tag{11}$$

Given the indifferences in equation (7) and (9), and applying Lemma 2, we have:

$$g_{[s_j^A, c_j)} b_{\tilde{A}} \mathbf{y} \sim_a g_{[s_k^B, t_k^B)} b_{\tilde{B}} \mathbf{z}$$

Using indifferences (8) and (11) and Lemma 2, we have:

$$g'_{[s_j^A+\tau, c_j+\tau)} b'_{\tilde{A}+\tau} \mathbf{y}' \sim_{a+\tau} g'_{[s_k^B+\tau, t_k^B+\tau)} b'_{\tilde{B}+\tau} \mathbf{z}'.$$

Combining these indifferences with (10), and the induction hypothesis that TTI* holds for the $m + n - 1$ case, we have:

$$g'_{[s^A_j+\tau, c_j+\tau)} g'_{\tilde{A}+\tau} \mathbf{y}' \sim_{a+\tau} g'_{[s^B_k+\tau, t^B_k+\tau)} g'_{\tilde{B}+\tau} \mathbf{z}'$$

which can be rewritten as $g'_{A+\tau}\mathbf{y}' \sim_{a+\tau} g'_{B+\tau}\mathbf{z}'$, and so TTI* holds. $\square$

Small period continuity on $\alpha(\mathcal{I})$ holds if, for all $\mathbf{x} \succ \mathbf{y}$ and $z \in X$ there exists finitely many $A_1, \ldots, A_n \in \alpha(\mathcal{I})$ such that $\{A_1, \ldots, A_n\}$ partitions T and such that $\mathbf{x} \succ z_{A_i}\mathbf{y}$ and $z_{A_j}\mathbf{y}$ for all $i, j \in \{1, \ldots, n\}$. This is Savage's (1954) P6 on $\alpha(\mathcal{I})$. Small period continuity on $\mathcal{I}$ is a stronger condition than on $\alpha(\mathcal{I})$, because the former requires the existence of suitable $A_1, \ldots, A_n$ drawn from a strictly smaller set, that is, intervals $[s_i, t_i)$ rather than time periods formed by finite unions of such intervals. Under the axioms assumed here, small period continuity holds on $\mathcal{I}$, hence on $\alpha(\mathcal{I})$, by Lemma 6 of Kopylov (2010, p.873).[6] It is known that P6 on an algebra of sets implies the solvability condition and the Archimedean condition of Abdellaoui and Wakker (2005, p.22). For each $a \geqslant 0$ and taking $\tau = 0$, the TTI* condition becomes formally equivalent to the likelihood consistency condition of Abdellaoui and Wakker (2005, p.11). Therefore, applying their Theorem 5.3 and our Lemma 3, for each $a \geqslant 0$ there exists a unique probability measure p_a on $\alpha(\mathcal{I}_a)$ and a cardinal utility $u_a : X \to \mathbb{R}$ such that preferences $\succcurlyeq_a$ over $\mathcal{C}_a$ are represented by $\mathbf{x} \mapsto \int_T u_a(\mathbf{x})\mathrm{d}p_a$. Because a subjective expected utility representations of each $\succcurlyeq_a$ exist, Savage's (1954) axioms must hold along with axiom A.1, Corollary 3 of Kopylov (2010, p.871), ensures that each p_a is countably additive, must have $p_a(t) = 0$ for all t, and that p_a must be convex ranged as each $\alpha(\mathcal{I}_a)$ is countably separated (see Kopylov, 2010, p.868 and Lemma 8). To complete the proof of Theorem 1, the following lemma is used to establish the relationship between the probability measures $\{p_a\}_{a \in T}$ derived above. The lemma, which extends Lemma 1 to weak preference and to the algebra $\alpha(\mathcal{I}_a)$, is similar to Lemma B.2 of Abdellaoui and Wakker (2005):

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[6] Lemma 6 of Kopylov (2010) assumes more axioms, and proves more, but the proof that P6 holds uses only axioms A.1 and A.1.

Lemma 4. *Suppose that* $\{\succcurlyeq_a\}_{a\in T}$ *satisfies axioms 1, 2, 3, and 4 (weak order, monotonicity, strong monotone continuity, and time tradeoff invariance) . Then, for all* $a, a+\tau \geqslant 0$, $g \succ_a b$ *and* $g' \succ_{a+\tau} b'$ *, and all* $A, B \in \alpha(\mathcal{I}_a)$*, we have* $g_A b \succcurlyeq_a g_B b$ *if and only if* $g'_{A+\tau} b' \succcurlyeq_{a+\tau} g'_{B+\tau} b$*.*

Proof. Reflexivity of indifferences implies $b_A b \sim_a b_B b$ and $b'_{A+\tau} b' \sim_{a+\tau} b'_{B+\tau} b'$. If either of $g_A b \sim_a g_B b$ or $g'_{A+\tau} b \sim_{a+\tau} g'_{B+\tau} b$ hold, then time tradeoff invariance implies that the other must hold. Hence the result holds for indifferences. Next suppose that $g_A b \succ_a g_B b$. Monotonicity implies that $g_B b \succcurlyeq_a b$. Solvability holds, and so there exists $C \subset A$ such that $g_C b \sim_a g_B b$. As the result holds for indifferences, we have $g'_{C+\tau} b' \sim_a g'_{B+\tau} b'$. Monotonicity implies $g'_{A+\tau} b' \succ_a g'_{B+\tau} b'$, and the result follows from transitivity. ☐

Define $\succcurlyeq_0^*$ over $\alpha(\mathcal{I})$ such that, for all $A, B \in \alpha(\mathcal{I})$ and all $w' \succ_0 w$, $A \succcurlyeq_0^* B$ if and only if $w'_A w \succcurlyeq_0 w'_B w$. The order $\succcurlyeq_0^*$ admits a unique probability measure representation p_0 over $\alpha(\mathcal{I})$, which was obtained above. Consider any a and recall the probability measure p_a obtained above. Note that p_a is defined over $\alpha(\mathcal{I}_a)$. We define $\tilde{p}_a$ over $\alpha(\mathcal{I})$ such that $\tilde{p}_a(A) = p_a(A+a)$ for all $A \in \alpha(\mathcal{I})$. By Lemma 4, $\succcurlyeq_0^*$ is represented by p_a. By the uniqueness of probability measure representations, we have $p_0 = \tilde{p}_a$. Let $g \succ_0 b$ and let $0 \leq s < t \leq +\infty$. Monotonicity requires that $g_{[s,t)} b \succ_a b$. Substituting the obtained representation, we have $p_0([s,t)) > 0$ for all $0 \leq s < t \leq +\infty$. We can now construct a discount function. Recall that elements of C_a are simple functions and so we can write that $\mathbf{x}$ gives outcome x_i on interval $[t_i, t_{i+1})$, where $a = t_0 < t_1 < \cdots < t_n = \infty$. Then:

$$\int_T u_a(\mathbf{x})\mathrm{d}p_a = \sum_{i=0}^{n-1} p_a([t_i, t_{i+1}))u(x_i) = \sum_{i=0}^{n-1} \big[p_a([t_i,\infty)) - p_a([t_{i+1},\infty))\big]u_0(x_i).$$

Define a continuous discount function $D(t) = p_0[t,\infty)$ for all t and we have $D(0) = p_0(T) = 1$, $\lim_{t\to\infty} D(t) = 0$, and D is strictly decreasing in t. Note also that, for all $t \geqslant a$, we have $p_a([t,\infty)) = p_0([t-a,\infty)) = D(t-a)$. and so we have:

$$\mathbf{x} \succcurlyeq_a \mathbf{y} \quad \Leftrightarrow \quad \int_a^\infty u_a(\mathbf{x}(t))\mathrm{d}D(t-a) \geq \int_a^\infty u_a(\mathbf{y}(t))\mathrm{d}D(t-a),$$

for all $\mathbf{x}, \mathbf{y} \in C_a$. Therefore $\{\succcurlyeq_a\}_{a \in T}$ admits a discounted current utility representation $(D, \{u_a\}_{a \in T})$, completing the proof of Theorem 1. ■

B.3 Proof of Theorem 2.

Suppose that dynamic preferences $\{\succcurlyeq_a\}_{a \in T}$ over streams satisfies weak order, monotonicity, strong monotone continuity, time tradeoff invariance, and time tradeoff consistency. The conditions of Theorem 2 hold, and $\{\succcurlyeq_a\}_{a \in T}$ admits a discounted current utility representation $(D, \{u_a\}_{a \in T})$.

Let s, s', t, t' be such that $D(s) - D(t) \geqslant D(s') - D(t')$ which is, by Proposition 1, equivalent to $[s, t) \succcurlyeq_0^* [s', t,)$. If there were σ such that $D(s + \sigma) - D(t + \sigma) < D(s' + \sigma) - D(t' + \sigma)$, equivalent to $[s, t) \prec_\sigma^* [s', t,)$, then time tradeoff consistency would be contradicted. Hence, for all s, s', t, t' and all σ, we have

$$D(s) - D(t) \geqslant D(s') - D(t') \Leftrightarrow \; D(s + \sigma) - D(t + \sigma) \geqslant D(s' + \sigma) - D(t' + \sigma).$$

Let $T^2_\downarrow := \{(s, t) \, : \, s \leqslant t\}$ denote the set of rank-ordered pairs of times. Define $\trianglerighteq$ over $T^2_\downarrow$ such that $(s, t) \trianglerighteq (s', t')$ if and only if $D(s) - D(t) \geqslant D(s') - D(t')$. That is, $(s, t) \mapsto D(s) - D(t)$ is a difference representation of $\trianglerighteq$, see Chapter 4 of Krantz et al. (1971). As shown above, for all s, t, s', t' and σ, we have $(s, t) \trianglerighteq (s', t')$ if and only if $(s + \sigma, t + \sigma) \trianglerighteq (s' + \sigma, t' + \sigma)$. Therefore, for all σ, $(s, t) \mapsto D(s + \sigma) - D(t + \sigma)$ is a difference representation of $\trianglerighteq$. It is already known that D is strictly decreasing and satisfies $D(0) = 1$. Theorem 1 of Miyamoto and Wakker (1996) ensures that D is an exponential function, that is, there exists $0 < \delta < 1$ such that $D(t) = \delta^t$ for all t. ■

B.4 Proof of Proposition 2.

Assume $\{\succcurlyeq_a\}_{a \in T}$ has a discounted current utility representation $(D, \{u_a\}_{a \in T})$. We show that a common u can be found such that (D, u) represents $\{\succcurlyeq_a\}_{a \in T}$. Note that the following holds for all a and all $\mathbf{x}$:

$$\int_0^\infty u_0(\mathbf{x}(t))\mathrm{d}D(t) = \int_a^\infty u_0(\mathbf{x}_a(t))\mathrm{d}D(t-a). \tag{12}$$

Consider the following equivalences, which hold for all a and all $\mathbf{x}, \mathbf{y} \in C_a$:

$$\begin{array}{ccl}
\mathbf{x} \succcurlyeq_a \mathbf{y} & \Leftrightarrow^a & \int_a^\infty u_a(\mathbf{x}(t))\mathrm{dD(t-a)} \geqslant \int_\mathrm{a}^\infty \mathrm{u_a}(\mathbf{y}(\mathrm{t}))\mathrm{dD(t-a)} \\
\Updownarrow^b & & \\
\mathbf{x}_{-a} \succcurlyeq_0 \mathbf{y}_{-a} & \Leftrightarrow^c & \int_0^\infty u_0(\mathbf{x}_{-a}(t))\mathrm{d}D(t) \geqslant \int_a^\infty u_0(\mathbf{y}_{-a}(t))\mathrm{d}D(t) \\
 & \Leftrightarrow^d & \int_a^\infty u_0(\mathbf{x}(t))\mathrm{d}D(t-a) \geqslant \int_a^\infty u_0(\mathbf{y}(t))\mathrm{d}D(t-a).
\end{array}$$

Equivalences (a) and (c) hold due to the discounted current utility representation of $\{\succcurlyeq_a\}_{a\in T}$. Equivalence (b) is time invariance. Equivalence (d) follows from identity (12). Hence, $\succcurlyeq_a$ is represented by both (D, u_a) and (D, u_0). By the uniqueness results associated with discounted current utility representations, u_a is a strictly positive affine transformation of u_0. This holds for all a and so, letting $u = u_0$, we have (D, u) is a discounted utility representation of $\{\succcurlyeq_a\}_{a\in T}$. ■

B.5 Proof of Theorem 4.

(1. implies 2.) Suppose that, given a, we have $\mu_A(s,t,a) \leqslant \mu_B(s,t,a)$ for all s, t. The generalised discount factors D_A and D_B are both strictly decreasing, are both continuous, and $D_A(T) = D_B(T) = (0,1]$. Hence, there exists a strictly increasing and continuous $\psi : (0,1] \to (0,1]$ such that $D_A = \psi \circ D_B$. Suppose, for the sake of contradiction, that $\alpha, \beta \in (0,1]$ existed such that $\psi(\frac{1}{2}\alpha + \frac{1}{2}\beta) > \frac{1}{2}\psi(\alpha) + \frac{1}{2}\psi(\beta)$. For such α, β there exist $s, t \in T$ so that $D_B(s-a) = \alpha$ and $D_B(t-a) = \beta$. Then, we have:

$$\begin{aligned}
\psi(\frac{1}{2}\alpha + \frac{1}{2}\beta) &= \psi(D_B(\mu_B(s,t,a) - a)) \\
> \frac{1}{2}\psi(\alpha) + \frac{1}{2}\psi(\beta) &= \frac{1}{2}\psi(D_B(s-a)) + \frac{1}{2}\psi(D_B(t-a)) \\
= \frac{1}{2}D_A(s-a) + \frac{1}{2}D_A(t-a) &= D_A(\mu_A(s,t,a) - a) = \psi(D_B(\mu_A(s,t,a) - a)),
\end{aligned}$$

and so, because $D_B(\mu_B(s,t,a) - a) > D_B(\mu_A(s,t,a) - a)$ and D_B decreases strictly, we have $\mu_A(s,t,a) > \mu_B(s,t,a)$, contradicting the assumed inequality. Hence, for all $\alpha, \beta \in (0,1]$, we have $\psi(\frac{1}{2}\alpha + \frac{1}{2}\beta) \leqslant \frac{1}{2}\psi(\alpha) + \frac{1}{2}\psi(\beta)$. That is, ψ is midpoint convex and so, by continuity, is convex.

(2. implies 1.) Suppose that $D_A = \psi \circ D_B$, where $\psi : (0,1] \to (0,1]$ is strictly increasing and convex. For all $\alpha, \beta, \kappa \in (0,1]$ we have $\psi(\beta) = \frac{1}{2}\psi(\alpha) + \frac{1}{2}\psi(\kappa)$ only if $\beta \geqslant \frac{1}{2}\beta + \frac{1}{2}\kappa$. That is, ψ-midpoints exceed objective midpoints on $(0,1]$. Consider any $s,t \in T$. By definition we have $D_A(\mu_A(s,t,a) - a) = \frac{1}{2}D_A(s-a) + \frac{1}{2}D_A(t-a)$ and so:

$$\psi(D_B(\mu_A(s,t,a) - a)) = \frac{1}{2}\psi(D_B(s-a)) + \frac{1}{2}\psi(D_B(t-a)).$$

This holds only if $D_B(\mu_A(s,t,a) - a) \geqslant \frac{1}{2}D_B(s-a) + \frac{1}{2}D_B(t-a) = D_B(\mu_B(s,t,a) - a)$ and so, because D_B decreases strictly, we have $\mu_A(s,t,a) \leqslant \mu_B(s,t,a)$. ■

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