The University of Manchester

Economics

Discussion Paper

Series

EDP-2101

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January 2021
Revised: July 2023

Economics

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Unbeatable Strategies*

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> "One may propose to investigate whether it is possible to determine a method of play better than all others; i.e., one that gives the player who adopts it a superiority over every player who does not adopt it."
>
> *Émile Borel*
> C. R. Acad. Sci. 173, 1921, p. 1304.

Abstract. The paper analyzes the notion of an unbeatable strategy as a game-theoretic solution concept. A general framework (games with relative preferences) suitable for the analysis of this concept is proposed. Basic properties of unbeatable strategies are presented and a number of examples and applications considered.

Keywords: unbeatable strategies, relative preferences, zero-sum games, evolutionary game theory, evolutionary finance.
JEL codes: C72; C73; D43.

1 Introduction

Nowadays Nash equilibrium is the most common solution concept in game theory. However, a century ago, when the discipline was in its infancy, the term "solving a game" was understood quite differently. The focus was not on finding a strategy profile that would

*The authors are grateful to Thorsten Hens, Jean-Jacques Herings, Alexander Matros, Alex Possajennikov, Larry Samuelson, and Mikhail Zhitlukhin for helpful comments and fruitful discussions.

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equilibrate conflicting interests of the players. The main goal was to find, if possible, an individual strategy enabling the player to win (or at least not to lose) the game, or in other words, to construct an *unbeatable strategy*. This question was considered in the paper by Bouton (1901-02), apparently the earliest mathematical paper in the field. Borel (1921) posed a general problem of investigating unbeatable strategies. When developing this idea, he introduced a now-famous class of games that later received the name "Colonel Blotto games".

It should be noted that the problem of constructing explicit unbeatable strategies turned out to be intractable for the vast majority of mind games of popular interest (such as chess). What mathematicians could achieve, at most, was to prove that games in certain classes were determinate. A game is called *determinate* if at least one of the players has an unbeatable strategy.

Questions related to the determinacy of chess were considered for the first time in the seminal paper by Zermelo (1913). Zermelo's determinacy theorem was analyzed and extended by Kalmár (1929). For a discussion of the history of these ideas, see Schwalbe and Walker (2001). A new treatment of Zermelo's theorem in the framework of games with relative preferences (see Section 2 below) has been suggested in a recent paper by Amir and Evstigneev (2017).

A deep mathematical analysis of the determinacy of infinite win-or-lose games of perfect information was initiated by Gale and Stewart (1953). This line of study has led to remarkable achievements in set theory and topology. The highlight in the field was Martin's determinacy axiom and a proof of its independence from the Zermelo–Fraenkel axioms of set theory (Martin 1975). For comprehensive surveys of research in this area see Telgársky (1987) and Kehris (1995); for reviews of topics related to unbeatable strategies in combinatorial game theory see Berlekamp et al. (1982).

The above achievements had for the most part purely theoretical value, having nothing to do with real-life applications. They dealt with elegant games created in the minds of mathematicians. A classical example is Bouton's (1902) game "Nim", a complete theory of which, including a construction of an unbeatable strategy, was developed in his pioneering paper. A new era began when novel, applications-oriented solution concepts based on the idea of optimal (saddle-point) strategies for zero-sum games came to the fore. Von Neumann proved his famous minimax theorem on the existence of a saddle-point solution for two-person zero-sum games in 1928. This result served as the basis for numerous applications in operations research and economics.

It should be noted that the existence of a saddle point in a zero-sum two-player game implies its determinacy. If the value of the game is non-negative (non-positive), then the saddle-point strategy of the first player (second player) is unbeatable. If the value is zero, then both players have unbeatable strategies. Thereby minimax theorems made it possible to construct unbeatable strategies via saddle points explicitly. At those times, most of the games where unbeatable strategies were investigated were zero-sum. This might be the reason why the topic of unbeatable strategies was absorbed for a while by the theory of zero-sum games and re-emerged only years after the seminal von Neumann (1928) paper.

In the 1950s, when game theory started developing primarily as a mathematical framework for economic modeling, non-zero-sum N-player games came to the fore. The notion of a Nash equilibrium (1950) became the fundamental solution concept for the study of strategic behavior. Subsequently, zero-sum games and unbeatable strategies faded to the background.

The next wave of interest in unbeatable strategies came from an unexpected side, theoretical biology. It served as a starting point for the development of evolutionary game theory (EGT). Hamilton (1967) used this notion, and the term "unbeatable strategy"—without a rigorous formalization—in his paper on the analysis of sex ratios in populations of some species, which turned out to be extremely influential. Maynard Smith and Price (1973) formalized Hamilton's idea, but at the same time somewhat changed its content. The notion they introduced, usually referred to as an evolutionary stable strategy (ESS), should be called, more precisely, a conditionally unbeatable strategy. It is indeed unbeatable, but only if the rival is "weak enough." In the context of evolutionary biology, ESS is a strategy that cannot be beaten if the fraction of rivals (mutants) in the population is sufficiently small. This definition requires the population to be infinite, since one has to speak of its arbitrarily small fractions. A version of ESS applicable to finite populations was suggested by Schaffer (1988, 1989). Schaffer's notion of ESS is also in a sense a conditionally unbeatable strategy: it requires the population to contain only *one* mutant.

It is not surprising that the notion of unbeatable strategy, rather than Nash equilibrium, turned out to be a key idea that fitted ideally the purposes of evolutionary modeling in biology. Nash equilibrium presumes full rationality of players, understood in terms of payoff maximization, and their ability to coordinate their actions (or the presence of Harsanyi's "mediator") to establish an equilibrium, especially if it is non-unique. In a biological context such possibilities are absent, and moreover the role of individual utilities, always having a subjective nature, is played in EGT by a *fitness function*, an objective characteristic reflecting the survival rate in the natural selection process.

It is standard to present EGT models in conventional game-theoretic terms, with utilities/payoffs and Nash equilibrium, but this is just a matter of convenience that makes it possible to employ the terminology and the results of conventional game theory. Moreover, EGT models are nearly exclusively symmetric, and as can be shown, the analysis of unbeatable strategies in the symmetric case boils down to the consideration of symmetric Nash equilibria (possessing some additional properties). At the same time, this kind of exposition, although convenient in some respects, might be misleading in others. In EGT, in contrast to conventional game theory, players do not select their strategies. Strategies are nothing but "genetic codes" of the players they have no influence on, while payoffs or utilities are not their individual characteristics (which are typically unobservable), but as has been noted, represent their fitness functions amenable to observation and statistical estimation.

The notion of ESS proposed by Maynard Smith and Price (1973) reigned in Evolutionary Game Theory for many years. An unconditional variant of ESS—fully corresponding to Hamilton's idea of an unbeatable strategy—was first revived in the context of economic applications of EGT in a remarkable paper by Kojima (2006), three decades after Maynard Smith and Price and four decades after Hamilton[4]. Kojima's study was motivated by economic applications, where the assumption of smallness of the population of "mutants" is obviously not realistic: a new technology or a new product may be thrown into the economy in any quantity.

Several years after Kojima's publication, it was discovered (Amir et al. 2011, 2013) that the concept of an unbeatable strategy represents a very convenient and efficient tool in the analysis of financial market models combining evolutionary and behavioral principles—Evolutionary Behavioral Finance. For most recent studies in the field see Amir et al. (2021, 2022) and Evstigneev et al. (2020, 2023). This motivated us to undertake a systematic study of unbeatable strategies, which is conducted in the present work.

The plan of the paper is as follows. Section 2 introduces and discusses a general modeling framework that covers, to the best of our knowledge, all models in which unbeatable strategies have been studied up to now. Section 3 analyzes in detail the classical case of a game with two players and cardinal preferences, presenting general basic facts on unbeatable strategies. Distinctions and similarities between unbeatable strategies and Nash equilibrium are discussed. A paradoxical example is provided in which rationality in terms of the relative and absolute criteria turn out to be wildly inconsistent with each other. Among other results, it is shown that in zero-sum games determinacy occurs in a sense substantially more often than a saddle point. Section 4 performs a comparative analysis of ESS and unbeatable strategies in the simplest, but at the same time sufficiently rich, framework of mixed strategies in two-player two-strategy games. The results obtained refute, or at least question, the common perception that in the evolutionary context unbeatable strategies are "rare" compared to ESS. Section 5 examines unbeatable strategies in an asymmetric Cournot duopoly and compares them with those resulting from a Nash equilibrium. The Appendix includes technical proofs of some results stated in Sections 3 and 5.

2 Game with relative preferences

Game description. In this section we introduce and discuss a general framework—*game with relative preferences*—serving as the basis for the analysis of unbeatable strategies. This framework extends the one proposed in Amir et al. (2013, Section 6). There

[4]Hamilton (1967) did not give a rigorous general definition of an unbeatable strategy, using this notion in the specific context of that particular paper. In later papers (Hamilton and May, 1977; Comins et al., 1980), he used the notion of an ESS and emphasized its "combination of simplicity and generality". However, in Hamilton (1996), three decades later, he stated that in his 1967 paper he had in mind indeed a genuine notion of an unbeatable strategy, without the additional assumption of a small fraction of mutants in the population. For a discussion of the history of these ideas see Sigmund (2001).

are N players $i = 1, ..., N$ choosing their *strategies* x^i from some given sets X^i. A set $Z \subseteq X^1 \times ... \times X^N$ of *admissible strategy profiles* is given. For each i there is a mapping $w^i : Z \to W^i$ from Z into the set W^i of *outcomes* of the game for player i. If players $i = 1, 2, ..., N$ select strategies $x^1, ..., x^N$ such that $(x^1, ..., x^N) \in Z$, then the outcome of the game for player i is $w^i(x^1, ..., x^N) \in W^i$.

We would like to define the notion of an unbeatable strategy of some player i. To this end we assume that for any pair of outcomes $w^i \in W^i$, $w^j \in W^j$ $(j \neq i)$ a preference relation $w^i \succcurlyeq_{ij} w^j$ is given. This preference relation is used to compare the game outcomes w^i and w^j of players i and j by estimating their relative performance.

Definition 2.1. A strategy x^* of player i is called *unbeatable* if for any admissible strategy profile $(x^1, x^2, ..., x^N) \in Z$ with $x^i = x^*$, we have

$$w^i(x^1, x^2, ..., x^N) \succcurlyeq_{ij} w^j(x^1, x^2, ..., x^N) \text{ for all } j \neq i. \quad (1)$$

According to this definition, player i adopting the strategy x^* cannot be outperformed by any other player $j \neq i$ irrespective of what strategies player i's rivals $j \neq i$ use.

Cardinal preferences (numerical measures of performance). Suppose that for each player i, a function $F_i(x^1, ..., x^N)$ of an admissible strategy profile $(x^1, x^2, ..., x^N) \in Z$ is given. One can interpret the number $F_i(x^1, ..., x^N)$ as the "score" or payoff which player i gets if the strategy profile of the all players is $(x^1, ..., x^N)$. This number characterizes the outcome of the game for player i. The sets of outcomes W^i for all the players i are the same and coincide with the real line R. The preference relations between the game outcomes are defined as usual non-strict inequalities between real numbers: $w^i \succcurlyeq_{ij} w^j$ if and only if $w^i \geq w^j$. In this setting, a strategy x^* of player i is unbeatable if for any admissible strategy profile $(x^1, x^2, ..., x^N) \in Z$ with $x^i = x^*$, we have

$$F_i(x^1, x^2, ..., x^N) \geq F_j(x^1, x^2, ..., x^N) \text{ for all } j \neq i,$$

or equivalently, $F_i(x^1, x^2, ..., x^N) \geq \max_{j \neq i} F_j(x^1, x^2, ..., x^N)$.

Symmetric N-player games. Let us say that the game introduced in the previous subsection is *symmetric* if $X^1 = X^2 = ... = X^N = X$ and for every permutation $\pi(i)$ of the numbers $1, ..., N$ we have $(x^1, x^2, ..., x^N) \in Z$ if and only if $(x^{\pi(1)}, x^{\pi(2)}, ..., x^{\pi(N)}) \in Z$ and

$$F_i(x^1, ..., x^i, ..., x^N) = F_{\pi(i)}(x^{\pi(1)}, ..., x^{\pi(i)}, ..., x^{\pi(N)}), \quad (2)$$

i.e. both the class of admissible profiles Z and the payoff functions $F_i(x^1, ..., x^i, ..., x^N)$ are permutation-invariant. In the general case, if we wish to verify that x^* is an unbeatable strategy of some player, say, player 1, then we need to check the validity of $N-1$ inequalities

$$F_1(x^*, x^2, ..., x^N) \geq F_j(x^*, x^2, ..., x^N) \text{ for all } j = 2, ..., N \text{ and } x^2, ..., x^N. \quad (3)$$

However, if the game is symmetric, it is sufficient to verify only one of these inequalities, for some particular j, say $j = 2$:

$$F_1(x^*, x^2, ..., x^N) \geq F_2(x^*, x^2, ..., x^N). \quad (4)$$

Indeed, assume (4) holds, consider any $j = 3, ..., N$, and observe that inequality (3) is equivalent to (4) because

$$F_j(x^*, x^2, ..., x^j, ..., x^N) = F_2(x^*, x^j, ..., x^2, ..., x^N)$$

by virtue of the symmetry of the game.

Schaffer's ESS. We define within the present framework the important notion of an evolutionary stable strategy (ESS) for finite populations introduced in the seminal works of Schaffer (1988, 1989). Consider the symmetric model with cardinal preferences described in the previous subsection. Let us define the class Z of admissible strategy profiles as follows. Let $(x^1, x^2, ..., x^N) \in Z$ if there exists $i = 1, 2, ..., N$ such that all the strategies x_j for j distinct from i coincide:

$$x_j = x_{j'} \text{ for all } j, j' \neq i.$$

Thus all the admissible strategy profiles are of the following form

$$z_i = (y, ..., y, x, y, ...y), \; x, y \in X, \; i = 1, 2..., N,$$

where "x" stands at the ith place. By symmetry, for such a strategy profile we have

$$F_j(y, ..., y, x, y, ...y) = F_{j'}(y, ..., y, x, y, ...y) \text{ for all } j, j' \neq i.$$

Definition 2.2. A strategy x^* is called a *Schaffer's ESS* if

$$F_j(x^*, ..., x^*, x, x^*, ..., x^*) \geq F_i(x^*, ..., x^*, x, x^*, ..., x^*) \text{ for all } x \in X \text{ and } j \neq i. \quad (5)$$

It is said in (5) that a group of $N - 1$ identical non-mutants x^* cannot be outperformed by a mutant x.

Games with relative preferences: Motivation and examples. This paper discusses a number of examples that can be included into the above general framework of games with relative preferences. The primary focus here is on those game-theoretic models that permit to perform a *comparative analysis* of unbeatable strategies and those resulting from a Nash equilibrium. A convenient vehicle for this analysis is a game with two players and numerical absolute preferences, in terms of which the relative ones are defined—see the next section. Examples in Sections 3 and 5 (but not in Section 4) can be directly embedded into this classical scheme. However, the main motivation for the general setting proposed above lies elsewhere. It has been developed primarily for the analysis of financial market models considered in Evolutionary Behavioural Finance (EBF), see, e.g., Amir et al. (2011, 2013, 2021, 2022), Evstigneev et al. (2015, 2020, 2023), and Zhitlukhin (2021, 2022a,b). In these models, there are no natural numerical preferences or utility functions, and the game solution concepts are distinct from Nash equilibrium. The present paper does not intend to provide a survey of results in this area. Here we will only outline very briefly the idea of relative preferences involved in EBF, referring the interested reader for details to the papers cited.

In a typical EBF model (see e.g. Amir et al. 2013), there are several players/investors $i = 1, 2, ..., N$ allocating at each time $t = 0, 1, ...,$ their wealth w_t^i across assets $k = 1, ..., K$ in proportions $\lambda_{t,k}^i$. These proportions, as well as all the other variables in the model, depend on an exogenous stochastic process of "states of the world" and are specified by a given investment strategy/portfolio rule Λ^i of market player i. A strategy profile $\Lambda = (\Lambda^1, ..., \Lambda^N)$ of all the investors determines at each time t short-run equilibrium asset prices, investors' portfolios and their payoffs w_{t+1}^i at the next moment of time $t+1$. Given the strategy profile Λ, the outcome $w^i = w^i(\Lambda)$ of the game for player i is the stochastic wealth process $w^i = (w_t^i)_{t=0}^{\infty}$. We write $w^i \succcurlyeq_{ij} w^j$ if there is a random variable C_{ij} such that $w_t^j \leq C_{ij} w_t^i$ almost surely (a.s.) for all $t = 0, 1, 2, ...$, i.e., the sequence w^j does not grow asymptotically faster than w^i. According to the general definition, a strategy Λ^* of player i is unbeatable if for any strategy profile $\Lambda = (\Lambda^1, ..., \Lambda^N)$ with $\Lambda^i = \Lambda^*$ we have $w^i(\Lambda) \succcurlyeq_{ij} w^j(\Lambda)$ for all $j \neq i$. Clearly (and remarkably!) a portfolio rule Λ^i is unbeatable in this sense if and only if it is a *survival strategy*, making it possible for player i using it to survive in competition with any strategies of i's rivals, i.e., to keep a.s. a strictly positive bounded away from zero share of market wealth $w_t^i/(w_t^1 + ... + w_t^N)$ over an infinite time horizon. This is the key concept of unbeatability/survival in EBF models. Conditional versions of it, with some flavour of the notions of ESS introduced by Maynard Smith and Price (small initial wealth of a "mutant") and Schaffer (identical "non-mutants") are considered in Amir et al. (2021) and Evstigneev et al. (2023). For related models with benchmarking see Leippold and Rohner (2011).

Relativity in Economics. The conceptual idea that individuals are motivated in significant ways by positional concerns and relative status, and thus relative payoffs, has emerged long ago in economic thought (Veblen, 1899). It has been revived periodically since then, e.g., in Duesenberry's (1949) relative income hypothesis and in Easterlin's (1974) paradox. The latter study (backed by much empirical evidence) inspired a new area of research, the so-called economics of happiness, see, e.g. Di Tella et. al (2001), Frey and Stutzer (2002a, 2002b), and Clark et al. (2008).

In experimental/behavioral economics, various types of relative preferences have been postulated as a key ingredient to rationalize laboratory subjects' failure to behave according to common theoretical models with independent preferences. Among others, Fehr and Schmidt (1999) formulate a simple model of inequality aversion in which individuals incur negative utility as the distribution of payoffs moves away from the egalitarian distribution in a way that depends on whether they are ahead of or behind others (also see Bolton and Ockenfels, 2000). Levine (1998) studies an extensive-form model that classifies people according to their (private information) type: spiteful or altruistic, and shows that a particular distribution of these types produces behavior that is consistent with the findings from some experiments, including ultimatum, auctions, and centipede games.

In line with this overall development, a number of theoretical studies have incorporated this basic insight into models in different strategic settings, in addition to the literature already cited in evolutionary game theory.[5] In the theory of strategic delegation, Fershtman and Judd (1987) have shown, in the context of Cournot models, that a firm whose objective includes a weight on relative profits or sales will outperform a classical firm in terms of absolute profits. Kockesen, Ok, and Sethi (2000) have identified classes of games in which players with interdependent preferences outperform, or have a strategic advantage, over those with classical preferences. Hopkins and Kornienko (2009) examine the effects of changes in the income distribution in a strategic model wherein agents' utility depends both on consumption and on status (or rank in conspicuous consumption). Finally, in the context of a general equilibrium setting with Cournot firms, Crès and Tvede (2023) discuss various possible firms' objectives including shareholder voting.

Ariely's (2008) book, addressed to a broader audience, discusses the main ideas, principles and paradoxes of behavioral economics, emphasizing, in particular, the relative nature of economic agents' preferences.

3 Two players, cardinal preferences

Two-player game with cardinal preferences. In this section, we examine in detail unbeatable strategies in the classical framework of static two-player games with preferences specified by payoff functions. Consider a game G with strategy sets A, B and payoff functions $u(a,b)$, $v(a,b)$ of players 1 and 2 which are interpreted as their *measures of performance* ("scores"). The goal of a player is to construct a strategy that cannot be outperformed in terms of higher payoffs by the rival, whatever the rival's strategy might be.

The general definition of an unbeatable strategy (see Section 2) takes on in this context the following form:

Definition 3.1. A strategy $a^* \in A$ of player 1 is said to be *unbeatable* if

$$u(a^*, b) \geq v(a^*, b) \tag{6}$$

for any strategy $b \in B$ of player 2. Analogously, a strategy $b^* \in B$ of player 2 is called *unbeatable* if

$$v(a, b^*) \geq u(a, b^*) \tag{7}$$

for any strategy $a \in A$ of player 1. The game is said to be *determinate* if at least one of the players has an unbeatable strategy.

According to (6), player 1 using the strategy a^* cannot be outperformed by player 2, irrespective of the strategy b of player 2. Condition (7) expresses the analogous property of the strategy b^* of player 2.

[5] A deep study aimed at providing a rationalization of this insight is conducted by Samuelson (2004), who examines a model of an evolutionary environment in which Nature optimally builds relative consumption effects into preferences in order to compensate for incomplete environmental information.

The associated zero-sum game. To analyze the concept of an unbeatable strategy we will associate with the original game G defined in terms of the strategy sets A, B and payoff functions $u(a,b)$, $v(a,b)$ a zero-sum game G^0 in which the strategy sets of players 1 and 2 are the same as above, A and B, while the payoff functions of players 1 and 2 are given by

$$f(a,b) = u(a,b) - v(a,b) \text{ and } g(a,b) = -f(a,b).$$

The game G^0 will be called *the zero-sum game associated with the game* G.

Remark 3.1. If the original game is zero-sum, then $v(a,b) = -u(a,b)$, and so $f(a,b) = u(a,b) - v(a,b) = 2u(a,b)$, which means that the associated zero-sum game G^0 is isomorphic to the original one.

Remark 3.2. If the original game is symmetric, i.e. $A = B$ and $v(a,b) = u(b,a)$, then $f(a,b) = u(a,b) - v(a,b) = v(b,a) - u(b,a) = -f(b,a)$, and consequently, the payoff function $f(a,b)$ in the associated zero-sum game G^0 is skew-symmetric: $f(a,b) = -f(b,a)$. Thus $f(a,b) = g(b,a)$, which means that the game G^0 is symmetric.

The associated game and unbeatable strategies. We reformulate the definition of an unbeatable strategy in the game G in terms of the zero-sum game G^0. A strategy a^* of player 1 is unbeatable if and only if $f(a^*,b) \geq 0$ for every strategy b of player 2, or equivalently, $\inf_{b\in B} f(a^*,b) \geq 0$. A strategy b^* of player 2 is unbeatable if and only if $f(a,b^*) \leq 0$ for every strategy a of player 1, or equivalently, $\sup_{a\in A} f(a,b^*) \leq 0$. Clearly, an unbeatable strategy of player 1 exists if and only if one of the following inequalities holds:

$$\underline{f} := \sup_{a\in A} \inf_{b\in B} f(a,b) > 0 \quad \text{or} \quad \max_{a\in A} \inf_{b\in B} f(a,b) \geq 0.$$

Player 2 possesses an unbeatable strategy if and only if one of these relations is valid:

$$\overline{f} := \inf_{b\in B} \sup_{a\in A} f(a,b) < 0 \quad \text{or} \quad \min_{b\in B} \sup_{a\in A} f(a,b) \leq 0.$$

Here and in what follows, we write "max" in place of "sup" and "min" in place of "inf" if the corresponding extremum is attained. The numbers $\overline{f}$ and $\underline{f}$ are called the *upper* and the *lower values* of the zero-sum game G^0, respectively. As is well-known, the former is always not less than the latter.

3.4. Existence of unbeatable strategies. Recall that a function $F(x)$ defined on a topological space X is called *upper semicontinuous* if for every real number r the upper level set of this function $\{x : F(x) \geq r\}$ is closed. This function is termed *lower semicontinuous* if every lower level set $\{x : F(x) \leq r\}$ of this function is closed. An upper semicontinuous function attains its maximum and a lower semicontinuous function attains its minimum on a compact set. Consider the following conditions:

(**A**) The strategy set A is a compact topological space, for each b the function $f(a,b)$ is upper semicontinuous with respect to a, and $\underline{f} \geq 0$.

(**B**) The strategy set B is a compact topological space, for each a the function $f(a,b)$ is lower semicontinuous with respect to b, and $\overline{f} \leq 0$.

Define

$$\underline{f}(a) = \inf_{b \in B} f(a,b), \quad \overline{f}(b) = \sup_{a \in A} f(a,b).$$

The following result provides general sufficient conditions for the existence of unbeatable strategies.

Theorem 3.1. *If assumption (**A**) holds, then the function $\underline{f}(a)$ attains its maximum on A, and any element a^* of the set A maximizing $\underline{f}(a)$ is an unbeatable strategy of player 1. If condition (**B**) is fulfilled, then the function $\overline{f}(b)$ attains its minimum on B, and any element b^* of the set B minimizing $\overline{f}(b)$ is an unbeatable strategy of player 2. If one of assumptions (**A**) and (**B**) holds, then the game is determinate.*

Proof. Suppose condition (**A**) is satisfied. Then the function $\underline{f}(a)$ is upper semicontinuous in a because the infimum of any family of upper semicontinuous functions is upper semicontinuous. Therefore $\underline{f}(a)$ attains its maximum on the compact set A. If $a^* \in A$ is a point where this maximum is attained, then

$$\underline{f}(a^*) = \max_{a \in A} \underline{f}(a) = \max_{a \in A} \inf_{b \in B} f(a,b) = \underline{f} \geq 0$$

by virtue of assumption (**A**), consequently, $\underline{f}(a^*) = \inf_{b \in B} f(a^*, b) \geq 0$, which means that a^* is an unbeatable strategy of player 1.

Let assumption (**B**) hold. In this case the function $\overline{f}(b)$ is lower semicontinuous in b since the supremum of an arbitrary family of lower semicontinuous functions is lower semicontinuous. Consequently, $\overline{f}(b)$ attains its minimum on the compact set B. If b^* minimizes $\overline{f}(b)$, then

$$\overline{f}(b^*) = \min_{b \in B} \overline{f}(b) = \min_{b \in B} \sup_{a \in A} f(a,b) = \overline{f} \leq 0.$$

according to condition (**B**). Thus $\overline{f}(b^*) = \sup_{a \in A} f(a, b^*) = \overline{f} \leq 0$, and so b^* is an unbeatable strategy of player 2. □

Unbeatable strategies vs. Nash equilibrium. Theorem 3.1 suggests that the analysis of unbeatable strategies in static two-player games (with standard numerical preferences) is a "much easier" task than the analysis of Nash equilibrium strategies. First of all, a general game G reduces to a zero-sum one, the associated game G^0. If G^0 satisfies some assumptions of semicontinuity and compactness, then in order to prove the existence of an unbeatable strategy for one player or another we have only to find the upper value $\overline{f}$ or the lower value $\underline{f}$ of the game G^0 and simply check its sign. If $\underline{f} \geq 0$ (resp. $\overline{f} \leq 0$), then player 1 (resp. player 2) has an unbeatable strategy. Moreover, results of the above type provide effective constructions of unbeatable strategies based on minimization and maximization procedures. They do not employ non-constructive mathematical tools like Brouwer's or Kakutani's theorems. Note that in Nash equilibrium analysis, such constructions are possible only for potential games (Monderer and Shapley, 1996).

Unbeatable strategies and saddle points. Let us examine relations between unbeatable strategies in the original game G and saddle points (Nash equilibria) in the associated zero-sum game G^0. Recall that a pair $(\bar{a}, \bar{b}) \in A \times B$ is called a *saddle point* of the function $f(a, b)$, or of the zero-sum game G^0, if

$$f(a, \bar{b}) \leq f(\bar{a}, \bar{b}) \leq f(\bar{a}, b) \text{ for all } a \text{ and } b. \tag{8}$$

Proposition 3.1. *If the associated zero-sum game G^0 has a saddle point $(\bar{a}, \bar{b})$, then the original game G is determinate. Specifically, if $f(\bar{a}, \bar{b}) \geq 0$, then $\bar{a}$ is an unbeatable strategy of player 1. If $f(\bar{a}, \bar{b}) \leq 0$, then $\bar{b}$ is an unbeatable strategy of Player 2. If $f(\bar{a}, \bar{b}) = 0$, then $\bar{a}$ and $\bar{b}$ are unbeatable strategies of Players 1 and 2, respectively.*

Proof. If $f(\bar{a}, \bar{b}) \geq 0$, then $\bar{a}$ is an unbeatable strategy of player 1 because by virtue of (8), $f(\bar{a}, b) \geq f(\bar{a}, \bar{b}) \geq 0$ for all b. If $f(\bar{a}, \bar{b}) \leq 0$, then $f(a, \bar{b}) \leq f(\bar{a}, \bar{b}) \leq 0$ for all a, and so $\bar{b}$ is an unbeatable strategy of player 2. Consequently, if $f(\bar{a}, \bar{b}) = 0$, then $\bar{a}$ and $\bar{b}$ are unbeatable strategies of players 1 and 2, respectively. □

Remark 3.3. By virtue of Proposition 3.1, the existence of a saddle point in the associated zero-sum game G^0 is a sufficient condition for the determinacy of the original game G. But this condition is by no means necessary—see, e.g., Example 3.1 below. Moreover, in Proposition 3.5 we will show that determinacy occurs in a sense "substantially more often" than a saddle point. As Theorem 3.1 demonstrates, the assumptions guaranteeing the existence of unbeatable strategies are quite general and make it possible to construct such strategies effectively. Existence theorems for saddle points relying upon *minimax theorems* are mathematically deeper and require stronger assumptions. However, such results might be applied when one needs to establish only the determinacy of G without finding unbeatable strategies, which can be done by proving the existence of a saddle point in the associated zero-sum game G^0. The literature contains a whole variety of minimax theorems, most of which pertain to zero-sum games with strategy sets in linear spaces and require some assumptions of convexity—see, e.g., Willem (1996). Results on the existence of saddle points that do not assume convexity and use other conditions (submodularity, increasing differences, finite actions, discrete quasi-convexity, existence of potentials, etc.) are contained in Duersch, Oechssler and Schipper (2012), where special attention is paid to symmetric relative payoff games.

Minimax theorems and unbeatable strategies. We cite here the classical Sion's (1958) minimax theorem that can be employed for proving the existence of saddle points in zero-sum games and, consequently, the existence of unbeatable strategies. Let us introduce the following condition:

(**S**) 1) A and B are convex sets in linear topological spaces; 2) the set A is compact; 3) the function $f(a, b)$ is quasi-concave and upper semicontinuous in a and quasi-convex and lower semicontinuous in b.

We recall that a real-valued function $\phi(x)$ defined on a linear space is termed *quasi-concave* (resp. *quasi-convex*) if for every real number r the upper level set $\{x : \phi(x) \geq r\}$ (resp. the lower level set $\{x : \phi(x) \leq r\}$) of this function is convex.

Theorem 3.2 (Sion 1958)**.** *(i) Under assumption (**S**), we have*

$$\max_{a \in A} \inf_{b \in B} \ f(a,b) = \inf_{b \in B} \max_{a \in A} \ f(a,b). \tag{9}$$

(ii) If additionally the set B is compact, then

$$\max_{a \in A} \min_{b \in B} \ f(a,b) = \min_{b \in B} \max_{a \in A} \ f(a,b).$$

and the zero-sum game at hand has a saddle point.

Clearly assertion (ii) in the above theorem is a direct consequence of (i).

Counter-strategies. Let us show how Theorem 3.2 can be used for proving the existence of unbeatable strategies. Consider the following hypothesis.

(**C**) For each strategy b of player 2, there exists a strategy $a^*(b)$ of player 1 satisfying $u(a^*(b), b) \geq v(a^*(b), b)$.

This hypothesis means that player 1 can respond to every strategy b of player 2 with a *counter-strategy* $a^*(b)$ that does not permit the latter to beat the former. Clearly condition (**C**) is necessary for the existence of an unbeatable strategy for player 1. Indeed, an unbeatable strategy is nothing but a counter-strategy $a^* = a^*(b)$ that *does not depend on* b. We will show that under assumption (**S**), condition (**C**) is not only necessary, but also sufficient for the existence of an unbeatable strategy of player 1.

Proposition 3.2. *Under assumptions (**S**) and (**C**), player 1 possesses an unbeatable strategy.*

Proof. From (**C**) we get $f(a^*(b), b) \geq 0$. Consequently, $\max_{a \in A} \ f(a,b) \geq 0$ for each b, and therefore $\inf_{b \in B} \max_{a \in A} \ f(a,b) \geq 0$. By using (9), we obtain $\max_{a \in A} \inf_{b \in B} \ f(a,b) = \inf_{b \in B} \max_{a \in A} \ f(a,b) \geq 0$. Let a^* be the point in A where the maximum on the left-hand side of this equality is attained. Then $\inf_{b \in B} \ f(a^*, b) \geq 0$, i.e., a^* is an unbeatable strategy of Player 1. □

When does determinacy imply a saddle point? As Proposition 3.1 shows, the existence of a saddle point in the zero-sum game G^0 implies the determinacy of the original game G. The proposition below lists some cases when the converse is true.

Proposition 3.3. *(a) If both players have unbeatable strategies, a^* and b^*, then (a^*, b^*) is a saddle point in the game G^0, and $f(a^*, b^*) = 0$. (b) If the game G is symmetric, then any unbeatable strategy a^* of one of the players is an unbeatable strategy of the other, the pair (a^*, a^*) is a saddle point in the game G^0, and $f(a^*, a^*) = 0$. Thus a symmetric game G is determinate if and only if the zero-sum game G^0 possesses a symmetric saddle point.*

Proof. To prove (a) we observe that $f(a, b^*) \leq 0 \leq f(a^*, b)$ for all a and b, which implies that $f(a^*, b^*) = 0$ and shows that (a^*, b^*) is a saddle point in the game G^0. Assertion (b) is a consequence of (a). □

Finite games. Suppose that the strategy sets A and B are finite: $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_l\}$. Put $u_{ij} = u(a_i, b_j)$ and $v_{ij} = v(a_i, b_j)$. Consider the payoff matrix (f_{ij}) of the associated zero-sum game G^0 defined by

$$f_{ij} = f(a_i, a_j) = u_{ij} - v_{ij}.$$

According to the above definition, a strategy a_n of player 1 is unbeatable if in the nth row in the payoff matrix of the original game G, the payoffs of the first player are not less than the payoffs of the second player: $u_{nj} \geq v_{nj}$ for all j. A strategy b_m of player 2 is unbeatable if in the mth column of the payoff matrix, the payoffs of the second player are not less than the payoffs of the first player: $v_{im} \geq u_{im}$ for all i. Consequently, a strategy a_n of player 1 is unbeatable if all the elements f_{nj} in the nth row of the matrix (f_{ij}) are non-negative: $f_{nj} \geq 0$ for all j. A strategy b_m of player 2 is unbeatable if all the elements f_{im} in the mth column of this matrix are non-positive: $f_{im} \leq 0$ for all i.

Thus we obtain the following result regarding unbeatable strategies in the original game G formulated in terms of the payoff matrix (f_{ij}) of the associated zero-sum game G^0.

Proposition 3.4. *Player 1 possesses an unbeatable strategy if and only if the matrix (f_{ij}) has a non-negative row. Player 2 possesses an unbeatable strategy if and only if the matrix (f_{ij}) has a non-positive column. The game is determinate if and only if the matrix (f_{ij}) has either a non-negative row or a non-positive column.*

Example 3.1. Consider the following game G with the payoffs $u_{ij} = u(a_i, b_j)$, $v_{ij} = v(a_i, b_j)$ of players 1 and 2 and the associated zero-sum game G^0 with the payoffs $f_{ij} = u_{ij} - v_{ij}$ of player 1:

G	b_1	b_2	b_3
a_1	$-3, -1$	$3, 1$	$2, 0$
a_2	$0, 1$	$4, 6$	$5, 4$
a_3	$1, 2$	$2, 4$	$3, 3$

G^0		b_1	b_2	b_3	$\min_j f_{ij}$
	a_1	-2	2	2	-2
	a_2	-1	-2	1	-2
	a_3	-1	-2	0	-2
$\max_i f_{ij}$		-1	2	2	

• The matrix (f_{ij}) has a non-positive column, and so player 2 has an unbeatable strategy.

• There are no non-negative rows in the matrix (f_{ij}), and so player 1 does not have unbeatable strategies.

• The game is determinate but the associated zero sum game does not have a saddle point because $\max_i \min_j f_{ij} = -2 \neq -1 = \min_j \max_i f_{ij}$.

Pyrrhic victory. Let us look at the game considered in the above example. Clearly the strategy b_1 of player 2 is unbeatable: it yields payoff greater than the payoff of player 1, irrespective of his strategy. Thus b_1 is good in terms of the *relative* payoffs. However, in terms of the *absolute* payoffs, b_1 is the worst (strictly dominated by any other!) strategy of player 2. This seeming paradox demonstrates that the rationality in terms of a relative criterion may be wildly inconsistent with the rationality in terms of the absolute one. The strategy b_1 allows player 2 to gain the victory over player 1, but this is a *Pyrrhic victory*—a victory that is so devastating for the victor that it is tantamount to defeat. It is achieved at the expense of a dramatic reduction in player 2's payoff, which is less, however, than the reduction in the payoff of player 1.

Random game: Determinacy vs saddle point. As we have seen above, the existence of a saddle point in the associated zero-sum game G^0 is sufficient but not necessary for the determinacy of the original game G. Let us show that in a natural probabilistic sense the determinacy of G is a much more frequent event than a saddle point in G^0. This assertion is formalized in Proposition 3.6 below.

Consider a zero-sum game G_n with a finite set of strategies $A = \{a_1, a_2, ..., a_n\}$, the same for both players, and payoffs $u_{ij} = u(a_i, a_j)$, $v_{ij} = -u(a_i, a_j)$ of the first and the second players, respectively. The payoff matrix (f_{ij}) of the associated zero-sum game G^0 is defined by $f_{ij} = f(a_i, a_j) = u_{ij} - v_{ij} = 2u_{ij}$.Suppose that u_{ij}, and therefore f_{ij}, are independent identically distributed random variables with a continuous distribution. Denote by p_1 the probability that $f_{ij} \geq 0$ and by p_2 the probability that $f_{ij} \leq 0$, in symbols, $p_1 = P\{f_{ij} \geq 0\}$ and $p_2 = P\{f_{ij} \leq 0\}$. We will assume that both numbers p_1 and p_2 are strictly positive.

Probabilities of determinacy and saddle point. Denote by Δ_n^i $(i = 1, 2)$ the probability that player i in the game G_n has an unbeatable strategy, by Δ_n the probability that this game is determinate, and by Σ_n the probability that it has a saddle point.

Proposition 3.5. *We have*

$$\Sigma_n = (n!)^2/(2n-1)!, \tag{10}$$

$$\Delta_n^i = 1 - (1 - p_i^n)^n, \ i = 1, 2, \tag{11}$$

$$\Delta_n = \Delta_n^1 + \Delta_n^2. \tag{12}$$

The ratio Σ_n/Δ_n. Clearly both the probability Σ_n of a saddle point and the probability Δ_n of determinacy tend to zero as $n \to \infty$, but the former tends to zero faster than the latter.

Proposition 3.6. *The ratio* Σ_n/Δ_n *tends to zero at an exponential rate.*

For proofs of Propositions 3.5 and 3.6 see the Appendix.

4 Unbeatable strategies in evolutionary game theory

Kojima's work. The main source for this section is Kojima (2006). However, to simplify the exposition we focus on a model based on symmetric two-player two-strategy games. This makes it possible to elucidate key concepts in an elementary but sufficiently rich setting. The material related to ESS is well-known, and we present it in the shortest possible way to prepare a setup for comparing in the model at hand ESS and unbeatable strategies.

Population model (see, e.g., Weibull 1995, Samuelson 1997). Members of a population of organisms (e.g. animals, human beings, plants, etc.) interact pairwise. Each organism can be of a certain type x. The set of possible types is X. There is a function $u(x, y)$, $x, y \in X$ (*fitness function*) that characterizes the ability of organisms to survive. If an organism is of a type x and faces the probability distribution β of types y in the population, then its ability to survive is characterized by the expectation of $u(x, y)$ with respect to β. In evolutionary biology, elements x in X might represent *genotypes* of species and $u(x, y)$ the (average) number of surviving offspring. In evolutionary economics, such models serve to describe interactions in large populations of economic agents. Types x can represent various characteristics of economic agents and/or patterns of their behavior.

Symmetric game. With the given model, we associate a symmetric two-player game in which the payoff functions of the players are $u(x, y)$ and $v(x, y) = u(y, x)$, and their common strategy set is X. In this context, the terms "types" and "strategies" are used interchangeably. The values of the fitness function $u(x, y)$ are interpreted as "payoffs".

Let us say that a strategy x^* is *strictly unbeatable* if

$$(1-\varepsilon)u(x^*, x^*) + \varepsilon u(x^*, x) > \ (1-\varepsilon)u(x, x^*) + \varepsilon u(x, x) \tag{13}$$

for all $x \neq x^*$ and all $0 < \varepsilon < 1$. We will omit "strictly" in what follows. For an unbeatable strategy x^*, inequality (13) must hold for all $x \neq x^*$ and all $0 < \varepsilon < 1$. For an *evolutionary stable strategy (ESS)* x^*, according to its definition, it must hold only for $\varepsilon > 0$ small enough, which means that "non-mutants" x^* outperform "mutants" x only if the fraction of the mutants is small enough. The definition of an unbeatable strategy requires that this be true always, not only when the fraction of mutants is sufficiently small.

Proposition 4.1. *A strategy x^* is unbeatable if and only if for each $x \neq x^*$ at least one of the following conditions is fulfilled:*

$$u(x, x^*) < u(x^*, x^*) \textit{ and } u(x, x) \leq u(x^*, x), \tag{14}$$

$$u(x, x^*) \leq u(x^*, x^*) \textit{ and } u(x, x) < u(x^*, x). \tag{15}$$

Proof. Suppose (14) holds. Multiply the first inequality in (14) by $1 - \varepsilon$, the second by ε, and add up. This will yield (13). The same argument shows that (15) implies (13). Conversely, observe that inequality (13) holds for each $0 < \varepsilon < 1$ if and only if it holds as a non-strict inequality both for $\varepsilon = 0$ and $\varepsilon = 1$ and as a strict inequality in at least one of the two cases: $\varepsilon = 0$ and $\varepsilon = 1$ The former case corresponds to (14) and the latter to (15). □

Remark 4.1. We compare (14) and (15) with the conditions characterizing ESS: for each $x \neq x^*$, we have either

$$u(x, x^*) < u(x^*, x^*) \tag{16}$$

(x^* is a strict symmetric Nash equilibrium) or

$$u(x, x^*) = u(x^*, x^*) \text{ and } u(x, x) < u(x^*, x). \tag{17}$$

Note that the assertion that at least one of the conditions (14) and (15) holds is equivalent to the assertion that for each $x \neq x^*$, one (and only one) of the following two requirements is fulfilled:

(I) $u(x, x^*) < u(x^*, x^*)$ and $u(x, x) \leq u(x^*, x)$;
(II) $u(x, x^*) = u(x^*, x^*)$ and $u(x, x) < u(x^*, x)$.

Indeed, the inequality $u(x, x^*) \leq u(x^*, x^*)$ involved in (15) can hold either as a strict inequality, and then we have (I), or as equality, which leads to (II). Note that condition (II) coincides with property (17) in the definition of ESS, but condition (I) contains together with the strict equilibrium property $u(x, x^*) < u(x^*, x^*)$ stated in (16) the additional requirement $u(x, x) \leq u(x^*, x)$. This shows, in particular, that a strict symmetric Nash equilibrium is always an ESS, but it is not necessarily an unbeatable strategy.

Mixed strategies in two-player two-strategy games. Let us consider the concept of an unbeatable strategy in the case where X is the set of mixed strategies in a symmetric two-player game with two strategies a_1, a_2 and the payoffs $u_{ij} = u(a_i, a_j)$ of the first player:

	a_1	a_2
a_1	$u_{11} = u(a_1, a_1)$	$u_{12} = u(a_1, a_2)$
a_2	$u_{21} = u(a_2, a_1)$	$u_{22} = u(a_2, a_2)$

. (18)

Note that unbeatable strategies, as well as ESS, are defined in terms of the differences $u(\alpha, \beta) - u(\beta, \beta)$, where $\alpha = (p, 1-p)$ and $\beta = (q, 1-q)$ are mixed strategies. It follows from this that unbeatable strategies and ESS are the same for the original game and the following *simple* game:

	a_1	a_2
a_1	$u_1 = u_{11} - u_{21}$	0
a_2	0	$u_2 = u_{22} - u_{12}$

. (19)

"Simple", by definition, means that non-diagonal payoffs are equal to zero. In the analysis of such games, we will assume (to exclude degenerate cases) that $u_1 \neq 0$ and $u_2 \neq 0$. The simple game (19) will be called the *reduced version* of the original one.

Our goal is to characterize those mixed strategies $\beta = (q, 1-q)$ which are unbeatable, i.e. satisfy for all $\alpha \neq \beta$ conditions (**I**) or (**II**):

(**I**) $u(\alpha, \beta) < u(\beta, \beta)$ and $u(\alpha, \alpha) \leq u(\beta, \alpha)$,
(**II**) $u(\alpha, \beta) = u(\beta, \beta)$ and $u(\alpha, \alpha) < u(\beta, \alpha)$.

ESS in simple games. As is known (see, e.g., Weibull 1995), the structure of ESS in the game at hand is as follows:

		ESS	$\beta = (q, 1-q)$
Case 1	$u_1 < 0, u_2 < 0$	one	$q = q^*$, $q^* = \dfrac{u_2}{u_1 + u_2}$, $1 - q^* = \dfrac{u_1}{u_1 + u_2}$,
Case 2	$u_1 > 0, u_2 > 0$	two	$q = 0,\ 1$,
Case 3	$u_1 < 0, u_2 > 0$	one	$q = 0$,
Case 4	$u_1 > 0, u_2 < 0$	one	$q = 1$.

In cases 2, 3 and 4, ESS are strict Nash equilibria: $u(\alpha, \beta) < u(\beta, \beta)$ for all $\alpha \neq \beta$.

Evolutionary stable and unbeatable strategies. Let us find out which of the ESS described above are unbeatable.

Case 1. Let us show that $\beta^* = (q^*, 1 - q^*)$ is an unbeatable strategy. We have

$$u(\alpha, \beta^*) = u(\beta^*, \alpha) = p \cdot q^* u_1 + (1 - p) \cdot (1 - q^*) u_2 = u_1 u_2 / (u_1 + u_2) = u(\beta^*, \beta^*)$$

for each $\alpha = (p, 1 - p)$ because $q^* u_1 = (1 - q^*) u_2 = u_1 u_2 / (u_1 + u_2)$. Thus $u(\alpha, \beta^*) = u(\beta^*, \beta^*)$, and so we need to verify the inequality in (**II**): $u(\alpha, \alpha) < u(\beta^*, \alpha)$ for all $\alpha \neq \beta^*$. Since $u(\alpha, \alpha) = p^2 u_1 + (1 - p)^2 u_2$, this inequality can be written: $p^2 u_1 + (1 - p)^2 u_2 < u_1 u_2 / (u_1 + u_2)$ $(p \neq q^*)$. But this is indeed true: the concave quadratic function $\psi(p) = p^2 u_1 + (1 - p)^2 u_2$ $(u_1 < 0,\ u_2 < 0)$ attains its maximum $u_1 u_2 / (u_1 + u_2)$ at the unique point $p = q^* = u_2 / (u_1 + u_2)$, where its derivative is equal to zero.

In all the other cases (2, 3 and 4) ESS are strict, and therefore we have to check the second inequality in (**I**), which can be written as

$$u(\alpha, \alpha) = p^2 u_1 + (1 - p)^2 u_2 \leq p q u_1 + (1 - p)(1 - q) u_2 = u(\beta, \alpha). \tag{20}$$

Case 2: neither $q = 0$, nor $q = 1$ are unbeatable. Indeed, if $q = 0$, then (20) becomes $p^2 u_1 + (1 - p)^2 u_2 \leq (1 - p) u_2$, which is not true for $p = 1$. If $q = 1$, then (20) yields $p^2 u_1 + (1 - p)^2 u_2 \leq p u_1$, which is wrong for $p = 0$.

Case 3: $q = 0$ is unbeatable because $p^2 u_1 + (1 - p)^2 u_2 \leq (1 - p)^2 u_2 \leq (1 - p) u_2$.

Case 4: $q = 1$ is unbeatable because $p^2 u_1 + (1 - p)^2 u_2 \leq p^2 u_1 \leq p u_1$.

We summarize the results obtained in the following table:

		unbeatable strategies	$(q, 1 - q)$
Case 1	$u_1 < 0, u_2 < 0$	one	$q = q^*$, $q^* = \dfrac{u_2}{u_1 + u_2}$,
Case 2	$u_1 > 0, u_2 > 0$	no unbeatable strategies	
Case 3	$u_1 < 0, u_2 > 0$	one	$q = 0$,
Case 4	$u_1 > 0, u_2 < 0$	one	$q = 1$.

Are unbeatable strategies that rare? There is a widespread view that unbeatable strategies are rare compared to ESS (see e.g. Nowak et al., 2004, p. 649). However, the model at hand does not confirm this view. Only in one of four cases (case 2) an ESS fails to be unbeatable. It is notable that in case 2 both ESS that are not unbeatable strategies are strict symmetric Nash equilibria.

Selection model. Our next goal is to examine unbeatable strategies in a different model. The previous one considered mutations, this one focuses on selection. Although they are different, there are deep connections between them. Consider the symmetric two-strategy game (18). There is a large population of players. Some of them select the strategy a_1 and the others a_2. The fraction of a_1 players is $x \in (0, 1)$, and the fraction of a_2 players is $1 - x$. The average payoff of those playing a_1 is $U_1 = x u_{11} + (1 - x) u_{12}$ because an a_1 player encounters another a_1 player in the population with probability x and an a_2

player with probability $1-x$. For similar reasons the average payoff of those playing a_2 is $U_2 = xu_{21} + (1-x)u_{22}$. The average payoff across the population can be expressed as $U = xU_1 + (1-x)U_2$.

Replicator Dynamics. The proportion of a_1 players goes up if on average they are doing better than the overall average, and down otherwise. This principle is expressed by the *replicator dynamics (RD)* equation (Taylor and Jonker 1978): $x'/x = U_1 - U$. Here $x' = x'(t)$ is the derivative of the function $x(t)$ with respect to time t, and x'/x is the growth rate of $x(t)$. By using the definitions of U_1, U_2 and U, the RD equation can be written in the following form:

$$x' = f(x), \text{ where } f(x) := x(1-x)[xu_1 - (1-x)u_2], \ u_1 = u_{11} - u_{21}, \ u_2 = u_{22} - u_{12}. \quad (21)$$

Note that the function $f(x)$ is the same for the original game and its reduced version (see (19)). Thus we can focus on the differential equation (21) for the simple game (19). As before it will be assumed that $u_1 u_2 \neq 0$. Note that the equation $f(x) = 0$, has always the roots $x = 0$ and $x = 1$. This equation has a root x in the interval $0 < x < 1$ if and only if u_1 and u_2 have the same signs, and then $x = q^* = u_2/(u_1 + u_2)$ (compare with the ESS in the previous model!).

Definition 4.1. An *evolutionary stable steady state (ESSS)* is defined as an asymptotically stable[6] steady state of the dynamical system (21).

Replicator dynamics and ESS. We wish to analyze the asymptotic behavior of paths of the dynamical system described by the differential equation (21).

Case 1: $u_1 < 0, u_2 < 0$. Then $0 < q^* < 1$, and for $0 < x < 1$, we have $xu_1 - (1-x)u_2 > 0$ if and only if $x < q^* = u_2/(u_1 + u_2)$. Consequently, $f(x) > 0$ when $0 < x < q^*$ and $f(x) < 0$ when $q^* < x < 1$. As it is demonstrated in Fig. 1, we have convergence to q^* from every starting point $0 < x < 1$.

Case 2: $u_1 > 0, u_2 > 0$. Then $0 < q^* < 1$, and for $0 < x < 1$, we have $xu_1 - (1-x)u_2 > 0$ if and only if $x > q^* = u_2/(u_1 + u_2)$. Thus $f(x) > 0$ when $q^* < x < 1$ and $f(x) < 0$ when $0 < x < q^*$. Consequently, paths of the RD system converge to 0 if they start from any $0 < x < q^*$ and to 1 if they start from any $q^* < x < 1$, see Fig. 2.

Case 3: $u_1 < 0$, $u_2 > 0$. We have $f(x) < 0$ for $0 < x < 1$, therefore the RD process starting from every initial state $0 < x < 1$ converges to 0. The dynamics of the RD process in this case is shown in Fig 3.

Case 4: $u_1 > 0$ and $u_2 < 0$. The function $f(x)$ is strictly positive for $0 < x < 1$, and therefore the RD process starting from every point $0 < x < 1$ converges to 1, see Fig. 4.

[6] A steady state $\bar{x}$ of a dynamical system is called *asymptotically stable* if trajectories of the system starting from points x sufficiently close to $\bar{x}$ converge to $\bar{x}$.

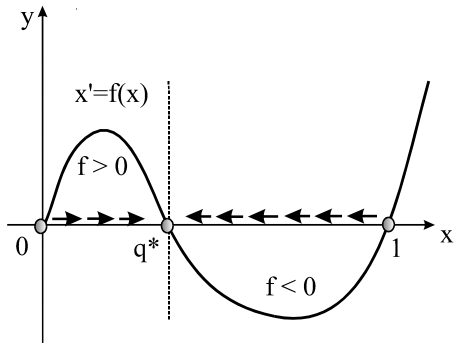

Fig.1: Case 1.

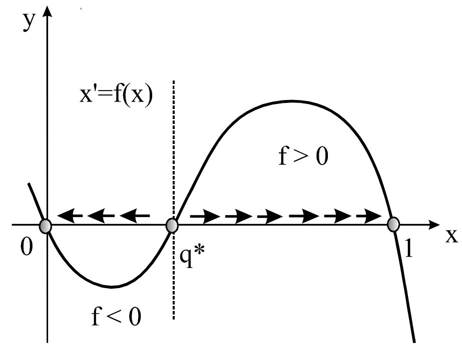

Fig. 2: Case 2

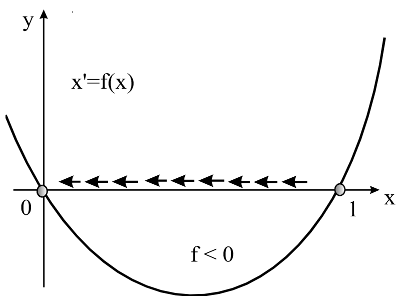

Fig. 3: Case 3

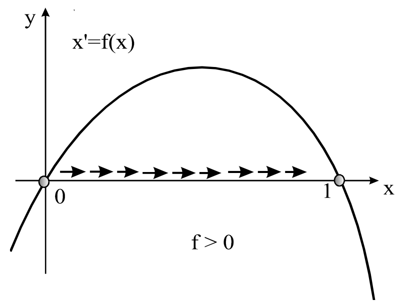

Fig. 4: Case 4

In all the cases considered, the ESSS of the replicator dynamics process coincide with the evolutionary stable strategies of the underlying game.

Replicator dynamics and unbeatable strategies. We will show that for the replicator dynamics process those ESSS which correspond to unbeatable strategies form globally evolutionary stable steady states.

Definition 4.2. A *globally evolutionary stable steady state (GESSS)* is a globally asymptotically stable steady state[7] of the dynamical system $x' = f(x)$.

Case 1: $u_1 < 0$, $u_2 < 0$. As it was shown, $q^* = u_1/(u_1 + u_2)$ corresponds to the unique ESS, which is an unbeatable strategy. We can see from Fig. 1 that the RD process converges to q^* starting from *any* initial point x in the interval $0 < x < 1$, and not only from points sufficiently close to q^*. Consequently, q^* is GESSS.

Case 2: $u_1 > 0, u_2 > 0$. The dynamics of the RD process is illustrated in Fig. 2. There are two ESS: $q = 0$ and $q = 1$, but *none* of them represents a GESSS. Convergence to 1

[7] A steady state $\bar{x}$ of a dynamical system is called *globally asymptotically stable* if trajectories of the system starting from *any* initial point in the domain of the system (and not only from points in some sufficiently small neighborhood of $\bar{x}$) converge to $\bar{x}$.

takes place only for trajectories of the RD process starting from initial states $q^* < x < 1$ and convergence to 0 takes place only for trajectories starting from $0 < x < q^*$ (recall that $q^* = u_2/(u_1 + u_2)$).

Case 3: $u_1 < 0$, $u_2 > 0$. According to Fig. 3, the ESS $q = 0$ is globally evolutionary stable since the RD process starting from *every* initial state $0 < x < 1$ converges to 0.

Case 4: $u_1 > 0$ and $u_2 < 0$. The function $f(x)$ is strictly positive for $0 < x < 1$, and therefore the RD process starting from *every* point $0 < x < 1$ converges to 1 (see Fig. 4).

In all the cases considered, *the globally evolutionary stable steady states of the replicator dynamics process correspond to the unbeatable strategies of the underlying game!*

5 Unbeatable strategies in an asymmetric Cournot duopoly

Asymmetric duopoly with capacity constraints. In this section we examine unbeatable strategies in an asymmetric Cournot duopoly model with a homogeneous good. The production units in this model feature different production costs and different capacity constraints. We conduct a comparative analysis of the Cournot-Nash equilibrium strategies and unbeatable strategies in this framework. In symmetric settings, questions of this kind were considered in the seminal paper by Schaffer (1989) and related works: Cressman and Hofbauer (2005), Duersch et al. (2012, 2012a, 2014), Hehenkamp et al. (2004), Hehenkamp et al. (2010), Lim and Matros (2009), Matros and Armanios (2009), Matros and Possajennikov (2016), Possajennikov (2003a,b), Rhode and Stegeman (2001), Schipper (2009), and Vega-Redondo (1997). In the context of public goods, see related recent work by Herings et al. (2021).[8] Asymmetry of the model adds interesting new aspects to the study and leads to results that to the best of our knowledge have no direct analogues in the existing literature.

Model description. A firm owns two production units/plants $i = 1, 2$ producing quantities $q_1 \geq 0$, $q_2 \geq 0$ of a homogeneous good, the inverse demand for which is $1 - q_1 - q_2$. The plant i's production cost is $c_i \in (0, 1)$ and its capacity (the maximum quantity it can produce) is $Q_i > 0$. The plants are run by two managers who select the quantities $q_i \in [0, Q_i]$, $i = 1, 2$, of the good to be produced representing strategies in the game at hand. If

[8] In many of the afore-mentioned models, imitation (of the most successful strategy) plays a key role in the modeling of evolutionary dynamics in biology and economics. This idea serves at the basis for the work of Eshel, Samuelson, and Shaked (1998) focusing on questions of evolutionary stability of altruistic behavior in a local interaction framework. The model analyzed by Eshel et. al (1998) abandons the assumption that economic agents are rational utility maximizers. Rather, it assumes that they take care primarily about relative performance of strategies, selecting those that are more successful in the long run. This idea has been further developed, also in a local interaction setting, in Eshel, Sansone and Shaked (1999), where direct links to Hamilton's unbeatable strategies are traced. This line of thinking, combined with others, has reached its culmination in the general program of synthesizing behavioral and mainstream economics proposed recently by Aumann (2019).

strategies q_1 and q_2 are chosen, then the profits of the plants are

$$\pi_i(q_1, q_2) = q_i(1 - q_1 - q_2 - c_i),\ i = 1, 2.$$

The goal of the firm, serving the whole market, is to maximize profits. To achieve this goal it contemplates an incentive scheme for the managers. A standard possibility would be to share with them a certain fraction of profits. This would lead to the conventional Cournot-Nash equilibrium outcome. However, the "parsimonious" firm, rather than allocating to the managers some fixed percentage of profits, sets up a contest. The principle of this contest is that the managers are rewarded for getting a *higher profit* than their rivals. What matters is not the absolute value π_i of the profit that plant i obtains, but the difference $\pi_i\,(q_1, q_2) - \pi_j(q_1, q_2)$ between the profits of plants i and j. The contest represents a game with relative preferences in which the players/managers strive to employ unbeatable strategies. In this section we examine this contest and compare its outcome with the conventional Cournot-Nash equilibrium outcome in the original game.

Contest setup. The firm establishes a *bonus fund* B some part of which is allocated to manager 1 and the rest to manager 2 depending on their relative performance. Specifically, there is a constant $\theta > 0$ such that player 1 gets $g_1(q_1, q_2) = B/2 + \theta f(q_1, q_2)$, where

$$f(q_1, q_2) = \pi_1(q_1, q_2) - \pi_2(q_1, q_2),\ q_i \in [0, Q_i],\ i = 1, 2, \tag{22}$$

and player 2 receives $g_2(q_1, q_2) = B/2 - \theta f(q_1, q_2)$. If the profits obtained by both plants are equal, both managers receive $B/2$, half the amount contained in the bonus fund.

This is a constant-sum game, in which the payoffs of the players sum up to B: $g_1(q_1, q_2)+ g_2(q_1, q_2) = B$. Clearly this game is isomorphic to the zero-sum game with the payoffs $f(q_1, q_2)$ and $-f(q_1, q_2)$. Player 1 will maximize the payoff function $g_1(q_1, q_2)$, or equivalently, maximize $f(q_1, q_2)$. Player 2 will maximize $g_2(q_1, q_2)$, or equivalently, minimize $f(q_1, q_2)$. Thus the solution to the contest game will be a saddle point $(\bar{q}_1, \bar{q}_2)$ of the zero-sum game with the payoffs $f(q_1, q_2)$ and $-f(q_1, q_2)$. The strategy $\bar{q}_1$ will be unbeatable for player 1 if $f(\bar{q}_1, \bar{q}_2) \geq 0$, and the strategy $\bar{q}_1$ will be unbeatable for player 2 if $f(\bar{q}_1, \bar{q}_2) \leq 0$ (see Section 3).

Saddle point in the associated zero-sum game. Put $\gamma_i = 1 - c_i \in (0, 1]$, $i = 1, 2$. Then the profit of production unit 1 can be expressed as

$$\pi_i(q_1, q_2) = q_i(\gamma_i - q_1 - q_2). \tag{23}$$

Define

$$\bar{q}_1 = \min\{Q_1, \gamma_1/2\},\ \bar{q}_2 = \min\{Q_2, \gamma_2/2\}. \tag{24}$$

Proposition 5.1. *The pair* $(\bar{q}_1, \bar{q}_2)$ *is the unique saddle point of the function* $f(q_1, q_2)$*, i.e.*

$$f(q_1, \bar{q}_2) \leq f(\bar{q}_1, \bar{q}_2) \leq f(\bar{q}_1, q_2) \text{ for all } q_i \in [0, Q_i],\ i = 1, 2; \tag{25}$$

$\bar{q}_1$ *is an unbeatable strategy of player* 1 *if* $f(\bar{q}_1, \bar{q}_2) \geq 0$*;* $\bar{q}_2$ *is an unbeatable strategy of player* 2 *if* $f(\bar{q}_1, \bar{q}_2) \leq 0$*.*

Proof. We have

$$f(q_1, q_2) = q_1(\gamma_1 - q_1 - q_2) - q_2(\gamma_2 - q_1 - q_2) = q_2^2 - \gamma_2 q_2 + q_1\gamma_1 - q_1^2. \tag{26}$$

For each q_1, this function attains its unique minimum with respect to q_2 at $\bar{q}_2 = \min\{Q_2, \gamma_2/2\}$, and for each q_2 it attains its unique maximum with respect to q_1 at $\bar{q}_1 = \min\{Q_1, \gamma_1/2\}$, which proves that $(\bar{q}_1, \bar{q}_2)$ is the unique saddle point of $f(q_1, q_2)$. If $f(\bar{q}_1, \bar{q}_2) \geq 0$, then from the second inequality in (25) we obtain $0 \leq f(\bar{q}_1, q_2) = \pi_1(\bar{q}_1, q_2) - \pi_2(\bar{q}_1, q_2)$, which means that $\bar{q}_1$ is an unbeatable strategy of player 1. If $f(\bar{q}_1, q_2) \leq 0$, then the first inequality in (25) implies $0 \geq f(q_1, \bar{q}_2) = \pi_1(q_1, \bar{q}_2) - \pi_2(q_1, \bar{q}_2)$, consequently, $\bar{q}_2$ is an unbeatable strategy of player 2. □

The value of the associated zero-sum game. By virtue of (25) and (22) we get

$$f(\bar{q}_1, \bar{q}_2) = \begin{cases} -\gamma_2^2/4 + \gamma_1^2/4 & \text{if } Q_1 \geq \gamma_1/2, Q_2 \geq \gamma_2/2, \\ -\gamma_2^2/4 + Q_1(\gamma_1 - Q_1) & \text{if } \gamma_2/2 \leq Q_2, \gamma_1/2 \geq Q_1, \\ Q_2(Q_2 - \gamma_2) + \gamma_1^2/4 & \text{if } \gamma_2/2 \geq Q_2,\ \gamma_1/2 \leq Q_1, \\ Q_2(Q_2 - \gamma_2) + Q_1(\gamma_1 - Q_1) & \text{if } Q_1 \leq \gamma_1/2, Q_2 \leq \gamma_2/2. \end{cases} \tag{27}$$

As we have seen above, the sign of this value determines who of the players has an unbeatable strategy.

Capacity constraints: an assumption. If there are no capacity constraints, or they are not binding, then as is easily seen, that plant which has the lower production cost can beat the rival. However, this might not necessarily be the case if this plant does not have a sufficient capacity to fully realize its potential of producing the good at the lower cost. The rival might have a greater capacity so that by producing more at a greater cost it may achieve a higher profit. To focus on the essence of the model, we will exclude the cases where the capacity constraints are "too binding": $Q_1 \leq \gamma_1/2$, $Q_2 \leq \gamma_2/2$ or "not binding enough": $Q_1 \geq \gamma_1/2, Q_2 \geq \gamma_2/2$. We will concentrate on the cases $Q_1 \leq \gamma_1/2$, $Q_2 \geq \gamma_2/2$ and $Q_1 \geq \gamma_1/2$, $Q_2 \leq \gamma_2/2$. Each of this cases can be reduced to the other by changing the notation, and therefore it can be assumed without loss of generality that the following condition is satisfied:

(**A1**) The following inequalities hold: $Q_1 \geq \gamma_1/2$, $Q_2 \leq \gamma_2/2$.

Under this assumption, we have

$$\bar{q}_1 = \min\{Q_1, \gamma_1/2\} = \gamma_1/2,\ \bar{q}_2 = \min\{Q_2, \gamma_2/2\} = Q_2 \tag{28}$$

(see (24)).

When is the strategy $\bar{q}_1$ unbeatable? By virtue of (27) and (28), $\bar{q}_1$ is an unbeatable strategy of player 1 when

$$f(\bar{q}_1, \bar{q}_2) = Q_2(Q_2 - \gamma_2) + \gamma_1^2/4 \geq 0. \tag{29}$$

If $\gamma_1 \geq \gamma_2$, then inequality (29) holds always (as long as (**A1**) is satisfied) because then $Q_2(Q_2 - \gamma_2) \geq -\gamma_2^2/4 \geq -\gamma_1^2/4$. Let us assume that the opposite inequality holds:

(**A2**) We have $\gamma_1 \leq \gamma_2$.

Recall that $\gamma_i = 1 - c_i$. Thus condition (**A2**) means that plant 1 has the production cost c_1 greater than the production cost c_2 of plant 2. As the following proposition shows, the strategy $\bar{q}_1$ of player 1 happens to be unbeatable if the capacity of production unit 2 is below a certain threshold.

Proposition 5.2. *Under assumptions (**A1**)and (**A2**), the strategy $\bar{q}_1 = \gamma_1/2$ of player 1 is unbeatable if and only if*

$$Q_2 \leq (\gamma_2 - \sqrt{\gamma_2^2 - \gamma_1^2})/2 \quad (\leq \gamma_2/2). \tag{30}$$

Proof. The value of the function $\phi(Q_2) = Q_2(Q_2-\gamma_2)+\gamma_1^2/4$ at the point $0 \leq Q_2 \leq \gamma_2/2$ is non-negative if and only if Q_2 does not exceed the smaller root of the quadratic equation $Q_2(Q_2 - \gamma_2) + \gamma_1^2/4 = 0$, which is equal to $(\gamma_2 - \sqrt{\gamma_2^2 - \gamma_1^2})/2$. □

Note that the greater is the asymmetry in costs (i.e. the greater is the difference between $\gamma_1 = 1 - c_1$ and $\gamma_2 = 1 - c_2$) the lower is the threshold for Q_2 in (30) guaranteeing that the strategy $\bar{q}_1 = \gamma_1/2$ is unbeatable.

Total output, price and profits. Suppose player 1 follows the unbeatable strategy $\bar{q}_1 = \gamma_1/2$ and player 2 employs the strategy $\bar{q}_2 = Q_2$. Then the total production output and the price of the good can be expressed as follows:

$$\bar{Q} = \bar{q}_1 + \bar{q}_2 = \gamma_1/2 + Q_2 = (1 - c_1)/2 + Q_2, \tag{31}$$

$$\bar{p} = 1 - \bar{Q} = 1 - \gamma_1/2 - Q_2 = (1 + c_1)/2 - Q_2. \tag{32}$$

Since $\pi_i(\bar{q}_1, \bar{q}_2) = \bar{q}_i(\gamma_i - \bar{Q})$, we get the following formulas for the profits $\bar{\pi}_i := \pi_i(\bar{q}_1, \bar{q}_2)$ of plants $i = 1, 2$:

$$\bar{\pi}_1 = \bar{q}_1(\gamma_1 - \bar{Q}) = \gamma_1(\gamma_1/2 - Q_2)/2 = (1 - c_1)[(1 - c_1)/2 - Q_2]/2, \tag{33}$$

$$\bar{\pi}_2 = \bar{q}_2(\gamma_2 - \bar{Q}) = Q_2(\gamma_2 - \gamma_1/2 - Q_2) = Q_2[(1 + c_1 - 2c_2)/2 - Q_2]. \tag{34}$$

Schaffer's paradox (Schaffer 1989). Consider the symmetric case. Suppose the production units have the same capacities $Q_1 = Q_2 = Q$ and the same production costs: $c_1 = c_2 = c$, so that $\gamma_i = (1 - c_i) = \gamma$, where $\gamma = 1 - c$. Then in view of (**A2**) we have $Q \geq \gamma/2$, $Q \leq \gamma/2$, which yields $Q = \gamma/2$. Furthermore, by virtue of (31) $\bar{Q} = \gamma = 1 - c$, and so the price $\bar{p} = 1 - \bar{Q}$ coincides with the production cost c. This implies that the profits $\pi_i(\bar{q}_1, \bar{q}_2)$ of both plants are equal to zero—an outcome disastrous for the profit maximizing firm!

Nash equilibrium. To find a Nash equilibrium in the Cournot game at hand observe that for each q_2 the function $\pi_1(q_1, q_2) = q_1(1 - q_1 - q_2 - c_1) = -q_1^2 + q_1(\gamma_1 - q_2)$ attains its maximum with respect to q_1 on $[0, Q_1]$ at $q_1 = \min\{(\gamma_1 - q_2)/2, Q_1\}$. Analogously for each

q_1 the function $\pi_2(q_1, q_2) = q_2(1 - q_1 - q_2 - c_2) = -q_2^2 + q_2(\gamma_2 - q_1)$ reaches its maximum with respect to q_2 on $[0, Q_2]$ when $q_2 = \min\{(\gamma_2 - q_1)/2, Q_2\}$. A Nash equilibrium (q_1^*, q_2^*) is a solution to the system of two equations

$$q_1^* = \min\{(\gamma_1 - q_2^*)/2, Q_1\}, \; q_2^* = \min\{(\gamma_2 - q_1^*)/2, Q_2\}. \tag{35}$$

Sufficiency of capacities. We introduce an assumption from which it will follow that the capacities Q_1 and Q_2 are not "too binding": they make it possible to produce those quantities of the good which correspond to the Nash equilibrium strategies in the Cournot game without capacity constraints.

(**A3**) The numbers $\gamma_i = 1 - c_i$ and Q_i satisfy the following inequalities:

$$0 \le (2\gamma_1 - \gamma_2)/3 \le Q_1, \; 0 \le (2\gamma_2 - \gamma_1)/3 \le Q_2. \tag{36}$$

Note that the inequality $2\gamma_2 - \gamma_1 \ge 0$ follows from (**A2**), and so it does not impose any new constraints. However, the inequality $2\gamma_1 - \gamma_2 \ge 0$ does. It says that the asymmetry in the model is not "too big". Although the number $\gamma_1 = 1 - c_1$ is smaller than $\gamma_2 = 1 - c_2$, it should not be smaller by more than two times.

Proposition 5.3. *Under (**A3**), the Nash equilibrium (q_1^*, q_2^*) is given by*

$$q_1^* = (2\gamma_1 - \gamma_2)/3, \; q_2^* = (2\gamma_2 - \gamma_1)/3. \tag{37}$$

Proof. To verify the first equation in (35) we write

$$\min\{(\gamma_1 - q_2^*)/2, Q_1\} = \min\{(3\gamma_1 - 2\gamma_2 + \gamma_1)/6, Q_1\} = \min\{(2\gamma_1 - \gamma_2)/3, Q_1\} = q_1^*.$$

The second equation in (35) is proved analogously. □

The assumptions are consistent. We introduced a number of assumptions on the data of the model that are needed for the comparative analysis of Nash equilibrium and unbeatable strategies. Are these assumptions consistent, i.e. do not some of them contradict the others? How to find a simple condition under which all of them are satisfied? The following proposition gives answers to these questions. Put $\bar{b} = (3\sqrt{3} - 1)/4$ (≈ 1.049) and

$$u(\gamma_1, \gamma_2) = (2\gamma_2 - \gamma_1)/3, \; v(\gamma_1, \gamma_2) = (\gamma_2 - \sqrt{\gamma_2^2 - \gamma_1^2}\,)/2. \tag{38}$$

Proofs of Propositions 5.4–5.6 we formulate below are given in the Appendix.

Proposition 5.4. *If*

$$0 < \gamma_1 \le \gamma_2 \le \bar{b}\gamma_1, \tag{39}$$

then

$$u(\gamma_1, \gamma_2) \le \; v(\gamma_1, \gamma_2), \tag{40}$$

and if

$$Q_1 \ge \gamma_1/2, \; u(\gamma_1, \gamma_2) \le Q_2 \le v(\gamma_1, \gamma_2). \tag{41}$$

*then conditions (**A1**) – (**A3**) and (30) are satisfied.*

Typically, asymmetric Cournot duopoly models are workable if their asymmetry is in a sense not too big. The most standard assumption expressing this idea is the inequality $\gamma_1 \leq 2\gamma_2$. (As we already noticed, it follows from (**A3**)). In the present context we need a stronger condition (39), meaning that the asymmetry of the model is small enough. Proposition 5.4 can be used as follows. Inequality (40) holding under assumption (b) implies that the segment $[u(\gamma_1, \gamma_2), v(\gamma_1, \gamma_2)]$ is not empty. Using this, we can select any Q_2 in this segment and any $Q_1 \geq \gamma_1/2$. Then according to Proposition 5.4 all the assumptions imposed above and hence all the assertions proved will be valid.

Nash equilibrium outcome. Suppose that the production units 1 and 2 select Nash equilibrium strategies (37). Then the total production output Q^*, the market price p^* of the good, and the profits π_1^*, π_2^* of plants 1 and 2 will be as follows:

$$Q^* = (\gamma_1 + \gamma_2)/3, \; p^* = 1 - Q^* = 1 - (\gamma_1 + \gamma_2)/3,$$

$$\pi_i^* = q_i^*(\gamma_i - Q^*) = (2\gamma_i - \gamma_j)^2/9, \; i \neq j.$$

Contest vs. Nash equilibrium. Define

$$\mu = (\gamma_1 + \gamma_2)/2 = 1 - (c_1 + c_2)/2 \in (0, 1), \; \sigma = (\gamma_2 - \gamma_1)/2 = (c_1 - c_2) \in [0, \mu).$$

Here $1 - \mu$ is the average production cost. The number σ may be regarded as a "measure of asymmetry" of the model: 2σ is equal to the difference between the production costs. In further analysis, it will be convenient to change the variables γ_1 and γ_2 to μ and σ. Clearly,

$$\gamma_1 = \mu - \sigma, \; \gamma_2 = \mu + \sigma. \tag{42}$$

We will show that the contest under consideration leads, compared with the Nash equilibrium, to a greater output, smaller price, and lower profits for both production units. It is important to note that the differences between the corresponding variables in the two settings grow as the index $\sigma = (\gamma_2 - \gamma_1)/2$ of the model asymmetry grows. Thus, although Schaffer's paradox can essentially be observed in the present context as well, the asymmetry of the model, surprisingly or not, makes it "milder".

Proposition 5.5. *The following inequalities hold:*

$$\bar{Q} \geq Q^* + (\mu + 3\sigma)/6, \tag{43}$$

$$\bar{p} \leq p^* - (\mu + 3\sigma)/6, \tag{44}$$

$$\bar{\pi}_1 \leq \pi_1^* - (\mu + 3\sigma)^2/36, \tag{45}$$

$$\bar{\pi}_2 \leq \pi_2^* - 7(\mu + 3\sigma)^2/144. \tag{46}$$

From symmetric to asymmetric case. We provide some estimates for the variables (31) - (34) showing how they may change when the degree of asymmetry of the model changes. Fix some $\mu \in (0,1)$ and consider the following functions of $\sigma \in [0,\mu]$:

$$\Phi(\sigma) = \mu - \sqrt{\mu\sigma},\ \Pi_1(\sigma) = (\mu-\sigma)(\sqrt{\mu\sigma}-\sigma)/2,\ \Pi_2(\sigma) = (\mu+3\sigma)(\sigma+\sqrt{\mu\sigma})/3.$$

Proposition 5.6. *The following inequalities hold:*

$$\bar{Q} \leq \Phi(\sigma),\ \bar{p} \geq 1-\Phi(\sigma),\ \bar{\pi}_1 \geq \Pi_1(\sigma),\ \bar{\pi}_2 \geq \Pi_2(\sigma).$$

For $0 < \sigma < \mu$ *we have* $\Phi(\sigma) > 0$, $\Pi_1(\sigma) > 0$, $\Pi_2(\sigma) > 0$, $\Phi'(\sigma) < 0$, $\Pi_2'(\sigma) > 0$, *and* $\Pi_1'(\sigma) > 0$ *if* σ *is small enough.*

This proposition shows that when the asymmetry of the model (measured in terms of σ) increases, the outcome of the asymmetric contest becomes more and more distinct from the outcome of its symmetric counterpart. The total output $\bar{Q}$ exhibits a tendency to decrease as it is bounded above by a decreasing function of σ. The price $\bar{p}$ and the profits $\bar{\pi}_1$ and $\bar{\pi}_2$ are bounded below by strictly positive increasing functions of σ, and so they tend to grow when σ grows.

Appendix

The appendix contains proofs of several results stated in the body of the paper.

Proof of Proposition 3.5. Formula (10) is obtained in Goldman (1957). As we have shown above (see Proposition 3.2), Δ_n^1 coincides with the probability that the random matrix (f_{ij}) has a non-negative row. The probability that some particular row is non-negative is equal to p_1^n, and so the probability that at least one row is non-negative is $1-(1-p_1^n)^n$. This yields (11) for $i=1$. By virtue of Proposition 3.2, the number Δ_n^2 represents the probability that the random matrix (f_{ij}) possesses a non-positive column. It is equal to $1-(1-p_2^n)^n$, which implies (11) for $i=2$. If the matrix f_{ij} has both a non-negative row and a non-positive column, it must have a zero element. Since the distribution of f_{ij} is continuous, the probability of this event is equal to zero. Consequently, $\Delta_n = \Delta_n^1 + \Delta_n^2$. □

Proof of Proposition 3.6. By using the Stirling formula $n! \sim \sqrt{2\pi n}n^n e^{-n}$, we get

$$\Sigma_n = \frac{(n!)^2}{(2n-1)!} \sim \frac{2\pi n \cdot n^{2n} e^{-2n}}{\sqrt{2\pi(2n-1)}(2n-1)^{2n-1}e^{-(2n-1)}} =$$

$$= \frac{\sqrt{2\pi}e^{-1} \cdot n \cdot (2n-1) \cdot n^{2n}}{\sqrt{2n-1}(2n-1)^{2n}} = \sqrt{2\pi}e^{-1}n\sqrt{2n-1}\left(\frac{n}{2n-1}\right)^{2n}, \tag{47}$$

where $[n/(2n-1)]^2 = 1/(2-1/n)^2 \leq 1/3$ for all sufficiently large n. Consequently, for some constant C and all n large enough, we have

$$\Sigma_n \leq Cn^2 3^{-n}. \tag{48}$$

Let us show that $g_n := 1 - (1 - p^n)^n \sim np^n$ for any $0 < p < 1$. We can represent $\exp x$ and $\ln(1+x)$ as $\exp x = 1 + x(1+\alpha(x))$ and $\ln(1+x) = x(1+\beta(x))$ with $\alpha(x) \to 0$, $\beta(x) \to 0$ as $x \to 0$. Therefore $g_n = 1 - \exp[n\ln(1-p^n)] = 1 - \exp[n\ln(1-p^n)] = 1 - \exp r_n$, where $r_n := -np^n(1+\beta_n) \to 0$ and $\beta_n := \beta(-p^n) \to 0$. By setting $\alpha_n := \alpha(r_n)$, we obtain $g_n = 1 - \exp r_n = -r_n(1+\alpha_n)$, which yields $g_n/np^n \to 1$. Since $p_1 + p_2 = 1$, one of the numbers p_1, p_2 is not less than $1/2$, and so

$$\Delta_n = \Delta_n^1 + \Delta_n^2 \geq 1 - [1 - (1/2)^n]^n \geq cn2^{-n} \tag{49}$$

for some constant c and all sufficiently large n. From (48) and (49) we obtain that $\Sigma_n/\Delta_n \leq Cn^2 3^{-n}/cn2^{-n} \leq (C/c)n(2/3)^n$ for all n large enough. □

Proof of Proposition 5.4. We have $v(\gamma_1,\gamma_2) - u(\gamma_1,\gamma_2) = -(\sqrt{\gamma_2^2 - \gamma_1^2})/2 - (\gamma_2 - 2\gamma_1)/6$. This expression is non-negative if and only if $8\gamma_2^2 - 13\gamma_1^2 + 4\gamma_1\gamma_2 \leq 0$, or equivalently, $8(\gamma_2/\gamma_1)^2 - 13 + 4(\gamma_2/\gamma_1) \leq 0$, which holds if and only if $\gamma_2/\gamma_1 \leq \bar{b}$.

To complete the proof we observe that (**A1**) holds because $Q_1 \geq \gamma_1/2$, as assumed in (41), and $Q_2 \leq \gamma_2/2$ by virtue of the second inequality in (41) and the definition of $v(\gamma_1,\gamma_2)$ in (38). Condition (**A2**) follows from (39). To check (**A3**) we observe the following. The first inequality in (36) holds because $\gamma_2 \leq \bar{b}\gamma_1 < 2\gamma_1$ by virtue of (39). The second is true since $\gamma_2 \geq \gamma_1$, and so $(2\gamma_1 - \gamma_2)/3 \leq \gamma_1/3 < \gamma_1/2 \leq Q_1$ (see (41)). The third is fulfilled by virtue of (**A2**) and the fourth follows from the second inequality in (41). Finally, (30) coincides with the third inequality in (41). □

Proof of Proposition 5.5. By using (31), the first inequality in (41) and (42), we write

$$\bar{Q} = \gamma_1/2 + Q_2 \geq \gamma_1/2 + (2\gamma_2 - \gamma_1)/3 = (3\gamma_1 + 4\gamma_2 - 2\gamma_1)/6 =$$

$$= (\gamma_1 + \gamma_2)/3 + (2\gamma_2 - \gamma_1)/6 = Q^* + (\mu + 3\sigma)/6,$$

which implies (43) and (44). Further, in view of (31), (33), the first inequality in (41) and (42), we have

$$\bar{\pi}_1 = \bar{q}_1(\gamma_1 - \bar{Q}) = \gamma_1(\gamma_1/2 - Q_2)/2 \leq \gamma_1[\gamma_1/2 - (2\gamma_2 - \gamma_1)/3]/2 = \gamma_1(5\gamma_1 - 4\gamma_2)/12,$$

and consequently,

$$\pi_1^* - \bar{\pi}_1 \geq (2\gamma_1 - \gamma_2)^2/9 - \gamma_1(5\gamma_1 - 4\gamma_2)/12$$

$$= [2(\mu - \sigma) - (\mu + \sigma)]^2/9 - (\mu - \sigma)[5(\mu - \sigma) - 4(\mu + \sigma)]/12 = (\mu + 3\sigma)^2/36,$$

and thus we obtain (45). Finally, we get

$$\bar{\pi}_2 = \bar{q}_2(\gamma_2 - \bar{Q}) = Q_2(\gamma_2 - \gamma_1/2 - Q_2) \leq \max_{Q_2} Q_2(\gamma_2 - \gamma_1/2 - Q_2)$$

$$= (\gamma_2 - \gamma_1/2)^2/4 = (\mu + 3\sigma)^2/16 = (\mu + 3\sigma)^2/9 - 7(\mu + 3\sigma)^2/144 = \pi_2^* - 7(\mu + 3\sigma)^2/144,$$

which yields (46). □

Proof of Proposition 5.6. Using the fact that $(\sqrt{\gamma_2^2-\gamma_1^2})/2 = \sqrt{\mu\sigma}$, we can write inequality (30) as $Q_2 \le \gamma_2/2 - \sqrt{\mu\sigma}$. By employing this relation and (42), we get

$$\bar{Q} = \gamma_1/2 + Q_2 \le \gamma_1/2 + \gamma_2/2 - \sqrt{\mu\sigma} = \mu - \sqrt{\mu\sigma} = \Phi(\sigma),$$

$$\bar{p} = 1 - \bar{Q} \ge 1 - \mu + \sqrt{\mu\sigma} = 1 - \Phi(\sigma),$$

$$\bar{\pi}_1 = \bar{q}_1(\gamma_1 - \bar{Q}) = \gamma_1(\gamma_1/2 - Q_2)/2 \ge (\mu - \sigma)(\sqrt{\mu\sigma} - \sigma)/2 = \Pi_1(\sigma),$$

$$\bar{\pi}_2 = \bar{q}_2(\gamma_2 - \bar{Q}) \ge Q_2(\gamma_2 - \mu + \sqrt{\mu\sigma}) = Q_2(\mu + \sigma - \mu + \sqrt{\mu\sigma}),$$

$$= Q_2(\sigma + \sqrt{\mu\sigma}) \ge (2\gamma_2 - \gamma_1)(\sigma + \sqrt{\mu\sigma})/3 = (\mu + 3\sigma)(\sigma + \sqrt{\mu\sigma})/3 = \Pi_2(\sigma),$$

where the last inequality in the above chain of relations follows from the first inequality in (41). Strict positivity of the functions in question follows from the inequalities $\sigma < \sqrt{\mu\sigma} < \mu$ (holding when $\sigma < \mu$). It is clear that $\Phi'(\sigma) < 0$ and $\Pi_2'(\sigma) > 0$. Finally, we can see from the equation $\Pi_1'(\sigma) = (\mu - \sigma)[\mu/(2\sqrt{\mu\sigma}) - 1]/2 - (\sqrt{\mu\sigma} - \sigma)/2$ that $\Pi_1'(\sigma) > 0$ for sufficiently small $\sigma > 0$ because the above expression tends to infinity as σ tends to zero.□

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