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Abstract. The paper provides necessary and sufficient conditions for the uniqueness of pure-strategy Nash equilibrium in Bertrand duopoly with a homogeneous product. The main condition is elementary, easy to interpret, and nests all known sufficient conditions in the literature.

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Keywords: Price competition, uniqueness of Nash equilibrium, weakly dominated strategy.

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1 Introduction

Although written in the form of a book review, Bertrand's (1883) critique of Cournot's (1838) oligopoly turned out to form the most widely used model of price competition. Indeed, nowadays, the Bertrand duopoly model is one of the cornerstones of microeconomics and game theory. The "Bertrand paradox" usually refers to the paradoxical equilibrium outcome of perfect competition in a market with only two firms. The strategies whose implementation leads to this outcome prescribe to set the minimal prices (equal to the marginal costs), resulting in zero profits. The fact that such strategies form a Nash equilibrium is a simple observation, and actually does not require any assumptions on the model. However, uniqueness holds only under additional assumptions, and its proof requires some (of course quite elementary) arguments.[1]

To the best of our knowledge the literature provides no necessary and sufficient conditions for the uniqueness of a pure-strategy Bertrand-Nash equilibrium. In the present note, we formulate such conditions and derive from them some common sufficient ones. This finding (unexpected for us) seems to be the first result of this kind after more than a century of studies on the Bertrand duopoly model and its versions.

Due to a number of special game-theoretic features, the classical Bertrand duopoly has generated broad interest extending far beyond oligopoly theory and modern industrial organization. The first feature of interest is that the Nash equilibrium is in weakly dominated strategies: Pricing at marginal cost yields zero profit to the two firms. In addition, the Nash equilibrium payoffs correspond to the individually rational payoffs of the players. Another unusual feature is that the Bertrand game is the oldest representative of the class of classical games with discontinuous payoff functions.[2] Finally, the Bertrand game belongs to a small family of classical games, for which a tie breaking rule

[1] For the more complex Bertrand-Edgeworth version of this model with unequal unit costs, see Deneckere and Kovenock (1996).

[2] The payoff functions are not even upper semi-continuous in the prices, so the firms' reaction curves are not even well defined. This makes the Bertrand duopoly lie outside of the commonly studied classes of games with discontinuous payoffs. In addition, the payoffs are not quasi-concave in own action. Therefore, the results from this literature (see e.g., Reny, 2016) cannot be applied. Nonetheless, an exception is Prokopovych and Yannelis (2017), who derive a general existence result that includes existence in the Bertrand model as a special case.

is a necessary part of the definition of the game, and its specification is often a critical part for the solution of the game.

Other well-known games in microeconomics and game theory share these properties in one form or other. These games include the Guess-the-average game (Moulin, 1986) and the Traveler's Dilemma (Basu, 1994). Both of these games, along with various extensions, have spurned an extensive literature in experimental economics, see e.g. Nagel, Bühren, and Björn (2017). One possible application of the present study of Bertrand duopoly is that it may pave the way for a systematic study of the afore-mentioned general class of games.

2 The main result

We consider a version of the Bertrand duopoly model with a homogeneous product. There are two profit-maximizing firms 1 and 2 producing a homogeneous good in a market whose demand function is given by $D(p) \geq 0$ ($p \geq 0$). The cost, $c \geq 0$, per unit produced is the same for both firms. The firms simultaneously set their prices p_1 and p_2. Sales for firm 1 are then given by

$$D_1(p_1, p_2) = \begin{cases} D(p_1), & \text{if } p_1 < p_2; \\ \frac{1}{2}D(p_1), & \text{if } p_1 = p_2; \\ 0, & \text{if } p_1 > p_2. \end{cases}$$

Analogously, for firm 2, we have

$$D_2(p_1, p_2) = \begin{cases} D(p_2), & \text{if } p_2 < p_1; \\ \frac{1}{2}D(p_2), & \text{if } p_2 = p_1; \\ 0, & \text{if } p_2 > p_1. \end{cases}$$

The firms' profits are

$$\pi_i(p_1, p_2) = (p_i - c)D_i(p_1, p_2), \; i = 1, 2.$$

We will assume that the firms never set prices that are less than c: if $p_i < c$, then firm i cannot have a strictly positive profit $\pi_i > 0$. Thus we have a game with the payoffs $\pi_1(p_1, p_2)$, $\pi_2(p_1, p_2)$ and the strategy set $P_c = [c, \infty)$ for both players. The game is symmetric: $\pi_2(p_1, p_2) = \pi_1(p_2, p_1)$.

Since we consider only those prices p_i that satisfy $p_i \geq c$, it is sufficient to assume that $D(p)$ is defined only for $p \geq c$.

We are interested in Nash equilibria of the above game (Bertrand-Nash equilibria), i.e., pairs of prices (p_1^*, p_2^*), $p_i^* \geq c$, such that

$$\pi_1(p_1^*, p_2^*) \geq \pi_1(p_1, p_2^*),\ \ \pi_2(p_1^*, p_2^*) \geq \pi_2(p_1^*, p_2)$$

for all $p_i \geq c$. The following result is well-known.

Theorem 1. *The prices*

$$p_1^* = p_2^* = c$$

form a symmetric Nash equilibrium in the Bertrand game.

Proof. Since the game is symmetric, it is sufficient to prove that $\pi_1(p, c) \leq \pi_1(c, c)$ for each $p \geq c$. We have $\pi_1(c, c) = 0$. If $p > c$, then $\pi_1(p, c) = (p - c)D_1(p, c) = (p - c) \cdot 0 = \pi_1(c, c)$, which completes the proof.

We provide a criterion for the uniqueness of equilibrium in the Bertrand duopoly model. Denote by $\pi(p)$ the monopoly profit

$$\pi(p) = (p - c)D(p)$$

that the firm gets by setting the price p and serving the entire demand $D(p)$ alone.

For the uniqueness result, we shall need the following simple and intuitive condition:

(D) For each $p > c$, there exists $q = q(p)$ such that $c < q < p$ and

$$\pi(q) > \pi(p)/2.$$

This condition means that by setting some price q smaller than p the firm can get a profit $\pi(q)$ higher than half of $\pi(p)$. An interesting interpretation of this condition is that it ensures that static collusion between the two firms in the form of equal market sharing at any price p is not sustainable, in that each firm has a profitable deviation q.[3] This is a very mild assumption, as will be seen through two sufficient easy-to-check conditions we provide for it below.

[3] In other words, this assumption is minimally needed to pave the way for the now-standard repeated games approach to the study of collusion (see e.g., Tirole, 1988, Chap. 6).

Theorem 2. *Condition (D) is necessary and sufficient for (c, c) to be the only Nash equilibrium in the Bertrand duopoly.*

Proof. Let (D) hold. Suppose there is another Nash equilibrium, (p_1, p_2), $p_i \geq c$, distinct from (c, c). We may assume without loss of generality that $p_1 \leq p_2$ (one can always swap p_1 and p_2).

We consider three cases, in all of which we arrive at a contradiction with the assumption that (p_1, p_2) is a Nash equilibrium.

1st case: $c < p_1 < p_2$. Observe that $D(p_1) > 0$. Indeed, if $D(p_1) = 0$, then firm 1 can replace the price p_1 by the price $q = q(p_1) < p_1$ described in (D), which will lead to a strict increase in its profit

$$(q - c)D(q) > (p_1 - c)D(p_1)/2 = 0.$$

But this is impossible as long as (p_1, p_2) is a Nash equilibrium.

Note that the profit of firm 2 is zero (since $p_1 < p_2$). Therefore by setting the price p_1 instead of p_2, it can get a strictly positive profit

$$\pi_2(p_1, p_1) = (p_1 - c)D(p_1)/2 > 0.$$

Consequently, (p_1, p_2) is not a Nash equilibrium.

2nd case: $c = p_1 < p_2$. Then firm 1's profit is zero, and it can obtain, by setting instead of the price $p_1 = c$ the price $q = q(p_2) < p_2$ (see condition (D)), a strictly positive profit:

$$\pi_1(q, p_2) = (q - c)D(q) > (p_2 - c)D(p_2)/2 \geq 0.$$

Therefore (p_1, p_2) is not a Nash equilibrium.

3rd case: $c < p_1 = p_2$. By virtue of (D), there exists a price $q = q(p_1) < p_1$ such that

$$(q - c)D(q) > (p_1 - c)D(p_1)/2 = \pi_1(p_1, p_2).$$

This means that firm 1 can increase its profit by charging the price q instead of p_1, which contradicts the assumption that (p_1, p_2) is Nash equilibrium.

Thus we have proved that condition (D) is sufficient for the uniqueness of Nash equilibrium in Bertrand duopoly. Let us prove that this condition is also necessary. Suppose (D) does not hold. This means that for some $\bar{p} > c$, the inequality

$$(q - c)D(q) \leq (\bar{p} - c)D(\bar{p})/2 \tag{1}$$

is satisfied for all $c < q < \bar{p}$. We claim that $(\bar{p}, \bar{p})$ is a Nash equilibrium, i.e., each firm, deviating unilaterally from $\bar{p}$ cannot strictly increase its profit. Indeed, if it sets a price $q > \bar{p}$, then its profit is zero. If it sets a price $c < q < \bar{p}$, its profit cannot be greater than $(\bar{p} - c)D(\bar{p})/2$ by virtue of (1), and the same is true of course if $q = c$.

The proof is complete.

3 Sufficient conditions

We now provide familiar sufficient conditions for the uniqueness of a Bertrand-Nash equilibrium. Consider the following alternative assumptions:

(D1) The function $D(p)$ is continuous on $[c, \infty)$, and $D(c) > 0$.

(D2) The function $D(p)$ is non-increasing on $[c, \infty)$, and there exists $p^* > c$ such that $D(p^*) > 0$.

In most textbook treatments of Bertrand duopoly, (D1) and (D2) are assumed together, along with the existence of a choke-off price p_0 such that $D(p_0) = 0$ (see e.g., Mas-Colell, Whinston and Green, 1995, p. 388). In textbooks on industrial organization, these assumptions are often not explicitly listed, but one infers from the context that the tacit assumptions on demand are the same: continuity and downward-monotonicity (see e.g., Tirole, 1988).

It is worth recalling that, to ensure that a demand function derived from the maximization of a utility (subject to a budget constraint) is decreasing (as in (D2) here), one needs the somewhat restrictive assumption of a quasi-linear utility in a numeraire good (e.g., Vives, 2000). There is thus some theoretical motivation for developing results in oligopoly theory that do not rely on a downward-sloping demand as a primitive (Vives, 1987).

We now show that either of (D1) and (D2) is sufficient for our assumption (D) to hold.

Theorem 3. *Each of the assumptions (D1) and (D2) implies (D).*

Proof. Let (D1) hold. Consider any $p > c$. We have either $D(p) = 0$ or $D(p) > 0$. If $D(p) = 0$, take any $q > c$ sufficiently close to c for which $D(q) > 0$ (q exists by virtue of the continuity of $D(p)$). Then

$$\pi(q) = (q-c)D(q) > 0 = (p-c)D(p)/2 = \pi(p)/2.$$

If $D(p) > 0$, then $(p-c)D(p) > 0$, and since $D(p)$ is continuous, we have

$$(q-c)D(q) > (p-c)D(p)/2$$

for all $q < p$ close enough to p. Thus we have proved that (D1) implies (D).

Suppose (D2) holds. Let $p > c$. If $D(p) = 0$, then $p > p^*$ because $D(\cdot)$ is a non-increasing function and $D(p^*) > 0$. We can define $q = q(p)$ as p^*. Indeed,

$$\pi(q) = (p^* - c)D(p^*) > 0 = (p-c)D(p)/2 = \pi(p)/2.$$

Let $D(p) > 0$. Define $q = q(p)$ as any number satisfying $(p+c)/2 < q < p$. Then we have

$$\pi(q) = (q-c)D(q) > \frac{p-c}{2}D(p) = \pi(p)/2$$

because $q - c > (p+c)/2 - c = (p-c)/2$ and $D(q) \geq D(p) > 0$.

The proof is complete.

REFERENCES

Basu, K. (1994). The Traveler's Dilemma: Paradoxes of rationality in game theory. *American Economic Review* 84, 391–395.

Bertrand, J.L.F. (1883). Compte-rendu de Théorie de la richesse sociale par Léon Walras et de Recherches sur les principes mathématiques de la théorie des richesses par Augustin Cournot. *Journal des Savants* 67, 499–508.

Cournot, A. (1838). *Recherches sur les Principes Mathématiques de la Théorie des Richesses.* Hachette Livre, Paris.

Deneckere, R.J. and Kovenock, D. (1996). Bertrand-Edgeworth duopoly with unit cost asymmetry. *Economic Theory* 8, 1-25.

Mas-Colell, A., Whinston, M.D., and Green, J.R. (1995). *Microeconomic Theory.* Oxford University Press, Oxford, U.K.

Moulin, H. (1986). *Game Theory for Social Sciences.* New York University Press, New York.

Nagel, R., Bühren, C., and Björn, F. (2017). Inspired and inspiring: Hervé Moulin and the discovery of the beauty contest game. *Mathematical Social Sciences,* to appear.

Prokopovych, P., and Yannelis, N.C. (2017). On strategic complementarities in discontinuous games with totally ordered strategies. Mimeo.

Reny, P.J. (2016). Introduction to the symposium on discontinuous games. *Economic Theory* 61, 423–429.

Tirole, J. (1988). *The Theory of Industrial Organization.* MIT Press, Cambridge, MA.

Vives, X. (1987). Small income effects: A Marshallian theory of consumer surplus and downward sloping demand. *Review of Economic Studies* 54, 87–103.

Vives, X. (2000). *Oligopoly Pricing: Old Ideas and New Tools.* MIT Press, Cambridge, MA.