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Identifying the Vulnerable: Concepts and Measurement*

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Abstract

Vulnerability to poverty, which broadly captures the susceptibility to becoming poor in the future, has become an integral part of any deprivation assessment. In this paper, we take a fresh look at measuring vulnerability, where we separate out the identification part of whether an individual is vulnerable, from their level of vulnerability. Given the substantial informational challenges that one faces in the context of measuring vulnerability, our framework allows for different information sets for identification and aggregation. These challenges lead us to propose identification rules based on the future probability of falling into poverty. We axiomatically characterise these identification rules along with a widely used measure of vulnerability. Using a simple societal measure we show that societal vulnerability can be decomposed into vulnerabilities arising from the different shocks. Further, we provide an empirical illustration of the identification and measurement rules proposed in this paper using real-world data from Bangladesh.

Key Words: Poverty, Vulnerability, Uncertainty.

JEL Classification: D80, I32, O12

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1 Introduction

Vulnerability to poverty, which captures the susceptibility of households and individuals to fall into poverty in the future, is fast becoming an integral part of any deprivation assessment. The Human Development Report (UNDP, 2014, p.10) noted that "Vulnerability threatens human development and unless it is systematically addressed, by changing policies and social norms, progress will be neither equitable nor sustainable." Goals to reduce vulnerability recognise the uncertainty individuals face in their daily livelihoods (O'Brien et al.,2018), however successful policy design and implementation require effective identification of the vulnerable and their level of vulnerability. Since vulnerability is associated with uncertainties regarding future outcomes, both the identification and the measurement of vulnerability is likely to be a complex and informationally demanding exercise.

This paper puts forth a systematic way of addressing these two distinct questions of (a) identifying the vulnerable, and (b) measuring the level of vulnerability that each individual faces. In recent years although there has been significant development in our understanding of how to measure and estimate vulnerability (see Calvo 2018; Gallardo, 2017 and Fuji, 2016) for comprehensive reviews), several empirical papers have used adhoc rules for identifying the vulnerable (Hohberg et al. 2018, Dang and Lanjouw 2018). Our identification rule is based on an individual's probability of falling into poverty in the future which has been widely used in vulnerability measurement (Vo, 2018; Imai et al., 2011; Chaudhuri et al. 2002; Pritchett et al. 2000). We explicitly take into consideration the informational constraints that are inherent in the nature of vulnerability. Our proposed rule is particularly attractive in a context where policy-makers choose a different and perhaps coarser set of information to screen the vulnerable and a more richer set of information to deduce their level of vulnerability.

Once we identify who the vulnerable are, we are faced with a second non-trivial problem of assessing their level of vulnerability to poverty. In our second contribution to the literature, we axiomatically characterise the widely used Foster Greer and Thorbecke (FGT) class of vulnerability measures (Chaudhuri 2003; Hoddinott and Quisumbing, 2003) to assess individual vulnerability.[1] Hence, our paper can be viewed as providing a

[1]The FGT class of vulnerability measure is essentially the *expected* Foster, Greer and Thorbecke (1984) poverty measure. Several studies employ the FGT class of vulnerability measures such as Ward (2016), Celidoni and Procidano (2015), Zhang and Wan (2006), among others.

theoretical underpinning to much of the applied work in the context of vulnerability.

Our proposed method for identification has broad similarities with the *counting* approach to multidimensional deprivation (Alkire and Foster, 2011; Atkinson, 2003), where an individual is identified as multidimensionally poor if they are above a certain threshold based on the weighted number of dimensions the individual is deprived. In a similar vein, we can identify vulnerable individuals using a threshold (or cut-off) based on the probability of income losses resulting in poverty in the future. The loss of income in the future can be related to adverse income shocks an individual may face, such as, sudden illness or death (health shocks), extreme weather events (natural shocks), or a combination of these shocks, among others. We can, therefore, map the deprivations arising from the future outcomes of income to possible adverse shocks faced by the individual. From this perspective, our proposed framework is similar to the choice under uncertainty literature (Gilboa, 2009; Luce and Raiffa, 1957), with each individual facing a probability distribution over future outcomes. Studies such as Knippenberg and Hoddinott (2017), Haq (2015), Gloede et al. (2013), and Heltburg and Lund (2009) have used information on discrete shocks to examine vulnerability.

In addition to identifying the vulnerable, we measure their level of vulnerability using the FGT vulnerability measure. In our framework, this measure is more suitable since, in line with the choice under uncertainty literature, it treats the different outcomes, which represents the various adverse shocks, as independent of each other. Thus, the impact of each of these adverse shocks on the vulnerability of the individual is independent of any other shocks. This, however, does not rule out the possibility of simultaneous multiple shocks since they can be represented as one future outcome. The greater the exposure to these adverse shocks and the income losses associated with them, the higher will be the vulnerability of the individual to poverty. In our framework, a simple average of the vulnerability across all individuals is chosen as a measure of societal vulnerability. We demonstrate that societal vulnerability can then be decomposed into vulnerability arising out of the different adverse income shocks. This should be particularly useful for shock-responsive social protection programmes (O'Brien et al. 2018) since policy-makers can now identify the key shocks that determine vulnerability in their society.

Recent studies such as Dang and Lanjouw (2017), Chakravarty (2016), and de la Fuente et al. (2015) have used a *vulnerability line*, which is similar in concept to a poverty line, to identify the vulnerable. While intuitively appealing, this approach ignores the ex-ante nature of vulnerability since it is operationalised using ex-post data.[2] It also requires longitudinal data which are not easily available for many developing countries.[3] A slightly different route to identifying the vulnerable, which we refer as the *standard approach*, is taken by many empirical studies such as Ward (2016), Imai et al. (2011), Jha and Dang (2010), Zhang and Wan (2009), Christiaensen and Subbarao (2005), Chaudhuri (2003), where the level of vulnerability for each individual is first computed using an estimated future income or consumption. Those with vulnerability above a certain threshold are identified as vulnerable.[4] Apart from arbitrarily setting the vulnerability threshold, this approach calibrates an individual's vulnerability level even for identifying whether the individual is vulnerable or not. As a result, identification of the vulnerable becomes informationally as demanding as measuring their level of vulnerability since for each individual we now need information on both the exposure to the different shocks and the associated income shortfalls with each shock to estimate their vulnerability. Such information can be both difficult and expensive to gather.[5]

In this paper, we contribute to the current literature in several ways. We propose and axiomatically characterise an identification rule based on probabilities associated with various adverse income shocks resulting in poverty.[6] Going a step further, if we consider a richer set of criteria which includes the probabilities and the income shortfall associated with each of these adverse shocks, then we can adopt the FGT vulnerability measure that we axiomatise in this paper as an identifying rule. This is exactly what the *standard approach* in the literature does. In that case, as mentioned before, the identification rule and the vulnerability measure for the individual are the same. Finally, we show that a simple societal measure has the advantage of being the sum of the vulnerabilities arising

[2]The issue of measuring vulnerability in an ex-post sense also arises when vulnerability is conceptualised as downside spell (see Asheim et al. 2020).

[3]The data issues concerning measurement of vulnerability have been discussed in Heltberg and Lund (2009) and Patel et al. (2017).

[4]Cisse and Barrett (2018) use a similar framework to estimate resilience which is just the opposite of vulnerability.

[5]Since this information is not easy to collect always, a coarser information set may be preferred for identification. For instance, in the current pandemic and consequent lockdown, for benefit transfer schemes in India, vulnerable households were identified using coarse information based on occupations prone to income shocks (Das and Mishra, 2020).

[6]Most vulnerability assessments, typically collect data on the exposure to adverse shocks that people experience (see Erman et al., 2018; Gerlitz et al., 2014; Korboe, 2011; World Bank, 2007), which can be used to calibrate the probability of different future outcomes, rather than the detailed income losses under those shocks.

from the different shocks. These theoretical results are then applied to a real-world data set from Bangladesh.

The plan of the paper is as follows. In the next section, we develop the notations and illustrate through numerical examples our proposed concept of identifying the vulnerable and measuring the vulnerability of those individuals and the whole society. In Section 3, we explore the identification rules in greater detail and discuss axioms to characterise our proposed identification rules. Section 4, provides an axiomatic characterisation of the FGT class of individual vulnerability measures. In the following section, we discuss the societal vulnerability measure and its decomposition into vulnerability arising from different shocks. An empirical illustration of our proposed framework, using data from Bangladesh, is provided in Section 6 of the paper. It demonstrates how we can apply the identification methods discussed in this paper to real-world data. The final section draws the arguments of the paper together with some brief remarks.

2 Concept of Vulnerability Measure

2.1 Notation

Consider a society of N individuals. Each individual i faces a future deprivation level d drawn from $[0,1]$ according to some probability distribution with finite support.[7] Let $\mathcal{P}$ denote the set of all probability distributions over $[0,1]$ with finite support. Whenever individual i's future deprivation is drawn according to a probability distribution $P^i \in \mathcal{P}$, individual i faces a corresponding lottery $L^i_m = (p^i_1, d^i_1; ..., p^i_m, d^i_m)$, where $\{d^i_1, ..., d^i_m\}$ is the support of P^i and p^i_s is the probability of facing the deprivation level d^i_s for each s, such that $p^i_s > 0$ for all s and $\sum_{s=1}^{m} p^i_s = 1$.[8] Let $\mathcal{L}$ denote the set of all lotteries corresponding to the probability distributions in $\mathcal{P}$. For ease of exposition, without any loss of generality, we will carry out the analysis in the rest of the paper using the lottery space $\mathcal{L}$ instead of the probability distribution space $\mathcal{P}$. For any lottery $L^i_m = (p^i_1, d^i_1; ..., p^i_m, d^i_m)$, the number of positive deprivations that individual i faces is either m or $m-1$. This is because individual i can also face no deprivation ($d = 0$) in the future with a positive probability. If an individual faces a deprivation, say d, with certainity, we denote it by

[7] In our context, $d = (z - y)/z$ if $y \leq z$, where y is the future income earned and z is the future poverty line, otherwise $d = 0$.

[8] Note that, for any d^i_k and d^i_l in the support of P^i, $d^i_k \neq d^i_l$ if $k \neq l$.

the degenerate lottery $(1, d)$. To reduce notational burden, when there is no scope for confusion, we shall drop the superscript i and subscript m from the lotteries and the associated probabilities and deprivations when dealing with an individual.

In identifying the vulnerable, we consider the degenerate lottery $(1, 0)$, where the individual will not fall into poverty for sure, as not vulnerable under all identification rules. Hence, our identification function will be on the domain $\mathcal{L}^I = \mathcal{L} \backslash (1, 0)$. A general identification rule is a function $\rho : \mathcal{L}^I \to [0, 1]$. For a given $\theta \in [0, 1]$, an individual i facing a lottery $L^i \in \mathcal{L}^I$ is identified as vulnerable based on an identification rule ρ if $\rho(L^i) \geq \theta$. If $\theta = 0$, an individual is identified as vulnerable if they fall into poverty in any of the future outcomes with positive probability. This is known as the *union* approach. On the other hand, when $\theta = 1$, we have the *intersection* approach where an individual is deemed vulnerable if the individual is deprived in all outcomes in the future with positive probability. Once we identify the individual as vulnerable, we then use the lottery the individual faces to come up with the level of vulnerability an individual faces. Thus, for any individual i, vulnerability is measured by $V : \mathcal{L} \to [0, 1]$. Finally, if $\rho(L) < \theta$ or $L = (1, 0)$, $V(L) = 0$.[9]

Individual vulnerability can be aggregated to form the societal vulnerability. At the societal level, with N individuals, the social vulnerability is measured by a function $V^S : \mathcal{L}^N \to [0, 1]$.

2.2 Vulnerability Measure

In this section, we present our vulnerability measure and illustrate it with an example. Similar to the theoretical and empirical literature on vulnerability (Calvo and Dercon, 2013; Dutta, Foster and Mishra, 2011; Chaudhuri, 2003) we have divided the assessment of vulnerability into three steps: (i) identify who is vulnerable, (ii) measure the vulnerability of those who are identified as vulnerable and (iii) aggregate the vulnerability of all individuals to measure the societal vulnerability. We view them as inter-linked steps to measure overall vulnerability.

The *P-rule* that we propose for identification formalizes a common understanding of

[9] While we distinguish between the identification rule $\rho(L)$ and the individual vulnerability measure $V(L)$, it is possible that the functional forms of $\rho(L)$ and $V(L)$ are the same. This is common in empirical applications as mentioned in the previous section. Hence, the *FGT* vulnerability measure, characterised in our paper, can also be seen as an identification rule.

vulnerability which is the *probability of an individual becoming deprived* in the future. For any lottery $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}$:

(i) let $\mathbf{r}^L = (r_1^L, ..., r_m^L)$ be the associated deprivation identification vector such that, for each s:

$$r_s^L = \begin{cases} 1 & \text{if } d_s > 0 \\ 0 & \text{otherwise} \end{cases}$$

(ii) let $\mathbf{p}^L = (p_1, ..., p_m)$ be the associated probability vector.

It is clear that $\mathbf{r}^L$ partitions the lottery into binary outcomes: deprived and the non-deprived, where those outcomes with positive deprivation are assigned the highest possible value of one. Thus, while identifying whether an individual is vulnerable or not, we do not distinguish between the different future positive deprivations by the extent of their deprivation. The *P-rule* identification function, $\rho^P : \mathcal{L}^I \to [0, 1]$, is defined as:

Definition 1 $\rho^P : \mathcal{L}^I \to [0, 1]$ *is a vulnerability identification rule such that, for any* $L, L' \in \mathcal{L}^I$*:*

(i) if $\mathbf{p}^L \cdot \mathbf{r}^L > \mathbf{p}^{L'} \cdot \mathbf{r}^{L'}$*, then* $\rho^P(L) > \rho^P(L')$*;*

(ii) if $\mathbf{p}^L \cdot \mathbf{r}^L = \mathbf{p}^{L'} \cdot \mathbf{r}^{L'}$*, then* $\rho^P(L) = \rho^P(L')$*.*

Individual i is vulnerable if the scalar product $\mathbf{p}^L \cdot \mathbf{r}^L \geq \theta$, where $\theta \in [0, 1]$ is the threshold (or cutoff) value. $\rho^P(L^i) = 1$ would mean that individual i is certainly going to be deprived in the future whereas $\rho^P(L^i) = 0$, will mean individual i will certainly not be deprived.

In the second step, we present the Foster-Greer-Thorbecke (FGT) vulnerability measure, which aggregates across all deprivation an individual faces in a lottery $L = (p_1, d_1; ...; p_m, d_m)$ to assess the individual's vulnerability using the following function:

$$V(L) = \begin{cases} \sum_{s=1}^{m} p_s (d_s)^\alpha & \text{if } \rho(L) \geq \theta \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $\alpha > 0$. For the $\alpha = 0$ case, there will be no difference between the aggregation rule proposed in (1) and the *P-rule* identification strategy given by $\rho^P(L)$.

In the third step, we take a simple aggregation of individual vulnerabilities to measure societal level of vulnerability. The societal vulnerability measure for each $(L^1, ..., L^N) \in \mathcal{L}^N$ is represented as

$$V^S(L^1, ..., L^N) = \frac{1}{N}\sum_{i=1}^{N} V(L^i) \tag{2}$$

Note that the societal average does not consider the distribution of vulnerability across the population. However, the simple formulation has the great advantage of being able to decompose vulnerability into different population subgroups. Hence, policy-makers would be able to target vulnerable groups based on region, gender, ethnicity, or any other population-based characteristics. One could allow more general functions such as a weighted average of individual vulnerabilities, or an average which takes into account the correlation among vulnerabilities (Calvo and Dercon, 2013). However, if we want to have some notion of decomposability in our societal measure, where the societal vulnerability is the sum of vulnerabilities of sub-populations, then we would be limited to a narrower set of functions (Foster and Shorrocks, 1991), which includes the simple average that we consider here.

2.3 Illustrated Examples

Example 1 *For a society with $N = 3$ individuals, consider the following lotteries faced by each of the individuals: $L^1 = (0.1, 0.1; 0.8, 0; 0.1, 0.2)$, $L^2 = (0.3, 0; 0.7, 0.8)$, and $L^3 = (0.5, 0.6; 0.25, 0.4; 0.25, 0.2)$. Thus $\mathbf{r}^{L^1} = (1, 0, 1)$, $\mathbf{r}^{L^2} = (0, 1)$, $\mathbf{r}^{L^3} = (1, 1, 1)$, $\mathbf{p}^{L^1} = (0.1, 0.8, 0.1)$, $\mathbf{p}^{L^2} = (0.3, 0.7)$ and $\mathbf{p}^{L^3} = (0.5, 0.25, 0.25)$.*
First, let us begin with Stage 1. Let the threshold value be $\theta = 0.5$. Then using P-rule identification strategy we find $\mathbf{p}^{L^1} \bullet \mathbf{r}^{L^1} = 0.2 < 0.5$. Hence, individual 1 is not considered vulnerable. On the other hand $\mathbf{p}^{L^2} \bullet \mathbf{r}^{L^2_2} = 0.7 > 0.5$, which implies that individual 2 is vulnerable. Similarly, for the third individual, $\mathbf{p}^{L^3} \bullet \mathbf{r}^{L^3} = 1 > 0.5$, which indicates that individual 3 is also vulnerable.
In stage 2, we assess individual vulnerability using (1) and $\alpha = 1$; $V(L^1) = 0$, $V(L^2) = 0.56$ and $V(L^3) = 0.45$.
In stage 3, the overall societal vulnerability will be $V^S(L^1, L^2, L^3) = 1/3(V(L^1) + V(L^2) + V(L^3)) = 0.34$.

In this example, although individual 2 is deprived only in one future outcome, compared to individual 1 who is deprived in two outcomes, we identify individual 2 as vulnerable but not individual 1. This is because the probability associated with the outcome in which individual 2 is deprived is far higher than the combined probability of the two outcomes in which individual 1 is deprived. Thus, individual 2 is more likely to be deprived in the future than individual 1.

The *P-rule* identification strategy, however, can sometimes lead to counter-intuitive results, where vulnerability rankings can change under different threshold values represented by θ. We demonstrate this by an example below.

Example 2 *Suppose the lotteries faced by two individuals are:*$L^1 = (5/6, 0.3; 1/6, 0)$*, and* $L^2 = (2/3, 0; 1/3, 0.95)$.
Note that $\mathbf{r}^{L^1} = (1, 0)$*,* $\mathbf{p}^{L^1} = (5/6, 1/6)$*,* $\mathbf{r}^{L^2} = (0, 1)$ *and* $\mathbf{p}^{L^2} = (2/3, 1/3)$*. Thus,* $\mathbf{p}^{L^1} \bullet \mathbf{r}^{L^1} = 0.83$ *and* $\mathbf{p}^{L^2} \bullet \mathbf{r}^{L^2} = 0.33$.
For $\theta = 0.5$*,* $\rho(L^1) > 0.5 > \rho(L^2)$*. Hence for* $\alpha = 1$*,* $V(L^1) = 0.25 > V(L^2) = 0$.
For $\theta = 0.3$*,* $\rho(L^1) > \rho(L^2) > 0.3$*. Hence for* $\alpha = 1$*,* $V(L^1) = 0.25 < V(L^2) = 0.32$.

In Example 2, the reversal of ranking of the vulnerability of the two individuals under different values of θ happens because of the identification rule, where an individual is identified as vulnerable based on the probability of deprivation in the future and not on their deprivation level. Such ranking reversal happens in other contexts too, such as for multidimensional poverty where this issue has been highlighted by Pattanaik and Xu (2018).

One way to overcome this problem, as suggested by Pattanaik and Xu (2018), is to take into account a broader set of information that includes the deprivation associated with each shock, along with the probability of the shock, in the identification function itself. This method is implicitly followed widely in the empirical literature on vulnerability where under the *standard approach* vulnerability of each individual based on the FGT measure is calibrated and a threshold applied, above which individuals are identified as vulnerable. In this paper we have axiomatically characterise the FGT measure mainly to capture the level of vulnerability of an individual, however, it can be considered an example of a vulnerability identification function that identifies individuals based on both the probability of an adverse shock and the resulting deprivation from these shocks. While the use of such function does not lead to 'inconsistencies', this approach uses the same function and data for identification of the vulnerable and for measuring their level of vulnerability.

Following Sen (1976, 1979, 1981), however, the identification of who is vulnerable could be considered as a distinct and separate exercise to measuring the level of vulnerability of individuals. This can be partly due to the different objectives of the two processes and partly driven by the different information set one may have. A policy-maker, for instance, may decide to have a broader and to some extent a looser set of criteria to identify the vulnerable for enrollment in social protection programmes and yet may undertake a different exercise using detailed information to measure the level of vulnerability of those identified as vulnerable for evaluative purposes.[10] The separate process of identification and measuring vulnerability can also arise from data constraints, with information on the exposure to different shocks more readily available, while data on the impact of these shocks requiring significantly more effort to gather. Hence, despite the 'inconsistencies', there may be reasonable grounds to consider separate functions for identification and measuring individual vulnerability.

3 Characterising Identification Rules

Using two new axioms we axiomatically characterise the *P-rule* which identifies vulnerable individuals based on the probability of falling into poverty in the future. The first axiom captures the notion that if each of two lotteries with two possible outcomes has only one outcome where an individual is deprived, then the lottery which has a higher probability of the deprived outcome should be ranked higher in terms of being identified as vulnerable compared to the other lottery. In other words, if we identify the lottery with lower probability of the deprived outcome as vulnerable, then so should the other lottery.

Axiom 1 *Probability Dominance (A1): Consider two lotteries* $L = (p_1, d_1; p_2, d_2), L' = (p'_1, d'_1; p'_2, d'_2) \in \mathcal{L}^I$ *such that* $d_1 = d'_1 = 0$ *(so* $d_2 > 0$ *and* $d'_2 > 0$*). Then:*

(i) $\rho(L') > \rho(L)$ *if* $p'_2 > p_2$*;*

(ii) $\rho(L') = \rho(L)$ *if* $p'_2 = p_2$*.*

[10]Vulnerability assessments typically collect data on the exposure to different hazards that people experience (see Erman et al. (2018), Gerlitz et al. (2014), Korboe (2011), World Bank (2007)) rather than the detailed income losses under those hazards.

Consider the following two lotteries faced by two different individuals, $L = (0.4, 0.8; 0.6, 0)$, and $L' = (0.7, 0.1; 0.3, 0)$, where each lottery has deprivation with positive probability. The axiom implies, L' should be identified as vulnerable relative to L, since the individual has a higher probability of falling into deprivation in lottery L'. Note that we are not taking into account the actual deprivation an individual may face. A plausible justification might be situations where policy-makers want to identify the vulnerable for targeting purposes and provide them the same level of support irrespective of their level of deprivation. This is the case for most social protection programmes such as unemployment benefits or disaster reliefs.

The intuition for the next axiom is quite straight forward. Consider two lotteries L and L' such that the probability of falling into poverty in the future is the same, then, both L and L' should be equally identified as vulnerable. On the other hand if the probability of falling in to poverty is higher in L', then L' should be ranked higher in terms of being identified as vulnerable compared to L. Before we state the axiom, we define the concept of probability transfer.

Definition 2 *Consider any $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}^I$ and any $d_{m+1} > 0$ such that $d_{m+1} \notin \{d_1, ..., d_m\}$. Let $L' \in \mathcal{L}^I$ be such that $L' = (p_1, d_1; ...; p_k - \delta, d_k; ...; \delta, d_{m+1})$ for some $\delta \in (0, p_k]$, where we let $L' = (p_1, d_1; ...; p_{k-1}, d_{k-1}; p_{k+1}, d_{k+1}; ...; p_m, d_{m+1})$ if $\delta = p_k$. Then we say that L' is derived through a probability transfer from L.*

Note that L' is derived through a probability transfer from L means that the probability of the additional deprivation outcome in L' has been transferred from one of the outcomes in L. Based on this definition we can now state the axiom formally.

Axiom 2 *Probability Transfer (A2): Consider $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}^I$. If $L' \in \mathcal{L}^I$ is a lottery derived through a probability transfer from L, then*

$$\rho(L') \begin{cases} > \rho(L) & \text{if } d_m = 0 \\ = \rho(L) & \text{if } d_m > 0 \end{cases}$$

Consider lotteries, $L = (0.3, 0; 0.7, 0.1)$ and $L' = (0.3, 0; 0.3, 0.1; 0.4, 0.5)$, where L' is derived from L through a probability transfer. Then according to the axiom of probability transfer, we should treat these two lotteries the same when it comes to the identification of being vulnerable or not. Note that, although in L' there is a new higher deprivation outcome (whose probability is transfered from another positive deprivation outcome), by considering the lotteries to be the same for identifying the vulnerable, this axiom ignores the distribution of the probabilities over the future positive deprivation outcomes.

Given axioms (A1) and (A2), we can establish the following result.

Proposition 1 *An identification rule ρ satisfies axioms of Probability Dominance (A1) and Probability Transfer (A2) if and only if $\rho = \rho^P$.*

Proof of Proposition 1: It can be verified that ρ^P satisfies axioms of Probability Dominance (A1) and Probability Transfer (A2). Suppose now that ρ satisfies axioms of Probability Dominance (A1) and Probability Transfer (A2). Pick any $L = (p_1, d_1; ...; p_m, d_m), L' = (p'_1, d'_1; ...; p'_n, d'_n) \in \mathcal{L}^I$. Then we only need to consider two possibilities: (i) $\mathbf{p}^L \cdot \mathbf{r}^L > \mathbf{p}^{L'} \cdot \mathbf{r}^{L'}$; (ii) $\mathbf{p}^L \cdot \mathbf{r}^L = \mathbf{p}^{L'} \cdot \mathbf{r}^{L'}$.

(i) $\underline{\mathbf{p}^L \cdot \mathbf{r}^L > \mathbf{p}^{L'} \cdot \mathbf{r}^{L'}}$: In this case there exists l such that $d'_l = 0$. Without loss of generality let $d'_1 = 0$. By applying (A2) repeatedly, $\rho(L') = \rho((p'_1, 0; (1 - p'_1), d'_2))$.

Suppose $d_s > 0$ for all s. By repeated application of (A2), $\rho(L) = \rho((1, d_1))$. Let $0 < d < d'_2$. Then (A2) implies that $\rho((1, d)) = \rho((p'_1, d; (1 - p'_1), d'_2)) > \rho((p'_1, 0; (1 - p'_1), d'_2))$. By (A2), we also have $\rho((1, d)) = \rho((1, d_1))$. Hence, $\rho(L) > \rho(L')$.

Suppose there exists k such that $d_k = 0$. Without loss of generality, let $d_1 = 0$. By applying (A2) repeatedly, we get $\rho(L) = \rho((p_1, 0; (1 - p_1), d_2))$. We also have $(1 - p_1) = \mathbf{p}^L \cdot \mathbf{r}^L > \mathbf{p}^{L'} \cdot \mathbf{r}^{L'} = (1 - p'_1)$. Then (A1) implies that $\rho((p_1, 0; (1 - p_1), d_2)) > \rho((p'_1, 0; (1 - p'_1), d'_2))$. Hence, $\rho(L) > \rho(L')$.

(ii) $\underline{\mathbf{p}^L \cdot \mathbf{r}^L = \mathbf{p}^{L'} \cdot \mathbf{r}^{L'}}$: We only need to consider $L \neq L'$.

Suppose $d_s > 0$ for all s and $d'_l > 0$ for all l. By applying (A2) repeatedly,we get $\rho(L) = \rho((1, d_1))$ and $\rho(L') = \rho((1, d'_1))$. Then $\rho(L) = \rho(L')$ if $d_1 = d'_1$. If $d_1 \neq d'_1$, then (A2) implies that $\rho((1, d_1)) = \rho((1, d'_1))$ and hence, $\rho(L) = \rho(L')$.

Suppose there exist i, j such that $d_i = d'_j = 0$. Without loss of generality let $d_1 = d'_1 = 0$. By repeated application of (A2), we have $\rho(L) = \rho((p_1, 0; (1 - p_1), d_2))$ and $\rho(L') = \rho((p'_1, 0; (1 - p'_1), d'_2))$. We also have $(1 - p_1) = \mathbf{p}^L \cdot \mathbf{r}^L = \mathbf{p}^{L'} \cdot \mathbf{r}^{L'} = (1 - p'_1)$. Then (A1) implies that $\rho((p_1, 0; (1 - p_1), d_2)) = \rho((p'_1, 0; (1 - p'_1), d'_2))$. Hence, $\rho(L) = \rho(L')$. ■

Note that the *P-rule* identification strategy uses only the information on the probability of the various adverse outcomes in the future. One can also consider other identification rules based on the information needed, such as including information about the maximum deprivation in the future that an individual will face along with the probability of future deprivations to identify whether the individual is vulnerable or not.

4 Characterising Individual's Vulnerability Measure

In this section we axiomatically characterise the popular FGT class of vulnerability measures. While discussing our axioms, we highlight where there are strong similarities with related literature in multidimensional poverty and choice under uncertainty.

In the context of multidimensional poverty, Alkire and Foster (2011), suggest a threshold based on the weighted number of dimensions to decide whether one is poor or not. In a similar spirit, we consider an identification rule ρ and a threshold θ which partitions the lottery space $\mathcal{L}$ into $\mathcal{L}^{\rho\theta}$ and $\mathcal{L} \setminus \mathcal{L}^{\rho\theta}$ such that, for any lottery L, $\rho(L) \geq \theta$ if $L \in \mathcal{L}^{\rho\theta}$ and $\rho(L) < \theta$ if $L \in \mathcal{L} \setminus \mathcal{L}^{\rho\theta}$. So, an individual with lottery L is identified as vulnerable if $L \in \mathcal{L}^{\rho\theta}$ and not vulnerable if $L \in \mathcal{L} \setminus \mathcal{L}^{\rho\theta}$. The cutoff value θ is exogenously determined by policy-makers, practitioners or analysts, based on what they think is appropriate for their context. In the empirical literature on vulnerability, $\theta = 0.5$ is often considered as the threshold value.

Let $\mathcal{L}_+$ be the set of lotteries whose supports contain only positive deprivations, i.e.

$$\mathcal{L}_+ = \{L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L} : \; d_s > 0 \text{ for all } s\}.$$

If an individual is going to suffer positive deprivations in all possible future outcomes, it seems reasonable to identify such an individual as vulnerable. In order to incorporate this idea in our characterisation of the individual vulnerability measure, we will require the vulnerability identification strategy to adopt a rule ρ and a threshold θ such that, an individual facing a lottery from $\mathcal{L}_+$ is always identified as vulnerable, i.e. $\mathcal{L}_+$ is a subset of $\mathcal{L}^{\rho\theta}$. Note that, this does not preclude an individual facing a lottery with zero deprivation in one of the outcomes from being identified as vulnerable.

In this section, we take for given that individuals are already identified as vulnerable or not, using identification rules such as ρ^E and threshold θ that we discussed in the previous section. We propose this identification aspect as the first axiom for characterising the vulnerability measure V.

Axiom 3 *Focus (A3):* $V(L) = 0$ *for all* $L \in \mathcal{L} \setminus \mathcal{L}^{\rho\theta}$.

The following axiom is similar to the monotonicity property under choice under uncertainty literature, where an increase in the probability of 'good' outcomes relative to 'bad' outcomes is preferable (Luce and Raiffa, 1957, p.29). In our context, it means vulnerability should increase when the probability associated with a high deprivation outcome increases relative to an outcome with lower deprivation. On the other hand, to reduce vulnerability we have to ensure that the individual becomes less likely to fall into high deprivation in the future.

Axiom 4 *Monotonicity (A4): Consider any* $L = (p_1, d_1; ...; p_k, d_k; p_l, d_l; ...; p_m, d_m) \in \mathcal{L}^{\rho\theta}$ *such that* $d_k > d_l$. *Let* $L' \in \mathcal{L}$ *be such that* $L' = (p_1, d_1; ...; p_k + \delta, d_k; p_l - \delta, d_l; ...; p_m, d_m)$ *for some* $\delta \in (0, p_l]$, *where we let* $L' = (p_1, d_1; ...; p_k + p_l, d_k; ...; p_m, d_m)$ *if* $\delta = p_l$. *Then* $V(L) < V(L')$ *if* $L' \in \mathcal{L}^{\rho\theta}$.

The above axiom also captures the spirit of the monotonicity property in multidimensional poverty literature (Alkire and Foster, 2011).

For the next axiom, we use the notion of convex combination of two lotteries which we define as follows:

Definition 3 *Consider* $L = (p_1, d_1; ...; p_m, d_m), L' = (p'_1, d'_1; ...; p'_n, d'_n) \in \mathcal{L}$. *Let* $D = \{d_1, ..., d_m\}$ *and* $D' = \{d'_1, ..., d'_n\}$. *For any* $\lambda \in (0, 1)$, $\lambda L + (1-\lambda)L'$ *is defined as follows:*

(a) Suppose $D \cap D' = \emptyset$. *Then* $\lambda L + (1 - \lambda)L' =$

$(\lambda p_1, d_1; ...; \lambda p_m, d_m; (1 - \lambda)p'_1, d'_1; ...; (1 - \lambda)p'_n, d'_n)$.

(b) Suppose $D \subseteq D'$. *W.l.o.g. let* $d_1 = d'_1, ..., d_m = d'_m$. *Then* $\lambda L + (1 - \lambda)L' =$

$(\lambda p_1 + (1 - \lambda)p'_1, d_1; ...; \lambda p_m + (1 - \lambda)p'_m, d_m; (1 - \lambda)p'_{m+1}, d'_{m+1}; ...; (1 - \lambda)p'_n, d'_n)$.

(c) Suppose $D' \subseteq D$. *W.l.o.g. let* $d_1 = d'_1, ..., d_n = d'_n$. *Then* $\lambda L + (1 - \lambda)L' =$

$(\lambda p_1 + (1 - \lambda)p'_1, d_1; ...; \lambda p_n + (1 - \lambda)p'_n, d_n; \lambda p_{n+1}, d_{n+1}; ...; \lambda p_m, d_m)$.

(d) Suppose $D \cap D' \neq \emptyset$, $D \nsubseteq D'$ *and* $D' \nsubseteq D$. *W.l.o.g. let* $d_1 = d'_1, ..., d_l = d'_l$ *and* $D \cap D' = \{d_1, ..., d_l\}$. *Then* $\lambda L + (1 - \lambda)L' =$

$(\lambda p_1 + (1 - \lambda)p'_1, d_1; ...; \lambda p_l + (1 - \lambda)p'_l, d_l; \lambda p_{l+1}, d_{l+1}; ...; \lambda p_m, d_m;$

$(1 - \lambda)p'_{l+1}, d'_{l+1}; ...; (1 - \lambda)p'_n, d'_n)$.

Thus, for each deprivation d_s belonging to the support of lottery L but not L', the probability associated with this deprivation in the convex combination will be λp_s. Similarly, for each deprivation d'_k belonging to the support of L' but not L, the probability associated with this deprivation in the convex combination will be $(1-\lambda)p'_k$. On the otherhand, for each deprivation d common to the support of L and L', the probability associated with this deprivation in the convex combination will be $\lambda p_i + (1-\lambda)p'_j$, where $d_i = d'_j = d$.

The next axiom is similar to the axiom of decomposability in Dutta, Foster and Mishra (2011) in the context of measuring vulnerability. The intuition is that any additional uncertainties that are not captured through the lotteries do not impact vulnerability. Suppose depending upon the policies government undertakes, one ends up with lottery L with probability λ or lottery L' with probability $(1-\lambda)$. One could claim that this extra layer of uncertainty arising out of government policy can lead to more vulnerability than the expected value of vulnerabilities arising from the two lotteries. The following axiom rules out such possibilities. This axiom is similar to the axiom of 'reduction of compound lotteries' in the choice under uncertainty literature (Luce and Raiffa, 1957, p.28). Hence, the vulnerability of a convex combination of lotteries should be the same as the convex combination of the vulnerability of each of the lotteries.[11]

Axiom 5 *Decomposability (A5): Consider any two lotteries* $L, L' \in \mathcal{L}^{\rho\theta}$ *and any* $\lambda \in (0,1)$. *Then* $V(\lambda L + (1-\lambda)L') = \lambda V(L) + (1-\lambda)V(L')$ *if* $\lambda L + (1-\lambda)L' \in \mathcal{L}^{\rho\theta}$.

As in the choice under uncertainty literature, the implication of this axiom would be to make the vulnerability measure linear in probabilities. Thus, it will allow us to generate the von Nueman-Morgenstern expected utility structure for the vulnerability measure.

Next we consider an axiom that addresses the issue of change in the gap between the vulnerability measures of two situations when all the deprivations in both situations are scaled up or down. Consider two situations represented by the lotteries L and L'. Suppose all deprivations in the two lotteries are reduced by the same proportion. This reduction in deprivation can happen due to better provision of insurance or social safety options to protect against adverse outcomes. Then the size of the change in the gap between the vulnerability measures of the two situations should depend only on the initial gap

[11] This axiom can be derived from more fundamental axioms (Gilboa, 2009, Chapter 8).

in the vulnerability measures and the proportionality factor by which the deprivations have been changed. To formulate this axiom, for each lottery $L = (p_1, d_1; ..; p_m, d_m) \in \mathcal{L}$, we denote the proportionate change in deprivations as $L_\lambda = (p_1, \lambda d_1; ..; p_m, \lambda d_m)$ for any $\lambda > 0$. The formal statement of the axiom is as follows:

Axiom 6 *Scale Invariance (A6): Consider any $L, L', \tilde{L}, \tilde{L}' \in \mathcal{L}^{\rho\theta}$. Suppose $V(L') - V(L) = V(\tilde{L}') - V(\tilde{L})$. Then $V(L'_\lambda) - V(L_\lambda) = V(\tilde{L}'_\lambda) - V(\tilde{L}_\lambda)$ for every $\lambda > 0$ such that $L_\lambda, L'_\lambda, \tilde{L}_\lambda, \tilde{L}'_\lambda \in \mathcal{L}^{\rho\theta}$.*

Note that, while the level of vulnerability might be reduced with better insurance or social safety net, the change in vulnerability would be the same in both situations. In other words, better social safety nets will not lead to differential impacts in terms of vulnerability faced by individuals.

The next axiom consists of two technical conditions about the continuity of the vulnerability measure on $\mathcal{L}^{\rho\theta}$ with respect to the deprivations. The first condition requires the vulnerability measure of degenerate lotteries $(1, d)$, where $0 < d \leq 1$, to be continuous. To formally state the second condition, for each $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}$ with $d_s > 0$ for all s, each $k \in \{1, ..., m\}$ and each $x \in [0, 1]$ with $x \neq d_s$ for all $s \neq k$, we will denote by L_{kx} the lottery that is obtained from L by replacing d_k with x. Then the condition requires the vulnerability measure of L_{kx} to be continuous at $x = 0$ if $L_{k0} \in \mathcal{L}^{\rho\theta}$.

Axiom 7 *Continuity (A7): The individual vulnerability measure V satisfies the following two conditions:*

(i) $V((1, d))$ is continuous on $(0, 1]$;

(ii) For each lottery $L = (p_1, d_1; ...; ...; p_m, d_m) \in \mathcal{L}_+$, and each $k \in \{1, ..., m\}$, $V(L_{k0}) = \lim_{x \to 0^+} V(L_{kx})$ if $L_{k0} \in \mathcal{L}^{\rho\theta}$ and $\lim_{x \to 0^+} V(L_{kx})$ exists.

Our final axiom is about the normalisation of the vulnerability measures of degenerate lotteries with positive deprivation.

Axiom 8 *Normalisation (A8): $V((1, 1)) = 1$ and $\lim_{d \to 0^+} V((1, d)) = 0$ if $\lim_{d \to 0^+} V((1, d))$ exists.*

The six axioms we have proposed in this section characterises the individual vulnerability measure based on the *FGT* class of vulnerability measures as stated in Proposition 2 below.

Proposition 2 *Suppose the vulnerability identification rule ρ and threshold θ are such that $\mathcal{L}_+ \subseteq \mathcal{L}^{\rho\theta}$. Then the individual vulnerability measure V satisfies axioms of Focus (A3), Monotonicity (A4), Decomposability (A5), Scale Invariance (A6), Continuity (A7) and Normalisation (A8) if and only if, for each lottery $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}$, $V(L)$ is of the following form:*

$$V(L) = \begin{cases} \sum_{s=1}^{m} p_s (d_s)^\alpha & \text{if } L \in \mathcal{L}^{\rho\theta} \\ 0 & \text{if } L \in \mathcal{L} \setminus \mathcal{L}^{\rho\theta} \end{cases}$$

where α is some positive real number.

Proof of Proposition 2: See Appendix A

When $\alpha < 1$, for a given probability distribution, the increase in deprivation by $0 < \delta < 1$ amount, increases overall vulnerability by less than δ units. The opposite would be true for $\alpha > 1$. Note that the vulnerability measure in Proposition 2 can also be interpreted as an expected poverty measure. The individual poverty measure is $f(d) = d^\alpha$, which has an Arrow-Pratt constant relative risk aversion value of $-(\alpha - 1)$. Thus, $\alpha > 1$ is the counterpart of a risk averse individual in the choice under uncertainty framework. Under risk aversion getting the expected values is better than facing the lottery itself, and here facing the lottery would make the individual more vulnerable than facing the expected value of the lottery with certainty.

5 Discussion: Shocks and Vulnerability

In this section we show that our measure of vulnerability can be linked to the adverse shocks individuals face. While we have formulated our vulnerability measure interms of the von-Neuman-Morgenstern framework, we can interpret the lotteries as arising out of different shocks, i.e. the deprivations and probabilities associated with them can be viewed respectively as the outcomes of the shocks and probabilities of those shocks.

For example, consider the following lottery $L_3 = (0.5, 0.3; 0.2, 0.4; 0.3, 0)$ faced by an individual. Each of the positive deprivations and their associated probabilities in lottery L_3 can be thought of as arising from adverse shocks and the probabilities of the occurrence of those shocks. Thus, the deprivation of 0.3 and the associated probability of 0.5 in

L_3 could be thought of a result of health shocks which can occur with probability 0.5. From Proposition 2 we know that for any individual i, faced with lottery L^i_m, if they are identified as vulnerable, the vulnerability level is given by $V(L^i_m) = \sum_{s=1}^{m} p_s(d^i_s)^\alpha$. Hence, for our lottery, if the individual is identified as vulnerable, then the level of vulnerability when $\alpha = 1$ is $V(L_3) = 0.5 \times 0.3 + 0.2 \times 0.4$, which is composed of the deprivation arising from the different shocks and their associated probabilities.

Under our framework, the deprivation associated with each shock need not be distinct. Suppose, for example, an individual faces health shocks, law and order shocks and natural shocks with the following probabilities of occurrence $(0.5, 0.1, 0.4)$ and associated deprivations of $(0.3, 0.3, 0.8)$. Note that for both health shocks and law and order shocks, the future deprivation is same, although they have different probabilites of occurence.This can be represented as lottery $L_4 = (0.6, 0.3; 0.4, 0.8)$. The vulnerability of lottery L_4, assuming the individual is identified as vulnerable, and $\alpha = 1$, is $V(L_4) = 0.6 \times 0.3 + 0.4 \times 0.8$, which is equivalent to a representation interms of the shocks where $V(L_4) = 0.5 \times 0.3 + 0.1 \times 0.3 + 0.4 \times 0.8 = \sum_{s=1}^{\pi} p_s d_s$, where π is the number of shocks that individual faces under lottery L_4. Thus at the individual level, vulnerability can be decomposed in terms of shocks. Note that the number of shocks captured in a lottery can be different from the number of future deprivations the lottery represents.

Since societal vulnerability is a simple average of individual vulnerabilities, we can decompose the societal vulnerability into different shocks. Given that for any individual i not identified as vulnerable $V(L^i_m) = 0$, societal vulnerability can then be represented as,

$$V^S = \frac{1}{N}\sum_{i=1}^{q}\sum_{s=1}^{\pi^i} p_s^{L^i_m}(d_s^{L^i_m})^\alpha, \tag{3}$$

where, N is the total population, q is the number of individuals identified as vulnerable, and π^i is the number of shocks that the individual i with lottery L^i_m faces. We can rewrite equation (3) as,

$$V^S = \sum_{sh=1}^{T} V^{sh}, \tag{4}$$

where $T = \max\{\pi^i : i = 1, .., N\}$, and

$$V^{sh} = (1/N) \sum_{i=1}^{q} p_s^{L^i_m} (d_s^{L^i_m})^\alpha \tag{5}$$

is the vulnerability associated with each shock at the societal level. This decomposition is particularly useful if we are interested in understanding how different shocks contribute to societal vulnerability. Note that within (4), there are two layers of decomposability. First, the axiom of Decomposability (A4) allows us to decompose the individual vulnerability into vulnerabilities arising from each of the different shocks, which essentially is the expected deprivation from the various adverse shocks. The second decomposability implicit in (2) allows us to separate the societal vulnerability in terms of individual vulnerabilities. These two layers of decomposability, thus, imply that we can separate overall societal vulnerability into vulnerabilities arising from each different shock for each individual. Failure of either of these two decomposability properties will mean that societal vulnerability cannot be represented as in (4).

The decomposability properties both at the individual and societal level, also allow us to decompose vulnerability interms of incidence and intensity as has been done by Alkire and Foster (2011) in the context of multidimensional poverty. Suppose $\alpha > 0$ and q is the number of individuals identified as vulnerable. Using individual vulnerability measure as given in Proposition 2, we can write the societal vulnerability as

$$V^S = \frac{q}{N} . \frac{\sum_{i=1}^{q} \sum_{s=1}^{\pi^i} p_s^{L^i_m} (d_s^{L^i_m})^\alpha}{q} = H.I_\alpha \tag{6}$$

where H is the head-count ratio and I_α is the intensity of vulnerability since it captures the average expected poverty of those who are identified as vulnerable. In this sense, the our vulnerability measure mirrors the adjusted head-count ratio proposed by Alkire and Foster (2011) in the context of multidimensional poverty. We can easily expand this analysis for $\alpha = 0$. Similar to 6 societal vulnerability can be written as

$$V^S = \frac{1}{N} \sum_{i=1}^{q} \sum_{s=1}^{\pi^i} p_s^{L^i_m}. \tag{7}$$

Equation (7) implies

$$V^S = \frac{q}{N} . \frac{\sum_{i=1}^{q} \sum_{s=1}^{\pi^i} p_s^{L_m^i}}{q} = H.I_0 \tag{8}$$

where H is the head-count ratio of the vulnerable, and I_0 is the average probability with which those identified as vulnerable will fall into poverty in the future and thus reflects the intensity of the vulnerability.

Finally, in our discussion we map various shocks into future probability of occurrence and deprivation, which can include correlated shocks. For instance, suppose future poverty arises out of the following : Income shocks; Natural shocks; and Health shocks or a combination of these shocks. The vulnerability framework proposed here can accommodate these specific and correlated shocks and would map them into the probability and deprivation associated with the full set of shocks which are:Income shocks; Natural shocks; Health shocks; Income and Natural shocks; Income and Health shocks; Natural and Health shocks; and Income, Natural, and Health shocks. For empirical applications, however, we may consider just a few shocks, which may not include the full set of correlated shocks.

6 Empirical Illustration

In this section, we apply the vulnerability assessment framework developed in this paper to real-world data. By the very nature of this exercise, this is not a full-scale empirical application. The intention is to demonstrate how the *P-rule* identification criteria along with the FGT vulnerability measure can be used in the assessment of vulnerability.

Data: Our data comes from the Hrishipada Daily Diary project in Bangladesh where main income earners of 70 households kept a record of the daily income and expenditures, from May 2015 to December 2018, resulting in over 567,000 data points.[12] In assessing vulnerability, we define income as any earnings during the sample period, which includes wages from jobs and profits from businesses, loans taken, savings withdrawn, and gifts and transfers received. For the self-employed, we do subtract from their income, the legitimate expenses that one has to incur to keep their business running. To smoothen

[12]Rutherford (2017, 2016) provides a detailed description of the data. More information on the financial diary project and data can be found at https://sites.google.com/site/hrishiparadailydiaries/home. We have dropped one household due to very few observations and another household due to extremely high income.

out the noise in daily income, we aggregate the daily information into weekly income. Per capita income is defined as household income divided by household size. Table A1 (Appendix A), provides some basic information on the data. There is considerable variation in terms of the number of observations that we have for each household. For our empirical illustration, we focus on 2017 and 2018 since they have the highest number of households.

Empirical Strategy: To capture negative income shocks that individuals face each year, we consider three possible future outcomes resulting from shocks of varying intensity based on Ferreira and Sanchez-Paramo (2017) poverty lines. These outcomes are deprivations due to (i) extreme shocks (income less than \$1.90 per day), (ii) moderate shocks (income between \$1.90 and \$3.20 per day), (iii) mild shocks (income between \$3.20 and \$5.50 per day).[13] Any income above \$5.50 per day, is not deprived.

Synthetic Distribution method: To capture the probability of different shocks and the associated losses we assume that weekly income of individual i, y_t^i, follows a log normal distribution, thus $\ln y_t^i \sim N(\mu_i, \sigma_i^2)$. Using regression methods one can then estimate the mean and variance of the income of the individual. Since our data does not have information on household characteristics, we estimate just a trend regression

$$\ln y_t^i = \alpha + \beta X_t^i + \varepsilon_i \tag{9}$$

where X_t^i represents the number of weeks till income y_t^i was earned.[14] For each individual we then generate a synthetic income distribution of over 550,000 data points from on a normal distribution of mean $\mu_i = E(\ln y_t^i \mid X_t^i)$ and variance $\sigma_i^2 = E(\varepsilon_i^2 \mid X_t^i)$.[15] The probability for individual i of experiencing shock s is $p_s^i = n_s^i / n_T^i$, where n_s^i is the number of income that comes under shock s, and n_T^i is the total number of observations. Individual i can then be identified as vulnerable using the *P-rule*, if $\rho^E = \sum_{s=1}^{m} p_s^i$ is above a certain threshold θ which takes the values 0, 0.25, 0.5, 0.75 and 1, as alternative

[13] Due to informational constraints, we have framed the different states in this nested manner. However, our theoretical model provides a general framework where there is no need for such restrictions. For instance, if we had information on health shocks, or unemployment shocks, or natural shocks (flood or drought), then there is no reason why these different states should be nested in the way that we see in our application.

[14] For each household, we take into account all available information till that point, including from previous years. While we use simple regression to demonstrate our method, one could use a more sophisticated estimation of mean and variance based on data availability.

[15] In our case, for each household we have drawn 567,216 incomes from a normal distribution with mean and variance specific to the household. This is the same number as the total observations we have.

thresholds in identifying the vulnerable. We also compare our results with the *standard approach*, where individual vulnerability is calibrated first and then those individuals whose vulnerability level is above 0.5 is identified as vulnerable

Next, the level of vulnerability of those who are identified as vulnerable is computed from the synthetic distribution. The average income under shock s is $\overline{y}_s^i = \sum_{t=1}^{n_s^i} y_{st}^i / n_s^i$, where y_{st}^i is a generated income under shock s, and n_s^i is the number of such incomes. To compute individual vulnerability arising from shocks, we use the FGT vulnerability measure which is given as follows

$$V^i = \sum_{s=1}^{\pi^i} p_s^i \left(\frac{z - \overline{y}_s^i}{z} \right)^{\alpha}. \tag{10}$$

In all our calibrations of vulnerability we consider $z = 1397.69$ taka.[16]

In assessing societal vulnerability, under the assumption that each member of a household has the same level of vulnerability, we multiply the individual vulnerability with the household size to get the total vulnerability of that household. The average of the vulnerability of all the individuals in society is the societal vulnerability.

Results: We undertake two calibration exercises. First, for the year 2017, we identify the vulnerable and measure the societal vulnerability for different levels of θ along with the *standard approach*. Second, for the years 2017 and 2018, we assess the contribution of the different shocks to societal vulnerability.

In our first exercise, individual vulnerability (10) is calibrated for $\alpha = 1$ and $\alpha = 2$. The results are presented in Table 1 below:

[16] We have taken 2011 as the base year and the PPP Exchange rate as US $ 1 = Taka 24.849. The deflators for 2017 used was 146.10 and for 2018 was 154.20. Hence, US $5.50 per person per day translated to 1397.69 Bangladeshi taka per person per week.

Table 1: Levels of Vulnerability in Hrishipada, Bangladesh: 2017

	Vulnerable Individuals	Head-count	Vulnerability (FGT, $\alpha = 1$)			Vulnerability (FGT, $\alpha = 2$)		
			Societal	Intensity	Median	Societal	Intensity	Median
$\theta = 0$	284	0.969	0.417	0.431	0.425	0.253	0.261	0.191
$\theta = 0.25$	245	0.836	0.416	0.498	0.425	0.252	0.302	0.191
$\theta = 0.50$	232	0.792	0.413	0.522	0.425	0.252	0.318	0.191
$\theta = 0.75$	228	0.778	0.412	0.530	0.425	0.251	0.323	0.191
$\theta = 1$	115	0.392	0.258	0.657	0.000	0.177	0.452	0.000
Standard Approach	127	0.433	0.295	0.680	0.000	0.206	0.475	0.000

As expected, the number of individuals who are identified as vulnerable, and the resulting headcount ratio reduces as θ increases. By making the identifying criteria for vulnerability more stringent through a higher threshold, fewer people are identified as vulnerable. Note that the number of vulnerable individuals and the headcount ratio for each threshold is the same under $\alpha = 1$ and $\alpha = 2$. This is because in our empirical illustration, the FGT vulnerability measure does not have any role to play in the identification stage. Its role is to assess an individual's level of vulnerability once the individual has been identified as vulnerable.

Based on equation (6), for both $\alpha = 1$ and $\alpha = 2$, we decompose societal vulnerability into the head-count ratio and the intensity of vulnerability. What is evident is that these two aspects work in the opposite direction. Thus, as θ increases, the headcount ratio decreases, whereas the intensity of vulnerability increases. With increasing θ we only identify individuals as vulnerable whose probability of being deprived in the future is very high. For instance, under $\alpha = 2$, as we move from $\theta = 0$ to $\theta = 0.25$, there is hardly any change in the societal vulnerability despite a 15 percentage points decline in the headcount ratio because the intensity of vulnerability had also increased. Beyond $\theta = 0.75$, we find that the reduction in the head-count dominates the increase in the intensity resulting in big changes to societal vulnerability.

We find large differences between the *standard approach* and results based on our framework. For $\alpha = 1$, under the *standard approach*, societal vulnerability is 0.30 and around 43 percent of the population is identified vulnerable, which is in sharp contrast to the method proposed in our paper, where based on a $\theta = 0.5$ threshold value, the head-count is 79 percent and societal vulnerability is 0.41. The *standard approach* thus significantly underestimates both the incidence and the level of vulnerability. The story is similar for $\alpha = 2$ too. We also find a difference between the *standard approach* and our proposed method for the vulnerability of the median individual, particularly for lower values of θ.

Another advantage of our framework is that we can decompose the overall vulnerability to investigate which shocks are contributing most to the societal vulnerability. Equation (4) shows how societal vulnerability V^S can be decomposed as the sum of the vulnerabilities arising from different shocks. The relative contribution of a shock m to societal vulnerability is V^m/V^S, where V^m is based on (5). Our second calibration exercise presents the relative contribution (in percentage) of extreme shocks, moderate shocks and mild shocks to societal vulnerability for 2017 and 2018 in the table below for different values of θ and $\alpha = 1$ for the FGT measure.

Table 2: Decomposition of Vulnerability based on Shocks: 2017-2018

	2017			2018		
	Extreme Shocks	Moderate Shocks	Mild Shocks	Extreme Shocks	Moderate Shocks	Mild Shocks
$\theta = 0$	47.50	36.53	15.97	41.39	42.86	15.75
$\theta = 0.25$	47.65	36.64	15.71	41.45	42.91	15.64
$\theta = 0.50$	47.95	36.72	15.33	41.57	43.02	15.41
$\theta = 0.75$	48.07	36.80	15.13	41.67	43.13	15.20
$\theta = 1$	66.26	30.09	3.66	48.59	46.57	4.84

Table 2 displays some interesting patterns. In 2017, for all values of θ, over 47 percent of the vulnerability is arising from extreme shocks, followed by another 36 percent from moderate shocks and around 16 percent from mild shocks. This strong contribution of the extreme shocks to vulnerability implies that most people in this sample are likely to fall into extreme poverty in the future. Further, comparing 2017 and 2018, we see that 2018 is better since, for all the threshold except $\theta = 1$, less of the vulnerability is emanating from extreme shocks. Instead, the majority of the vulnerability in 2018 is coming from moderate shocks reflecting the fact that per capita income and median income in 2018 are higher compared to 2017. In a more general setting, where vulnerability arises from various shocks such as health, economic or natural shocks, this ability to decompose vulnerability in terms of the contribution of the shocks, can be extremely useful from a policy perspective.

7 Conclusion

The main innovation of the paper is in bringing a clear identification part to the measurement of vulnerability within a unified framework of measuring individual and societal vulnerability. We have proposed a probability based identification criteria, which can be easily applied. The analytical framework is based on mapping the different outcomes an individual faces in the future to the shocks. In a broader sense, our framework could be considered to be closely linked to the *counting approach* in the literature on multidimensional poverty since the analytical structure of measurement of vulnerability and multidimensional poverty is very similar with the former based on different adverse shocks that an individual faces and the latter based on the different dimensions of deprivation an individual is evaluated on.

The computation of vulnerability of those identified as vulnerable is an integral part of our proposed framework to measure vulnerability. For that purpose, we have axiomatically characterised the FGT vulnerability measure which is widely used in empirical applications in this area. Our characterisation, thus, provides the axiomatic foundations for many of the empirical work on vulnerability. As in the *standard* empirical literature on vulnerability, in our proposed framework the FGT vulnerability measure can also be viewed as an identification rule which uses a richer set of information to identify the vulnerable. In addition, we demonstrate that societal vulnerability can be decomposed into vulnerabilities arising from different shocks. This should help policy-makers to understand the sources of vulnerability in their society and prioritise policy accordingly.

Through a real-world application using data from Bangladesh, we demonstrate how the *P-rule* identification strategy along with the FGT measure can be used to identify the vulnerable and measure societal vulnerability. By considering identification to be a distinct part of the measurement of vulnerability we obtain very different results compared to the *standard approach* in calibrating vulnerability. A realistic approach towards identifying the vulnerable and measuring their vulnerability has to keep in mind the substantial informational challenges when estimating future shocks. This paper is a step towards addressing these challenges.

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A Appendix

Before we proceed with the proof of Proposition 2, we present a lemma which will be used in the proof of the proposition.

Lemma: *Suppose the vulnerability identification rule ρ and threshold θ are such that $\mathcal{L}_+ \subseteq \mathcal{L}^{\rho\theta}$, and the individual vulnerability measure V satisfies axioms of Focus (A3), Monotonicity (A4), Decomposability (A5), Continuity (A7) and Normalisation (A8). Then there exists a continuous and increasing function $f : [0,1] \to [0,1]$ such that $f(1) = 1$, $f(0) = 0$ and*

$$V(L) = \sum_{s=1}^{m} p_s f(d_s) \text{ for each } L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}^{\rho\theta}. \tag{Eq-A1}$$

Proof of Lemma: For each $m \geq 1$, let $\mathcal{L}_{+m}$ be the set of lotteries in $\mathcal{L}_+$ with exactly m different deprivations. Also, for each $m \geq 2$, given any $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}_{+m}$, let $L_{m-1} = (p_1/(1-p_m), d_1; ...; p_{m-1}/(1-p_m), d_{m-1}) \in \mathcal{L}_{+m-1}$. Then we have $(1-p_m)L_{m-1} + p_m(1, d_m) = L$.

Since $(1, d) \in \mathcal{L}_{+1}$ for each $d \in (0, 1]$ and we know from (A7) that $\lim_{d \to 0^+} V((1, d))$ exists, for each $d \in [0, 1]$, define $f(d)$ as follows:

$$f(d) = \begin{cases} V((1,d)) & \text{if } d \neq 0 \\ \lim_{x \to 0^+} V((1,x)) & \text{if } d = 0 \end{cases}$$

Then (A7) and (A8) imply that f is continuous on $[0, 1]$, $f(1) = 1$ and $f(0) = 0$.

We will now prove the following:

$$V(L) = \sum_{s=1}^{m} p_s f(d_s) \text{ for all } L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}_+, \tag{Eq-A2}$$

which is equivalent to showing that

$$V(L) = \sum_{s=1}^{m} p_s f(d_s) \text{ for each } m \geq 1 \text{ and each } L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}_{+m}. \tag{Eq-A3}$$

For any $L = (1, d) \in \mathcal{L}_{+1}$, we have

$$V(L) = V((1, d)) = f(d).$$

So (Eq-A3) holds for $m = 1$. For each $L = (p_1, d_1; p_2, d_2) \in \mathcal{L}_{+2}$, we have

$$\begin{aligned} V(L) &= V((1-p_2)L_1 + p_2(1, d_2)) \\ &= (1-p_2)V(L_1) + p_2 V((1, d_2)) \quad \text{[by (A5)]} \\ &= p_1 f(d_1) + p_2 f(d_2). \end{aligned}$$

So (Eq-A3) also holds for $m = 2$. We will now complete the proof of (Eq-A3) for any $m \geq 1$ by induction. Consider any $m \geq 3$ and assume that (Eq-A3) holds for $m-1$. Then it suffices to show that (Eq-A3) holds for m. Pick any $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}_{+m}$. Then we have

$$\begin{aligned} V(L) &= V((1-p_m)L_{m-1} + p_m(1, d_m)) \\ &= (1-p_m)V(L_{m-1}) + p_m V((1, d_m)) \quad \text{[by (A5)]} \\ &= (1-p_m)\left[\left(\frac{p_1}{1-p_m}\right) f(d_1) + ... + \left(\frac{p_{m-1}}{1-p_m}\right) f(d_{m-1})\right] + p_m f(d_m) \\ &= \sum_{s=1}^{m} p_s f(d_s). \end{aligned}$$

We will next show that f is increasing on $[0, 1]$. Consider any $L = (p_1, d_1; ...; p_k, d_k; p_l, d_l; ...; p_m, d_m) \in \mathcal{L}_+$ such that $d_k > d_l$. Let $L' \in \mathcal{L}_+$ be such that $L' = (p_1, d_1; ...; p_k + \delta, d_k; p_l - \delta, d_l; ...; p_m, d_m)$ for some $\delta \in (0, p_l)$. Then it follows from (A4) that $V(L) < V(L')$, which together with (Eq-A2) imply

$$\begin{aligned} p_k f(d_k) + p_l f(d_l) &< (p_k + \delta) f(d_k) + (p_l - \delta) f(d_l) \\ \Longrightarrow \quad f(d_l) &< f(d_k). \end{aligned} \tag{Eq-A4}$$

Since the lottery L was picked arbitrarily, d_k and d_l can be any deprivations from $(0, 1]$ such that $d_k > d_l$. Hence, we can conclude from (Eq-A4) that, for all $d, d' \in (0, 1]$ with $d > d'$, $f(d) > f(d')$. It then follows from $\lim_{d \to 0^+} f(d) = f(0)$ that f is increasing on $[0, 1]$.

For each $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}_+$ and each $k \in \{1, ..., m\}$, (Eq-A2) implies that

$\lim_{x \to 0^+} V(L_{kx})$ exists and

$$\lim_{x \to 0^+} V(L_{kx}) = \sum_{s \in \{1,...,m\} \setminus \{k\}} p_s f(d_s) + p_k f(0).$$

Hence, it follows from (A7) that, for each lottery $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}^{\rho\theta} \setminus \mathcal{L}_+$, $V(L) = \sum_{s=1}^{m} p_s f(d_s)$. This completes the proof of the Lemma. ∎

Proof of Proposition 2: It can be checked that, if V has the form given in the proposition, then V satisfies axioms of Focus (A3), Monotonicity (A4), Decomposability (A5), Scale Invariance (A6), Continuity (A7) and Normalisation (A8). Suppose now that V satisfies axioms of Focus (A3), Monotonicity (A4), Decomposability (A5), Scale Invariance (A6), Continuity (A7) and Normalisation (A8). Then we know from the Lemma that V satisfies (Eq-A1), with f as defined in the proof of the Lemma, such that f is continuous and increasing on $[0, 1]$, $f(1) = 1$ and $f(0) = 0$.[17]

Since $(1, d) \in \mathcal{L}^{\rho\theta}$ for each $d \in (0, 1]$, it follows from (A6) that, for all $a, b, \hat{a}, \hat{b} \in (0, 1]$ and any $\lambda > 0$ such that $\lambda a, \lambda b, \lambda\hat{a}, \lambda\hat{b} \in (0, 1]$, f satisfies the following:

$$\left[f(a) - f(b) = f(\hat{a}) - f(\hat{b}) \right] \implies \left[f(\lambda a) - f(\lambda b) = f(\lambda\hat{a}) - f(\lambda\hat{b}) \right]. \qquad \text{(Eq-A5)}$$

Equation (Eq-A5) implies that there is some function H such that, for all $\lambda > 0$,

$$f(\lambda a) - f(\lambda b) = H(f(a) - f(b), \lambda) \text{ whenever } a, b, \lambda a, \lambda b \in (0, 1]. \qquad \text{(Eq-A6)}$$

Since f is continuous and increasing on $[0, 1]$, $f(1) = 1$ and $f(0) = 0$, it can be checked that the function H is continuous in the first argument and

$$\{u : \ u = f(a) - f(b) \text{ for some } a, b \in (0, 1]\} = (-1, 1).$$

Let $u, v \in (-1, 1)$ be such that $u + v \in (-1, 1)$. We will show that there exist $a, b, c \in (0, 1]$ such that $u = f(a) - f(b)$, $v = f(b) - f(c)$ and $u + v = f(a) - f(c)$ by considering four exhaustive possibilities.

(i) Suppose $u, v \geq 0$. Since $u + v \in (-1, 1)$, there exist $a, c \in (0, 1]$ such that

[17] It is worth pointing out that, without (A7), the remaining axioms would still imply that $V((1, d))$ is increasing on $(0, 1]$, and hence continuous almost everywhere on $(0, 1]$.

$f(a) - f(c) = u + v \geq 0$. Then $f(a) - f(c) \geq u$ and $f(a) - f(c) \geq v$. So there exist $b, b' \in [c, a]$ such that $f(a) - f(b) = u$ and $f(b') - f(c) = v$ because f is continuous and increasing. Then $f(a) - f(b) + f(b') - f(c) = u + v$, which implies $f(b) = f(b')$. Hence, $b = b'$ because f is increasing.

(ii) Suppose $u \geq 0 \geq v$ and $u + v \geq 0$ (the $v \geq 0 \geq u$ and $u + v \geq 0$ case can be treated similarly). Clearly, $-v \geq 0$ and $-v \in (-1, 1)$. Also, $-v + (u + v) = u \in (-1, 1)$ and $-v + (u + v) \geq 0$. Then we know from (i) that there exist $a, b, c \in (0, 1]$ such that $-v = f(c) - f(b)$, $u + v = f(a) - f(c)$ and $u = -v + (u + v) = f(a) - f(b)$. Hence, we get $v = f(b) - f(c)$, $u = f(a) - f(b)$ and $u + v = f(a) - f(c)$.

(iii) Suppose $u \geq 0 \geq v$ and $u+v \leq 0$ (the $v \geq 0 \geq u$ and $u+v \leq 0$ case can be treated similarly). Clearly, $-v \geq 0 \geq -u$, $-(u+v) \geq 0$ and $-u, -v, -(u+v) \in (-1, 1)$. Then we know from (ii) that there exist $a, b, c \in (0, 1]$ such that $-v = f(c)-f(b)$, $-u = f(b)-f(a)$ and $-(u + v) = f(c) - f(a)$. Hence, we get $u = f(a) - f(b)$, $v = f(b) - f(c)$ and $u + v = f(a) - f(c)$.

(iv) Suppose $u, v \leq 0$. Then $-u, -v, -(u + v) \geq 0$ and $-u, -v, -(u + v) \in (-1, 1)$. Then we know from (i) that there exist $a, b, c \in (0, 1]$ such that $-u = f(b) - f(a)$, $-v = f(c)-f(b)$ and $-(u+v) = f(c)-f(a)$. Hence, we get $u = f(a)-f(b)$, $v = f(b)-f(c)$ and $u + v = f(a) - f(c)$.

Consider any $u, v \in (-1, 1)$ such that $u + v \in (-1, 1)$, and let $a, b, c \in (0, 1]$ be such that $u = f(a) - f(b)$, $v = f(b) - f(c)$ and $u + v = f(a) - f(c)$. It follows from (Eq-A6) that, for all $\lambda > 0$ such that $\lambda a, \lambda b, \lambda c \in (0, 1]$,

$$\begin{aligned} f(\lambda a) - f(\lambda b) &= H(u, \lambda); \\ f(\lambda b) - f(\lambda c) &= H(v, \lambda); \\ f(\lambda a) - f(\lambda c) &= H(u + v, \lambda). \end{aligned}$$

Thus, we get $H(u + v, \lambda) = H(u, \lambda) + H(v, \lambda)$. This is the Cauchy equation and its solution is given by $H(u, \lambda) = \phi(\lambda)u$ (see, for example, Corollary 8 in Section 1 of Aczel (1987)). Then equation (Eq-A6) can be rewritten as:

$$f(\lambda a) - f(\lambda b) = \phi(\lambda)[f(a) - f(b)] \text{ whenever } a, b, \lambda a, \lambda b \in (0, 1].$$

Note that $\phi(\lambda) \neq 0$ because f is increasing. Since f is continuous and

$\lim_{b\to 0^+} f(\lambda b) = \lim_{b\to 0^+} f(b) = f(0) = 0$, we get the following from the above equation:

$$f(\lambda a) = \phi(\lambda) f(a) \ \text{ whenever } a, \lambda a \in (0,1]. \tag{Eq-A7}$$

Given any $u > 0$, it is clear that $u = a/b$ for some $a, b \in (0,1]$. Define the function F by:

$$F(u) = \frac{f(a)}{f(b)} \ \text{ for all } u > 0, \text{ where } a, b \in (0,1] \text{ and } a/b = u.$$

Suppose $a, b, a', b' \in (0,1]$ are such that $a/b = a'/b'$. Let $\eta_a = a'/a > 0$ and $\eta_b = b'/b > 0$. Then we have $\eta_a a = a'$ and $\eta_b b = b'$. This implies that $\eta_a = \eta_b$ because $\eta_a a/\eta_b b = a'/b' = a/b$. Let $\eta = \eta_a = \eta_b$. Then we get $f(\eta a)/f(\eta b) = f(a')/f(b')$. From equation (Eq-A7) we also have

$$\frac{f(\eta a)}{f(\eta b)} = \frac{\phi(\eta) f(a)}{\phi(\eta) f(b)} = \frac{f(a)}{f(b)}.$$

Thus, we can conclude that $f(a)/f(b) = f(a')/f(b')$. This proves that F is well-defined. Furthermore, since f is continuous and increasing, it can be verified that F is also continuous and increasing.

Next, we will show that F satisfies the following:

$$[F(u) - F(v) = F(\hat{u}) - F(\hat{v})] \Longrightarrow [F(\lambda u) - F(\lambda v) = F(\lambda \hat{u}) - F(\lambda \hat{v})] \text{ for all } u, v, \hat{u}, \hat{v}, \lambda > 0. \tag{Eq-A8}$$

Suppose $u, v, \hat{u}, \hat{v}, \lambda > 0$ are such that $F(u) - F(v) = F(\hat{u}) - F(\hat{v})$. Let $\eta \in (0,1]$ be such that $\eta w, \eta\lambda w \in (0,1]$ for each $w \in \{u, v, \hat{u}, \hat{v}\}$. For each $w \in \{u, v, \hat{u}, \hat{v}\}$, we then have the following:

$$\begin{aligned} F(w) = F\left(\frac{\eta w}{\eta}\right) &= \frac{f(\eta w)}{f(\eta)}; \\ F(\lambda w) = F\left(\frac{\lambda \eta w}{\eta}\right) &= \frac{f(\lambda \eta w)}{f(\eta)}. \end{aligned}$$

So we get

$$\begin{aligned} \frac{f(\eta u) - f(\eta v)}{f(\eta)} &= \frac{f(\eta \hat{u}) - f(\eta \hat{v})}{f(\eta)} \\ \Longrightarrow \ f(\eta u) - f(\eta v) &= f(\eta \hat{u}) - f(\eta \hat{v}). \end{aligned}$$

It then follows from (Eq-A5) that

$$\begin{aligned} f(\lambda\eta u) - f(\lambda\eta v) &= f(\lambda\eta\hat{u}) - f(\lambda\eta\hat{v}) \\ \Longrightarrow \quad \frac{f(\lambda\eta u) - f(\lambda\eta v)}{f(\eta)} &= \frac{f(\lambda\eta\hat{u}) - f(\lambda\eta\hat{v})}{f(\eta)}. \end{aligned}$$

Hence, $F(\lambda u) - F(\lambda v) = F(\lambda\hat{u}) - F(\lambda\hat{v})$

We know from Theorem 1 in Section 2 of Aczel (1987) that the general solutions of any continuous and non-constant function F satisfying (Eq-A8) are given by either

$$F(u) = \beta \log u + \gamma \quad \text{for all } u > 0, \tag{Eq-A9}$$

where $\beta \neq 0$ and γ are constants, or

$$F(u) = \beta u^{\alpha} + \gamma \quad \text{for all } u > 0, \tag{Eq-A10}$$

where $\alpha \neq 0$, $\beta \neq 0$ and γ are constants.

From the definition of F it can be checked that $F(1) = 1$. Then we get $\gamma = 1$ and $\beta > 0$ in (Eq-A9). This implies that $F(u) < 0$ for u sufficiently close to 0 if F is given by (Eq-A9). However, it can be verified from the definition that $F(u) > 0$ for all $u > 0$. Hence, (Eq-A9) can be ruled out.

From the definition of F it can be verified that $\lim_{u \to 0^+} F(u) = 0$. Then we must have $\gamma = 0$ in (Eq-A10). This together with $F(1) = 1$ imply that $\beta = 1$ in (Eq-A10). Also, $\alpha > 0$ in (Eq-A10) because F is increasing. Thus, we get

$$f(d) = \frac{f(d)}{f(1)} = F(d) = d^{\alpha} \text{ for all } d \in (0, 1].$$

Then we have $f(d) = d^{\alpha}$ for all $d \in [0, 1]$ because $f(0) = 0 = 0^{\alpha}$.

Therefore, it follows from (A3) and equation (Eq-A1) that, for each lottery $L = (p_1, d_1; ...; p_m, d_m) \in \mathcal{L}$, $V(L)$ is of the following form:

$$V(L) = \begin{cases} \sum_{s=1}^{m} p_s (d_s)^{\alpha} & \text{if } L \in \mathcal{L}^{\rho\theta} \\ 0 & \text{if } L \in \mathcal{L} \setminus \mathcal{L}^{\rho\theta} \end{cases}$$

where α is some positive real number. ■

Empirics: We present below a Table providing descriptive information on the data we have used for our empirical illustration.

Table A1: Descriptive Statistics of Hrishipada Daily Diary Project, Bangladesh: 2015-2018

	2015	2016	2017	2018
Weekly Income per capita (BangladeshTaka)	1629.99	1703.27	2020.85	2532.73
Variance of Weekly Income per capita	5523.89	4202.40	7512.63	11234.10
Weekly Median Income (BangladeshTaka)	588.00	616.83	662.50	665.00
Average number of weeks per household	14.28	50.94	37.13	51.97
Number of Households	40	49	70	59
Average Household Size	4.25	4.23	4.19	4.15