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Evolutionary Finance: Models with Short-Lived Assets

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Abstract. Evolutionary Finance explores the "survival and extinction" questions of investment strategies (portfolio rules) in the market selection process. It studies the stochastic dynamics of financial markets, where asset prices are determined endogenously by short-run equilibrium between supply and demand, which is formed each period as a result of the interaction of strategies employed by competing market participants. This paper focuses on "short-lived" risky securities that are traded at the beginning of each period and yield payoffs at the end of it (which live only one period), with the cycle then repeating. We review some key models developed in this area, which address the following problems in order: 1) introducing the central results that we are primarily interested in under substantially more general assumptions; 2) exploring the Nash equilibrium properties of survival strategies and the single survivor problem within the framework of independent and identically distributed states of the world and fixed-mix portfolio rules; 3) extending the discussion on the single survivor problem to a considerably broader scope, emphasizing its Markovian nature; 4) including a risk-free asset into the market; 5) allowing for short-selling in the market. The two main results of these studies are: i) the existence of survival strategies that can be expressed by explicit formulas, i.e., Kelly's rule of "betting one's beliefs"; and, ii) the asymptotic uniqueness (within a specific class of strategies called basic) of such survival strategies.

Key words: Evolutionary finance, Survival portfolio rules, Short-lived assets, Stochastic dynamic games, Evolutionary game theory.

JEL Classification: C73, D53, G11, C73, D58.

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Chapter 1

Introduction

> "...the perception of potential threats to survival may be much more important in determining behavior than the perceptions of potential profits, so that profit maximization is not really the driving force. It is fear of loss rather than hope of gain that limits our behavior."
>
> Kenneth E. Boulding
> *Evolutionary Economics*, 1981, p.108.

Conventional models of equilibrium and dynamics of asset markets are based on the classical Walrasian general equilibrium theory[1], which assumes that investors behave fully rationally and maximize utilities of consumption subject to budget constraints (i.e., a well-defined and precisely stated constrained optimization problem). In contrast, the present work in Evolutionary Finance (EF)[2] develops an alternative equilibrium concept – *behavioral equilibrium* – admitting that market participants may be boundedly rational and have a whole variety of patterns of behavior determined by their individual psychology. Strategies may involve, for instance, mimicking, satisficing, rules of thumb (based on experience), etc.; and, may be interactive, depending on the behavior of the others. This approach, inspired by the insights of behavioral economics and finance[3], overcomes several drawbacks of traditional theory, particularly by eliminating the need for the "perfect foresight" assumption to establish equilibrium and the reliance on knowledge of unobservable individual utilities and beliefs. In this sense, it opens new possibilities for the modern financial markets modeling, especially on the global level, where the main objectives are: *domination* in a market segment, fastest capital *growth*, or simply *survival* (especially in crisis environments).

We examine game-theoretic models of financial markets with endogenous asset prices determined by short-run equilibrium of supply and demand. Equi-

[1] For a textbook treatment of this subject, see Magill and Quinzii [111].

[2] For surveys describing the state of the art in this filed, see Evstigneev et al. [61, 64]; for an elementary textbook treatment of this subject, see Evstigneev et al. [63], Chapter 20.

[3] See, e.g., Tversky and Kahneman [158], Shleifer [149], Shiller [148], and Thaler [154].

librium is formed consecutively in each time period by the interaction of general, adaptive portfolio rules adopted by competing investors, depending on the exogenous random factors and observed history of the game. Uncertainty in asset payoffs at each period is modeled via an exogenous discrete-time stochastic process that governs the evolution of the states of the world, which aims to capture various macroeconomic and business cycle variables that may affect investors' behavior. There are two fundamental types of models in EF: with short-lived (one-period) assets and long-lived dividend-paying assets, see, e.g., Amir et al. [4] and [3], respectively. This paper focuses on models of the former type. Short-lived assets refer to standardized contracts for various real assets (e.g., energy or natural resources, agricultural products, real estate, etc.) that are traded (at the market prices) at the beginning of each time period $[t, t+1]$ and yield payoffs at the end of it, contingent on random events that might occur by date $t+1$.

The primary aim of the study is to identify investment strategies that guarantee "long-run survival", i.e., keeping a strictly positive, bounded away from zero, share of market wealth over an infinite time horizon, irrespective of what strategies used by others. The main result demonstrates that Kelly's [95] well-known portfolio rule of "betting one's beliefs" guarantees the property of unconditional survival[4]. Moreover, the strategy satisfying this property is essentially unique: any other strategy of this kind (within a specific class of strategies called *basic*) converges asymptotically to the Kelly rule. This result on asymptotic uniqueness, can be regarded as an analogue of turnpike theorems (see Remark 2.4).

The models we examine in this paper, through a game-theoretic lens, bridge two fundamental paradigms: stochastic dynamic games (strategic frameworks, as pioneered by Shapley [145])[5] and evolutionary game theory (solution concepts).

The modeling framework postulates a dynamic non-cooperative market game, in which the mechanisms of short-term price formation and market clearing follow those of one-shot strategic market games (see Shapley and Shubik [146], Amir et al. [7], Erickson and Pakes [57]). On the conceptual level, this approach relates to the *games of survival*[6] first explored by Milnor and Shapley [120] as a constant-sum stochastic game, which can be treated as a natural game-theoretic analogue of the famous gambler's ruin decision problem (Dubins and Savage [53]). A zero-sum matrix game is played by two players at each stage over an infinite time horizon, the outcome of which determines their wealth dynamics (as a state variable), and the final outcome shows either bankruptcy of one player or a draw. Likewise, Shubik and Whitt [151] analyzed a N-player

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[4] The state of the art in this area related to the Kelly investment criterion is surveyed by MacLean et al. [109].

[5] For more recent developments, see Neyman and Sorin [125]. In view of this paper's focus on long-run survival only, the most closely relevant class of stochastic games would be the one with undiscounted rewards, as discussed by Vieille [160, 161, 162].

[6] For textbook treatments of this class of games, see Luce and Raiffa ([107], Section A8.4), Maitra and Sudderth ([113], Section 7.16). For more recent studies on similar classes of games see Secchi and Sudderth [143] and references therein. Related questions are discussed in Borch [24], Shubik and Thompson [150], Karni and Schmeidler [94].

dynamic market game with one unit of a durable good available per period and a fixed total wealth distributed across the players in exogenous fixed shares. Finally, Giraud and Stahn [77] extended the basic Shapley-Shubik model to a two-period financial economy with uncertainty, allowing short selling, akin to the work presented in Chapter 6 in our setting.

The closest game-theoretic models to our setting are those in capital growth theory considered by Bell and Cover [15, 16], which demonstrate the Kelly portfolio rule as "competitively optimal," established via an appropriate zero-sum game.[7] However, there are two crucial differences among them: 1) their models assume exogenous asset prices within a standard framework of capital growth theory, while we extend that theory to a setting with endogenous price formation mechanism; 2) the fundamental game solution concept we adopt is defined in terms of a property holding almost surely, rather than the traditional notion of a Nash equilibrium involving payoff maximization (defined in terms of expectations).

This study's focus on survival aligns it with the evolutionary approach in the social sciences, a tradition that traces back to Malthus, who inspired Darwin (on the history of these ideas, see Hodgeson [90]). Significant contributions to the field in the 20th century were made by Schumpeter [139], Nelson and Winter [123], and others.

For decades, these ideas were mainly theoretical, until the recent crisis turned them from abstract theoretical considerations into the realm of vital practical importance. The emphasis on safeguarding the survival of financial institutions became a policy priority for governments and regulators.

The focus on the "survival and extinction" of investment strategies in the market selection process connects our research to evolutionary game theory (Weibull [165], Vega-Redondo [159], Samuelson [132], Hofbauer and Sigmund [91], Sandholm [134]), which was initially designed for modeling biological systems and later found fruitful applications in economics. The concept of a survival portfolio rule, stable within the market selection process, is akin to the concepts of evolutionary stable strategies (ESS) introduced by Maynard Smith and Price [118], and Schaffer [136, 137]. However, the market selection mechanism in our research is radically different from the typical schemes in evolutionary game theory, where species or agents undergo repeated random matchings in large populations, leading to their long-run survival or extinction.[8] Standard frameworks in that field deal with models based on a given static game that defines the process of evolutionary dynamics, where players follow relatively simple, predefined algorithms that completely characterize their behavior.

Our model differs in its essence. Although the solution concept we deal with

[7] For related research in capital growth theory, see Kelly [95], Latané [99], Breiman [34], Algoet and Cover [6], Hakansson and Ziemba [82], MacLean et al. [109], Kuhn and Luenberger [98], Ziemba and Vickson [169], MacLean and Ziemba [110], etc. For the history of these ideas initially expressed by Claude Shannon in his lectures on investment problems [144], and for relevant discussions, see Cover [42]. For textbook treatments towards capital growth theory, see Cover and Thomas [43], Chapter 16; Evstigneev et al. [63], Chapter 17.

[8] The evolutionary process may involve random noise (Foster and Young [70], Fudenberg and Harris [71], and Cabrales [39]) and the underlying game may be random (Germano [75]).

is of an evolutionary nature, the notion of the strategy we use is characteristic of the conventional setting in dynamic stochastic games, where a general rule prescribes what action to take based on the history of random states of the world and the observation of all previous play. Players are allowed to employ any such rule, possess all information needed for this purpose, and pursue a clear goal on guaranteed survival. Moreover, it has been shown that the objective of survival in this context aligns with the objective of winning a specific game linked to the original one, i.e., in the sense that "in order to survive you have to win" (for a detailed explanation, see Section 2.4.2).

This paper reviews the study of the following models with short-lived assets in EF.

Firstly, we introduce a model with substantially more general assumptions, serving as the basic framework for the subsequent, more specialized models. A comprehensive exposition of the model structure — e.g., the key concepts, the evolutionary dynamics in the asset market, etc. — is presented. The main focus of this study is to identify the portfolio rules that guarantee an investor to "survive" in the long-run market selection process. It turns out that the existence and asymptotic uniqueness (in a specific class of basic strategies) of survival strategies have been revealed. However, an interesting and more delicate question that we do not consider in this chapter is the analysis of the *single* survivor problem, which will be explored in the subsequent two chapters, under the corresponding, more restrictive assumptions, respectively.

Then, we examine a model with much stronger assumptions: (i) independent and identically distributed (i.i.d.) states of the world, and (ii) a restriction to fixed-mix portfolio rules (prescribing to select investment proportions initially and remain them fixed throughout the entire duration). Under this setting, we obtain two main results: (i) by viewing the model from a different, game-theoretic, perspective and analyzing the Nash equilibrium properties of survival strategies, a unique symmetric almost sure Nash equilibrium can be formed; and as a consequence, (ii) the survival strategy of “betting one’s beliefs” leads an investor employing it to be the single survivor with global stability.

Next, we present a model under Markovian nature, which removes the two strong restrictions mentioned above, and instead, uses a homogeneous discrete-time Markov process to describe the states of the world. The central result shows that an investor who distributes wealth across available assets according to their relative conditional expected payoffs (a direct analogue of the Kelly rule above) is a single survivor, provided this strategy is asymptotically distinct from the CAPM rule.

Further, we extend the model to a market where risky securities and a risk-free asset (which can be regarded as cash and serves as a numeraire) are traded. A key feature of the model is that the payoffs of the risky assets are assumed to depend linearly on the total cash in the previous period invested by all the market participants. And the main discussion is on the questions of existence and asymptotic uniqueness of the survival strategies.

Finally, we consider a model to incorporate short selling (or endogenous asset supply). In this model, investors’ decisions are specified by a pair of vectors: the vector of long positions defined by investment proportions and the vector of short positions defined by physical units. Our primary interest is also in the fundamental questions of the existence and asymptotic uniqueness of a survival strategy. The result shows, the strategy (without short selling) guarantees survival in a market where the rivals of this investor can sell short, and this strategy is asymptotically unique.

The paper is organized as follows. Chapter 1 delivers an introduction for this research direction – EF with short-lived assets. Chapter 2 studies the results of a basic model under general assumptions. Chapter 3 discusses a model with i.i.d. states of the world and fixed-mix strategies setting. Chapter 4 examines a model with Markov nature. Chapter 5 presents a model including a risk-free asset. Chapter 6 deals with a model with short selling. And Chapter 7 concludes.

Chapter 2

Model with General Assumptions

In this chapter, we introduce a model under the most general assumptions, which serves as the basic framework for each special model analyzed in the following chapters. The primary aim is to identify the portfolio rules that allow an investor to "survive" in the long-run market selection process, i.e., to keep a positive, bounded away from zero, share of market wealth over an infinite time horizon. The key results presented in this chapter demonstrate the existence and asymptotic uniqueness of a survival strategy using Kelly portfolio rule. [1]

2.1 The Basic Model

The Model Settings. We consider a market where $K \geq 2$ risky *assets* (*securities*) are traded among $N \geq 2$ *investors* (*traders*). A *portfolio* of investor i at date $t \geq 0$ is represented by a vector $x_t^i := (x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^K$, where $x_{t,k}^i$ stands for the quantity ("physical units") of asset k held in the portfolio x_t^i. All the coordinates of x_t^i are assumed to be non-negative, indicating that short selling is ruled out in this model. The vector $p_t := (p_{t,1}, ..., p_{t,K}) \in \mathbb{R}_+^K$ denotes the market prices of all assets. For each asset $k = 1, ..., K$, the coordinate $p_{t,k}$ in vector p_t expresses the price of one unit of asset k at date t. The market value of investor i's portfolio x_t^i at date t is given by the scalar product $\langle p_t, x_t^i \rangle := \sum_{k=1}^K p_{t,k} x_{t,k}^i$.

The sequence $s_1, s_2, ...$ is an exogenous discrete-time stochastic process with values in a measurable space S, where s_t indicates the state of the world at date t and $s^t := (s_1, ..., s_t)$ represents the history of the process (s_t) up to date t. Suppose that we have $V_{0,k} > 0$ and $V_{t,k}(s^t) > 0$ total units of each asset k available in the market at date 0 and each subsequent period $t = 1, 2, ...,$

[1] This chapter reviews the study by Amir, Evstigneev, and Schenk-Hoppé [4], a fundamental evolutionary finance model for short-lived assets with substantially more general assumptions.

respectively. Assets live for only one period (*short-lived assets*): they are traded at the start of each period $[t, t+1]$ and yield payoffs $A_{t+1,k}$ at the end of it; then the cycle repeats. The payoff $A_{t,k}(s^t) \geq 0$ for each asset k at date $t = 1, 2, ...$ depends, generally, on t and s^t, with the property that these measurable payoff functions satisfy

$$\sum_{k=1}^{K} A_{t,k}(s^t) > 0 \text{ for all } t, s^t. \tag{2.1}$$

Inequality (2.1) implies that in each random scenario, at least one asset yields a strictly positive payoff.

Investors begin with initial endowments $w_0^i > 0$ $(i = 1, 2, ..., N)$, which serve as their budgets at date $t = 0$. For $t \geq 1$, investor i's budget is given by $w_t^i = w_t^i(s^t) := \langle A_t(s^t), x_{t-1}^i(s^{t-1}) \rangle$, where $A_t(s^t) := (A_{t,1}(s^t), ..., A_{t,K}(s^t))$. This budget is formed by the payoffs $A_t(s^t)$ of the assets in investor i's previous portfolio x_{t-1}^i, and is re-invested into the assets available at date t, which will generate payoffs $A_{t+1,k}(s^{t+1})$, $k = 1, ..., K$, at date $t + 1$.

For each $t \geq 0$, each investor $i = 1, 2, ..., N$ selects a vector of *investment proportions* $\lambda_t^i := (\lambda_{t,1}^i, ..., \lambda_{t,K}^i)$, indicating how the available budget w_t^i is allocated across assets. Vectors λ_t^i belong to the unit simplex

$$\Delta^K := \{(a_1, ..., a_K) \in \mathbb{R}_+^K : a_1 + ... + a_K = 1\}.$$

The investment proportions at each $t \geq 0$ are chosen by N investors simultaneously and independently (i.e., a simultaneous-move N-person dynamic game). In this game under consideration, they represent the players' *actions* or *decisions.* For $t \geq 1$, the decisions may depend, generally, on the history $s^t := (s_1, ..., s_t)$ of the process of states of the world and the *history of the game*

$$\lambda^{t-1} := (\lambda_l^i),\ i = 1, ..., N,\ l = 0, ..., t-1,$$

which includes information about all the previous actions taken by all the players. A *portfolio rule*, or an *investment (trading) strategy* Λ^i of investor i, is formed by a vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions $\Lambda_t^i(s^t, \lambda^{t-1})$, $t \geq 1$ with values in Δ^K, prescribing his/her choice of investment proportions at each time period $t \geq 0$. This framework provides a *general* game-theoretic definition of a pure strategy, assuming full information about the game history (containing all players' previous actions) and the knowledge of all the past and present states of the world. Within such broad class of general portfolio rules, we will distinguish those *basic* portfolio rules: Λ_t^i depends only on s^t, and not on the market history λ^{t-1}.

A Description for the Asset Market Dynamics.

1. For date $t = 0$:

   Suppose each investor i has chosen his/her investment proportions $\lambda_0^i = (\lambda_{0,1}^i, ..., \lambda_{0,K}^i) \in \Delta^K$. Then the amount that trader i invests in asset k

is $\lambda^i_{0,k} w^i_0$, and the total amount invested in asset k by all traders equals $\sum_{i=1}^N \lambda^i_{0,k} w^i_0$. The equilibrium price $p_{0,k}$ of each asset k can be determined from

$$p_{0,k} V_{0,k} = \sum_{i=1}^{N} \lambda^i_{0,k} w^i_0, \ k = 1, 2, ..., K. \tag{2.2}$$

The left-hand side of (2.2) shows the total value, expressed in terms of the prevailing price $p_{0,k}$, of the kth asset purchased by all the market participants at $t = 0$ (recall that the quantity of each asset k at date 0 is $V_{0,k}$). On the right-hand side, we have the total money invested in asset k by all the traders.

Then, the investors' portfolios $x^i_0 = (x^i_{0,1}, ..., x^i_{0,K}),\ i = 1, 2, ..., N$, can be determined from the following equations:

$$x^i_{0,k} = \frac{\lambda^i_{0,k} w^i_0}{p_{0,k}}, \ k = 1, 2, ..., K, \ i = 1, ..., N, \tag{2.3}$$

indicating that the current market value $p_{0,k} x^i_{0,k}$ of the kth position in portfolio x^i_0 is equal to the proportion $\lambda^i_{0,k}$ of trader i's budget w^i_0.

2. For date $t \geq 1$:

   Suppose now that each investor i has selected $\lambda^i_t = (\lambda^i_{t,1}, ..., \lambda^i_{t,K}) \in \Delta^K$ at date t. Then the balance between aggregate monetary asset supply and demand leads to the formula that determines the equilibrium prices $p_{t,k}$:

$$p_{t,k} V_{t,k} = \sum_{i=1}^{N} \lambda^i_{t,k} \langle A_t, x^i_{t-1} \rangle, \ k = 1, ..., K, \tag{2.4}$$

   which, in turn, gives the investors' portfolios $x^i_t = (x^i_{t,1}, ..., x^i_{t,K})$:

$$x^i_{t,k} = \frac{\lambda^i_{t,k} \langle A_t, x^i_{t-1} \rangle}{p_{t,k}}, \ k = 1, ..., K, \ i = 1, ..., N. \tag{2.5}$$

   Here, in contrast with the case $t = 0$, the investors' budgets $w^i_t = \langle A_t, x^i_{t-1} \rangle$ at date $t \geq 1$ are not given exogenously as initial endowments, rather they are generated by the payoffs of the previous date's portfolios x^i_{t-1}.

Given a strategy profile $(\Lambda^1, ..., \Lambda^N)$, $\Lambda^i = \{\lambda^i_t(s^t)\}_{t=0}^{\infty}$, of the investors, a path of the market game can be generated by setting $\lambda^i_0 = \Lambda^i_0,\ i = 1, ..., N$,

$$\lambda^i_t = \Lambda^i_t(s^t, \lambda^{t-1}), \ t = 1, 2, ..., \ i = 1, ..., N, \tag{2.6}$$

and by defining p_t and x^i_t recursively according to Eqs. (2.2)–(2.5).[2] The random dynamical system described determines the state of the market at each

[2] Note that one might propose a seemingly more general definition of a strategy, assuming that the traders can use information not about their own and their rivals' decisions, but also about the past prices $p_0, ..., p_{t-1}$ and portfolios x^i_l, $i = 1, ..., N$, $l = 0, ..., t-1$. However, all these data are determined by the history s^{t-1} of the states of the world and the traders' decisions (λ^i_l), $i = 1, ..., N$, $l = 0, ..., t-1$ up to date $t-1$. Thus this broader definition does not lead, in fact, to a more general notion of a strategy.

date $t \geq 1$ as a measurable vector function of s^t:

$$(p_t(s^t); x_t^1(s^t), ..., x_t^N(s^t); \lambda_t^1(s^t), ..., \lambda_t^N(s^t)), \tag{2.7}$$

where $p_t(s^t)$, $x_t^i(s^t)$ and $\lambda_t^i(s^t)$ are the vectors of prices, investors' portfolios and their investment proportions, respectively. (For $t = 0$, these vectors are constant.)

Remark 2.1 Portfolio positions $x_{t,k}^i$ are well-defined by Eqs. (2.3) and (2.5) only if the prices $p_{t,k}$ are strictly positive for all $t \geq 0$, or equivalently, if the aggregate value of demand $\sum_{i=1}^N \lambda_{t,k}^i w_t^i$ for each asset k is strictly positive at each $t \geq 0$:

$$\sum_{i=1}^N \lambda_{0,k}^i w_0^i > 0 \text{ and } \sum_{i=1}^N \lambda_{t,k}^i \langle A_t, x_{t-1}^i \rangle > 0. \tag{2.8}$$

Strategy profiles that satisfy condition (2.8) for all assets throughout the entire recursive procedure defined above are termed *admissible*, and only such strategy profiles will be examined in the following discussion. This assumption guarantees that the random dynamical system under consideration is well-defined, with $p_{t,k} > 0$ for all t and k. By summing up Eqs. (2.3) and (2.5) over $i = 1, ..., N$, we obtain:

$$\sum_{i=1}^N x_{0,k}^i = \frac{\sum_{i=1}^N \lambda_{0,k}^i w_0^i}{p_{0,k}} = \frac{p_{0,k} V_{0,k}}{p_{0,k}} = V_{0,k}, \tag{2.9}$$

and

$$\sum_{i=1}^N x_{t,k}^i = \frac{\sum_{i=1}^N \lambda_{t,k}^i \langle A_t, x_{t-1}^i \rangle}{p_{t,k}} = \frac{p_{t,k} V_{t,k}}{p_{t,k}} = V_{t,k}, \ t \geq 1, \tag{2.10}$$

indicating that the market clears for each asset k at each date $t \geq 0$.

A sufficient condition for a strategy profile to be admissible is provided in hypothesis (2.14) below, which applies to all the strategy profiles we consider. Suppose that one of the traders, e.g., trader 1, follows a *fully diversified* portfolio rule, prescribing to allocate strictly positive proportions $\lambda_{t,k}^1$ to each asset $k = 1, ..., K$. Then, any strategy profile that includes such a portfolio rule is admissible. Indeed, for $t = 0$, we obtain $p_{0,k} \geq V_{0,k}^{-1} \lambda_{0,k}^1 w_0^1 > 0$ from (2.2) and $x_0^1 = (x_{0,1}^1, ..., x_{t,K}^1) > 0$ (coordinatewise) from (2.3). For $t \geq 1$, assuming $x_{t-1}^1 > 0$ and arguing by induction, it gives $\langle A_t, x_{t-1}^1 \rangle > 0$ by virtue of (2.1), which in turn yields $p_t > 0$ from (2.4) and $x_t^1 > 0$ from (2.5), as long as $\lambda_{t,k}^1 > 0$.

Remark 2.2 Some comments on the stochastic control framework underlying the game at hand. There are two general modeling approaches in discrete-time stochastic control theory—both in its conventional, single-agent version, and its game-theoretic setting, where decisions are made by multiple players with different objectives. Models of the first kind are characterized in terms of transition functions (stochastic kernels) specifying the distribution of the state of the system at time $t + 1$ for each given state and control at time t; see e.g. Shapley [145], Bertsekas and Shreve [18], Dynkin and Yushkevich [52]. In models of the second kind (such as the one in this paper), random factors influencing the system are described in terms of an exogenous random process of states of the world, the distribution of which does not depend on the actions of the players. This approach is often associated with the term "stochastic programming" (e.g., Dynkin [51], Rockafellar and Wets [129], Wallace and Ziemba [164], Birge and Louveaux [19]). Although theoretically both approaches are equivalent in many cases (see Dynkin and Yushkevich [52], Section 2.2), in varying contexts one is more natural and convenient than the other. Generally, the latter is preferable when the model possesses properties of convexity, which is characteristic for economic and financial applications. In a stochastic game setting, the second approach has been pursued by Haurie and coauthors (see, e.g., Haurie et al. [86]), with a primary focus on strategies that depend only on the exogenous states of the world. Haurie et al. [86] call them *S-adapted* (adapted to the given filtration S); we call them *basic* in this paper.

Remark 2.3 Some comments on the modeling of short-run equilibrium in this work. The dynamics of the asset market described above are similar to the dynamics of the commodity market as outlined in the classical treatise by Alfred Marshall [115] (Book V, Chapter II "Temporary Equilibrium of Demand and Supply"). Samuelson ([133], pp. 321–323) introduced Marshall's ideas into formal economics, who noticed that in order to study the process of market dynamics by using the Marshallian "temporary equilibrium method," one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our context, investors' portfolios) can be temporarily fixed, while the other set (in our context, the asset prices) can be assumed to rapidly reach the unique state of partial equilibrium.

The above notion of temporary, or moving, equilibrium was first introduced in economics by Marshall. However, in the last decades the term "temporary equilibrium" has been by and large understood differently. For the most part, it was associated with a different concept suggested by Lindahl [100] and Hicks [88], which was further developed in formal settings by Hildenbrand, Grandmont, and others (see, e.g., Grandmont and Hildenbrand [80], Grandmont [78], and Magill and Quinzii [112]). The characteristic feature of the Lindahl-Hicks temporary equilibrium is its formulation in terms of forecasts or beliefs of market participants about the future states of the world. Mathematically, beliefs of economic agents are represented by transition functions (stochastic kernels) that condition the distributions of future states of the world upon the agents' private information.

The models studied in EF do not use information about individual utilities, beliefs and other unobservable agents' characteristics. What matters is the investment strategy as such, rather than the data and the logic on which its choice is based. The results obtained are stated in the form of investment recommendations that use only some fundamental information about the market, in the same spirit as, for example, in the well-known principles of derivative securities pricing (Black-Scholes, Merton, etc.; see, e.g., Evstigneev, et al. [63], Part II).

2.2 The Main Results: Existence and Asymptotic Uniqueness of Survival Portfolio Rules

The Notion of Survival Strategies. Consider an admissible strategy profile $(\Lambda^1, ..., \Lambda^N)$ of the traders and the corresponding path (2.7) of the random dynamical system generated by it. For $t \geq 1$, trader i's wealth is

$$w_t^i = w_t^i(s^t) := \langle A_t(s^t), x_{t-1}^i(s^{t-1}) \rangle, \tag{2.11}$$

and the *total market wealth* equals

$$W_t := \sum_{i=1}^N w_t^i = \sum_{k=1}^N A_{t,k}(s^t) V_{t-1,k}(s^{t-1}) \ (> 0).$$

Our main focus is on the long-run behavior of the *relative wealth,* or the *market shares,*

$$r_t^i := \frac{w_t^i}{W_t}$$

of the investors, i.e., on the asymptotic properties of the sequence of vectors $r_t = (r_t^1, ..., r_t^N)$ as $t \to \infty$.

Given an admissible strategy profile $(\Lambda^1, ..., \Lambda^N)$, we say that the strategy Λ^i (or investor i using it) *survives* with probability one if $\inf_{t \geq 0} r_t^i > 0$ almost surely (a.s.), i.e., for almost all realizations of the process of states of the world (s_t), the relative wealth of investor i is bounded away from zero by a strictly positive random constant. A portfolio rule Λ is called a *survival strategy* if investor i using it survives with probability one regardless of what portfolio rules Λ^j, $j \neq i$, are employed by the other investors.

Explicit Expression for the Survival Strategy Λ^***.** To formulate the central result on survival strategies, define the *relative payoffs* of assets $k = 1, ..., K$ at time t as:

$$R_{t,k}(s^t) := \frac{A_{t,k}(s^t) V_{t-1,k}(s^{t-1})}{\sum_{m=1}^K A_{t,m}(s^t) V_{t-1,m}(s^{t-1})}, \tag{2.12}$$

which collectively form the vector $R_t(s^t) = (R_{t,1}(s^t), ..., R_{t,K}(s^t))$. Consider the investment strategy $\Lambda^* = (\lambda_t^*)$ defined by

$$\lambda_t^*(s^t) := E_t R_{t+1}(s^{t+1}), \tag{2.13}$$

where $E_t(\cdot) = E(\cdot|s^t)$ represents the conditional expectation given s^t (if $t = 0$, then $E_0(\cdot) = E(\cdot)$, i.e., unconditional expectation). The portfolio rule (2.13), depending only on the history s^t of the process (s_t), prescribes to allocate wealth according to the proportions of the conditional expectations of the relative payoffs. It is a generalization of the Kelly portfolio rule of "betting your beliefs" well-known in capital growth theory — see Kelly [95], Breiman [34], Algoet and Cover [6], Hakansson and Ziemba [82], and Thorp [156].

Suppose that for each asset $k = 1, 2, ..., K$,

$$E \ln E_t R_{t+1,k}(s^{t+1}) > -\infty. \tag{2.14}$$

This assumption implies that the conditional expectation $E_t R_{t+1,k} = E(R_{t+1,k}|s^t)$ is strictly positive a.s., and so we can choose a version of this conditional expectation that is strictly positive for all s^t. This version, $\lambda_t^*(s^t)$, will be used in the definition of the portfolio rule (2.13).

Two Theorems on the Existence and Asymptotic Uniqueness of the Survival Strategy Λ^*.

Theorem 2.1 *The portfolio rule Λ^* is a survival strategy.*

Note that Λ^* belongs to the class of basic portfolio rules, where $\lambda_t^*(s^t)$ depends only on the history s^t of the process of states of the world and not on the market history λ^{t-1}. The following theorem demonstrates that within this class, the survival strategy $\Lambda^* = (\lambda_t^*)$ is essentially unique: any other basic survival strategy is asymptotically similar to Λ^*.

Theorem 2.2 *If $\Lambda = (\lambda_t)$ is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \text{ (a.s.)}. \tag{2.15}$$

Here, $||\cdot||$ denotes the Euclidean norm in a finite-dimensional space.

Remark 2.4 Theorem 2.2 is akin to various *turnpike* results in the theory of economic growth, suggesting that all optimal or "quasi-optimal" optimal paths of an economic system incline in the long run to essentially the same route, see, e.g., Dorfman et al. [49], Nikaido [126], McKenzie [119], Arkin and Evstigneev [8]. Survival strategies Λ can be characterized by the property that the wealth w_t^j of any other investors j cannot grow asymptotically faster, with strictly positive probability, than the wealth of investor i using Λ — see Section 2.4.2. The class of such investment strategies shown in Theorem 2.2 is a direct counterpart of Gale's turnpike theorem (Gale [72], Theorem 8) for "good" paths of economic dynamics — paths that cannot be "infinitely worse" than the turnpike; a stochastic version of this result is established in Arkin and Evstigneev [8], Chapter 4, Theorem 6.

The class of basic strategies is *sufficient* in the following sense. Any sequence of market share vectors $r_t = (r_t^1, ..., r_t^N)$ (where $r_t = r_t(s^t)$) generated by strategy profile $(\Lambda^1, ..., \Lambda^N)$ can also be generated by a strategy profile $(\lambda_t^1(s^t), ..., \lambda_t^N(s^t))$ composed of basic portfolio rules, with the corresponding vector functions $\lambda_t^i(s^t)$ defined recursively by (2.6). Thus it is sufficient to prove Theorem 2.1 only for basic portfolio rules; this will imply that the portfolio rule (2.13) survives in competition with any, not necessarily basic, strategies. However, this reasoning cannot be applied to the problem of asymptotic uniqueness of Λ^* in the class of general survival strategies, and Theorem 2.2 fails to hold in this class — a counterexample is provided in Section 2.4.1.

2.3 Proofs of the Main Results

Proof of Theorem 2.1. *1st step.* We start by deriving a system of equations that describe the dynamics of the market shares r_t^i. From Eqs. (2.2)–(2.5) and (2.11), we have

$$p_{t,k}V_{t,k} = \langle \lambda_{t,k}, w_t \rangle,\ x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i V_{t,k}}{\langle \lambda_{t,k}, w_t \rangle}, \tag{2.16}$$

where $\lambda_{t,k} := (\lambda_{t,k}^1, ..., \lambda_{t,k}^N)$ and $w_t := (w_t^1, ..., w_t^N)$. Consequently,

$$w_{t+1}^i = \sum_{k=1}^K A_{t+1,k} x_{t,k}^i = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}. \tag{2.17}$$

By summing up (2.17) over investors $i = 1, ..., N$, we get

$$W_{t+1} = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\sum_{i=1}^N \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle} = \sum_{k=1}^K A_{t+1,k} V_{t,k}. \tag{2.18}$$

Dividing the left-hand side of (2.17) by W_{t+1}, the right-hand side of (2.17) by $\sum_{m=1}^K A_{t+1,m} V_{t,m}$, and applying (2.18) and (2.12), we arrive at the system of equations

$$r_{t+1}^i = \sum_{k=1}^K R_{t+1,k} \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle},\ i = 1, ..., N. \tag{2.19}$$

2nd step. Observe that it is sufficient to prove Theorem 2.1 in the case of $N = 2$ investors. Consider the random dynamical system (2.19) and define

$$\tilde{\lambda}_{t,k}^2(s^t) = \begin{cases} (\lambda_{t,k}^2 r_t^2 + ... + \lambda_{t,k}^N r_t^N)/(1 - r_t^1) & \text{if } r_t^1 < 1, \\ 1/K & \text{if } r_t^1 = 1. \end{cases} \tag{2.20}$$

Then it gives

$$\lambda_{t,k}^2 r_t^2 + ... + \lambda_{t,k}^N r_t^N = (1 - r_t^1)\tilde{\lambda}_{t,k}^2,$$

$$\langle \lambda_{t,k}, r_t \rangle = r_t^1 \lambda_{t,k}^1 + (1 - r_t^1)\tilde{\lambda}_{t,k}^2,$$

and so

$$r_{t+1}^1 = \sum_{k=1}^K R_{t+1,k} \frac{\lambda_{t,k}^1 r_t^1}{r_t^1 \lambda_{t,k}^1 + (1 - r_t^1)\tilde{\lambda}_{t,k}^2}. \tag{2.21}$$

By summing up formula (2.19) over $i = 2, ..., N$, we have

$$1 - r_{t+1}^1 = \sum_{k=1}^K R_{t+1,k} \frac{\tilde{\lambda}_{t,k}^2 (1 - r_t^1)}{r_t^1 \lambda_{t,k}^1 + (1 - r_t^1)\tilde{\lambda}_{t,k}^2}. \tag{2.22}$$

Therefore, the sequence of market share vectors $(r_t^1(s^t))$ generated by the original N-dimensional system (2.19) is identical to the analogous sequence generated by the two-dimensional system (2.21)–(2.22), which corresponds to the game involving two investors $i = 1, 2$ with investment proportions $\lambda^1_{t,k}(s^t)$ and $\tilde{\lambda}^2_{t,k}(s^t)$, respectively.

3rd step. Assume $N = 2$ and $\lambda^1_{t,k} = \lambda^*_{t,k}$. Since $\lambda^*_{t,k} > 0$, our standing hypothesis on the strict positivity of investor 1's investment proportions holds. Defining $\kappa_t = \kappa_t(s^t) := r^1_t(s^t)$, we obtain from (2.19) with $N = 2$:

$$\kappa_{t+1} = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda^1_{t,k}\kappa_t}{\lambda^1_{t,k}\kappa_t + \lambda^2_{t,k}(1-\kappa_t)}.$$

Observe that the process $\ln \kappa_t$ is a submartingale. Indeed, we have

$$\begin{aligned}
E_t \ln \kappa_{t+1} - \ln \kappa_t &= E_t \ln \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda^1_{t,k}}{\lambda^1_{t,k}\kappa_t + \lambda^2_{t,k}(1-\kappa_t)} \\
&\geq E_t \sum_{k=1}^{K} R_{t+1,k} \ln \frac{\lambda^1_{t,k}}{\lambda^1_{t,k}\kappa_t + \lambda^2_{t,k}(1-\kappa_t)} \\
&= \sum_{k=1}^{K} \lambda^1_{t,k} \ln \frac{\lambda^1_{t,k}}{\lambda^1_{t,k}\kappa_t + \lambda^2_{t,k}(1-\kappa_t)} \\
&= \sum_{k=1}^{K} \lambda^1_{t,k} \ln \lambda^1_{t,k} - \sum_{k=1}^{K} \lambda^1_{t,k} \ln[\lambda^1_{t,k}\kappa_t + \lambda^2_{t,k}(1-\kappa_t)] \geq 0 \text{ (a.s.)}.
\end{aligned}$$

Here, we applied Jensen's inequality for the concave function $\ln x$ along with the elementary inequality

$$\sum_{k=1}^{K} a_k \ln a_k \geq \sum_{k=1}^{K} a_k \ln b_k \quad [\ln 0 := -\infty] \tag{2.23}$$

holding for any vectors $(a_1, ..., a_K) > 0$ and $(b_1, ..., b_K) \geq 0$ with $\sum a_k = \sum b_k = 1$ (see Lemma 2.2 below).

Further, we have

$$\begin{aligned}
\kappa_{t+1} &= \kappa_t \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda^1_{t,k}}{\lambda^1_{t,k}\kappa_t + \lambda^2_{t,k}(1-\kappa_t)} \\
&\geq \kappa_t \sum_{k=1}^{K} R_{t+1,k} (\min_m \lambda^1_{t,m}) = \kappa_t (\min_m \lambda^1_{t,m}).
\end{aligned}$$

Since $E \min_m \ln \lambda^1_{t,m} > -\infty$ from assumption (2.14) and κ_0 is a strictly positive non-random number, each of the random variables $0 < \kappa_t \leq 1$ satisfies $E|\ln \kappa_t| < \infty$.

The non-positive submartingale $\ln \kappa_t$ has a finite limit a.s., which implies $\kappa_t \to \kappa_\infty$ (a.s.), where κ_∞ is a strictly positive random variable. Therefore, the sequence $\kappa_t > 0$ is bounded away from zero with probability one, indicating that investor 1 survives almost surely. ■

The proof of Theorem 2.2 is based on the following two lemmas.

Lemma 2.1 *Suppose ξ_t is a submartingale such that $\sup_t E\xi_t < \infty$. Then, the sequence of non-negative random variables $\sum_{t=0}^{\infty}(E_t\xi_{t+1} - \xi_t)$ converges a.s.*

Proof. From the definition of submartingale, we get $\zeta_t := E_t\xi_{t+1} - \xi_t \geq 0$. Further, we obtain

$$\sum_{t=0}^{T-1} E\zeta_t = \sum_{t=0}^{T-1}(E\xi_{t+1} - E\xi_t) = E\xi_T - E\xi_0,$$

and so the *series* $\sum_{t=0}^{T-1} E\zeta_t$ is bounded because $\sup_T E\xi_T < \infty$. Consequently, the series of expectations $\sum_{t=0}^{\infty} E\zeta_t$ of the non-negative random variables ζ_t converges, which in turn implies $\sum_{t=0}^{\infty} \zeta_t < \infty$ a.s., since $E\sum_{t=0}^{\infty} \zeta_t = \sum_{t=0}^{\infty} E\zeta_t$ (see, e.g., Saks [131], Theorem I.12.3). ■

Lemma 2.2 [3]*For any vectors $(a_1, ..., a_K) > 0$ and $(b_1, ..., b_K) \geq 0$ that satisfy $\sum a_k = \sum b_k = 1$, the following inequality holds:*

$$\sum_{k=1}^{K} a_k \ln a_k - \sum_{k=1}^{K} a_k \ln b_k \geq \frac{1}{4}\sum_{k=1}^{K}(a_k - b_k)^2. \tag{2.24}$$

Proof. We have $\ln x \leq x - 1$, which gives $(\ln x)/2 = \ln\sqrt{x} \leq \sqrt{x} - 1$, and thus $-\ln x \geq 2 - 2\sqrt{x}$. From this inequality, we obtain

$$\begin{aligned}\sum_{k=1}^{K} a_k(\ln a_k - \ln b_k) &= -\sum_{k=1}^{K} a_k \ln\frac{b_k}{a_k} \geq \sum_{k=1}^{K} a_k(2 - 2\frac{\sqrt{b_k}}{\sqrt{a_k}}) \\ &= 2 - 2\sum_{k=1}^{K}\sqrt{a_k b_k} = \sum_{k=1}^{K}(a_k - 2\sqrt{a_k b_k} + b_k) \\ &= \sum_{k=1}^{K}(\sqrt{a_k} - \sqrt{b_k})^2.\end{aligned}$$

This leads to (2.24) because $(\sqrt{a} - \sqrt{b})^2 \geq (a-b)^2/4$ for $0 \leq a, b \leq 1$. ■

Proof of Theorem 2.2. Let $\Lambda = (\lambda_t)$ be a basic survival strategy. Assume that investors $i = 1, 2, ..., N-1$ follow the strategy $\Lambda^* = (\lambda_t^*)$ and investor N uses Λ. By summing up (2.19) with $\lambda_t^i = \lambda_t^*$ over $i = 1, ..., N-1$, it gives

$$\hat{r}_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k}\frac{\lambda_{t,k}^*\hat{r}_t^1}{\lambda_{t,k}^*\hat{r}_t^1 + \lambda_{t,k}(1 - \hat{r}_t^1)},$$

[3]Lemma 2.2 follows from an inequality between the *Kullback-Leibler divergence* (generalizing the left-hand side of (2.24)) and the *Hellinger distance* (which reduces to $[\sum(\sqrt{a_k} - \sqrt{b_k})^2]^{1/2}$) — see, e.g., Borovkov [26], Section II.31.

where $\hat{r}_t^1 := r_t^1 + ... + r_t^{N-1}$ is the market share of the group of investors $i = 1, 2, ..., N-1$ and $1 - \hat{r}_t^1 = r_t^N$ is that of investor N. Further, we get

$$1 - \hat{r}_{t+1}^1 = \sum_{k=1}^K R_{t+1,k} \frac{\lambda_{t,k}(1 - \hat{r}_t^1)}{\lambda_{t,k}^* \hat{r}_t^1 + \lambda_{t,k}(1 - \hat{r}_t^1)}.$$

Hence, the dynamics of $\hat{r}_t^1 = r_t^1 + ... + r_t^{N-1}, 1 - \hat{r}_t^1 = r_t^N$ is exactly the same as those of $\hat{r}_t^1, \hat{r}_t^2 = 1 - \hat{r}_t^1$ of two investors $i = 1, 2$ ($N = 2$) with the strategies $(\lambda_t^1) = (\lambda_t^*)$ and $(\lambda_t^2) = (\lambda_t)$, respectively. Since (λ_t) is a survival strategy, the random series $r_t^N = 1 - \hat{r}_t^1 = \hat{r}_t^2$ is bounded away from zero a.s.

As we establishd in the proof of Theorem 2.1 (step 3), the random sequence $\ln \kappa_t := \ln \hat{r}_t^1$ is a non-positive submartingale satisfying

$$\begin{aligned} &E_t \ln \kappa_{t+1} - \ln \kappa_t \\ &\quad \geq \sum_{k=1}^K \lambda_{t,k}^* \ln \lambda_{t,k}^* - \sum_{k=1}^K \lambda_{t,k}^* \ln[\lambda_{t,k}^* \kappa_t + \lambda_{t,k}(1 - \kappa_t)] \text{ (a.s.)}. \end{aligned} \tag{2.25}$$

According to Lemma 2.1, the sequence $\sum(E_t \ln \kappa_{t+1} - \ln \kappa_t)$ of non-negative random variables converges almost surely. Based on inequalities (2.24) and (2.25), this suggests that the sum

$$\sum_{t=0}^\infty \sum_{k=1}^K [\lambda_{t,k}^* - \lambda_{t,k}^* \kappa_t - \lambda_{t,k}(1 - \kappa_t)]^2 = \sum_{t=0}^\infty (1 - \kappa_t)^2 ||\lambda_t^* - \lambda_t||^2 \tag{2.26}$$

is finite with probability one. Since $\inf(1 - \kappa_t) = \inf \hat{r}_t^2 > 0$ a.s., the fact that the sequence in (2.26) converges a.s. implies (2.15). ■

2.4 Discussions on the Main Results

2.4.1 A Counterexample

In this section, an example is provided to illustrate that Theorem 2.2 cannot be naturally extended to the class of general, not necessarily basic, portfolio rules. One might conjecture that the following extension of Theorem 2.2 is valid: if $\Lambda_t^1(\cdot)$ is a general survival strategy of player 1, then for any set of strategies $\Lambda_t^2(\cdot)$,...,$\Lambda_t^N(\cdot)$ employed by players $2, 3, ..., N$, the investment proportion vectors $\lambda_t^1(s^t)$ of investor 1 generated by the strategy profile $(\Lambda_t^1(\cdot), \Lambda_t^2(\cdot), ..., \Lambda_t^N(\cdot))$ will converge to $\lambda_t^*(s^t)$ (in one sense or another) as $t \to \infty$. Below is a counterexample to this conjecture.[4]

A Constructed Model Serving as Counterexample. There are two traders ($N = 2$). The random states of the world $s_1, s_2, ...$ are independent and identically distributed in the state space S consisting of two elements: $S = \{1, 2\}$,

[4]The strategy in this counterexample is constructed in a way that is akin to the *trigger strategies* described in the "folk theorems" of game theory (see, e.g., Myerson [122]).

with probabilities $P\{s_t = 1\} = 1/3$ and $P\{s_t = 2\} = 2/3$. There are two assets $k = 1, 2$ with $V_{t,k}(s^t) = 1$ for all $t, s^t, k = 1, 2$, and $A_{t,k}(s^t) = A_k(s_t)$, where $A(1) = (A_1(1), A_2(1)) = (1, 0)$, $A(2) = (A_1(2), A_2(2)) = (0, 1)$. In this model, $R_{t,k}(s^t) = R_k(s_t)$, where

$$R(1) = (R_1(1), R_2(1)) = (1, 0),\ \ R(2) = (R_1(2), R_2(2)) = (0, 1), \tag{2.27}$$

and

$$\lambda^* = E_t R(s_{t+1}) = ER(s_{t+1}) = (1/3, 2/3).$$

Consider some vector $0 < b \neq \lambda^*$ in the unit simplex Δ^2 and a set $\Gamma \subseteq \Delta^2$ such that

$$E\frac{\langle a, R(s)\rangle}{\langle b, R(s)\rangle} \leq 1,\ \ a \in \Gamma. \tag{2.28}$$

As an example, let $b = (b_1, b_2) = (1/4, 3/4)$ and

$$\Gamma = \{a = (a_1, 1 - a_1) :\ 0 \leq a_1 \leq 1/4\}.$$

Then (2.28) holds, since

$$E\frac{\langle a, R(s)\rangle}{\langle b, R(s)\rangle} = \frac{1}{3}\frac{a_1}{b_1} + \frac{2}{3}\frac{a_2}{b_2} = \frac{4}{3}a_1 + \frac{8}{9}(1 - a_1) = \frac{4a_1 + 8}{9} \leq 1,$$

as long as $a_1 \leq 1/4$.

Define the strategy $\Lambda_t^1(s^t, \lambda^{t-1})$, $t = 0, 1, ...$, of player 1 as follows. For $t = 0$ put $\Lambda_0^1 = b$. For $t \geq 1$, define

$$\Lambda_t^1(\lambda_0^2, ..., \lambda_{t-1}^2) = \begin{cases} b\ , & \text{if } \lambda_l^2 \in \Gamma,\ l = 0, ..., t-1; \\ \lambda^*\ , & \text{otherwise.} \end{cases} \tag{2.29}$$

The action prescribed by this strategy is independent of s^t and player 1's previous actions λ_l^1, $l = 0, ..., t-1$, depending only on the previous actions λ_l^2, $l = 0, ..., t-1$ of player 2, the rival of player 1. This strategy says, if the rival at least once selects a vector of investment proportions outside the set Γ, then 1 immediately adopts λ^* and sticks with it forever. Otherwise, 1 always plays b.

Proposition 2.1 *The portfolio rule $\Lambda_t^1(\lambda_0^2, ..., \lambda_{t-1}^2)$ is a survival strategy. There exists a whole class $\mathcal{L}$ of strategies $\Lambda_t^2(\cdot)$ of player 2 such that the vectors $\lambda_t^1(s^t)$ of investment proportions of player 1 generated by the strategy profile $(\Lambda_t^1(\cdot), \Lambda_t^2(\cdot))$ coincide for all t with the constant vector $b \neq \lambda^*$ (and hence do not converge to λ^* in any sense). The class $\mathcal{L}$ includes all the basic strategies $(\lambda_t^2(s^t))$ such that $\lambda_t^2(s^t) \in \Gamma$ for all t, s^t.*

Proof. Consider any strategy $\Lambda_t^2(\cdot)$ of player 2. The strategy profile $(\Lambda_t^1(\cdot), \Lambda_t^2(\cdot))$ is admissible because $\Lambda_t^1(\cdot) > 0$. Therefore, the path of the game is well-defined and the market shares r_t^1, r_t^2 of players $i = 1, 2$ satisfy

$$r_{t+1}^i = \sum_{k=1}^{2} R_k(s_{t+1}) \frac{\lambda_{t,k}^i r_t^i}{\lambda_{t,k}^1 r_t^1 + \lambda_{t,k}^2 r_t^2},\ \ i = 1, 2 \tag{2.30}$$

(see (2.19)), where $(\lambda^i_{t,1}, \lambda^i_{t,2}) = (\lambda^i_{t,1}(s^t), \lambda^i_{t,2}(s^t))$ are the sequences of vectors of investment proportions of players $i = 1, 2$ generated by the strategy profile $(\Lambda^1_t(\cdot), \Lambda^2_t(\cdot))$. We need to show that r^1_t is bounded away from zero (a.s.), or equivalently, that the random sequence $\gamma_t := r^2_t / r^1_t$ is bounded above (a.s.).

By virtue of (2.27), we obtain

$$\gamma_{t+1} = \frac{\lambda^2_{t,s_{t+1}} r^2_t / (\lambda^1_{t,s_{t+1}} r^1_t + \lambda^2_{t,s_{t+1}} r^2_t)}{\lambda^1_{t,s_{t+1}} r^1_t / (\lambda^1_{t,s_{t+1}} r^1_t + \lambda^2_{t,s_{t+1}} r^2_t)} = \frac{\lambda^2_{t,s_{t+1}} r^2_t}{\lambda^1_{t,s_{t+1}} r^1_t} = \frac{\langle \lambda^2_t, R(s_{t+1}) \rangle}{\langle \lambda^1_t, R(s_{t+1}) \rangle} \gamma_t. \quad (2.31)$$

Denote by $\tau = \tau(s_1, s_2, ...)$ the moment of time such that $\lambda^2_\tau \notin \Gamma$,$\lambda^2_{\tau-1} \in \Gamma$, ...,$\lambda^2_0 \in \Gamma$, if such a moment of time exists; otherwise, put $\tau = \infty$. Hence, $\tau < \infty$ is the first moment of time when the random sequence λ^2_t leaves the set Γ, and $\tau = \infty$ if this sequence always stays in Γ.

If $\tau = \infty$, then $\gamma_{t+1} = \beta_{t+1} \gamma_0$, where

$$\beta_{t+1} = \frac{\langle \lambda^2_t, R(s_{t+1}) \rangle}{\langle b, R(s_{t+1}) \rangle} ... \frac{\langle \lambda^2_0, R(s_1) \rangle}{\langle b, R(s_1) \rangle}$$

(see (2.29) and (2.31)). If $\tau < \infty$, then for each $t > \tau$ we have

$$\gamma_{t+1} = \frac{\langle \lambda^2_t, R(s_{t+1}) \rangle}{\langle \lambda^*, R(s_{t+1}) \rangle} ... \frac{\langle \lambda^2_{\tau+1}, R(s_{\tau+2}) \rangle}{\langle \lambda^*, R(s_{\tau+2}) \rangle} \cdot \frac{\langle \lambda^2_\tau, R(s_{\tau+1}) \rangle}{\langle b, R(s_{\tau+1}) \rangle} ... \frac{\langle \lambda^2_0, R(s_1) \rangle}{\langle b, R(s_1) \rangle} \gamma_0 = \alpha_{t+1} \delta,$$

where

$$\alpha_{t+1} := \frac{\langle \lambda^2_t, R(s_{t+1}) \rangle}{\langle \lambda^*, R(s_{t+1}) \rangle} ... \frac{\langle \lambda^2_0, R(s_1) \rangle}{\langle \lambda^*, R(s_1) \rangle}, \ \delta := \frac{\langle \lambda^*, R(s_{\tau+1}) \rangle}{\langle b, R(s_{\tau+1}) \rangle} ... \frac{\langle \lambda^*, R(s_1) \rangle}{\langle b, R(s_1) \rangle} \gamma_0.$$

Fix some element $\bar{a} \in \Gamma$, and define $\bar{\lambda}^2_t := \bar{a}$ if $\lambda^2_t \notin \Gamma$ and $\bar{\lambda}^2_t := \lambda^2_t$ if $\lambda^2_t \in \Gamma$. Then if $\tau = \infty$, we obtain $\beta_{t+1} = \bar{\beta}_{t+1}$ $(t = 0, 1, ...)$, where

$$\bar{\beta}_{t+1} := \frac{\langle \bar{\lambda}^2_t, R(s_{t+1}) \rangle}{\langle b, R(s_{t+1}) \rangle} ... \frac{\langle \bar{\lambda}^2_0, R(s_1) \rangle}{\langle b, R(s_1) \rangle}$$

because in this case $\lambda^2_t \in \Gamma$, and so $\bar{\lambda}^2_t = \lambda^2_t$, for all t. Observe that the process $\bar{\beta}_t$ is a non-negative supermartingale (with respect to the system of σ-algebras generated by s^t). Indeed, in view of (2.28),

$$E(\bar{\beta}_{t+1} | s^t) = \bar{\beta}_t E \frac{\langle a, R(s) \rangle}{\langle b, R(s) \rangle} \leq \bar{\beta}_t$$

where $a := \bar{\lambda}^2_t(s^t) \in \Gamma$. Thus the process $\bar{\beta}_t$ converges a.s., and hence is bounded a.s.. Consequently, if $\tau = \infty$, the process $\gamma_t = \beta_t \gamma_0 = \bar{\beta}_t \gamma_0$ is bounded a.s..

Further, observe that the sequence α_t is a non-negative martingale. Indeed, if we fix s^t and put $a := (a_1, a_2) = \lambda^2_t(s^t)$, then

$$E(\alpha_{t+1} | s^t) = \alpha_t E \frac{\langle a, R(s) \rangle}{\langle \lambda^*, R(s) \rangle} = \alpha_t$$

because

$$E\frac{\langle a, R(s)\rangle}{\langle \lambda^*, R(s)\rangle} = \frac{1}{3}\frac{\langle a, R(1)\rangle}{\langle \lambda^*, R(1)\rangle} + \frac{2}{3}\frac{\langle a, R(2)\rangle}{\langle \lambda^*, R(2)\rangle} = \frac{1}{3}\frac{a_1}{1/3} + \frac{2}{3}\frac{a_2}{2/3} = 1.$$

Therefore α_t is bounded a.s., and so if $\tau < \infty$, then

$$\sup_{t>\tau} \gamma_{t+1} = \delta \sup_{t>\tau} \alpha_{t+1} < \infty \text{ (a.s.)}.$$

Thus we have obtained that the process γ_t is bounded a.s. both when $\tau = \infty$ and when $\tau < \infty$. This proves that $\Lambda_t^1(\cdot)$ is a survival strategy.

It remains to observe that if player 2 uses a strategy (e.g., a basic one) for which $\lambda_t^2(s^t) \in \Gamma$ for all t, s^t, then we have $\lambda_t^1 = b \neq \lambda^*$ for all t, s^t. ■

2.4.2 Strategies of Survival as Unbeatable Strategies

The Notion of Unbeatable Strategies. Consider an abstract game of N players $i = 1, ..., N$ choosing strategies Λ^i from some given sets. Suppose $w^i = w^i(\Lambda^1, ..., \Lambda^N)$ is the outcome of the game for player i corresponding to the strategy profile $(\Lambda^1, ..., \Lambda^N)$, where possible outcomes w^i are elements of a set $\mathcal{W}^i$. Assume that a preference relation

$$w^i \succcurlyeq_{ij} w^j, w^i \in \mathcal{W}^i,\ w^j \in \mathcal{W}^j,\ i \neq j,$$

is given, allowing for the comparison of the relative performance of players i and j. A strategy Λ of player i is called *unbeatable* if for any admissible strategy profile $(\Lambda^1, \Lambda^2, ..., \Lambda^N)$ in which $\Lambda^i = \Lambda$, we have

$$w^i(\Lambda^1, \Lambda^2, ..., \Lambda^N) \succcurlyeq_{ij} w^j(\Lambda^1, \Lambda^2, ..., \Lambda^N) \text{ for all } j \neq i. \tag{2.32}$$

Thus, if player i employs the strategy Λ, then this investor cannot be outperformed by any of his/her rivals $j \neq i$, irrespective of what strategies they use.

Let us return to the game we consider. Denote by $\mathcal{W}$ the set of sequences of positive random variables, and define the following relation $\succcurlyeq$ between two sequences $\alpha = (\alpha_t)$ and $\beta = (\beta_t)$ in $\mathcal{W}$. We write $\beta \succcurlyeq \alpha$ (or $\alpha \preccurlyeq \beta$) if there exists a random variable $H > 0$ such that $\alpha_t \leq H\beta_t$ (a.s.), meaning that almost surely, the sequence α does not grow asymptotically faster than β. Consider the asset market game with N players, each having initial endowments $w_0^1 > 0, ..., w_0^N > 0$. Given an admissible strategy profile $(\Lambda^1, ..., \Lambda^N)$ of N investors, we obtain a sequence $w^i := (w_0^i, w_1^i, ...)$ of positive random variables for each $i = 1, 2, ..., N$, where w_t^i stands for investor i's wealth at date t. As we have seen, the evolution of the vectors $w_t := (w_t^1, ..., w_t^N)$ is governed by the random dynamical system (2.17). We can view the sequence $w^i = w^i(\Lambda^1, ..., \Lambda^N)$ as an outcome of the game for player i corresponding to the strategy profile $(\Lambda^1, ..., \Lambda^N)$, and apply the above definition with $\mathcal{W}^i = \mathcal{W}$ and the preference relation $\succcurlyeq$ (independent of i and j) to this specific game at hand. Its meaning in the present context is as follows: a portfolio rule Λ is an unbeatable strategy of investor i if it guarantees the asymptotically fastest growth rate of investor i's wealth, irrespective of what strategies used by the other investors.

The Relation between "Survival" and "Unbeatable".

Proposition 2.2 *A portfolio rule is a survival strategy if and only if it is unbeatable.*

Proof. Since the market shares are expressed as $r_t^i = w_t^i / W_t$, relations (2.32) hold if and only if

$$r^i \succcurlyeq r^j \text{ for all } j \neq i. \tag{2.33}$$

If the market share r_t^i of investor i satisfies $r_t^i \geq c$ (a.s.), where c is a strictly positive random variable, then $w_t^i \geq cW_t \geq cw_t^j$ (a.s.) for all j. Thus $w_t^j \leq c^{-1} w_t^i$ (a.s.), and so (2.32) holds. Conversely, if $w_t^j \leq Hw_t^i$ (a.s.) for some random variable $H > 0$, then $W_t \leq [(N-1)H + 1]w_t^i$ (a.s.), which yields $r_t^i \geq [(N-1)H+1]^{-1}$ (a.s.). ■

Remark 2.5 The idea of an unbeatable (or *winning*) strategy was central in the pre-Nash game theory. At those times, solving a game meant primarily finding a winning strategy that yields the player who uses it a superiority over any other players who do not use it, see e.g., Bouton [33], Borel [25]. Extensive work has been done around the notion of *determinacy* of a game, which is formulated in terms of winning strategies: a game is determined if one of the players has a winning strategy.[5] The problem of finding a winning strategy in two-person zero-sum games essentially reduces to finding an equilibrium (minimax) strategy. For this reason, questions of this kind were typically formulated and examined in terms of minimax strategies in zero-sum games, especially after the seminal paper by von Neumann [163]. In the 1950s, when game theory started developing primarily as a mathematical framework of economic modeling, non-zero sum N-player games came to the fore, and the notion of Nash equilibrium became central to the field.

The concept of an unbeatable strategy as such reemerged in theoretical biology, serving as the starting point for the development of evolutionary game theory. Hamilton [84] used this notion, and the term "unbeatable strategy" — though without rigorous formalization — in his paper on the analysis of sex ratios in populations of certain species. Later, Maynard Smith and Price [118] formalized Hamilton's idea, but at the same time somewhat modified its content. The notion commonly known as the Maynard Smith's evolutionary stable strategy (ESS) can be described as a *conditionally unbeatable* strategy. It cannot be beaten as long as the rival is "weak enough." In the context of evolutionary biology, Maynard Smith's ESS refers to a strategy that cannot be beaten if the fraction of the rivals (mutants) in the population is sufficiently small.[6] This definition assumes an infinite population, as it involves considering arbitrarily

[5] For relavant studies, see, e.g., Zermelo [166], Schwalbe and Walker [140], Gale and Stewart [73], Martin [116], Telgársky [153].

[6] An unconditional variant of the Maynard Smith's ESS was considered by Kojima [97].

small fractions of the population. Versions of Maynard Smith's ESS applicable to finite populations were considered by Schaffer [136, 137]. Schaffer's notions of ESS, the weaker and the stronger ones, are also conditionally unbeatable strategies. The former assumes that the population contains only *one* mutant, while the latter considers the presence of several identical mutants. Having in mind this, evolutionary, branch of the history of unbeatable strategies, it is not surprising that they have appeared in our EF context.

Chapter 3

Model with I.I.D. States of the World and Fixed-Mix Strategies

An interesting and more delicate question, which we did not consider in Chapter 2, is the analysis of conditions on the strategy profile of investors under which the Kelly rule Λ^* is the *single* survivor in the market selection process. More precisely, suppose investor 1 uses Λ^*, while all the others use some other strategies $\Lambda^2, ..., \Lambda^N$ distinct from Λ^*. Under what conditions on $(\Lambda^2, ..., \Lambda^N)$, the market share of investor 1 tends to one almost surely (so that he/she not only survives, but drives all the others out of the market) in the long run?

This question was examined by Blume and Easley [21] in the case of Arrow securities, fixed-mix strategies and i.i.d. states of the world s_t. In this case, when the market is complete, the analysis essentially reduces to the classical capital growth theory with exogenous asset returns (Algoet and Cover [6]), from which the results on the single survivor follow. In models of incomplete markets, the analogous question was studied by Evstigneev et al. [58] (a finite space S with i.i.d. s_t, fixed-mix strategies only) and Amir et al. [2] (a finite-state stationary Markov s_t, basic strategies only), which we will present in this chapter and the next chapter, respectively. However, the analysis of the single survivor problem in the most general framework given in Chapter 2, is still open for further research.

In this chapter, a special case of the model (as mentioned above, i.i.d. states of the world and fixed-mix strategies) is given in Section 3.1. Then, we analyze it from a different, game-theoretic, perspective in Section 3.2. We consider a game associated with this model in which the payoffs of the players are defined in terms of the growth rates of their relative wealth. It turns out that in the game under consideration the survival strategy Λ^* forms with probability one a unique symmetric Nash equilibrium.

This study establishes a link between the EF models and the classical ones, e.g., game-theoretic models of asset markets dealing with relative wealth of investors considered by Bell and Cover [15, 16]). In those models, the notion of Nash equilibrium was defined in terms of the expectations of random payoffs. Whereas, we consider a different (stronger) solution concept: almost sure Nash equilibrium. According to this definition of an equilibrium strategy, any unilateral deviation from it leads to a decrease in the random payoff with probability one, not just in the expected payoff. For formulating this evolutionary solution concept, a strategy strictly dominating the market at an exponential rate, we introduce the Lyapunov exponent of the relative growth of wealth of an investor as the objective function. Here, "relative" refers to the fraction of wealth of the investor and the total wealth of the group of his/her rivals.

And so, the result of a single survivor with global stability becomes a direct consequence of the conclusion of exponential market domination in the almost sure Nash equilibrium that we proved. The relevant introduction for this part is outlined in Section 3.3. [1]

3.1 The Model

A Summary on Changes in Model Settings. The model framework is substantially consistent with the general model given in Section 2.1, only with the addition of two stronger assumptions as follows:

- uncertainty is modeled by a sequence s_t, $t = 1, 2, ...,$ of independent and identically distributed (i.i.d.) random variables (states of the world), taking values in a finite space S. In each period the states are drawn in accordance with the probability distribution $p = (p_s)_{s\in S}$, $p_s > 0$. We use P to denote the product measure of p constructed on the space of sample paths $(s_t)_{t=1,2,...}$, and E denotes the expectation with respect to the measure P.

  Accordingly, the model has the following changes:

  - the total amount of security k in the market is $V_k > 0$.
  - The payoff $A_{t,k} = A_k(s_t) \geq 0$ of asset $k = 1, 2, ..., K$ at date $t = 1, 2, ...$ depends only on the state of the world s_t at that date. The functions $A_k(s)$, $s \in S$, are measurable and satisfy

    $$\sum_{k=1}^{K} A_k(s) > 0 \text{ for all } s. \tag{3.1}$$

  - Assume the *no redundant assets* assumption holds: the functions $A_1(s), ..., A_K(s)$ (equivalently, the functions $R_1(s), ..., R_K(s)$) are linearly independent with respect to the probability distribution of s_t,

[1] The analysis in Section 3.2 is based on the work of Belkov, Evstigneev, Hens, and Xu [14], and the analysis in Section 3.3 is based on the paper by Evstigneev, Hens, and Schenk-Hoppé [58].

i.e. the equality $\sum \gamma_k A_k(s_t) = 0$ holding a.s. for some constants γ_k implies that $\gamma_1 = ... = \gamma_K = 0$.

- we focus exclusively on fixed-mix strategies, and a "strategy" ("portfolio rule") will always mean a "fixed-mix strategy" ("fixed-mix portfolio rule").

For the asset market dynamics, the equilibrium prices $p_{t,k}$ of each asset k and the investors' portfolios $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i)$, $i = 1, 2, ..., N$ for $t \geq 0$ can be determined from the recursive process in terms of Eqs. (2.2)–(2.5) as we stated in Chapter 2. Note that here $V_k = V_{t,k}$, $\lambda_k^i = \lambda_{t,k}^i$, $k = 1, 2, ..., K$, $i = 1, 2, ..., N$ for $t \geq 0$.

It follows from (2.10) that the total market wealth, $W_t := w_t^1 + ... + w_t^N$, is expressed as follows:

$$W_t = \sum_{i=1}^{N} \langle A_t, x_{t-1}^i \rangle = \sum_{i=1}^{N} \sum_{k=1}^{K} A_{t,k} x_{t-1,k}^i = \sum_{k=1}^{K} A_{t,k} \sum_{i=1}^{N} x_{t-1,k}^i = \sum_{k=1}^{K} A_{t,k} V_k,$$

which implies in view of (3.1) that $W_t > 0$. Note that in view of $A_{t,k} = A_k(s_t)$, so we have $W_t(s^t) = W(s_t)$, similarly $R_k(s^t) = R_k(s_t)$, as what we gives below.

Since we are now considering i.i.d. states of the world, there will be a minor change (i.e., from the conditional expectation form to the unconditional one) to the sufficient condition for a strategy profile to be admissible, which follows from the assumption that

$$ER_k(s_t) > 0$$

for each $k = 1, 2, ..., K$. Assume that one of the investors, e.g., investor 1, adopts a fully diversified portfolio rule, prescribing to invest into all the assets in strictly positive proportions $\lambda_{t,k}^1$, $k = 1, 2, ..., K$, then any strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we have $p_{0,k} \geq V_k^{-1} \lambda_{0,k}^1 w_0^1 > 0$ and $x_0^1 = (x_{0,1}^1, ..., x_{0,K}^1) > 0$ (coordinatewise). Assuming that $x_{t-1}^1 > 0$ and arguing by induction, we obtain

$$w_t^1 = \langle A_t, x_{t-1}^1 \rangle > 0,$$

which in turn yields $p_t > 0$ and $x_t^1 > 0$, as long as $\lambda_{t,k}^1 > 0$.

As previously defined, a strategy Λ^i of trader i is called a survival strategy (portfolio rule) if for any strategies Λ^j of investors $j \neq i$, the *market share* $r_t^i := \frac{w_t^i}{W_t}$ of trader i is strictly positive and bounded away from zero almost surely:

$$\inf_t r_t^i > 0 \quad \text{(a.s.)}.$$

Explicit Expression for the Survival Strategy Λ^*. Define the *relative payoffs* by

$$R_k(s) := \frac{A_k(s) V_k}{\sum_{m=1}^{K} A_m(s) V_m} \tag{3.2}$$

and put

$$\lambda_k^* = ER_k(s_t)\Big[= \sum_{s\in S} p_s R_k(s) > 0\Big],\ k = 1, .., K. \tag{3.3}$$

Consider the constant (independent of t and s^t) strategy $\lambda_t^* = \lambda^* = (\lambda_1^*, ..., \lambda_K^*)$, where the numbers λ_k^*, $k = 1, ..., K$, are given by formula (3.3). The portfolio rule specified by (3.3) prescribes to allocate wealth across assets according to the proportions of the expected relative payoffs, which do not depend on t because the random elements s_t are i.i.d.. Strategies of this type are called *fixed-mix* (or *constant proportions*) portfolio rules, prescribing to choose investment proportions at time 0 and keep them fixed over the whole infinite time horizon.

Same Results for the Existence and Asymptotic Uniqueness of the Survival Strategy Λ^*.

Theorem 3.1 *The portfolio rule Λ^* is a survival strategy.*

Note that although Λ^* is a fixed-mix strategy, it guarantees unconditional survival in competition with all, not necessarily fixed-mix strategies. And the following theorem shows that the survival strategy $\Lambda^* = (\lambda_t^*)$ is essentially unique: any other survival portfolio rule is in a sense asymptotically similar to Λ^*.

Theorem 3.2 *If $\Lambda = (\lambda_t)$ is a survival strategy, then*

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \quad \text{(a.s.)}. \tag{3.4}$$

For proofs of these results, see Theorem 2.1 and Theorem 2.2 in Section 2.2.

3.2 A Classical Game-theoretic Perspective: Nash Equilibrium Properties of Survival Portfolio Rules

3.2.1 The Main Result: Almost Sure Nash Equilibrium

The Notion of Symmetric Almost Sure Nash Equilibrium. Given an admissible strategy profile $(\lambda^1, ..., \lambda^N)$, the performance of a strategy λ^i employed by investor i will be characterized by the following random variable

$$\xi^i := \limsup_{t\to\infty} \frac{1}{t} \ln \frac{w_t^i}{\sum_{j\neq i} w_t^j}, \tag{3.5}$$

generally, taking values in $[-\infty, +\infty]$. The expression $w_t^i / \sum_{j\neq i} w_t^j$ in (3.5) is the *relative wealth* of player i against the group $\{j : j \neq i\}$ of i's rivals. The random variable $\xi^i = \xi^i(s^\infty; \lambda^1, ..., \lambda^N)$ depends on the strategy profile $(\lambda^1, ..., \lambda^N)$ and on the whole history $s^\infty := (s_1, s_2, ...)$ of states of the world from time 1 to ∞. In the game under consideration, ξ^i plays the role of the (random) *payoff function* of player i.

We shall say that a strategy $\bar{\lambda}$ forms a *symmetric Nash equilibrium almost surely* (a.s.) if

$$\xi^i(s^\infty; \bar{\lambda}, ..., \bar{\lambda}) \geq \xi^i(s^\infty; \bar{\lambda}, ..., \lambda, ..., \bar{\lambda}) \quad \text{(a.s.)} \tag{3.6}$$

for each i, each strategy λ of investor i and each set of initial endowments $w_0^1 > 0, ..., w_0^N > 0$. The Nash equilibrium is termed *strict* if the inequality in (3.6) is strict for any $\bar{\lambda}$.

Recall that we consider only those admissible strategy profiles. Clearly, if all the players use the same strategy $\bar{\lambda}$, then the strategy profile $(\bar{\lambda}, ..., \bar{\lambda})$ is admissible if and only if the vector $\bar{\lambda}$ is strictly positive (see defining formula (2.8) in Section 2.1).

A Unique Symmetric Almost Sure Nash Equilibrium Formed by Λ^*.

Theorem 3.3 *The portfolio rule λ^* forms a unique symmetric Nash equilibrium almost surely. If an investor i uses any strategy λ distinct from λ^*, then*

$$\xi^i(s^\infty; \lambda^*, ..., \lambda, ..., \lambda^*) < \xi^i(s^\infty; \lambda^*, ..., \lambda^*) = 0 \quad \textit{(a.s.)}, \tag{3.7}$$

and so the Nash equilibrium formed by strategy λ^* is strict.

The uniqueness of λ^* is understood in the following sense: if λ is a portfolio rule that forms a symmetric Nash equilibrium a.s., then $\lambda = \lambda^*$.

Remark 3.1 We conclude this section with a comment on the definition of the Lyapunov exponent (3.5) in terms of the variables $w_t^i / \sum_{j \neq i} w_t^j$. In the Evolutionary Finance literature, relative wealth is typically defined as $r_t^i = w_t^i / \sum_{j=1}^N w_t^j$ (the market share of investor i), which measures the relative wealth of player i compared to all investors, not just i's rivals. In many cases, results can be equivalently formulated both in terms of relative wealth—as it is defined in this paper—and market shares. However, this is *not* the case in the context of the present work. The consideration of the Lyapunov exponent

$$\eta^i := \limsup_{t \to \infty} \frac{1}{t} \ln \frac{w_t^i}{\sum_{j=1}^N w_t^j} \tag{3.8}$$

leads to a trivial notion of a Nash equilibrium. If the payoff functions of the players $i = 1, .., N$ in the market game are characterized by (3.8), then *any* completely mixed strategy λ forms a symmetric Nash equilibrium. Indeed, $\eta^i(\lambda, ..., \lambda) = 0$ because if all the investors adopt the same strategy, their market shares remain constant. On the other hand, η^i is always non-positive, and so $\eta^i(\lambda^1, ..., \lambda^K) \leq 0 = \eta^i(\lambda, ..., \lambda)$, which implies that the strategy profile $(\lambda, ..., \lambda)$ is a Nash equilibrium.

3.2.2 Proof of the Main Result

The proof of Theorem 3.3 is based on an auxiliary result, Lemma 3.1 below. For any $\lambda = (\lambda_1, ..., \lambda_K) \in \Delta^K$ and $\kappa \in (0, 1]$ put

$$F(\lambda, \kappa, s) := \frac{\sum_{k=1}^K R_k(s) \dfrac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k(1-\kappa)}}{\sum_{k=1}^K R_k(s) \dfrac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)}}, \tag{3.9}$$

where $\lambda_k^* > 0$ are defined by (3.3). Observe that the nominator A of the fraction $F(\lambda, \kappa, s) =: A/B$ is not less than

$$c := \min\{\lambda_1^*, ..., \lambda_K^*\},$$

while the denominator B is not greater than c^{-1}. Consequently,

$$c^2 \le F(\lambda, \kappa, s) \le +\infty. \tag{3.10}$$

Lemma 3.1 For any $\lambda \in \Delta^K$ distinct from λ^*, there exist numbers $H > 0$ and $\delta > 0$ such that

$$E \min\{H, \ln F(\lambda, \kappa, s)\} \ge \delta \tag{3.11}$$

for all $\kappa \in (0, 1]$.

The proof of Lemma 3.1 is routine but rather lengthy, thus it is relegated to the Appendix A.

Remark 3.2 A comment about the function $F(\lambda, \kappa, s)$ is in order. An expression involving F appears in the random dynamical system (3.12) analyzed in the course of the proof of Theorem 3.3. A key step in the proof lies in the application of the law of large numbers to the sequence of martingale differences $G_t^H - E_{t-1} G_t^H$, where G_t^H is defined in terms of the truncation of F by some constant H (see (3.15)). As commonly understood, merely the finiteness of expectations is not enough for the validity of this version of the law of large numbers, but the boundedness of the random variables is fully sufficient – hence the truncation of F by H.

Proof of Theorem 3.3. To prove Theorem 3.3, it suffices to consider the case of two investors ($N = 2$) because the group of $N - 1$ investors who all use λ^* is equivalent in terms of wealth dynamics to one, using the strategy λ^* and possessing the aggregate wealth of this group (see Section 2.3, Proof of Theorem 2.1, 2^{nd} step). Now suppose investors 1 and 2 use the strategies λ^* and $\lambda \ne \lambda^*$, respectively. Put

$$z_t := w_t^1 / w_t^2 = r_t^1 / r_t^2.$$

Since $\lambda^* > 0$, the variable z_t takes on its values in $(0, +\infty]$. The sequence (z_t) satisfies

$$z_{t+1} = z_t \frac{\sum_{k=1}^K R_{t+1,k} \dfrac{\lambda_k^*}{\lambda_k^* r_t^1 + \lambda_k(1 - r_t^1)}}{\sum_{k=1}^K R_{t+1,k} \dfrac{\lambda_k}{\lambda_k^* r_t^1 + \lambda_k(1 - r_t^1)}} \tag{3.12}$$

(see Eqs. (2.21)(2.22)). To verify (3.7), it is sufficient to show

$$\liminf_{t\to\infty} t^{-1} \ln z_t > 0 \text{ (a.s.)}. \tag{3.13}$$

Indeed, if inequality (3.13) holds, then

$$\begin{aligned}\xi^2(\lambda^*, \lambda) &= \limsup_{t\to\infty} t^{-1} \ln \frac{w_t^2}{w_t^1} = \limsup_{t\to\infty} t^{-1} \ln \frac{r_t^2}{r_t^1} = \limsup_{t\to\infty}(-t^{-1} \ln z_t) \\ &= -\liminf_{t\to\infty}(t^{-1} \ln z_t) < 0 = \xi^2(\lambda^*, \lambda^*) \text{ (a.s.)}.\end{aligned}$$

We shall now verify (3.13). Put $G_t = \ln z_t - \ln z_{t-1}$, then

$$\sum_{t=1}^{T} G_t = \sum_{t=1}^{T}(\ln z_t - \ln z_{t-1}) = \ln z_T - \ln z_0.$$

Thus it is sufficient to show that

$$\liminf_{T\to\infty} T^{-1} \sum_{t=1}^{T} G_t > 0 \text{ (a.s.)}.$$

For any constant H, define $G_t^H := \min(G_t, H)$. Since $G_t^H \leq G_t$ it suffices to prove that

$$\liminf_{T\to\infty} T^{-1} \sum_{t=1}^{T} G_t^H > 0 \text{ (a.s.)} \tag{3.14}$$

for some H. We have

$$G_{t+1} = \ln z_{t+1} - \ln z_t = \ln \frac{\sum_{k=1}^{K} R_{t+1,k} \dfrac{\lambda_k^*}{\lambda_k^* r_t^1 + \lambda_k(1 - r_t^1)}}{\sum_{k=1}^{K} R_{t+1,k} \dfrac{\lambda_k}{\lambda_k^* r_t^1 + \lambda_k(1 - r_t^1)}} = \ln F(\lambda, r_t^1; s_{t+1})$$

and

$$G_{t+1}^H(s^{t+1}) = \min\{H, \ln F(\lambda, r_t^1(s^t); s_{t+1})\}. \tag{3.15}$$

In view of Lemma 3.1, there exist $H > 0$ and $\delta > 0$, such that $E_t G_{t+1}^H \geq \delta$, where $E_t(\cdot) = E(\cdot|s^t)$ denotes the conditional expectation given s^t. When computing $E_t G_{t+1}^H$, we fix s^t and take the unconditional expectation of G_{t+1}^H with respect to s_{t+1}, which is valid since s^t and s_{t+1} are independent.

Finally, we have

$$\frac{1}{T}\sum_{t=1}^{T} G_t^H = \frac{1}{T}\sum_{t=1}^{T} E_{t-1} G_t^H + \frac{1}{T}\sum_{t=1}^{T}(G_t^H - E_{t-1} G_t^H). \tag{3.16}$$

By virtue of (3.10) and (3.15), for each H the sequence of random variables $G_t^H(s^t)$ is bounded uniformly in t and $(s_t)_{t\geq 1}$. Therefore, we can apply the strong law of large numbers for martingale differences (see, e.g., [83]), which gives

$$\frac{1}{T}\sum_{t=1}^{T}(G_t^H - E_{t-1}G_t^H) \to 0 \text{ (a.s.)}.$$

Consequently,

$$\liminf_{t\to\infty} T^{-1}\sum_{t=1}^{T} G_t^H \geq \delta,$$

which proves (3.14).

It remains to show the uniqueness of a symmetric almost sure equilibrium. Assume a strategy $\lambda \neq \lambda^*$ forms such an equilibrium. Then,

$$0 = \xi^i(s^\infty; \lambda, ..., \lambda) \geq \xi^i(s^\infty; \lambda, ..., \lambda, \lambda^*) \text{ (a.s.)}, \tag{3.17}$$

where

$$\xi^i(s^\infty; \lambda, ..., \lambda, \lambda^*) = \limsup_{t\to\infty} t^{-1} \ln \frac{r_t^N}{1 - r_t^N}.$$

As we have proved above that

$$\liminf_{t\to\infty} t^{-1} \ln \frac{r_t^N}{1 - r_t^N} > 0 \text{ (a.s.)},$$

which yields the inequality "$<$" in (3.17). This is a contradiction. ■

3.3 Global Stability of Single Survivor

The Main Result: A Theorem for Single Survivor Problem. The market selection process of this model can be described by the evolution of the market shares from the following random dynamical system:

$$r_{t+1}^i(s^{t+1}) = \sum_{k=1}^{K} R_k(s_{t+1}) \frac{\lambda_k^i r_t^i(s^t)}{\sum_{i=1}^{N} \lambda_k^i r_t^i(s^t)},\ i = 1, 2, ..., N. \tag{3.18}$$

As previously defined, an investor i (or the strategy λ^i employed by i) is a single survivor if, for any strictly positive initial vector r_0, we have $r_t^i \to 1$ a.s. and $r_t^j \to 0$ a.s. when $j \neq i$. Denote by e_i the vector in $\mathbb{R}^N$, whose coordinates are equal to 0 except for the ith coordinate which equals 1. If investor i is a single survivor, then all the random paths (3.18) are attracted to e_i a.s., regardless of the initial state. In this sense, e_i is a *globally stable* fixed point of the random dynamical system under consideration. Theorem 3.4 below indicates a strategy $\lambda^* \in \Delta^K$ such that, under quite general assumptions, the following assertion holds: if investor i uses the strategy λ^*, whereas all the other investors adopt distinct strategies $\lambda^j \neq \lambda^*$, then investor i is a single survivor. And this theorem is valid for any strictly positive vector of initial wealth r_0, and so yields the global stability of the state e_i, as mentioned above.

Theorem 3.4 *Let investor i use the strategy $\lambda^i = \lambda^*$, while all the other investors $j \neq i$ use strategies $\lambda^j \neq \lambda^*$. Then investor i is the single survivor.*

Theorem 3.4 is a direct consequence of Theorem 3.3 (for the proof of Theorem 3.3, see last section). This is valid in the following sense. By virtue of Theorem 3.3, if all the investors except one, say investor i, use the strategy λ^* and i uses any other strategy λ distinct from λ^*, then the relative wealth $w_t^i / \sum_{j \neq i} w_t^j$ of i tends to zero at the exponential rate $\xi^i < 0$ (a.s.). In other words, the group of investors using λ^* drives all the others who adopt λ out of the market, which is interpreted as the property of global (holding for all initial states) evolutionary stability of λ^* in Evolutionary Finance. Thus, we arrive at Theorem 3.4: investor i uses the strategy λ^* while all the other investors $j \neq i$ employ strategies $\lambda^j \neq \lambda^*$, then investor i will eventually drive all the other investors out of the market, accumulate the wealth of the entire market and become the single survivor.

An earlier version of the proof for Theorem 3.4 was provided by Evstigneev et al. [58], Section 3, containing the redundant assumption of completely mixed strategies (strict positivity of all investment proportions). The proof was completed by utilizing an auxiliary result on an inequality (Evstigneev et al. [58], Lemma 1). Importantly, this inequality serves as the foundation for establishing Lemma 3.1 and Lemma 4.1, which are essential for proving Theorem 3.3 and Theorem 4.1, respectively. Consequently, these elementary inequalities, along with their proofs, are collected in Appendix A.

Remark 3.3 It is important to note that the assumption of non-existence of redundant assets (as we described in Section 3.1) is crucial. This assumption can be fulfilled only if the number of elements in S is not less than K. As a result, it cannot hold in the deterministic case – when S consists of only a single point. In the latter case, it is straightforward to construct an example when the investor i using strategy λ^* is not a single survivor. Indeed, assume $i = 1$ and the strategy $\lambda^1 = \lambda^*$ is representable in the form $\lambda^* = \sum_{j=2}^{I} b_j \lambda^j$, where $b = (b_j) \in \Delta^{I-1}$, $b_j > 0$. Then, for any $\kappa \in (0, 1)$, λ^* can be represented as $\sum_{j=1}^{I} a_j \lambda^j$, where $a_1 = 1 - \kappa$ and $a_j = \kappa b_j$, $j \geq 2$. In this deterministic case, the values $R_k(s) = \lambda_k^*$ do not depend on s, and it follows from (3.18) that the vector $r = (a_j) \in \Delta^I$ is a fixed point of the dynamical system in question for each $\kappa \in (0, 1)$. This leads to the failure of (global as well as local) stability of the point e_1.

Chapter 4

Model with Markov Nature

This chapter continues the study shown in previous chapter, for which fixed-mix portfolio rules and i.i.d. distributed states of the world have been considered. The goal for this chapter is to remove these two simplifying assumptions that substantially reduce the scope of the models. That is, (i) basic, rather than fixed-mix strategies, will be considered; (ii) we use a homogeneous discrete-time Markov process, instead of the i.i.d. distributed random variables, to describe the states of the world. With this more general set of assumptions, the model becomes more aligned with real-life behavior, where investors tend to make adjustments in response to unfolding events and newly disclosed information, rather than committing to a single constant strategy throughout the entire duration. Additionally, some serial dependence, at least of a Markov nature, can be postulated to address the limitation that such a complex evolution of many relevant state variables can hardly be captured by a random process with i.i.d. values. Consequently, it is reasonable that this model contributes to a considerably enlarged scope of theory at hand, with enhanced realism and greater potential applicability.

A central result is that in any–complete or incomplete–market for short-lived assets, a trader distributing wealth across available assets in accordance with their relative conditional expected payoffs (the portfolio rule λ^*) eventually accumulates the entire market wealth, i.e. the single survivor in the market selection process, provided that λ^* is asymptotically distinct from the CAPM rule (prescribing investment in the market portfolio). And the λ^*-trader accumulates the entire market wealth at an exponential rate, if λ^* remains bounded away from the CAPM rule for "sufficiently many" time periods. [1]

4.1 The Model

A Summary on Changes in Model Settings. Founded on the basic model framework stated in Section 2.1, a summary of the changes that occur in this

[1] This chapter reviews the work by Amir, Evstigneev, Hens, and Schenk-Hoppé [2].

model is as follows:

- Consider a market where $K \geq 2$ short-lived risky assets (securities) are traded by $N \geq 2$ investors (traders). Let S be a finite set and s_t, $t = 0, 1, 2, ...$(the "state of the world" at time t), a homogeneous Markov chain with transition function $p(\sigma|s)$, specifying the conditional probabilities $P\{s_{t+1} = \sigma|\ s_t = s\}$.

- The total amount of units of each asset $k = 1, ..., K$ available in the market is $V_k(s) > 0$ of $s \in S$ (exogenously given). Each trader i selects a portfolio $x_t^i = x_t^i(s^t) := (x_{t,1}^i, ..., x_{t,K}^i)$ at each date $t = 0, 1, 2, ...$, which depends on the history $s^t = (s_0, ..., s_t)$ of the process s_t up to date t (the argument s^t is often omitted as long as this does not cause ambiguity). For each date $t \geq 0$, each asset $k = 1, ..., K$, and in every random scenario s^t, the asset market clears:
$$\sum_{i=1}^{N} x_{t,k}(s^t) = V_k(s_t), \tag{4.1}$$
where the demand for physical units of each asset k (left-hand side of (4.1)) is equal to its supply (right-hand side of (4.1)). One unit of asset k issued at date t yields payoff $A_k(s_{t+1}, s_t) \geq 0$ at date $t+1$, and we assume
$$\sum_{k=1}^{K} A_k(\sigma, s) > 0 \tag{4.2}$$
for all $\sigma, s \in S$.

- If trader i selects a portfolio $x_t^i = (x_{t,k}^i)$ at date $t \geq 0$, then the market wealth w_{t+1}^i of the portfolio x_t^i at date $t+1$ can be expressed as
$$w_{t+1}^i = \sum_{k=1}^{K} A_k(s_{t+1}, s_t)\, x_{t,k}^i.$$
For each trader i, the initial wealth is given by a strictly positive number $w_0^i > 0$. In view of (4.1), we obtain
$$\sum_{i=1}^{N} w_{t+1}^i = \sum_{k=1}^{K} A_k(s_{t+1}, s_t)\, V_k(s_t), \quad t \geq 0. \tag{4.3}$$

- For each $t \geq 0$, each investor $i = 1, ..., N$ selects a vector of investment proportions
$$\lambda_t^i := (\lambda_{t,1}^i, ..., \lambda_{t,K}^i),\ \lambda_t^i = \lambda_t^i(s^t), \quad t \geq 0, \tag{4.4}$$
of the budget w_t^i, such that
$$\lambda_{t,k}^i > 0,\ \sum_{k=1}^{K} \lambda_{t,k}^i = 1. \tag{4.5}$$

Note that (4.5) implies that vectors λ_t^i belong to the relative interior (expressed as Δ_+^K) of the unit simplex Δ^K, where

$$\Delta_+^K := \{(a_1, ..., a_K) \in \mathbb{R}_+^K : \, a_k > 0, \, \sum_{k=1}^K a_k = 1\},$$

and such strategies are sometimes called *completely mixed*.

A Description for the Asset Market Dynamics. Given each investor i has chosen a strategy $(\lambda_{t,k}^i)$, the equilibrium price $p_{t,k} = p_{t,k}(s^t)$ of asset k at date $t \geq 0$ can be determined from the equation

$$p_{t,k} = \frac{1}{V_k(s_t)} \sum_{i=1}^N \lambda_{t,k}^i \, w_t^i.$$

Then, the portfolio x_t^i of investor i at date $t \geq 0$ will be determined by

$$x_{t,k}^i = \frac{\lambda_{t,k}^i \, w_t^i}{p_{t,k}}.$$

From the last two equations, we find

$$x_{t,k}^i = V_k(s_t) \, \frac{\lambda_{t,k}^i \, w_t^i}{\sum_{j=1}^N \lambda_{t,k}^j \, w_t^j}. \tag{4.6}$$

This leads to the following equation expressing the wealth w_{t+1}^i of investor i at date $t+1$ through w_t^i:

$$w_{t+1}^i = \sum_{k=1}^K A_k(s_{t+1}, s_t) \, V_k(s_t) \, \frac{\lambda_{t,k}^i \, w_t^i}{\sum_{j=1}^N \lambda_{t,k}^j \, w_t^j}. \tag{4.7}$$

Since $w_0^i > 0$, by way of induction we obtain $w_t^i > 0$ for each t (see (4.2) and (4.5)). As a result, we conclude that the evolution of the relative market shares of the investors, $r_t^i := \frac{w_t^i}{W_t}$ (where $W_t = \sum_{i=1}^N w_t^i$), is governed by the equations

$$r_{t+1}^i = \sum_{k=1}^K R_k(s_{t+1}, s_t) \, \frac{\lambda_{t,k}^i \, r_t^i}{\sum_{j=1}^N \lambda_{t,k}^j \, r_t^j}, \quad i = 1, ..., N, \tag{4.8}$$

where

$$R_k(s_{t+1}, s_t) = \frac{A_k(s_{t+1}, s_t) \, V_k(s_t)}{\sum_{m=1}^K A_m(s_{t+1}, s_t) \, V_m(s_t)}.$$

The numbers $R_k(s_{t+1}, s_t)$ characterize the relative (normalized) payoff of each asset k, which satisfy $R_k(s_{t+1}, s_t) \geq 0$ and

$$\sum_{k=1}^K R_k(s_{t+1}, s_t) = 1. \tag{4.9}$$

We are primarily interested in strategies that (i) allow an investor to *survive*, i.e., to maintain a strictly positive relative market share r_t^i in the limit, and (ii) enable the investor to *dominate the market*, i.e., to gather in the limit the entire market wealth. We say that an investor i (or the strategy $\lambda^i = (\lambda_{t,k}^i)$) is a *single survivor* in the market selection process (4.8) if

$$\lim r_t^i = 1 \quad \text{a.s.} \tag{4.10}$$

Condition (4.10) implies $\lim r_t^j = 0$ a.s. for all $j \neq i$, meaning that investor i in the limit accumulates all the market wealth. If the sequence r_t^i involved in (4.10) converges to 1 at an exponential rate, we shall say that the strategy λ^i *dominates* the others *exponentially.*

Remark 4.1 It is an important problem to identify those strategies which lead an investor employing them to be the single survivor. Hens and Schenk-Hoppé [87] and Evstigneev, et al. [58] (the special case shown in Section 3.3) considered this problem within two different settings (local and global, respectively). In the latter model, Theorem 3.4 generalizes the result of Blume and Easley [21], which dealt with the case of Arrow securities ($S = \{1, 2, ..., K\}$, $A_k(s) = 0$ if $s \neq k$ and $A_k(s) = 1$ if $s = k$). Furthermore, the strategy (3.3) defined in terms of the expected payoffs in Theorem 3.4 may be regarded as a development of the Kelly rule of "betting one's beliefs" (Kelly [95]). This rule was originally designed in connection with gambling problems, but later on it was successfully applied to portfolio theory (Thorp [155], Aurell et al. [10]). In next section, a version of Theorem 3.4 applicable to the more general model described in current section will be given, i.e., we can define a strategy λ^* of "betting one's beliefs"—a direct analogue of the one considered in Theorem 3.4. As it turns out, we cannot, generally, guarantee λ^* to be the single survivor. Nevertheless, this conclusion does obtain under a natural sufficient condition, having a clear economic meaning. And a necessary and sufficient condition for an investor employing λ^* to be the single survivor dominating the others exponentially will be provided.

4.2 The Main Results: Conditions for Single Survivor

Assumptions on the Random Dynamical System. Consider the random dynamical system (4.8) describing the evolution of the relative market shares $r_t^i(s^t)$ of the investors $i = 1, 2, ..., N$. Note that if $r_t = (r_t^i)$ is a strictly positive vector, then, from (4.8), (4.9) and (4.5) we know that r_{t+1} is a strictly positive vector as well. Thus $r_t = r_t(s^t)$ is a random process with values in the relative interior Δ_+^N of the unit simplex. The initial state $r_0 = (r_0^1, ..., r_0^N) \in \Delta_+^N$, from which this process starts, is fixed ($r_0^i = w_0^i / \sum w_0^j$).

The random dynamical system (4.8) will be analyzed under the following assumptions.

(A1). *The functions*

$$R_k^*(s) := \sum_{\sigma \in S} p(\sigma|s)\, R_k(\sigma, s), \quad k = 1, 2, ..., K, \tag{4.11}$$

take on strictly positive values for each $s \in S$.
(A2). *For each* $s \in S$, *the functions* $R_1(\cdot, s), ..., R_K(\cdot, s)$ *restricted to the set*

$$\Pi(s) = \{\sigma \in S : \; p(\sigma|s) > 0\}$$

are linearly independent.

According to (A1), the conditional expectation

$$R_k^*(s) = E[R_k(s_{t+1}, s_t) \mid s_t = s] \tag{4.12}$$

of the relative payoff $R_k(s_{t+1}, s_t)$ of each asset k given $s_t = s$ is strictly positive at each state s. Assumption (A2) indicates the absence of *conditionally redundant* assets. The term "conditionally" refers to the fact that the functions $R_k(\cdot, s)$, $k = 1, ..., K$, are linearly independent on the set $\Pi(s)$—the support of the conditional distribution $p(\sigma|s)$.

In what follows, we will focus exclusively on those investment strategies $\lambda = (\lambda_{t,k})$ that satisfy the additional assumption outlined below.

(B). *The coordinates* $\lambda_{t,k}(s^t)$ *of the vectors* $\lambda_t(s^t)$ *are bounded away from zero by a strictly positive non-random constant* ρ *(which might depend on the strategy* λ*, but not on* k, t *and* s^t*), i.e.* $\inf_{i,k,t,s^t} \lambda_{t,k}^i(s^t) > \rho > 0$.

In (4.5), we included the completely mixed strategies assumption $\lambda_{t,k} > 0$. Now that (B) contains the additional requirement of uniform strict positivity of $\lambda_{t,k}$.

The Kelly Rule. It is crucial to emphasize the strategy $\lambda^* = (\lambda_{t,k}^*(s_t))$ defined by the formula

$$\lambda_{t,k}^*(s_t) = R_k^*(s_t), \tag{4.13}$$

where $R_k^*(s)$ is the conditional expectation of $R_k(s_{t+1}, s_t)$ given $s_t = s$ (see (4.11) and (4.12)). As mentioned, this is a direct analog of the Kelly rule of "betting one's beliefs", which takes on the form (3.3) in the i.i.d. distributed s_t case. Note that $\lambda_k^*(s_t) = \lambda_{t,k}^*(s_t)$ depends only on the current state s_t and not on t (explicitly) or the whole history s^t, which implies, by virtue of (A1) and in view of finiteness of S, that the strategy λ^* satisfies (B).

The CAPM Rule. To proceed further, a recursive method of constructing strategies based on (Markovian) decision rules will be given. Suppose one of the investors, say investor 1, has a privilege of making the investment decision at time t with full information about the current market structure r_t and the actions $\lambda_t^2(s^t)$, $\lambda_t^3(s^t)$, ... , $\lambda_t^N(s^t)$ that have just been undertaken by all the other investors $2, 3, ..., N$. Formally, the decision of investor 1 is specified by a function

$$f_t(r, l^2, ..., l^N), \ r \in \Delta_+^N, \ l^j \in \Delta_+^K \quad (j = 2, 3, ..., N)$$

taking values in Δ_+^K. Suppose that such functions (i.e. *decision rules*) are given for all $t = 0, 1, 2, ...$, and that investors $2, ..., N$ have chosen some strategies $\lambda_t^2, ..., \lambda_t^N$ $(t = 0, 1, 2, ...)$. Then one can construct a strategy $\lambda_t^1(s^t)$, $t = 0, 1, 2, ...$, of investor 1 by using

$$\lambda_t^1(s^t) = f_t(r_t, \lambda_t^2, ..., \lambda_t^N), \tag{4.14}$$

where $r_t = r_t(s^t)$ and $\lambda_t^j = \lambda_t^j(s^t)$, $j = 2, ..., N$.

Let us consider a particular decision rule $f = (f_1, ..., f_K)$ (which does not explicitly depend on t) defined by

$$f(r, l^2, ..., l^N) = \sum_{j=2}^{N} \frac{r^j}{1 - r^1} \, l^j, \tag{4.15}$$

where $r = (r^1, ..., r^N) \in \Delta_+^N$, $l^j = (l_1^j, ..., l_K^j) \in \Delta_+^K$, thus the vector $f = (f_1, ..., f_K)$ belongs to Δ_+^K. Note that the vector f is a convex combination of the vectors $l^2, ..., l^N$ with weights $r^j(1-r^1)^{-1}$, which implies: if the coordinates l_k^j of the vectors l^j are bounded away from 0 by a constant $\rho > 0$, then the coordinates f_k of f are bounded away from 0 by the same constant. Consequently, if the strategies $\lambda_t^2, ..., \lambda_t^N$ satisfy (B), the strategy (4.14) satisfies (B) as well. In what follows, we will use the notation $f = (f_k)$ for the particular decision rule described in (4.15).

The decision rule (4.15) exhibits several remarkable properties. First of all, observe the following. Suppose investor 1 uses the strategy $\lambda_t^1(s^t)$ defined by (4.14) in terms of the decision rule (4.15). Then we obatin

$$\lambda_{t,k}^1 = \sum_{j=1}^{N} \lambda_{t,k}^j \, r_t^j, \tag{4.16}$$

which, in view of (4.8) and (4.9), yields

$$r_{t+1}^1 = r_t^1.$$

Hence, if investor 1 employs the strategy generated by the decision rule (4.15), then, regardless of what strategies are adopted by the others, the relative market share of investor 1 remains constant over time. This observation results in the following conclusion. If one of the other traders $2, ..., N$ employs λ^*, then this trader cannot be a single survivor, as long as trader 1 uses (4.14)–(4.15), and thus maintains a constant strictly positive market share $r_t^1 = r_0^1$ for all t.

Further, the portfolio of investor 1, who employs the strategy λ_t^1 defined in terms of (4.14)–(4.15), is given by

$$x_{t,k}^1 = V_k \, \frac{\lambda_{t,k}^1 w_t^1}{\sum_{j=1}^{N} \lambda_{t,k}^j \, w_t^j} = V_k \frac{\lambda_{t,k}^1 \, r_t^1}{\sum_{j=1}^{N} \lambda_{t,k}^j \, r_t^j} = V_k \, r_t^1,$$

for each $k = 1, ..., K$ (see (4.6) and (4.16)). Hence, the vector $x_t^1 = (x_{t,1}^1, ..., x_{t,K}^1)$ turns out to be proportional to the *market portfolio*, represented by the vector $(V_1, ..., V_k)$, whose components indicate the amount of assets $k = 1, ..., K$ currently traded in the market. According to the well-known Tobin mutual fund theorem (Magill and Quinzii [111], Proposition 16.15), portfolios with this structure arise from the mean-variance optimization in the Capital Asset Pricing Model (CAPM). Therefore, it is natural to refer to the decision rule (4.15) as the *CAPM decision rule* and the strategy generated by it as the *CAPM strategy.*

Three Theorems on Single Survivor Problem. A sufficient condition for the strategy (4.13) to be a single survivor is given in Theorem 4.1 below, based on the dynamical system (4.8) and assuming that all the investors $i \in \{1, 2, ..., N\}$ employ some strategies $\lambda^i = (\lambda_t^i)$ satisfying (B). We define

$$\zeta_t = (\zeta_{t,1}, ..., \zeta_{t,K}) = f(r_t, \lambda_t^2, ..., \lambda_t^N),$$

where f is the CAPM decision rule (4.15). The symbol $|\cdot|$ denotes the sum of the absolute values of the coordinates of a finite-dimensional vector.

Theorem 4.1 *Suppose investor* 1 *uses the strategy* $\lambda^1 = \lambda^*$ *defined by (4.13). Let the following condition be fulfilled:*
(C). *With probability 1, we have*

$$\liminf_{t\to\infty} |\lambda^*(s_t) - \zeta_t| > 0. \tag{4.17}$$

Then investor 1 is a single survivor, and, moreover,

$$\liminf_{t\to\infty} \frac{1}{t} \ln \frac{r_t^1}{1 - r_t^1} > 0 \tag{4.18}$$

almost surely.

Property (4.18) indicates that the relative market share of investor 1 tends to one at an exponential rate, whereas the relative market shares of all the other investors vanish at such rates, and so the strategy λ^* dominates the others exponentially.

Assumption (C) can be restated as follows: there exists a strictly positive random variable κ such that,

$$|\lambda^*(s_t) - \zeta_t(s^t)| \geq \kappa \tag{4.19}$$

a.s. for all t large enough. Inequality (4.19) requires that the actions $\lambda^*(s_t)$ prescribed by the strategy λ^* should differ by not less than $\kappa > 0$ from the actions

$$\zeta_t(s^t) = (\zeta_{t,1}(s^t), ..., \zeta_{t,K}(s^t)), \;\; \zeta_{t,k}(s^t) = \sum_{j=2}^{N} \frac{r_t^j(s^t)}{1 - r_t^1(s^t)} \, \lambda_{t,k}^j(s^t),$$

prescribed by the CAPM decision rule. In this context, we do not assume that any of the market participants indeed employs the CAPM rule; we need it only as an indicator, a proper deviation of which from λ^* guarantees λ^* to be a single survivor.

In concrete instances, verifying (C) directly might not be easy. Thus we introduce another hypothesis, (C1), which is stronger than (C) but can be more conveniently checked in various examples.

(C1). *There exists a strictly positive random variable κ such that, with probability 1, the distance between the vector $\lambda^*(s_t) \in \mathbb{R}^K$ and the convex hull of the vectors $\lambda_t^2(s^t), ..., \lambda_t^N(s^t) \in \mathbb{R}^K$ is not less than κ for all t large enough.*

Clearly (C1) implies (C) because $\zeta_t = f(r_t, \lambda_t^2, ..., \lambda_t^N)$ is a convex combination of $\lambda_t^2, ..., \lambda_t^N$. Assumption (C), which suffices for investor i to be a single survivor, turns out to be close to a necessary one. The theorem below provides a version of (C) that is both necessary and sufficient for the conclusion of Theorem 4.1 to hold.

Theorem 4.2 *Investor 1 using the strategy (4.13) is a single survivor in the market selection process, and, moreover, dominates the others exponentially, if and only if the following condition is fulfilled:*
(C2). *There exists a random variable $\kappa > 0$ such that*

$$\liminf_{T\to\infty} \frac{1}{T} \#\{t \in \{0, ..., T\} : |\lambda^*(s_t) - \zeta_t(s^t)| \geq \kappa\} > 0 \tag{4.20}$$

with probability 1.

The symbol # represents the number of elements in a finite set. Observe that (C2) follows from (C). Indeed, hypothesis (C) is equivalent to the existence of a random variable κ for which, inequality (4.19) is fulfilled a.s. for all t large enough. In this case, the limit in (4.20) is equal to 1, and may be regarded as a *density* (in the set of natural numbers) of those natural numbers t for which inequality (4.19) holds. Hypothesis (C2) only requires this density to be strictly positive, whereas (C) asserts that (4.19) should hold from some t on.

Recall Theorem 4.1, from which it follows immediately that if the relation

$$\liminf_{t\to\infty} \frac{1}{t} \ln \frac{r_t^1}{1 - r_t^1} \leq 0 \tag{4.21}$$

holds with positive probability, then, there exists a (random) sequence t_k such that

$$|\lambda^*(s_{t_k}) - \zeta_{t_k}(s^{t_k})| \to 0 \tag{4.22}$$

with positive probability. By appropriately strengthening (4.21), we can make a stronger statement about convergence in (4.22), as provided below.

Theorem 4.3 *Let the following condition be fulfilled:*
(D1). *There exists a random variable* $0 < \gamma < 1$ *such that* $E \ln \gamma > -\infty$ *and*

$$r_t^1 < 1 - \gamma$$

a.s. for all t.

Then we have

$$|\lambda^*(s_t) - \zeta_t| \to 0 \quad a.s.$$

We will actually prove Theorem 4.3 under a weaker condition:

(D2). *The expectations*

$$E[\ln(1 - r_t^1)]$$

do not converge to $-\infty$.

Clearly (D1) is stronger than both (D2) and (4.21), but (D2) does not necessarily imply (4.21). Assumption (D1) holds, e.g., if one of the investors $i = 2, ..., N$ employs the CAPM strategy (and so this investor's relative market share keeps constant). Then, as Theorem 4.3 shows, the difference between the budget shares of investor 1 prescribed by the strategy λ^* and the budget shares prescribed by the CAPM decision rule converges to 0 almost surely.

4.3 Proofs of the Main Results

Theorem 4.1 is a direct consequence of Theorem 4.2.

Proof of Theorem 4.2. By virtue of (4.8), we write

$$\begin{aligned} \frac{1 - r_{t+1}^1}{1 - r_t^1} &= \frac{\sum_{i=2}^N r_{t+1}^i}{1 - r_t^1} = \sum_{k=1}^K R_k(s_{t+1}, s_t) \frac{(1 - r_t^1)^{-1} \sum_{i=2}^N \lambda_{t,k}^i r_t^i}{q_{t,k}} \\ &= \sum_{k=1}^K R_k(s_{t+1}, s_t) \frac{\zeta_{t,k}}{q_{t,k}}, \end{aligned}$$

where

$$q_{t,k} = \sum_{m=1}^N \lambda_{t,k}^m r_t^m = \lambda_{t,k}^1 r_t^1 + (1 - r_t^1) \frac{\sum_{i=2}^N \lambda_{t,k}^i r_t^i}{1 - r_t^1} = \lambda_{t,k}^1 r_t^1 + \zeta_{t,k}(1 - r_t^1).$$

Consequently,

$$1 - r_{t+1}^1 = \sum_{k=1}^K R_k(s_{t+1}, s_t) \frac{\zeta_{t,k}(1 - r_t^1)}{\lambda_{t,k}^1 r_t^1 + \zeta_{t,k}(1 - r_t^1)}, \tag{4.23}$$

and

$$r_{t+1}^1 = \sum_{k=1}^K R_k(s_{t+1}, s_t) \frac{\lambda_{t,k}^1 r_t^1}{\lambda_{t,k}^1 r_t^1 + \zeta_{t,k}(1 - r_t^1)}. \tag{4.24}$$

For each $t = 1, 2, ...$, consider

$$D_t = \ln \frac{r_t^1 (r_{t-1}^1)^{-1}}{(1 - r_t^1)(1 - r_{t-1}^1)^{-1}}.$$

We have

$$D_1 + ... + D_T = \ln \frac{r_T^1}{(1 - r_T^1)} - \ln \frac{r_0^1}{(1 - r_0^1)}. \tag{4.25}$$

Therefore, (4.18) holds if and only if

$$\liminf_{T \to \infty} \frac{1}{T} (D_1 + ... + D_T) > 0 \quad \text{a.s.}$$

According to hypothesis (B), for each set of strategies $(\lambda_{t,k}^i)$, $i = 1, ..., N$, there exists a constant $H > 1$, such that $(\inf_{i,k,t} \lambda_{t,k}^i)^{-1} \leq H$. For this H, we have

$$H^{-1} \leq \frac{r_{t+1}^i}{r_t^i} \leq H, \quad i = 1, ..., N,$$

which implies

$$H^{-1} \leq \frac{1 - r_{t+1}^1}{1 - r_t^1} \leq H$$

because $1 - r_t^1 = \sum_{m=2}^N r_t^m$. As a result, the random variables D_t are uniformly bounded.

We have the following identity

$$\frac{1}{T} \sum_{t=1}^T D_t = \frac{1}{T} \sum_{t=1}^T E(D_t | s^{t-1}) + \frac{1}{T} \sum_{t=1}^T [D_t - E(D_t | s^{t-1})].$$

In view of uniform boundedness of D_t, we can apply to the process of martingale differences $B_t := D_t - E(D_t | s^{t-1})$ the strong law of large numbers (Hall and Heyde [83], Theorem 2.19), which yields $T^{-1}(B_1 + ... + B_T) \to 0$ with probability 1. Hence, we obtain

$$\liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^T D_t = \liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^T E(D_t | s^{t-1}), \tag{4.26}$$

and so (4.18) is equivalent to

$$\liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^T E(D_t | s^{t-1}) > 0 \quad \text{a.s.} \tag{4.27}$$

By virtue of (4.23), (4.24), we have

$$E[D_t|s^{t-1}] = E[\ln \frac{r_t^1(r_{t-1}^1)^{-1}}{(1-r_t^1)(1-r_{t-1}^1)^{-1}}|s^{t-1}]$$

$$= \sum_{\sigma \in S} p(\sigma|s_{t-1}) \ln \frac{\sum_k R_k(\sigma, s_{t-1}) \frac{\lambda_{t-1,k}^1}{\lambda_{t-1,k}^1 r_{t-1}^1 + \zeta_{t-1,k}(1-r_{t-1}^1)}}{\sum_k R_k(\sigma, s_{t-1}) \frac{\zeta_{t-1,k}}{\lambda_{t-1,k}^1 r_{t-1}^1 + \zeta_{t-1,k}(1-r_{t-1}^1)}}, \tag{4.28}$$

where

$$\zeta_{t-1,k} = \zeta_{t-1,k}(s^{t-1}) = \frac{\sum_{i=2}^N \lambda_{t-1,k}^i r_{t-1}^i}{1 - r_{t-1}^1}, \tag{4.29}$$

$$\lambda_{t-1,k}^i = \lambda_{t-1,k}^i(s^{t-1}),\ r_{t-1}^i = r_{t-1}^i(s^{t-1}), \tag{4.30}$$

$$\lambda_{t-1,k}^1 = \lambda_{t-1,k}^1(s_{t-1}) = R_k^*(s_{t-1}).$$

Let us use Lemma 4.1 below (for the proof, see Appendix A) to estimate the expression in (4.28).

Lemma 4.1 *There exists a constant L_ρ and a function $\delta_\rho(\gamma) \geq 0$ of $\gamma \in [0, \infty)$ satisfying the following conditions:*

1. *The function $\delta(\cdot)$ is non-decreasing, and $\delta_\rho(\gamma) > 0$ for all $\gamma > 0$.*
2. *For any $s \in S$, $\kappa \in [0,1]$ and $\mu = (\mu_k) \in \Delta_\rho^K$, we have*

$$L_\rho |R^*(s) - \mu| \geq \Phi(s, \kappa, \mu) \geq \delta_\rho(|R^*(s) - \mu|). \tag{4.31}$$

In light of this lemma, we get

$$E(D_t|s^{t-1}) \geq \delta_\rho(|R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|), \tag{4.32}$$

where ρ is the strictly positive constant that bounds the coordinates of λ_t^i away from 0. We use $N(T) = N(T, s^T)$ to denote the set of those $t \in [0, T]$ for which $|R^*(s_t) - \zeta_t(s^t)| \geq \kappa$. We have

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^T E(D_t|s^{t-1}) \geq \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^T \delta_\rho(|R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|)$$

$$\geq \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \delta_\rho(|R^*(s_t) - \zeta_t(s^t)|) \geq \liminf_{T\to\infty} \frac{1}{T} \sum_{t \in N(T-1)} \delta_\rho(|R^*(s_t) - \zeta_t(s^t)|)$$

$$\geq \delta_\rho(\kappa) \cdot \liminf_{T\to\infty} \frac{1}{T} \#\{N(T-1)\} > 0,$$

where the last inequality follows from (C2). Therefore, we have established (4.27), which is equivalent to (4.18).

Now, assume (4.18), and so (4.27), hold. According to Lemma 4.1, we find

$$E(D_t|s^{t-1}) \leq L_\rho \cdot |R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|,$$

and hence (4.27) gives

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} d_t > 0 \quad \text{a.s.}, \tag{4.33}$$

where $d_t = |R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|$.

We use $\bar{\kappa}$ to denote the strictly positive random variable that equals the $\liminf$ in (4.33) a.s., and set $\kappa = \bar{\kappa}/2$. We claim that

$$\liminf \frac{1}{T} \#\big\{t \in \{1, ..., T\} : d_t \geq \kappa\big\} > 0 \quad \text{a.s.}, \tag{4.34}$$

which is equivalent to (C2). Indeed, assume the contrary (for the sake of contradiction). In that case, there is a sequence $(T_k)_k$ (which depends on (s_t)), such that

$$\frac{1}{T_k} \#\big\{t \in \{1, ..., T_k\} : d_t \geq \kappa\big\} \to 0 \tag{4.35}$$

with positive probability. For each k, denote by M_k (resp. N_k) the set of those $t \in \{1, ..., T_k\}$ for which $d_t \geq \kappa$ (resp. $d_t < \kappa$). Then, it yields, for all events where (4.35) holds,

$$\frac{1}{T_k} \sum_{t=1}^{T_k} d_t = \frac{1}{T_k} \sum_{t\in M_k} d_t + \frac{1}{T_k} \sum_{t\in N_k} d_t \leq 2 \cdot \frac{1}{T_k} \#(M_k) + \kappa, \tag{4.36}$$

since $d_t \leq 2$. According to (4.35), we have $(T_k)^{-1} \cdot \#(M_k) \to 0$. Therefore, for all events where (4.35) holds,

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} d_t \leq \liminf_{k\to\infty} \frac{1}{T_k} \sum_{t=1}^{T_k} d_t \leq \kappa < \bar{\kappa},$$

which contradicts the definition of $\bar{\kappa}$. ■

Proof of Theorem 4.3. Consider the nonnegative random variables $v_t = \delta_\rho(|R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|)$. By using (4.32), we have $Ev_t \leq E[E(D_t|s^{t-1})] = ED_t$, which gives, by virtue of (4.25),

$$\sum_{t=1}^{T} Ev_t \leq E \ln \frac{r_T^1}{1 - r_T^1} + C \leq -E \ln(1 - r_T^1) + C,$$

where C is some constant. From the hypothesis (D2), the expectations $-E\ln(1-r_T^1)$ do not converge to $+\infty$. Thus the non-negative sums $Ev_1 + ... + Ev_T$ are bounded by some constant C_1. Consequently, in light of the Fatou lemma,

$$E \lim_{T\to\infty} \sum_{t=0}^{T} v_t = E \liminf_{T\to\infty} \sum_{t=0}^{T} v_t \leq \liminf_{T\to\infty} \sum_{t=0}^{T} Ev_t \leq C_1.$$

Therefore, we get $\sum_{t=0}^{\infty} v_t < \infty$ a.s., thus $v_t \to 0$ a.s., and so $|R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})| \to 0$ a.s. ■

Chapter 5

Model with Risk-free Asset

As shown in Chapters 2–4, the majority of studies in this field up to now have been concentrated on the case that all the assets are purely risky, which means that asset prices are random and endogenous, determined by the strategy profiles of all the investors as a result of the asset market dynamics in short-run equilibrium. In this chapter, we examine a model that incorporates a risk-free asset, in the sense that its price is exogenous (which can be normalized to 1 and serves as a numeraire). A key feature of the model is that asset payoffs are assumed to be scalable, depending linearly on the total cash in all participants' bank accounts (no currency outside banks), which makes the numeraire effectively invariant under redenominations. If the asset payoffs and prices grow in scale, the total cash required to function the system has to increase accordingly. The primary aim of the study is also to identify whether there exists an (asymptotically unique) investment portfolio rule that allows an investor to "survive" in the long-run market selection process.[1]

5.1 The Model

A Summary on Changes in Model Settings. Based on the general model framework stated in Section 2.1, a summary of the changes that occur in this model is as follows:

- Consider a market where $K \geq 2$ short-lived risky *assets* (*securities*) $k = 1, 2, ..., K$ and a risk-free asset $k = 0$ are traded by $N \geq 2$ *investors* (*traders*). We use $(K+1)$-dimensional vectors $x_t^i := (x_{t,0}^i, x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^{K+1}$ (where $x_{t,k}^i \geq 0$, $k = 0, 1, ..., K$, i.e. short sales are ruled out) and $p_t = (p_{t,0}, ..., p_{t,K}) \in \mathbb{R}_+^{K+1}$ to denote the portfolio of investor i and asset prices at date $t \geq 0$, respectively. Accordingly, the market value of investor i's portfolio at date t is given by $\langle p_t, x_t^i \rangle := \sum_{k=0}^{K} p_{t,k} x_{t,k}^i$. These

[1] This chapter discusses the study conducted by Belkov, Evstigneev, and Hens [13].

vectors x_t^i and p_t depend on s^t:

$$p_t = p_t(s^t),\ x_t^i = x_t^i(s^t),$$

and all functions of s^t are assumed to be measurable. Note that the prices $p_{t,1}, ..., p_{t,K}$ of the risky assets $k = 1, 2, ..., K$ are determined endogenously through asset market dynamics by short-run equilibrium over each time period $[t-1, t)$. The price $p_{t,0}$ of the risk-free asset $k = 0$, which can be interpreted as cash, is exogenous and normalized to one: $p_{t,0} = 1$. That explains why asset $k = 0$ is considered risk-free. However, the interest rate for cash, which we introduce below, might be random and time dependent.

- Suppose that we have $V_{0,k} > 0$ and $V_{t,k}(s^t) > 0$ total units of each risky asset $k = 1, 2, ..., K$ available in the market at date 0 and each subsequent period $t = 1, 2, ...$, respectively. However, there are no exogenously set limits for the total amount of the risk-free asset $k = 0$ (cash), which will be used as a numeraire to express the market values of all the assets. Risk-free holdings $x_{t,0}^i$ in the portfolios $x_t^i = (x_{t,0}^i, x_{t,1}^i, ..., x_{t,K}^i)$ of investors $i = 1, 2, ..., N$ can be regarded as balances in their bank accounts, with some given interest rate $\beta_t = \beta_t(s^t) \geq 0$. It is assumed that all the money in the system is deposited in the bank accounts of the market participants, so that
$$\bar{w}_t := \sum_{i=1}^{N} x_{t,0}^i \tag{5.1}$$
represents the aggregate amount of money at date t.

- The payoffs $A_{t+1,k}$ of the risky assets $k = 1, 2, ..., K$ depend linearly on the ("narrow") money supply $\bar{w}_t$ at date t and on the history of states of the world s^{t+1}:
$$A_{t+1,k} = a_{t+1,k}(s^{t+1})\bar{w}_t\ . \tag{5.2}$$
The linear dependence of $A_{t+1,k}$ on $\bar{w}_t$ postulated in equation (5.2) reflects the idea of scalability of the asset payoffs and the relative nature of the numeraire value. The functions $a_{t,k}(s^t)$, $k = 1, 2, ..., K$, satisfy
$$\sum_{k=1}^{K} a_{t,k}(s^t) > 0 \text{ for all } t, s^t, \tag{5.3}$$
which means that in each random scenario, there is at least one of the assets $k = 1, 2, ..., K$ yields a strictly positive payoff (as long as $\bar{w}_t > 0$).

- Investors have initial endowments $w_0^i > 0$ $(i = 1, 2, ..., N)$ as their budgets at date 0. For date $t \geq 1$, investor i's budget is
$$w_t^i := \langle A_t, x_{t-1}^i \rangle = \sum_{k=0}^{K} A_{t,k} x_{t-1,k}^i\ , \tag{5.4}$$

where

$$A_t := (A_{t,0}, ..., A_{t,K}) \text{ and } A_{t,0} := 1 + \beta_t. \tag{5.5}$$

The budget (5.4) is formed by the payoffs

$$A_{t,k} x^i_{t-1,k} = a_{t,k} \bar{w}_{t-1} x^i_{t-1,k} \tag{5.6}$$

of the risky assets $k = 1, ..., K$ held in investor i's previous portfolio x^i_{t-1}, along with the amount of money $[1 + \beta_t(s^t)]x^i_{t-1,0}$ in this investor's bank account at the beginning of period $[t, t+1)$. The budget $\langle A_t, x^i_{t-1} \rangle$ is re-invested in the assets available at time t, which subsequently give the payoffs $A_{t+1,k}$, $k = 1, ..., K$, and interest with the rate β_{t+1} at time $t + 1$.

- At each $t \geq 0$, each investor $i = 1, 2, ..., N$ selects a vector of investment proportions $\lambda^i_t := (\lambda^i_{t,0}, ..., \lambda^i_{t,K})$, which belongs to the unit simplex

$$\Delta^{K+1} := \{(b_0, ..., b_K) \in \mathbb{R}^{K+1}_+ : b_0 + ... + b_K = 1\}.$$

A Description for the Asset Market Dynamics. The asset market dynamics of this model, analogous to those presented in Section 2.1, Eqs. (2.2)–(2.5), are outlined below.

1. For date $t = 0$:

   The equilibrium prices $p_{0,k}$ of the risky assets $k = 1, 2, ..., K$ are determined from

$$p_{0,k} V_{0,k} = \sum_{i=1}^{N} \lambda^i_{0,k} w^i_0, \; k = 1, 2, ..., K. \tag{5.7}$$

   The investors' portfolios $x^i_0 = (x^i_{0,1}, ..., x^i_{0,K})$, $i = 1, 2, ..., N$ are determined from

$$x^i_{0,k} = \frac{\lambda^i_{0,k} w^i_0}{p_{0,k}}, \; k = 1, 2, ..., K, \; i = 1, ..., N. \tag{5.8}$$

   The 0th position $x^i_{0,0}$ of portfolio x^i_0 (the amount of money deposited into the bank account by investor i), is given by

$$x^i_{0,0} = \frac{\lambda^i_{0,0} w^i_0}{p_{0,0}} = \lambda^i_{0,0} w^i_0, \tag{5.9}$$

   where $p_{0,0} = 1$.

2. For date $t \geq 1$:

   The equilibrium prices $p_{t,k}$ of the risky assets $k = 1, 2, ..., K$ are determined from

$$p_{t,k} V_{t,k} = \sum_{i=1}^{N} \lambda^i_{t,k} w^i_t, \; k = 1, ..., K. \tag{5.10}$$

The investors' portfolios $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i),\ i = 1, 2, ..., N$ are determined from

$$x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}},\ k = 1, ..., K, \tag{5.11}$$

with the 0th position $x_{t,0}^i$ of portfolio x_t^i:

$$x_{t,0}^i = \frac{\lambda_{t,0}^i w_t^i}{p_{t,0}} = \lambda_{t,0}^i w_t^i. \tag{5.12}$$

Remark 5.1 A comment regarding the admissibility of strategy profiles in this model is given. Portfolio positions $x_{t,k}^i$ are well-defined by (5.11) only if for each $t \geq 0$, the prices $p_{t,k},\ k = 1, .., K$ are strictly positive, or equivalently, if the total demand for each risky asset $k = 1, ..., K$ is strictly positive:

$$\sum_{i=1}^N \lambda_{t,k}^i w_t^i > 0. \tag{5.13}$$

It is crucial that

$$\bar{w}_t = \sum_{i=1}^N x_{t,0}^i = \sum_{i=1}^N \lambda_{t,0}^i w_t^i > 0 \tag{5.14}$$

for each $t \geq 0$, i.e., the toal demand $\sum_{i=1}^N \lambda_{t,0}^i w_t^i$ for the risk-free asset $k = 0$ is strictly positive as well. In fact, if $\bar{w}_t = \sum_{i=1}^N x_{t,0}^i = 0$, then it gives $x_{t,0}^i = 0$ for every $i = 1, ..., N$ and $A_{t+1,k} = a_{t+1,k}\bar{w}_t = 0$ for all $k = 1, ..., K$. Consequently,

$$w_{t+1}^i = \langle A_{t+1}, x_t^i \rangle = \sum_{k=0}^K A_{t+1,k} x_{t,k}^i = (1 + \beta_{t+1}) x_{t,0}^i + \sum_{k=1}^K a_{t+1,k} \bar{w}_t x_{t,k}^i = 0$$

due to $x_{t,0}^i = \bar{w}_t = 0$. Therefore, if $\bar{w}_t = 0$ for some date t, then at date $t+1$ the wealth w_{t+1}^i of each trader $i = 1, ..., N$ vanishes, and so the market collapses.

Those strategy profiles for which conditions (5.13) and (5.14) hold for all $k = 0, 1, ..., K$ and $t \geq 0$, will be called *admissible*. In what follows, we will consider only strategy profiles that meet this assumption, as it guarantees the random dynamical system under consideration is well-defined with $\bar{w}_t > 0$ and $p_{t,k} > 0$ for all $t \geq 0$ and $k = 1, ..., K$.

A sufficient condition for a strategy profile to be admissible, following from hypothesis (5.20) below, is provided, which will hold for all the strategy profiles we consider. Suppose that one of the traders, e.g., trader 1, follows a fully diversified portfolio rule, which prescribes $\lambda_{t,k}^1 > 0$ for all assets $k = 0, 1, ..., K$ and all $t \geq 0$. By induction, we get

$$w_t^1 = \sum_{k=0}^K A_{t,k} x_{t-1,k}^1 \geq A_{t,0} x_{t-1,0}^1 = (1 + \beta_t)\, \lambda_{t-1,0}^1 w_{t-1}^1 > 0,$$

which, in turn, gives

$$\bar{w}_t = \sum_{i=1}^{N} \lambda_{t,0}^i w_t^i \geq \lambda_{t,0}^1 w_t^1 > 0$$

and

$$\sum_{i=1}^{N} \lambda_{t,k}^i w_t^i \geq \lambda_{t,k}^1 w_t^1 > 0 \text{ for } k = 1, ..., K.$$

This implies that any strategy profile containing a fully diversified portfolio rule is admissible.

Assuming the admissibility of strategy profiles, and summing up Eqs. (5.11) over $i = 1, ..., N$, we get

$$\sum_{i=1}^{N} x_{t,k}^i = \frac{\sum_{i=1}^{N} \lambda_{t,k}^i w_t^i}{p_{t,k}} = \frac{p_{t,k} V_{t,k}}{p_{t,k}} = V_{t,k}, \;\; k = 1, ..., K, \tag{5.15}$$

indicating that the market clears for each risky asset $k = 1, ..., K$ and each date $t \geq 0$.

5.2 The Main Results: Existence and Asymptotic Uniqueness of Survival Portfolio Rules

Consider an admissible strategy profile $(\Lambda^1, ..., \Lambda^N)$ of the investors and the corresponding path of the random dynamical system generated by it. We are still primarily interested in the long-run behavior of the relative wealth (or the market shares), $r_t^i := w_t^i / W_t$ of the investors, i.e., in the asymptotic properties of the series of vector $r_t = (r_t^1, ..., r_t^N)$ as $t \to \infty$, where $w_t^i = \langle A_t, x_{t-1}^i \rangle$, $W_t = \sum_{i=1}^{N} w_t^i$. Note that r_t^i, $i = 1, 2, ..., N$ are well-defined since $W_t \geq \bar{w}_t > 0$, as long as the strategy profile is admissible. Recall that the strategy Λ^i (or investor i using it) survives with probability one if $\inf_{t \geq 0} r_t^i > 0$ a.s., and this portfolio rule is called a survival strategy if investor i using it survives with probability one regardless of what portfolio rules $\Lambda^j, j \neq i$, are used by the other investors.

Explicit Expression for the Survival Strategy Λ^*. To formulate the main results, we define

$$R_{t+1,k} = \frac{a_{t+1,k} V_{t,k}}{\sum_{l=0}^{K} a_{t+1,l} V_{t,l}}, \;\; k = 0, ..., K, \tag{5.16}$$

where

$$a_{t+1,0} := 1 + \beta_{t+1}, \;\; V_{t,0} := 1. \tag{5.17}$$

Clearly,

$$\sum_{k=0}^{K} R_{t+1,k} = 1. \tag{5.18}$$

Consider the investment strategy $\Lambda^* = (\lambda_t^*)$ for which the vector $\lambda_t^* = (\lambda_{t,0}^*, ..., \lambda_{t,K}^*)$ is given by

$$\lambda_{t,k}^*(s^t) := E[R_{t+1,k}(s^{t+1})|s^t]. \tag{5.19}$$

The strategy Λ^* shows the idea to distribute wealth by the proportions of the conditional expectations of the relative asset payoffs.

Assume that for each $k = 0, 1, 2, ..., K$,

$$E \ln E_t R_{t+1,k}(s^{t+1}) > -\infty. \tag{5.20}$$

This assumption implies that the conditional expectation $E_t R_{t+1,k} = E(R_{t+1,k}|s^t)$ is strictly positive a.s., and so we can choose a version of this conditional expectation that is strictly positive for all s^t. This version will be used in the definition (5.19) of the portfolio rule $\Lambda^* = (\lambda_t^*)$.

Two Theorems for the Existence and Asymptotic Uniqueness of the Survival Strategy Λ^*.

Theorem 5.1 *The portfolio rule Λ^* is a survival strategy.*

Note that the portfolio rule Λ^* belongs to the class of basic portfolio rules.

Theorem 5.2 *If $\Lambda = (\lambda_t)$ is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \text{ (a.s.)}. \tag{5.21}$$

Here, "$||\cdot||$" represents any norm in a finite-dimensional linear space. This theorem shows that in the class of basic portfolio rules, the survival strategy $\Lambda^* = (\lambda_t^*)$ is essentially unique: any other basic survival strategy Λ is asymptotically similar to Λ^*. Recall that according to Section 2.4.2, survival strategies Λ can be characterized by the property that the wealth w_t^j of any other investors j cannot grow asymptotically faster, with strictly positive probability, than the wealth of investor i using Λ.

In fact, Theorems 5.1 and 5.2 are direct consequences of Theorems 2.1 and 2.2 proved in Section 2.3. The proof procedures of Theorems 5.1 and 5.2 are routine and analogous to those in Section 2.3; therefore, the detailed proofs of the main results of this model are relegated to Appendix B.

Chapter 6

Model with Short Selling

Up to now, all the models we have reviewed are established on the assumptions of excluding short selling and exogenous asset supply, which are the foundations upon which the comprehensive theory of evolutionary dynamics has been developed. In the model examined in this chapter, market participants are allowed to construct their portfolios not only with long, but also with short positions. To create short positions, investors issue "replicas" of the original assets, which share the same equilibrium prices and payoffs. These newly issued assets are sold at the equilibrium prices, generating the short-selling income for the issuer, which in turn increases the investment budget available for purchasing other assets (i.e., creating long portfolio positions). On the other hand, selling each unit of an asset short obligates the seller to pay the buyer the same payoff as the original asset. Moreover, short selling results in an increase of the exogenously given total quantity $V_{t,k}$ of each asset k in the market, thereby influencing the equilibrium prices.

As with all the Evolutionary Finance models considered above, our primary focus is on the fundamental questions of the existence and (asymptotic) uniqueness of a survival strategy. First, we ask whether the strategy Λ^* (without short selling) guarantees survival in a market where the rivals of the Λ^*-investor can sell short, and if this strategy is (at least asymptotically) unique. The findings shown in Section 6.2 confirm that the strategy Λ^* (which rules out short selling) indeed guarantees survival in a market where short sales are allowed. However, are there strategies including short selling that also guarantee survival? If so, are they asymptotically distinct from Λ^*? The answers are negative. [1]

6.1 The Model

A Summary on Changes in Model Settings. A starting point for this model remains the basic model presented in Section 2.1, and the changes that occur in this model are summarized as follows:

[1] This chapter reviews the work by Amir, Belkov, Evstigneev, and Hens [1].

- Consider a market where $K \geq 2$ risky assets (securities) are traded by $N \geq 2$ investors (traders) at dates $t = 0, 1, ...,$ and the market is influenced by random factors captured by an exogenous stochastic process $s_1, s_2, ...$(states of the world), where s_t is a random element in a measurable space S_t. At each date t, the total number of units of asset k available is $V_{t,k} = V_{t,k}(s^t) > 0$ (with the constant $V_{0,k} > 0$), and each investor $i = 1, ..., N$ holds some wealth $w_t^i = w_t^i(s^t)$ (with the initial endowments $w_0^i > 0$). The payoff functions $A_{t,k} = A_{t,k}(s^t) \geq 0$ for each asset $k = 1, ..., K$ at date $t = 1, 2, ...,$ are measurable and satisfy

$$\sum_{k=1}^{K} A_{t,k}(s^t) > 0. \tag{6.1}$$

- At each date $t \geq 0$, investor i purchases $x_{t,k}^i \geq 0$ units of asset $k = 1, ..., K$ and/or sells short (at this date) $y_{t,k}^i \geq 0$ units of this asset k. At time period $[t, t+1]$, the payoff that investor i receives at date $t+1$ from $x_{t,k}^i$ units of asset k will be $A_{t+1,k}x_{t,k}^i = A_{t+1,k}(s^{t+1})x_{t,k}^i$; if, investor i sells short $y_{t,k}^i$ units of asset k at date t, then this investor has an obligation to pay $A_{t+1,k}(s^{t+1})y_{t,k}^i$ at date $t+1$.

- Denote the vectors of asset prices by $p_t = p_{t,k}(s^t) = (p_{t,1}, ..., p_{t,K})$, which are determined *endogenously* by the *market equilibrium*, reached when total supply of each asset k is equal to its total demand (i.e., the market clears):

$$V_{t,k} + \sum_{i=1}^{N} y_{t,k}^i = \sum_{i=1}^{N} x_{t,k}^i, \; k = 1, ..., K. \tag{6.2}$$

  Meanwhile, the amount $y_{t,k}^i$ of asset k that the investor has sold short provides the investor with extra budget for further investments, with this amount being $p_{t,k}y_{t,k}^i$ (the short-selling income defined below). The total investment budget of trader i at date t, represented as $w_t^i + v_t^i$, consists of this trader's wealth w_t^i and the *short selling income*

$$v_t^i := \sum_{k=1}^{K} p_{t,k}y_{t,k}^i. \tag{6.3}$$

  At each date t, each trader spends the entire available budget $w_t^i + v_t^i$ to purchase assets:

$$w_t^i + v_t^i = \sum_{k=1}^{K} p_{t,k}x_{t,k}^i.$$

  The wealth w_{t+1}^i of investor i at the end of the time period $[t, t+1]$ can be calculated by

$$w_{t+1}^i := \sum_{k=1}^{K} A_{t+1,k}x_{t,k}^i - \sum_{k=1}^{K} A_{t+1,k}y_{t,k}^i = \sum_{k=1}^{K} A_{t+1,k}\left(x_{t,k}^i - y_{t,k}^i\right). \tag{6.4}$$

The quantities $x^i_{t,k} \geq 0$, $k = 1, ..., K$ of the assets purchased by investor i form the vector $x^i_t := \left(x^i_{t,1}, ..., x^i_{t,K}\right)$, and the volumes $y^i_{t,k} \geq 0$, $k = 1, ..., K$ of the assets sold short by investor i form the vector $y^i_t := \left(y^i_{t,1}, ..., y^i_{t,K}\right)$. The portfolio of investor i is given by the pair of vectors $(x^i_t, -y^i_t)$.

- For each $t \geq 0$, each investor $i = 1, 2, ..., N$ selects a vector of investment proportions $\gamma^i_t := (\gamma^i_{t,1}, ..., \gamma^i_{t,K})$, which belongs to the unit simplex

$$\gamma^i_t \in \Delta^K := \{(a_1, ..., a_K) \in \mathbb{R}^K : a_1 + ... + a_K = 1,\ a_k \geq 0,\ k = 1, ..., K\}. \tag{6.5}$$

  When selling short $y^i_{t,k}$ units of asset $k = 1, ..., K$ at date t, investor i issues and sells $y^i_{t,k}$ "replicas" of asset k, which have the same price and yield the same payoff for the buyer at the next date $t+1$ as the original asset k. Though this operation increases investor i's investment budget by $p_{t,k} y^i_{t,k}$ at date t, it leads to an obligation to pay $A_{t+1,k} y^i_{t,k}$ at date $t+1$. Formally, in this model investor i's decision (or action) at date t is specified by a pair of vectors $\xi^i_t = (\gamma^i_t, y^i_t)$, where γ^i_t is the vector of investment proportions and y^i_t is the vector whose coordinates define the short positions. Note that long positions of a portfolio are specified in terms of *investment proportions*, while its short positions are defined in terms of *units* of assets!

- In the following analysis, we will consider only those decisions that do not permit to open simultaneously a long and a short position for the same asset, i.e.,

$$\gamma^i_{t,k} y^i_{t,k} = 0,\ k = 1, ..., K,\ t \geq 0. \tag{6.6}$$

  This property will be included into the definition of investors' decisions.

- Similar to the most general setting, the investment decisions ξ^i_t at each date $t \geq 0$ are selected by N investors simultaneously and independently (as in a simultaneous-move N-person dynamic game). For $t \geq 1$, this decision making usually depends on s^t and the history of the game (or the history of the market) $\xi^{t-1} := \left\{\xi^i_l,\ i = 1, ..., N,\ l = 0, ..., t-1\right\}$. A pair of vectors $\Xi^i_0 = \left(\Gamma^i_0, Y^i_0\right) \in \Delta^K \times \mathbb{R}^K_+$ and a sequence of measurable functions $\Xi^i_t(s^t, \xi^{t-1}) = \left(\Gamma^i_t(s^t, \xi^{t-1}), Y^i_t(s^t, \xi^{t-1})\right)$, $t = 1, 2, ...,$ taking values in $\Delta^K \times \mathbb{R}^K_+$ form a portfolio rule Ξ^i of investor i, according to which investor i makes the decision

$$\xi^i_t = \Xi^i_t\left(s^t, \xi^{t-1}\right) \tag{6.7}$$

  at each date $t \geq 0$. This provides a game-theoretic framework for defining general portfolio rules. Within this framework, we will distinguish those for which Ξ^i_t depends only on s^t and not on the game history ξ^{t-1}, and such portfolio rules $\Xi^i_t(s^t)$ are termed basic.

A Description for the Asset Market Dynamics. Given a decision $\xi^i_t = (\gamma^i_t, y^i_t)$ of investor i, the long positions $x^i_{t,k}$ of i's portfolio $(x^i_t, -y^i_t)$ can be computed by following formulas:

$$x^i_{t,k} = \frac{\gamma^i_{t,k}(w^i_t + v^i_t)}{p_{t,k}} = \frac{1}{p_{t,k}}\gamma^i_{t,k}\left(w^i_t + \sum_{m=1}^{K} p_{t,m}y^i_{t,m}\right),\ k = 1, ..., K, \quad (6.8)$$

and the short positions of this portfolio are specified by the vector $-y^i_t$, where $y^i_t = (y^i_{t,1}, ..., y^i_{t,K})$.

In the system of equations (6.2), $x^i_{t,k}$ and v^i_t can be expressed by applying formulas (6.8) and (6.3), respectively. This leads to the following system of equations from which we can determine the vector $p_t = (p_{t,1}, ..., p_{t,K})$ of equilibrium asset prices:

$$\sum_{i=1}^{N} \gamma^i_{t,k}\left(w^i_t + \sum_{m=1}^{K} p_{t,m}y^i_{t,m}\right) = p_{t,k}\left(V_{t,k} + \sum_{i=1}^{N} y^i_{t,k}\right),\ k = 1, ..., K. \quad (6.9)$$

Proposition 6.1 *Let the following conditions hold:*

$$w^i_t > 0,\ i = 1, ..., N;\ \sum_{i=1}^{N} \gamma^i_{t,k}w^i_t > 0,\ k = 1, ..., K. \quad (6.10)$$

Then the system of equations (6.9) has a unique strictly positive solution $p_t = (p_{t,1}, ..., p_{t,K})$, $p_{t,k} > 0$, *for each k.*

According to this proposition, if at date t the wealth w^i_t of each investor $i = 1, 2, ..., N$ is strictly positive and for each asset $k = 1, ..., K$ at least one of the investors selects a strictly positive investment proportion $\gamma^i_{t,k} > 0$, then the asset market has a unique equilibrium with strictly positive prices. Note that if the first inequality in (6.10) is satisfied and at least one of the investors has a strictly positive vector of investment proportions $\gamma^i_t = (\gamma^i_{t,1}, ..., \gamma^i_{t,K})$, then the second inequality in (6.10) holds as well.

We conclude this section with remarks on the design of the model at hand.

Remark 6.1 Some comments on modeling long and short portfolio positions in our context. The approach to short selling that involves "replicas" of assets with the same exogenous payoffs is widely employed in mathematical models within Financial Economics (Magill and Quinzii [111]) and Mathematical Finance (Pliska [128], Ross [130], Föllmer and Schied [69], and Zierhut [170]). However, quite often it is not explicitly spelled out, as a deeper analysis of this question is usually unnecessary. Here, we wish to discuss this method in more detail, particularly due to a certain asymmetry in our model–long portfolio positions are specified in terms of investment proportions, while short ones are in "physical" units of assets–an asymmetry that is conceptually important and has a clear meaning, as it reflects the substantial difference between the operations of creating long and short portfolio positions in our context. The former is concerned with purchasing available assets by distributing wealth across them according to the given investment strategy. The latter operation, understood as creating new one-period assets that replicate the initial ones, is nothing but endogenous asset supply. In the purely financial context, endogenous asset supply may be regarded as the analogue of production in models with real assets. (This analogy becomes especially transparent if we examine the creation of derivative securities, rather then identical replicas of the basic assets.) It is worth noting that the liabilities arising from the creation of new securities–copying the basic ones–can be precisely estimated only if the number of units issued is known: for each unit of asset k sold short at time t, the seller must pay the buyer the amount (denoted in our model by) $A_{t+1,k}(s^{t+1})$ at time $t+1$. Furthermore, since the exogenous component $V_{t,k}$ in the equilibrium pricing equations (6.9) is expressed in terms of units of assets, the total asset supply must also be represented in the same way, which justifies the approach used in this paper for specifying short positions in investors' portfolios. As regards the long ones, theoretically one can describe them either in terms of units of assets or in terms of their monetary values and investment proportions. The latter approach is traditional in classical capital growth theory (see, e.g., Evstigneev et al. [63], Chapter 17,18), and since EF may be viewed as an extension of this theory to the case of endogenous asset prices, it is natural to design the model in a way similar to the classical one in order to use, whenever possible, similar machinery, notation, etc.[2]

6.2 The Main Results: Existence and Asymptotic Uniqueness of Survival Portfolio Rules

We focus on the analysis of the stochastic dynamics of investors' wealth depending on their strategies. Let $w_t = (w_t^1, ..., w_t^N)$ be the vector of investors' wealth, and let $\Xi = (\Xi^1, ..., \Xi^N)$ be the strategy profile of N investors. The dynamics of w_t will be defined recursively, step by step from t to $t+1$, given that the initial state (at $t = 0$) is the vector $w_0 = (w_0^1, ..., w_0^N)$, which stands for the initial endowment of each investor. Suppose $w_0, w_1, ..., w_t$ are defined for some $t \geq 0$. Assume that the following condition holds:

(A). The vector $w_t = (w_t^1, ..., w_t^N)$ and the investment proportions $\gamma_{t,k}^i$ (generated the strategy profile Ξ) satisfy (6.10).

Then according to Proposition 6.1, there exists a unique strictly positive vector $p_t = (p_{t,1}, ..., p_{t,K})$ of equilibrium asset prices, in terms of which we can

[2] For further reading on other models in capital growth theory and EF involving short selling and endogenous asset supply, see Bucher and Woehrmann [38], Horváth and Urbán [92], Schenk-Hoppé and Sokko [138].

express the wealth w^i_{t+1} of each investor i:

$$\begin{aligned} w^i_{t+1} &= \sum_{k=1}^{K} A_{t+1,k} x^i_{t,k} - \sum_{k=1}^{K} A_{t+1,k} y^i_{t,k} \\ &= \sum_{k=1}^{K} A_{t+1,k} \frac{1}{p_{t,k}} \gamma^i_{t,k} \left(w^i_t + \sum_{m=1}^{K} p_{t,m} y^i_{t,m} \right) - \sum_{k=1}^{K} A_{t+1,k} y^i_{t,k} \\ &= \sum_{k=1}^{K} A_{t+1,k} \left(\frac{1}{p_{t,k}} \gamma^i_{t,k} \left(w^i_t + \sum_{m=1}^{K} p_{t,m} y^i_{t,m} \right) - y^i_{t,k} \right). \end{aligned} \tag{6.11}$$

Definition 6.1 *If in the course of this dynamical process, condition* (**A**) *happens to hold almost surely (a.s.)*[3] *for all* $t \in [0, T)$ $(T \leq \infty)$, *we say that the given strategy profile* Ξ *is admissible for the time interval* $[0, T)$.

Remark 6.2 Assume that one of the investors, e.g., investor 1, adopts a *fully diversified* portfolio rule that prescribes investing into all assets in strictly positive proportions $\gamma^1_{t,k} > 0$ for all $k = 1, ..., K$ and all $t \geq 0$. Then, given that no investor goes bankrupt during the time interval $[0, T)$, the strategy profile is admissible for the time interval $[0, T)$.

By summing up (6.11) over $i = 1, ..., N$ and taking into account the pricing equations (6.9), we get the following formula for the total market wealth W_{t+1}:

$$\begin{aligned} W_{t+1} := \sum_{i=1}^{N} w^i_{t+1} &= \sum_{i=1}^{N} \sum_{k=1}^{K} A_{t+1,k} \left(\frac{1}{p_{t,k}} \gamma^i_{t,k} \left(w^i_t + \sum_{m=1}^{K} p_{t,m} y^i_{t,m} \right) - y^i_{t,k} \right) \\ &= \sum_{k=1}^{K} A_{t+1,k} \left(\frac{1}{p_{t,k}} \sum_{i=1}^{N} \left[\gamma^i_{t,k} \left(w^i_t + \sum_{m=1}^{K} p_{t,m} y^i_{t,m} \right) \right] - \sum_{i=1}^{N} y^i_{t,k} \right) \\ &= \sum_{k=1}^{K} A_{t+1,k} \left(V_{t,k} + \sum_{i=1}^{N} y^i_{t,k} - \sum_{i=1}^{N} y^i_{t,k} \right) = \sum_{k=1}^{K} A_{t+1,k} V_{t,k}. \end{aligned} \tag{6.12}$$

As before, denote by $r^i_{t+1} := \frac{w^i_{t+1}}{W_{t+1}}$ the relative wealth (market share) of investor i and put $r_{t+1} := \left(r^1_{t+1}, ..., r^N_{t+1} \right)$.

Definition 6.2 *We say that a strategy* Ξ^i *employed by investor i can be driven out of the market at a (finite) time* $T < \infty$ *if there exists a strategy profile* $(\Xi^1, ..., \Xi^N)$ *including the strategy* Ξ^i *and admissible for* $t \in [0, T)$ *such that* $P \left\{ w^i_T \leq 0 \right\} > 0$.

[3] In this chapter, we will identify random variables coinciding almost surely, and we might often omit "a.s." if this does not lead to ambiguity. In fact the random variables under consideration may be defined not everywhere, but only almost everywhere: with probability one. If not otherwise stated, all relations between them (equalities, inequalities, etc.) will be supposed to hold almost surely.

Definition 6.3 *We say that a strategy Ξ^i employed by investor i can be driven out of the market in an infinite time if there exists a strategy profile $(\Xi^1, ..., \Xi^N)$ including the strategy Ξ^i and admissible for $t \in [0, \infty)$ such that $P\left\{\inf_{t \geq 0} r_t^i = 0\right\} > 0$.*

Definition 6.4 *We call a strategy Ξ a survival strategy if for any number $N \geq 2$ of agents acting in the market an investor using Ξ cannot be driven out of the market in any (finite or infinite) time.*

Explicit Expression for the Survival Strategy Ξ^*. Define the *relative payoffs* by

$$R_{t+1,k} := \frac{A_{t+1,k}V_{t,k}}{\sum_{l=1}^{K} A_{t+1,l}V_{t,l}} \tag{6.13}$$

and put $R_{t+1} := (R_{t+1,1}, ..., R_{t+1,K})$. Consider the investment strategy $\Xi^* = (\xi_t^*)_{t=0}^{\infty} = (\gamma_t^*, y_t^*)_{t=0}^{\infty}$ for which $y_t^* = 0$ and $\gamma_t^*(s^t) := (\gamma_{t,1}^*(s^t), ..., \gamma_{t,K}^*(s^t))$, where

$$\gamma_{t,k}^*(s^t) := E_t R_{t+1,k}(s^{t+1}). \tag{6.14}$$

Throughout this chapter, we will assume that for each $k = 1, ..., K$,

$$E \ln E_t R_{t+1,k}(s^{t+1}) > -\infty. \tag{6.15}$$

This assumption implies that the conditional expectations $E_t R_{t+1,k} = E(R_{t+1,k}|s^t)$ $(k = 1, ..., K)$ are strictly positive (a.s.). Consequently, we can choose their versions $\gamma_{t,k}^*(s^t)$ which are strictly positive for all s^t. In following analysis, the notation $\gamma_{t,k}^*(s^t)$ will refer to such versions of the above conditional expectations.

Three Theorems for the Existence and Asymptotic Uniqueness of the Survival Strategy Ξ^*.

Theorem 6.1 *The portfolio rule Ξ^* is a survival strategy.*

It is important to note that the portfolio rule Ξ^* is basic and it does not include short selling, but it survives in competition with *all*, not necessarily basic, strategies with short selling. The results we formulate below show that in the class of basic strategies, the survival portfolio rule Ξ^* is (at least asymptotically) unique.

Theorem 6.2 *If a basic strategy prescribes to sell short at least one asset at some moment of time with strictly positive probability, then it can be driven out of the market in a finite time.*

Hence, basic survival portfolio rules can exist only in the class of basic strategies that do not involve short selling (a.s.). It should be emphasized that a short seller can be driven out of the market in a finite time by a basic strategy profile of the rivals, which will be shown in the course of the proof of Theorem 6.2.

Theorem 6.3 *If Ξ is a basic survival strategy defined by a sequence of decisions $(\gamma_t(s^t), y_t(s^t))$, $t = 0, 1, 2...$, with $y_t(s^t) = 0$ (a.s.), then*

$$\sum_{t=0}^{\infty} ||\gamma_t^* - \gamma_t||^2 < \infty \text{ (a.s.)}. \tag{6.16}$$

This theorem (pertaining to a version of the present model without short selling) follows easily from Theorem 2.2, Section 2.2. Notice that vectors of investment proportions γ_t^* coincide with vectors λ_t^* generated by the strategy Λ^* considered in Section 2.2. In that model (where no short-selling is allowed), Λ^* is also an asymptotically unique basic survival strategy.

Proofs of Proposition 6.1, and Theorems 6.1, 6.2 and 6.3 are given in the next section.

6.3 Proofs of the Main Results

6.3.1 Short-run Equilibrium

Proof of Proposition 6.1. Fix t and omit it in the notation. For all $k = 1, ..., K$, define

$$u_k := p_k \left(V_k + \sum_{i=1}^{N} y_k^i \right), \ \sigma_k := \frac{\sum_{i=1}^{N} y_k^i}{V_k + \sum_{i=1}^{N} y_k^i},$$

$$\theta_k^i := \begin{cases} \dfrac{y_k^i}{\sum_{j=1}^{N} y_k^j}, & \text{if } \sum_{j=1}^{N} y_k^j > 0, \\ N^{-1}, & \text{otherwise.} \end{cases}$$

Note that $0 \leq \sigma_k < 1$ because $V_k > 0$, and we have $\sum_{i=1}^{N} \theta_k^i = 1$ and $\theta_k^i \geq 0$. If $\sum_{j=1}^{N} y_k^j > 0$, then for each $m = 1, ..., K$, the following identity holds:

$$y_m^i = \left(V_m + \sum_{j=1}^{N} y_m^j \right) \frac{(\sum_{j=1}^{N} y_m^j) y_m^i}{(V_m + \sum_{j=1}^{N} y_m^j) \cdot \sum_{j=1}^{N} y_m^j} = \left(V_m + \sum_{j=1}^{N} y_m^j \right) \sigma_m \theta_m^i,$$

which gives

$$p_m^i y_m^i = u_m \sigma_m \theta_m^i, \ m = 1, ..., K.$$

If $\sum_{j=1}^{N} y_k^j = 0$, the above equality is valid as well, since in that case $y_m^i = \sigma_m = 0$. Consequently, the system of equations (6.9) can be written as

$$\sum_{i=1}^{N} \gamma_k^i w^i + \sum_{i=1}^{N} \gamma_k^i \sum_{m=1}^{K} \theta_m^i \sigma_m u_m = u_k, \ k = 1, ..., K. \tag{6.17}$$

A vector $u = (u_1, ..., u_K)$ solves (6.17) if and only if u is a fixed point of the operator

$$F(u) := \left(\sum_{i=1}^{N} \gamma_k^i w^i + \sum_{i=1}^{N} \gamma_k^i \sum_{m=1}^{K} \theta_m^i \sigma_m u_m \right)_{k=1}^{K}$$

which is determined by the left-hand side of (6.17). This operator transforms the cone

$$\mathbb{R}_+^K = \{u \,|\, u_k \geq 0,\, k = 1, ..., K\}$$

of non-negative K-dimensional vectors into itself. And to establish that it has a unique fixed point in $\mathbb{R}_+^K$, it suffices to show that it is contracting in the norm $\|u\| = \sum_{k=1}^K |u_k|$. This follows from the chain of relations:

$$\begin{aligned}\|F(u) - F(u')\| &= \sum_{k=1}^K |\sum_{i=1}^N \gamma_k^i \sum_{m=1}^K \theta_m^i \sigma_m (u_m - u'_m)| \\ &\leq (\max_m \sigma_m) \sum_{i=1}^N (\sum_{k=1}^K \gamma_k^i) \sum_{m=1}^K \theta_m^i |u_m - u'_m| \\ &= (\max_m \sigma_m) \sum_{m=1}^K |u_m - u'_m| \sum_{i=1}^N \theta_m^i = (\max_m \sigma_m) \|u - u'\|,\end{aligned}$$

where the equalities hold because $\sum_{k=1}^K \gamma_k^i = 1$ and $\sum_{i=1}^N \theta_k^i = 1$. Here, $\max_m \sigma_m < 1$ since $\sigma_m < 1$ for all m, and thus the operator F is contracting. Therefore, the system (6.17) has a unique solution $u = (u_1, ..., u_K) \geq 0$. Moreover, if condition (6.10) is satisfied, then

$$u_k = \sum_{i=1}^N \gamma_k^i w^i + \sum_{i=1}^N \gamma_k^i \sum_{m=1}^K \theta_m^i \sigma_m u_m \geq \sum_{i=1}^N \gamma_k^i w^i > 0$$

and the system (6.9) also has the unique strictly positive solution

$$p = (p_1, ..., p_K),\ p_k = \frac{u_k}{V_k + \sum_{i=1}^N y_k^i}$$

and $p_k > 0$ for each k. ■

6.3.2 Survival Portfolio Rules

Proof of Theorem 6.1. Consider a market with $N \geq 2$ investors. Suppose that investor 1 employs the strategy $(\gamma_t^*, 0) = \left(E_t R_{t+1}(s^{t+1}), 0\right)$. Agent 1, using the strategy Ξ^*, cannot be driven out of the market at any finite time $0 < T < \infty$ because assumption (6.15) guarantees that investor 1's portfolio is fully diversified (a.s.), thus, $w_T^1 > 0$ (a.s.). We shall show that ξ_t^* cannot be driven out of the market for the time interval $[0, \infty)$.

Consider a strategy profile $\Xi = (\Xi^1, ..., \Xi^N)$ that is admissible over $[0, \infty)$, for which $w_t^i > 0$ a.s. for all $t \geq 0$ and $i = 1, ..., N$. Let $(\xi_t^1(s^t), ..., \xi_t^N(s^t))_{t=0}^\infty$ be the set of investors' decisions generated by Ξ. For all $t \geq 0$, $k = 1, ..., K$, $i = 1, ..., N$, define the numbers

$$\lambda_{t,k}^i = \lambda_{t,k}^i\left(s^t\right) = \frac{w_t^i + \sum_{m=1}^K p_{t,m} y_{t,m}^i}{w_t^i} \gamma_{t,k}^i - \frac{p_{t,k} y_{t,k}^i}{w_t^i}, \tag{6.18}$$

and the vectors $\lambda_{t,k} = \left(\lambda^1_{t,k}, ..., \lambda^N_{t,k}\right)$ and $\lambda^i_t = \left(\lambda^i_{t,1}, ..., \lambda^i_{t,K}\right)$. Note that $\lambda^1_t = \gamma^*_t$ since $\xi^1_t = (\gamma^*_t, 0)$.

We shall formulate a system of equations that describes the dynamics of the market shares r^i_t in terms of the sequence of vectors $(\lambda^1_t, ..., \lambda^N_t)$. We have

$$p_{t,k}V_{t,k} = \langle \lambda_{t,k}, w_t \rangle, \ \ k = 1, ..., K,$$

where $\langle \lambda_{t,k}, w_t \rangle := \sum_{i=1}^N \lambda^i_{t,k} w^i_t$. Indeed,

$$\sum_{i=1}^N \lambda^i_{t,k} w^i_t = \sum_{i=1}^N (w^i_t + \sum_{m=1}^K p_{t,m} y^i_{t,m}) \gamma^i_{t,k} - \sum_{i=1}^N p_{t,k} y^i_{t,k} = p_{t,k} V_{t,k}$$

in view of (6.18) and (6.9). Proposition 6.1 and the definition of an admissible strategy profile over time interval $[0, \infty)$ guarantee that

$$p_{t,k} = \frac{\langle \lambda_{t,k}, w_t \rangle}{V_{t,k}} > 0 \text{ (a.s.)}, \ \ k = 1, ..., K. \tag{6.19}$$

By virtue of (6.18) and (6.8), we obtain

$$\frac{\lambda^i_{t,k} w^i_t V_{t,k}}{\langle \lambda_{t,k}, w_t \rangle} = \frac{\lambda^i_{t,k} w^i_t}{p_{t,k}} = \frac{w^i_t + \sum_{m=1}^K p_{t,m} y^i_{t,m}}{p_{t,k}} \gamma^i_{t,k} - \frac{p_{t,k} y^i_{t,k}}{p_{t,k}} = x^i_{t,k} - y^i_{t,k} \tag{6.20}$$

($k = 1, ..., K$). Hence, the wealth w^i_{t+1} of investor i given by (6.4) can be expressed as follows:

$$w^i_{t+1} = \sum_{k=1}^K A_{t+1,k} \left(x^i_{t,k} - y^i_{t,k}\right) = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle}. \tag{6.21}$$

By summing up the equations in (6.21) over $i = 1, ..., N$, we get

$$W_{t+1} = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\sum_{i=1}^N \lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle} = \sum_{k=1}^K A_{t+1,k} V_{t,k}. \tag{6.22}$$

Dividing the left-hand side of (6.21) by W_{t+1}, and the right-hand side of (6.21) by

$$\sum_{m=1}^K A_{t+1,m} V_{t,m},$$

and applying (6.13), we arrive at the system of equations

$$r^i_{t+1} = \sum_{k=1}^K R_{t+1,k} \frac{\lambda^i_{t,k} r^i_t}{\langle \lambda_{t,k}, r_t \rangle}, \ \ i = 1, ..., N. \tag{6.23}$$

Since we consider a strategy profile $(\Xi^1, ..., \Xi^N)$ that is admissible for the time interval $[0, \infty)$, all the market shares are strictly positive:

$$r^i_t > 0, \ \ i = 1, ..., N, \ \ t \geq 0. \tag{6.24}$$

Observe that

$$\sum_{k=1}^{K} \lambda_{t,k}^{i} = \frac{w_t^i + \sum_{m=1}^{K} p_{t,m} y_{t,m}^i}{w_t^i} \sum_{k=1}^{K} \gamma_{t,k}^{i} - \frac{\sum_{k=1}^{K} p_{t,k} y_{t,k}^i}{w_t^i} = 1,$$

and thus

$$\sum_{k=1}^{K} \lambda_{t,k}^{i} = 1, \; t \geq 0. \tag{6.25}$$

From (6.19) we conclude that

$$\langle \lambda_{t,k}, r_t \rangle > 0, \; k = 1, ..., K \tag{6.26}$$

for each $t \geq 0$.

To complete the proof, we shall use Proposition 6.2 from which we conclude that $\inf_{t\geq 0} r_t^1 > 0$ (a.s.) and investor 1 cannot be driven out of the market at moment $T = \infty$. Therefore, Ξ^* is a survival strategy. ■

Proposition 6.2 *If $\lambda_t^1(s^t) = E_t R_{t+1}(s^{t+1})$ and dynamical system (6.23) satisfies the conditions (6.24)-(6.26) then*

$$\inf_{t\geq 0} r_t^1 > 0 \; \textit{(a.s.)}.$$

The proof for Proposition 6.2 is exactly the same as the 2^{nd} and 3^{rd} steps of the proof for Theorem 2.1, presented in Section 2.3.

6.3.3 Asymptotic Uniqueness of Survival Strategies

Proof of Theorem 6.2. Put $\bar{K} = \{1, ..., K\}$ and consider a market with $N = 2$ investors. Assume that investor 2 adopts a strategy Ξ^2 prescribing to open a short position with strictly positive probability for some asset at some moment of time, and let $T \geq 0$ be the smallest among such moments of time. Then, there exists a non-random M-element subset $\bar{M} \subset \bar{K}$ such that $M \geq 1$ and the event

$$\bar{S}^T := \{s^T : y_{T,k}^2\left(s^T\right) > 0 \text{ for } k \in \bar{M} \text{ and } y_{T,k}^2\left(s^T\right) = 0 \text{ for } k \in \bar{K}\backslash\bar{M}\}$$

has a strictly positive probability. (Note that $k \in \bar{K}\backslash\bar{M} \neq \emptyset$ because investor 2 cannot sell short *all* the assets.)

Without loss of generality, we can assume that investor 2 cannot be driven out of the market at any time $t \leq T$; otherwise, the theorem is proved. Under this condition, we construct a *spiteful strategy* Ξ^1 of investor 1, i.e. a strategy driving investor 2 out of the market at time $T+1$. With the strategy Ξ^1, the strategy profile $\left(\Xi^1, \Xi^2\right)$ will be admissible for the time interval $[0, T+1)$ and the wealth w_{T+1}^2 of investor 2 will be negative with strictly positive probability. The negativity of w_{T+1}^2 with strictly positive probability will be established under an additional assumption that the initial wealth w_0^2 of investor 2 is sufficiently small comparative to the initial wealth w_0^1 of his/her rival.

Fix some $\mu \in \bar{M}$. For any $\delta > 0$ and $\delta' > 0$, denote by $\bar{S}^{T+1}\left(\delta, \delta^{'}\right)$ the following event:

$$\bar{S}^{T+1}\left(\delta, \delta^{'}\right) = \{s^{T+1} : s^T \in \bar{S}^T,\ V_{T,\mu}^{-1} R_{T+1,\mu}(s^{T+1}) > \delta',\ y_{T,\mu}^2\left(s^T\right) > \delta\}. \tag{6.27}$$

Since $P\{s^T \in \bar{S}^T\} > 0$ and $P\{V_{T,\mu}^{-1} R_{T+1,\mu} > 0 \mid s^T\} > 0$ a.s. (in view of assumption (6.15)), there exist $\delta > 0$ and $\delta' > 0$ such that $P\left\{s^{T+1} \in \bar{S}^{T+1}\left(\delta, \delta^{'}\right)\right\} > 0$.

Fix a positive number $\varepsilon < \min\left(\delta'\delta/(M \cdot K), 1/K\right)$ and define the strategy Ξ^1 as follows. For each $t \in [0, T+1)$, put $y_t^1 = 0$ (no short selling) and define:

$$e = (1, 1, ..., 1),$$

$$\gamma_t^1 = \frac{1}{2}\gamma_t^2 + \frac{1}{2K}e, \tag{6.28}$$

for $t < T$ and

$$\gamma_{T,k}^1 = \begin{cases} \varepsilon & \text{if } k \in \bar{M}, \\ \frac{1-M\varepsilon}{K-M} & \text{if } k \in \bar{K} \backslash \bar{M}. \end{cases} \tag{6.29}$$

Since investor 2 cannot be driven out of the market at any time $t \leq T$, we conclude that $w_t^2 > 0$ (a.s.) for $t \leq T$. Further, observe that $\gamma_{t,k}^1 > 0$, which implies $w_t^1 > 0$ and $\sum_{i=1}^N \gamma_{t,k}^i w_t^i \geq \gamma_{t,k}^1 w_t^1 > 0$ for each $k \in \bar{K}$. Therefore, the strategy profile $\left(\Xi^1, \Xi^2\right)$ is admissible for the time interval $[0, T+1)$.

Note that $y_t^1 = y_t^2 = 0$ and $\gamma_t^1 \geq \frac{1}{2}\gamma_t^2$ (see (6.28)) for each $t < T$. Consequently, by virtue of (6.11) and (6.8), we get

$$\frac{w_{t+1}^2}{w_{t+1}^1} = \frac{\sum_{k \in \bar{K}} A_{t+1,k} x_{t,k}^2}{\sum_{k \in \bar{K}} A_{t+1,k} x_{t,k}^1} = \frac{\sum_{k \in \bar{K}} A_{t+1,k} \frac{\gamma_{t,k}^2 w_t^2}{p_{t,k}}}{\sum_{k \in \bar{K}} A_{t+1,k} \frac{\gamma_{t,k}^1 w_t^1}{p_{t,k}}} \leq 2\frac{w_t^2}{w_t^1}, \quad t < T. \tag{6.30}$$

Now assume that the initial wealth w_0^2 of investor 2 is small enough compared to the initial wealth w_0^1 of investor 1, specifically,

$$\frac{w_0^2}{w_0^1} \leq \frac{1}{2^T}\left(\frac{\delta'\delta}{K} - M\varepsilon\right). \tag{6.31}$$

Then, the inequality in (6.30) yields

$$\frac{w_T^2}{w_T^1} \leq 2^T \frac{w_0^2}{w_0^1} \leq \frac{\delta'\delta}{K} - M\varepsilon. \tag{6.32}$$

For $s^{T+1} \in \bar{S}^{T+1}\left(\delta, \delta^{'}\right)$, we obtain

$$\gamma_{T,k}^2 = 0 \text{ for } k \in \bar{M} \tag{6.33}$$

because $y_{T,k}^2 > 0$ for $k \in \bar{M}$. In view of (6.9), (6.33) and (6.29), the prices $p_{T,k}$ satisfy the following conditions:

$$p_{T,k} = \frac{w_T^1 \gamma_{T,k}^1 + \left(w_T^2 + v_T^2\right) \gamma_{T,k}^2}{V_{T,k} + y_{T,k}^2} = \frac{w_T^1 \varepsilon}{V_{T,k} + y_{T,k}^2}, \text{ if } k \in \bar{M}, \tag{6.34}$$

$$p_{T,k} \geq \frac{w_T^1 \cdot \gamma_{T,k}^1}{V_{T,k}} \geq \frac{w_T^1}{V_{T,k}} \frac{1}{K}, \text{ if } k \in \bar{K} \backslash \bar{M}. \tag{6.35}$$

By applying relations (6.8), (6.34), (6.35) and (6.32) to estimate $x_{T,k}^2$, $k \in \bar{K} \backslash \bar{M}$, we find that

$$x_{T,k}^2 \leq \frac{w_T^2 + \sum_{m \in \bar{M}} p_{T,m} y_{T,m}^2}{p_{T,k}} \leq \left(w_T^2 + \sum_{m \in \bar{M}} \frac{w_T^1 \varepsilon \cdot y_{T,m}^2}{V_{T,m} + y_{T,m}^2} \right) \cdot \left(\frac{w_T^1}{V_{T,k}} \frac{1}{K} \right)^{-1}$$
$$\leq \left(\frac{w_T^2}{w_T^1} + M\varepsilon \right) V_{T,k} K \leq \delta' \delta V_{T,k}. \tag{6.36}$$

Let $s^{T+1} \in \bar{S}^{T+1}\left(\delta, \delta^{'}\right)$. By using (6.27), (6.36), and (6.13), we arrive at the wealth of investor 2 expressed by the following sequence of inequalities:

$$w_{T+1}^2\left(s^{T+1}\right) = -\sum_{k \in \bar{M}} A_{T+1,k} y_{T,k}^2 + \sum_{k \in \bar{K} \backslash \bar{M}} A_{T+1,k} x_{T,k}^2$$
$$\leq -A_{T+1,\mu}\delta + \delta'\delta \sum_{k \in \bar{K} \backslash \bar{M}} A_{T+1,k} V_{T,k} + \delta'\delta \sum_{k \in \bar{M}} A_{T+1,k} V_{T,k}$$
$$= -A_{T+1,\mu}\delta + \delta'\delta \sum_{k \in \bar{K}} A_{T+1,k} V_{T,k} = \delta \cdot \sum_{k \in \bar{K}} A_{T+1,k} V_{T,k} \cdot (\delta' - \frac{R_{T+1,\mu}}{V_{T,\mu}}) \leq 0.$$

Therefore, if condition (6.31) holds and investor 1 employs the strategy defined by (6.29) and (6.28), then

$$P\left\{w_{T+1}^2 \leq 0\right\} \geq P\left\{s^{T+1} \in \bar{S}^{T+1}\left(\delta, \delta^{'}\right)\right\} > 0,$$

which means that the strategy Ξ^2 does not survive. ■

Proof of Theorem 6.3. Consider the basic strategy Ξ' defined by the sequence of decisions $(\gamma_t(s^t), 0)$ $(t = 0, 1, ...)$. Denote by w_t, $t = 0, 1, ...$ the wealth of investor using Ξ and by w_t', $t = 0, 1, ...$ the wealth of investor using Ξ'. Then $w_t' = w_t$ (a.s.), and Ξ' is a survival strategy. Strategies Ξ^* and Ξ' do not allow for short selling (not just a.s., but everywhere). From Theorem 2.2 given in Section 2.2, we obtain (6.16). ■

Chapter 7

Conclusion

This paper reviews some key models with short-lived assets in the field of Evolutionary Finance (EF). EF focuses on "survival and extinction" questions of investment portfolio rules in the market selection process. In contrast with the conventional theory of equilibrium and dynamics of asset markets that based on the Walrasian equilibrium paradigm, the EF theory we examined depicts a radically different world: 1) we consider a notion of behavioral equilibrium defined in consecutive short-run terms with a whole variety of strategy patterns, rather than maximizing small investors' individual utilities of consumption subject to budget constraints [1]; 2) we eliminate the need for the "perfect foresight" assumption to establish equilibrium; 3) we get rid of the reliance on knowledge of unobservable individual utilities and beliefs, which is hard to capture in investment practice; 4) a world of large, even super large (primarily institutional) investors who may operate on a global level and pursue objectives of an evolutionary nature: domination, fastest growth and survival (especially in crisis environments). Indeed, fastest growth is often related, and as shown in Section 2.4.2 is equivalent, to survival. Within this framework, each market actor's investment decisions may considerably influence equilibrium prices, as opposed to many classical market models where individual influence is generally negligible. Moreover, from game-theoretic aspects, the models we considered in this paper involve elements of both stochastic dynamic games (strategic frameworks) and evolutionary game theory (solution concepts).

The primary goal of the studies is to identify investment strategies that guarantee "long-run survival", i.e., keeping a strictly positive, bounded away from zero, share of market wealth over an infinite time horizon, irrespective of what strategies used by others. It turns out that in all the models we considered, there always exists a portfolio rule ensuring unconditional long-run survival, which is given by Kelly's well-known portfolio rule of "betting one's beliefs" and can be expressed with explicit formulas. Additionally, we demonstrate the

[1] In Walrasian equilibrium paradigm, market equilibrium can be understood as a situation in which the objectives (interests) of such economic agents are equilibrated by the market clearing prices (see, e.g., Flåm [68]).

asymptotic uniqueness (within a specific class of basic strategies) of such a rule.

We started with the basic modeling framework under substantially more general assumptions in Chapter 2. The results regarding the existence of a survival strategy and its asymptotic uniqueness within the class of basic portfolio rules are proved. However, as demonstrated by the counterexample shown in Section 2.4.1, this property of asymptotic uniqueness does not naturally extend to the entire class of general portfolio rules. Moreover, we also discussed that under such model settings, a portfolio rule guarantess survival if and only if it is unbeatable – in the sense that "in order to survive you have to win".

Chapter 3 presented an approach to formulating an evolutionary solution concept in terms of a non-conventional (stronger) version of Nash equilibrium that holds almost surely, where a strategy strictly dominates the market at an exponential rate. This can be achieved by introducing the Lyapunov exponent of the relative growth of wealth of a market participant–"relative" referring to the ratio of the player's wealth to the total wealth of the group of his/her rivals–as the objective function. It is a stronger version in the sense that, according to this definition of an equilibrium strategy, any unilateral deviation from the unique symmetric Nash equilibrium leads to a decrease in the random payoff with probability one, not just in the expected payoff. Hence, by replacing the standard Nash equilibrium conditions (formulated using expected utilities) by holding almost surely, we obtained a solution concept equivalent to the exponential domination of the market. As a consequence, those investors who adopt the Kelly portfolio rule achieve the highest growth rate of wealth in any population of fixed-mix strategies, ultimately becoming the single survivor who accumulates the entire market wealth in the long run.

Chapter 4 described a model with a Markovian nature, which removed two strong assumptions from Chapter 3, and instead, used a homogeneous discrete-time Markov process to describe the states of the world. The central result indicated that an investor who distributes wealth across available assets according to their relative conditional expected payoffs (a direct analogue of "betting one's beliefs" rule above) is a single survivor, provided this strategy is asymptotically distinct from the CAPM rule.

In Chapter 5 and Chapter 6, we examined models extended to the market involving a risk-free asset (typically represented as cash) and short selling, respectively. Our primary interest was also in the fundamental questions of the existence and (asymptotic) uniqueness of a survival strategy, and the answers to both questions are affirmative in both models.

EF is a promising yet still emerging field of research, with many questions remaining unanswered. In this paper, our focus is on the class of short-lived assets, which is more amenable to mathematical analysis and enable the development of a more complete and transparent theory. Because of this, it often serves as a "proving ground" for testing new conjectures in this field. For instance, in Chapter 5 and 6, we examined models involving a risk-free asset (cash) and short-selling, respectively. These two problems have been addressed in models of short-lived assets, but for long-lived assets they remain open.[2] In particular, the classical theory of derivative securities pricing, such as the Black-Scholes formula, is grounded in hypotheses which can be rigorously formulated only in the asset market models where short selling is allowed. This makes further investigation of this direction both theoretically and practically significant in quantitative financial applications.

[2] For a different evolutionary finance model, dealing with long-lived assets (see Evstigneev et al. [59]). And note that the counterpart of the global evolutionary stability that we stated in Chapter 3, was obtained for models with long-lived assets in Bahsoun et al. [12], thereby this gap has been filled.

Appendix A

Some Elementary Inequalities

In this paper, there are some elementary inequalities: (i) the inequality given in Evstigneev et al. [58], Lemma 1, (ii) Lemma 3.1, and (iii) Lemma 4.1, that play a key role in proving some main results: Theorem 3.4, Theorem 3.3, and Theorem 4.1, respectively. In fact, the last two inequalities, (ii) and (iii), are both based on the result of (i). To underpin and simplify the derivations, we begin with inequality (i) below, then (ii) and (iii).

Proof of Evstigneev et al. [58], Lemma 1

Lemma A.1 (Evstigneev et al. [58], Lemma 1) *For any* $\mu = (\mu_k) \in \Delta^K$ *with* $\mu > 0$ *and any* $\kappa \in [0, 1]$*, we have*

$$E \ln \sum_{k=1}^{K} R_k(s) \frac{\lambda_k^*}{\lambda_k^* \kappa + \mu_k (1-\kappa)} - E \ln \sum_{k=1}^{K} R_k(s) \frac{\mu_k}{\lambda_k^* \kappa + \mu_k (1-\kappa)} \geq 0. \quad \text{(A.1)}$$

Furthermore, if $\lambda^* \neq \mu$*, then the difference on the left-hand side of (A.1) is strictly positive.*

Proof. Clearly, if $\mu = \lambda^*$, inequality (A.1) turns into an equality. We shall show that the expression on the left-hand side of (A.1), which is denoted by $\Upsilon(\mu, \kappa)$, is strictly positive for all $\kappa \in [0, 1]$ and $\mu \neq \lambda^*$. By applying Jensen's inequality, we find

$$E \ln \sum_{k=1}^{K} R_k(s) \frac{\lambda_k^*}{\lambda_k^* \kappa + (1-\kappa) \mu_k} \geq \sum_{k=1}^{K} \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k^* \kappa + (1-\kappa) \mu_k} \quad \text{(A.2)}$$

$$E \ln \sum_{k=1}^{K} R_k(s) \frac{\mu_k}{\lambda_k^* \kappa + (1-\kappa) \mu_k} \leq \ln E \sum_{k=1}^{K} R_k(s) \frac{\mu_k}{\lambda_k^* \kappa + (1-\kappa) \mu_k} \quad \text{(A.3)}$$

and so

$$\Upsilon(\kappa,\mu) \;\geq\; \sum_{k=1}^{K} a_k \ln \frac{a_k}{a_k\,\kappa + (1-\kappa)\,\mu_k} - \ln \sum_{k=1}^{K} a_k \frac{\mu_k}{a_k\,\kappa + (1-\kappa)\,\mu_k} \qquad \text{(A.4)}$$

where $a_k = \lambda_k^*$.

Let $\kappa = 0$. Then the right-hand side of inequality (A.4) reduces to

$$\sum a_k \ln a_k - \sum a_k \ln \mu_k.$$

In view of Lemma 2.2, this difference is strictly positive, since $(a_k) \neq (\mu_k)$.

If $\kappa \in (0,1]$, then we have a strict inequality in (A.3). To prove this, it is sufficient to show that the function

$$\phi(s) = \sum_{k=1}^{K} R_k(s)\mu_k[\lambda_k^*\kappa + (1-\kappa)\mu_k]^{-1},\ s \in S,$$

is not a constant. Suppose $\phi(s)$ is constant, i.e., $\phi(s) \equiv \gamma$. Then

$$\sum_{k=1}^{K} R_k(s)\ \big(\mu_k\,[\lambda_k^*\,\kappa + (1-\kappa)\,\mu_k]^{-1} - \gamma\big) = 0,\ s \in S,$$

which implies $\mu_k = \gamma\,(\lambda_k^*\,\kappa + (1-\kappa)\,\mu_k)$, since the functions $R_k(\cdot)$, $k = 1, 2, ..., K$, are linearly independent. We can see that $\gamma = 1$, and so $\kappa\,(\lambda_k^* - \mu_k) = 0$. Since $\kappa > 0$, this implies $\lambda_k^* = \mu_k$, $k = 1, 2, ..., K$, which, however, is ruled out by our assumptions.

It remains to show that the expression on the right-hand side of (A.4) is non-negative. It is equal to zero if $\kappa = 1$. If $\kappa < 1$, we can write it in the form

$$g(u) = \sum_{k=1}^{K} a_k \ln \frac{a_k}{a_k u + \mu_k} - \ln \sum_{k=1}^{K} a_k \frac{\mu_k}{a_k u + \mu_k} \qquad \text{(A.5)}$$

where $u = \kappa(1-\kappa)^{-1}$. We can see that $g(u) \to 0$ as $u \to \infty$. Thus it remains to prove the inequality $g'(u) \leq 0$ for all $u > 0$. We have

$$g'(u) = -\sum_{k=1}^{K} a_k^2\,(a_k\,u + \mu_k)^{-1} + \frac{\sum_{k=1}^{K} a_k^2\,\mu_k\,(a_k\,u + \mu_k)^{-2}}{\sum_{k=1}^{K} a_k\,\mu_k\,(a_k\,u + \mu_k)^{-1}}$$

The sign of $g'(u)$ is the same as the sign of the expression

$$J := -\Big[\sum_{k=1}^{K} a_k^2\,(a_k\,u + \mu_k)^{-1}\Big]\ \sum_{k=1}^{K} a_k\,\mu_k\,(a_k\,u + \mu_k)^{-1} + \sum_{k=1}^{K} a_k^2\,\mu_k\,(a_k\,u + \mu_k)^{-2}.$$

By setting $w_k = a_k\, u + \mu_k$, we find $\mu_k = w_k - a_k\, u$ and

$$\begin{aligned} J &= -\Big[\sum_{k=1}^{K} a_k^2\, w_k^{-1}\Big] \sum_{k=1}^{K} a_k\,(w_k - a_k\, u)\, w_k^{-1} + \sum_{k=1}^{K} a_k^2\,(w_k - a_k\, u)\, w_k^{-2} \\ &= -\Big[\sum_{k=1}^{K} a_k^2\, w_k^{-1}\Big] \Big[1 - \sum_{k=1}^{K} a_k^2\, u\, w_k^{-1}\Big] + \sum_{k=1}^{K} a_k^2\, w_k^{-1} - \sum_{k=1}^{K} a_k^3\, u\, w_k^{-2} \\ &= u\,\Big[\big(\sum_{k=1}^{K} a_k\, v_k\big)^2 - \sum_{k=1}^{K} a_k\, v_k^2\Big], \end{aligned}$$

where $v_k = a_k w_k^{-1}$. The last expression is non-positive in view of the Schwartz inequality. The lemma is proved. ■

Proof of Lemma 3.1

Lemma A.2 (Lemma 3.1) *For any $\lambda \in \Delta^K$ distinct from λ^*, there exist numbers $H > 0$ and $\delta > 0$ such that*

$$E \min\{H, \ln F(\lambda, \kappa, s)\} \geq \delta$$

for all $\kappa \in (0, 1]$, where

$$F(\lambda, \kappa, s) := \frac{A}{B} = \frac{\sum_{k=1}^{K} R_k(s) \dfrac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k(1-\kappa)}}{\sum_{k=1}^{K} R_k(s) \dfrac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)}}.$$

Proof. Let us first assume that at least one of the coordinates of λ is zero, i.e. $\mathbf{K} := \{k : \lambda_k = 0\} \neq \emptyset$. Denote by S' the set of those s for which $\sum_{k=1}^{K} R_k(s)\lambda_k = 0$. If $s \in S'$, then $R_k(s)\lambda_k = 0$ for all k. Thus,

$$B = \sum_{k=1}^{K} R_k(s) \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)} = 0, \;\; s \in S',$$

and so $F(\lambda, \kappa; s) = +\infty$, $s \in S'$. Hence for any $H > 1$, we have

$$\begin{aligned} &E \min[H, \ln F(\lambda, \kappa; s)] \\ &= E \min[H, \ln F(\lambda, \kappa; s)] \mathbf{1}_{S'} + E \min[H, \ln F(\lambda, \kappa; s)] \mathbf{1}_{S \setminus S'} \\ &\geq H P(S') + 2(\ln c)(1 - P(S')), \end{aligned} \tag{A.6}$$

where $\mathbf{1}_\Gamma$ is the indicator function of the set Γ.

Suppose that $P(S') > 0$. Then for all H large enough, the expression in (A.6) is greater than 1. The assertion of Lemma 3.1 will be true for any H for which (A.6) is greater than 1 and $\delta = 1$.

Now assume that $P(S') = 0$, i.e. $\sum_{k=1}^{K} R_k(s)\lambda_k > 0$ (a.s.). In this case, we will proceed with the proof of the lemma in three steps.

Step 1: Let us show that

$$E \ln F(\lambda, \kappa; s) > 0 \text{ for all } \kappa \in (0, 1]. \tag{A.7}$$

By applying Jensen's inequality, we obtain

$$E \ln \sum_{k=1}^{K} R_k(s) \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k(1-\kappa)} \geq E \sum_{k=1}^{K} R_k(s) \ln \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k(1-\kappa)} \tag{A.8}$$

$$= \sum_{k=1}^{K} \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k(1-\kappa)} \tag{A.9}$$

and

$$E \ln \sum_{k=1}^{K} R_k(s) \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)} < \ln E \sum_{k=1}^{K} R_k(s) \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)} \tag{A.10}$$

$$= \ln \sum_{k=1}^{K} \lambda_k^* \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)}. \tag{A.11}$$

The inequality in (A.10) is strict because its right-hand side is finite (all λ_k^* and some λ_k are strictly positive) and there is no constant γ such that

$$\sum_{k=1}^{K} R_k(s) \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)} = \gamma \text{ (a.s.)}. \tag{A.12}$$

Indeed, if (A.12) holds, then

$$\sum_{k=1}^{K} R_k(s) \nu_k = 0 \text{ (a.s.)}, \tag{A.13}$$

where $\nu_k := \lambda_k[\lambda_k^* \kappa + \lambda_k(1-\kappa)]^{-1} - \gamma$. Observe that at least one of the numbers ν_k is not equal to zero. Otherwise, $\lambda_k = [\lambda_k^* \kappa + \lambda_k(1-\kappa)]\gamma$ for all k, and by summing up these equations over k, we obtain $\gamma = 1$, which yields $\lambda_k = [\lambda_k^* \kappa + \lambda_k(1-\kappa)]$. Hence $\lambda_k = \lambda_k^*$ for all k (recall that $\kappa \neq 0$). This is a contradiction because $\lambda \neq \lambda^*$. From this we conclude that at least one of the numbers v_k is not equal to zero, which indicates that (A.13) cannot hold since the functions $R_k(s)$ are linearly independent with respect to the given distribution on S.

By combining (A.8)-(A.11), we have

$$E \ln F(\lambda, \kappa; s) > \Phi_\kappa(\lambda), \tag{A.14}$$

where

$$\Phi_\kappa(\lambda) := \sum_{k=1}^{K} \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k(1-\kappa)} - \ln \sum_{k=1}^{K} \lambda_k^* \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k(1-\kappa)}.$$

Following from Evstigneev et al. [58], Lemma 1 (the first lemma that we proved in this appendix), we have that $\Phi_\kappa(\lambda) > 0$ for all $\lambda = (\lambda_1, ..., \lambda_K) > 0$.

Thus $\Phi_\kappa(\lambda(1-\varepsilon) + \varepsilon\lambda^*) > 0$ for each $\varepsilon > 0$. The function $\Phi_\kappa(\lambda)$ is finite and continuous on Δ^K. Consequently, $\Phi_\kappa(\lambda) = \lim_{\varepsilon\downarrow 0} \Phi_\kappa(\lambda(1-\varepsilon) + \varepsilon\lambda^*)) \geq 0$. This inequality combined with (A.14) yields (A.7).

Step 2: Let us show that for all $\kappa \in (0, 1]$,

$$F(\lambda, \kappa, s) \geq F(\lambda, 1, s) = [\sum_{k=1}^{K} R_k(s)\frac{\lambda_k}{\lambda_k^*}]^{-1} \tag{A.15}$$

($0^{-1} := +\infty$). Fix any $s \in S$. If $R_k(s)\lambda_k = 0$ for all k, then $F(\lambda, \kappa, s) = F(\lambda, 1, s) = +\infty$, and so inequality (A.15) is valid. Let us suppose that $R_k(s)\lambda_k \neq 0$ for some k. Then the denominator in (3.9) is strictly positive, and so $c^2 \leq F(\lambda, \kappa, s) < \infty$. For $\kappa \in (0, 1)$, we get $F(\lambda, \kappa; s) = G(\lambda, u, s)$, where

$$G(\lambda, u, s) := \frac{\sum_{k=1}^{K} R_k(s)\dfrac{\lambda_k^*}{\lambda_k^* u + \lambda_k}}{\sum_{k=1}^{K} R_k(s)\dfrac{\lambda_k}{\lambda_k^* u + \lambda_k}} = \frac{\sum_{k=1}^{K} R_k(s)\dfrac{\lambda_k^*}{\lambda_k^* + \lambda_k/u}}{\sum_{k=1}^{K} R_k(s)\dfrac{\lambda_k}{\lambda_k^* + \lambda_k/u}},$$

and $u := \kappa(1-\kappa)^{-1}$. When the variable κ ranges through $(0, 1)$, the variable u ranges through $(0, +\infty)$, and $G(\lambda, u, s) \to [\sum_{k=1}^{K} R_k(s)\lambda_k/\lambda_k^*]^{-1}$ as $u \to \infty$. Hence, to verify (A.15), it suffices to show that the derivative $G(\lambda, u; s)'$ of the function $G(\lambda, u; s)$ with respect to u is non-positive for all $u > 0$. Assuming that the parameter s is fixed, we omit "s" in the notation and write

$$G(\lambda, u)' = \frac{(\sum R_k \dfrac{\lambda_k^*}{\lambda_k^* u + \lambda_k})'(\sum R_k \dfrac{\lambda_k}{\lambda_k^* u + \lambda_k}) - (\sum R_k \dfrac{\lambda_k^*}{\lambda_k^* u + \lambda_k})(\sum R_k \dfrac{\lambda_k}{\lambda_k^* u + \lambda_k})'}{(\sum R_k \dfrac{\lambda_k}{\lambda_k^* u + \lambda_k})^2},$$

where $\sum = \sum_{k=1}^{K}$. The sign of the above fraction is the same as the sign of its nominator

$$J := -\sum R_k \frac{(\lambda_k^*)^2}{(\lambda_k^* u + \lambda_k)^2} \sum R_k \frac{\lambda_k}{\lambda_k^* u + \lambda_k} + \sum R_k \frac{\lambda_k^*}{\lambda_k^* u + \lambda_k} \sum R_k \frac{\lambda_k \lambda_k^*}{(\lambda_k^* u + \lambda_k)^2}$$

By setting $w_k := \lambda_k^* u + \lambda_k$ and $v_k := \lambda_k^* w_k^{-1}$, we obtain $\lambda_k = w_k - \lambda_k^* u$ and

$$\begin{aligned}
J =& - \sum R_k (\lambda_k^*)^2 w_k^{-2} \sum R_k \lambda_k w_k^{-1} + \sum R_k \lambda_k^* w_k^{-1} \sum R_k \lambda_k \lambda_k^* w_k^{-2} \\
=& - \sum R_k (\lambda_k^*)^2 w_k^{-2} \sum R_k (w_k - \lambda_k^* u) w_k^{-1} \\
& + \sum R_k \lambda_k^* w_k^{-1} \sum R_k (w_k - \lambda_k^* u) \lambda_k^* w_k^{-2} \\
=& - \sum R_k v_k^2 \sum (R_k - R_k u v_k) + \sum R_k v_k \sum R_k (v_k - v_k^2 u) \\
& - \sum R_k v_k^2 (1 - \sum R_k u v_k) + \sum R_k v_k (\sum R_k v_k - \sum R_k v_k^2 u) \\
=& - \sum R_k v_k^2 + (\sum R_k v_k^2) \sum R_k v_k u + (\sum R_k v_k)^2 - (\sum R_k v_k) \sum R_k v_k^2 u \\
=& (\sum R_k v_k)^2 - \sum R_k v_k^2 \leq 0.
\end{aligned}$$

The last inequality is non-positive by virtue of the Schwartz inequality (we apply here the fact that $R_k \geq 0$ and $\sum R_k = 1$). This completes the proof of inequality (A.15).

Step 3. Let us demonstrate that there exists a natural number m, such that $E \min\{m, \ln F(\lambda, 1, s)\} > 0$. The series of random variables

$$\phi_m := \min\{m, \ln F(\lambda, 1, s)\}, m = 1, 2, ...,$$

is bounded below by $2 \ln c$ (see (3.10)), is nondecreasing and tends to $\ln F(\lambda, 1, s)$ for each s. Therefore, we get $E\phi_m \to E \ln F(\lambda, 1, s) > 0$ (see (A.7)), and so $E\phi_m > 0$ for some $m = m_0$. By setting $H := m_0$, $\delta := E \min\{H, \ln F(\lambda, 1; s)\} > 0$ and applying (A.15), we find

$$E \min\{H, \ln F(\lambda, \kappa; s)\} \geq E \min\{H, \ln F(\lambda, 1; s)\} = \delta,$$

which proves Lemma 3.1 in the case when at least one of the coordinates of λ is zero.

Now assume that λ has no zero coordinates: $\lambda_k > 0$ for each k. Then the function $\ln F(\lambda, \kappa; s)$, $\kappa \in [0, 1]$, is uniformly bounded:

$$2 \ln c \leq \ln F(\lambda, \kappa; s) \leq \ln(\min_k \lambda_k)^{-2},$$

and so $E \ln F(\lambda, \kappa; s)$ is continuous in $\kappa \in [0, 1]$. It is sufficient to show that the infimum of $E \ln F(\lambda, \kappa; s)$ with $\kappa \in [0, 1]$ is strictly positive (then δ can be defined as this infimum and H as $2|\ln c| + 2|\ln \min \lambda_k|$). By virtue of the continuity of $E \ln F(\lambda, \kappa; s)$, this will be proved if we establish the inequality $E \ln F(\lambda, \kappa; s) > 0$ for each $\kappa \in [0, 1]$. For $\kappa \in (0, 1]$ this was proved above (see (A.7)) under the condition that $\sum_{k=1}^K R_k(s) \lambda_k > 0$ (a.s.), which holds if $\lambda > 0$. In the case of $\kappa = 0$, we have

$$E \ln F(\lambda, 0, s) = E \ln \sum_{k=1}^K R_k(s) \frac{\lambda_k^*}{\lambda_k} \geq \ln \sum_{k=1}^K \lambda_k^* \frac{\lambda_k^*}{\lambda_k} \geq \sum_{k=1}^K \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k} > 0$$

as long as λ^*,$\lambda > 0$ and $\lambda \neq \lambda^*$. ■

Proof of Lemma 4.1

Let S be a finite set, and, for each $s, \sigma \in S$, let $p(\sigma|s)$ be a probability distribution on S:

$$p(\sigma|s) \geq 0, \ \sum_{\sigma} p(\sigma|s) = 1.$$

For each $\sigma \in S$, let $R(\sigma, s) = (R_1(\sigma, s), ..., R_k(\sigma, s))$ be a vector in unit simplex Δ^K satisfying (A1) and (A2) for all $s \in S$.

Let $\rho > 0$ be a number, such that $R_k^*(s) > \rho$, $s \in S$ (see (4.11)). Denote by Δ_ρ^K the set of those vectors $(b_1, ..., b_K)$ in Δ^K that satisfy $b_k \geq \rho$, $k = 1, ..., K$. Consider the function

$$\begin{aligned} \Psi(s, \kappa, \mu) \quad = \quad & \sum_{\sigma \in S} p(\sigma|s) \ln \sum_{k=1}^{K} R_k(\sigma, s) \frac{R_k^*(s)}{R_k^*(s)\kappa + (1-\kappa)\mu_k} \\ & - \sum_{\sigma \in S} p(\sigma|s) \ln \sum_{k=1}^{K} R_k(\sigma, s) \frac{\mu_k}{R_k^*(s)\kappa + (1-\kappa)\mu_k} \end{aligned}$$

of $s \in S$, $\kappa \in [0, 1]$ and $\mu = (\mu_k) \in \Delta_\rho^K$.

Lemma A.3 (Lemma 4.1) *There exists a constant L_ρ and a function $\delta_\rho(\gamma) \geq 0$ of $\gamma \in [0, \infty)$ satisfying the following conditions:*

1. *The function $\delta(\cdot)$ is non-decreasing, and $\delta_\rho(\gamma) > 0$ for all $\gamma > 0$.*
2. *For any $s \in S$, $\kappa \in [0, 1]$ and $\mu = (\mu_k) \in \Delta_\rho^K$, we have*

$$L_\rho |R^*(s) - \mu| \geq \Phi(s, \kappa, \mu) \geq \delta_\rho(|R^*(s) - \mu|). \tag{A.16}$$

Proof. It follows from Evstigneev et al. [58], Lemma 1 (the first lemma that we proved in this appendix) that, for all $s \in S$, $\kappa \in [0, 1]$ and any $\mu \in \Delta_+^K$, $\mu \neq R^*(s)$, the value of $\Psi(s, \kappa, \mu)$ is strictly positive.

Fix some $\gamma_0 > 0$ for which the set $W(s, \gamma) = \{\mu \in \Delta_\rho^K : |R^*(s) - \mu| \geq \gamma\}$ is non-empty for all $s \in S$, $\gamma \in [0, \gamma_0]$ and define

$$\delta_\rho(s, \gamma) = \inf\{\Psi(s, \kappa, \mu) : \ \kappa \in [0, 1], \ \mu \in W(s, \gamma)\}.$$

if $\gamma \in [0, \gamma_0]$ and $\delta_\rho(s, \gamma) = \delta_\rho(s, \gamma_0)$ if $\gamma > \gamma_0$. Since $\Psi(s, \kappa, \mu)$ is continuous and strictly positive on the compact set $[0, 1] \times W(s, \gamma)$ $(\gamma > 0)$, the function $\delta_\rho(s, \gamma)$ takes on strictly positive values for $\gamma > 0$. Clearly this function is non-decreasing in γ. Fix some s, consider any $\mu \in \Delta_\rho^K$ and define $\gamma = |R^*(s) - \mu|$. Then we have $\mu \in W(s, \gamma)$, and so

$$\Psi(s, \kappa, \mu) \geq \delta_\rho(s, \gamma) = \delta_\rho(s, |R^*(s) - \mu|).$$

From this we can see that the sought-for function $\delta_\rho(\gamma)$ can be defined as

$$\delta_\rho(\gamma) = \min_{s \in S} \delta_\rho(s, \gamma).$$

We can write $\Psi(s, \kappa, \mu) = \Psi(s, \kappa, \mu) - \Psi(s, \kappa, R_k^*(s))$ since the latter term is zero. The function $\Psi(s, \kappa, \mu)$ is differentiable in $\mu \in \Delta_+^K$ and its gradient $\Psi'_\mu(s, \cdot, \cdot)$ is continuous, and hence bounded, on the compact set $[0, 1] \times \Delta_\rho^K$. This implies the existence of the Lipschitz constant L_ρ in (A.16). ■

Appendix B

Proofs of the Main Results in Section 5.2

To prove Theorems 5.1 and 5.2, we derive a random dynamical system describing the dynamics of the market shares r_t^i of investors $i = 1, ..., N$.

Theorem B.1 *The following equations hold:*

$$r_{t+1}^i = \sum_{k=0}^{K} R_{t+1,k} \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle}, \ t = 0, 1, 2, ..., \ i = 1, 2, ..., N. \tag{B.1}$$

Proof. Put

$$\bar{V}_{t,0} := \bar{w}_t \ [= \sum_{i=1}^{N} x_{t,0}^i = \sum_{i=1}^{N} \lambda_{t,0}^i w_t^i], \ \bar{V}_{t,k} = V_{t,k}, \ k = 1, 2, ..., K. \tag{B.2}$$

Using the notation $A_{t+1,0} = 1 + \beta_{t+1}$, we write

$$\begin{aligned} W_{t+1} &= \sum_{i=1}^{N} w_{t+1}^i = \sum_{i=1}^{N} \langle A_{t+1}, x_t^i \rangle = \sum_{i=1}^{N} \sum_{k=0}^{K} A_{t+1,k} x_{t,k}^i \\ &= \sum_{k=0}^{K} \left(A_{t+1,k} \sum_{i=1}^{N} x_{t,k}^i \right) = (1 + \beta_{t+1}) \bar{w}_t + \sum_{k=1}^{K} A_{t+1,k} V_{t,k} = \sum_{k=0}^{K} A_{t+1,k} \bar{V}_{t,k}. \end{aligned} \tag{B.3}$$

Define $\lambda_{t,k} := (\lambda_{t,k}^1, ..., \lambda_{t,k}^N)$ and $w_t := (w_t^1, ..., w_t^N)$. From Eqs. (5.7) and (5.10) we obtain

$$p_{t,k} V_{t,k} = \langle \lambda_{t,k}, w_t \rangle, \ k = 1, ..., K, \ t \geq 0,$$

which implies

$$x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}} = \frac{\lambda_{t,k}^i w_t^i V_{t,k}}{\langle \lambda_{t,k}, w_t \rangle} = \frac{\lambda_{t,k}^i w_t^i \bar{V}_{t,k}}{\langle \lambda_{t,k}, w_t \rangle}, \ k = 1, ..., K, \ t \geq 0 \tag{B.4}$$

(see Eq. (5.11)). Further, we find

$$p_{t,0}\bar{V}_{t,0} = \bar{V}_{t,0} = \bar{w}_t = \langle \lambda_{t,0}, w_t \rangle, \tag{B.5}$$

where the first equality holds because $p_{t,0} = 1$, the second follows from (B.2), and the third is a consequence of (5.1) and (5.12). Applying (5.12) and (B.5), we get

$$x^i_{t,0} = \frac{\lambda^i_{t,0} w^i_t}{p_{t,0}} = \frac{\lambda^i_{t,0} w^i_t \bar{V}_{t,0}}{\langle \lambda_{t,0}, w_t \rangle}. \tag{B.6}$$

From (B.6) and (B.4), we obtain

$$\begin{aligned} w^i_{t+1} &= \sum_{k=0}^{K} A_{t+1,k} x^i_{t,k} = A_{t+1,0} \bar{V}_{t,0} \frac{\lambda^i_{t,0} w^i_t}{\langle \lambda_{t,0}, w_t \rangle} + \sum_{k=1}^{K} A_{t+1,k} \bar{V}_{t,k} \frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle} \\ &= \sum_{k=0}^{K} A_{t+1,k} \bar{V}_{t,k} \frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle}. \end{aligned}$$

Dividing the first and the last terms in this chain of equalities by W_{t+1} and using (B.3), we have

$$r^i_{t+1} = \sum_{k=0}^{K} \frac{A_{t+1,k} \bar{V}_{t,k}}{\sum_{l=0}^{K} A_{t+1,l} \bar{V}_{t,l}} \frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle}. \tag{B.7}$$

Observe that

$$\frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle} = \frac{\lambda^i_{t,k} w^i_t / W_t}{\langle \lambda_{t,k}, w_t / W_t \rangle} = \frac{\lambda^i_{t,k} r^i_t}{\langle \lambda_{t,k}, r_t \rangle}. \tag{B.8}$$

Further, we obtain

$$\begin{aligned} \frac{A_{t+1,k} \bar{V}_{t,k}}{\sum_{l=0}^{K} A_{t+1,l} \bar{V}_{t,l}} &= \frac{a_{t+1,k}\, \bar{w}_t \bar{V}_{t,k}}{\left(1 + \beta_{t+1}\right) \bar{w}_t + \sum_{l=1}^{K} a_{t+1,l}\, \bar{w}_t \bar{V}_{t,l}} \\ &= \frac{a_{t+1,k}\, V_{t,k}}{\sum_{l=0}^{K} a_{t+1,l}\, V_{t,l}} = R_{t+1,k} \end{aligned} \tag{B.9}$$

$(k = 1, ..., K)$ and

$$\begin{aligned} \frac{A_{t+1,0} \bar{V}_{t,0}}{\sum_{l=0}^{K} A_{t+1,l} \bar{V}_{t,l}} &= \frac{\left(1 + \beta_{t+1}\right)\, \bar{w}_t}{\left(1 + \beta_{t+1}\right) \bar{w}_t + \sum_{l=1}^{K} a_{t+1,l}\, \bar{w}_t V_{t,l}} \\ &= \frac{a_{t+1,0}\, V_{t,0}}{\sum_{l=0}^{K} a_{t+1,l}\, V_{t,l}} = R_{t+1,0} \end{aligned} \tag{B.10}$$

(see (5.16) and (5.17)). By substituting Eqs. (B.8)-(B.10) into (B.7), we arrive at (B.1). The proof is complete. ■

We have shown that the dynamics of r^i_t of the investors $i = 1, 2, ..., N$, in this model is governed by the random dynamical system (B.1). A detailed analysis of this system was carried out in Section 2.3. As we mentioned in Section 5.2, Theorems 5.1 and 5.2 are direct consequences of Theorems B.2 and B.3 (correspondingly, Theorem 2.1 and Theorem 2.2 proved in Section 2.3), which we formulate below.

Let $R_{t,k}(s^t) \geq 0$ ($t = 0, 1, 2, ...; k = 0, ..., K$) be measurable real-valued functions satisfying (5.18) and (5.20). Consider sequences of measurable vector functions $\Lambda = (\lambda_t(s^t))_{t=0}^{\infty}$, where $\lambda_t(s^t) = (\lambda_{t,0}(s^t), ..., \lambda_{t,K}(s^t)) \in \Delta^{K+1}$. Denote by $\mathcal{L}$ the set of N-tuples $(\Lambda^1, ..., \Lambda^N)$ of such sequences for which the random dynamical system (B.1) generates well-defined vectors $r_t^i = (r_{t,0}^i, ..., r_{t,K}^i) \in \Delta^{K+1}$, $i = 1, ..., N$, $t = 0, 1, ...$, i.e., the validity of the inequality

$$\sum_{i=1}^{N} \lambda_{t,k}^i r_t^i > 0, \; k = 0, 1, ..., K,$$

is guaranteed for each $t \geq 0$. Define $\Lambda^* = (\lambda_t^*(s^t))_{t=0}^{\infty}$ by (5.19).

Theorem B.2 *For each* $(\Lambda^1, ..., \Lambda^N) \in \mathcal{L}$ *with* $\Lambda^1 = \Lambda^*$*, we have* $\inf_{t \geq 0} r_t^1 > 0$ *(a.s.).*

Theorem B.3 *Let* $\Lambda = (\lambda_t(s^t))_{t=0}^{\infty}$ *be a sequence of measurable vector functions with values in* Δ^{K+1}*. If* $\inf_{t \geq 0} r_t^1 > 0$ *for any* $(\Lambda^1, ..., \Lambda^N) \in \mathcal{L}$ *with* $\Lambda^1 = \Lambda$*, then (5.21) holds.*

For proofs of Theorems B.2 and B.3, see Section 2.3.

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