The University of Manchester

Economics Discussion Paper Series EDP-1501

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Wonki Jo Cho

Alejandro Saporiti

January 2015

Economics
School of Social Sciences
The University of Manchester
Manchester M13 9PL

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Wonki Jo Cho* Alejandro Saporiti†

January 15, 2015

Abstract

We study the problem of identifying members of a group based on individual opinions. Since agents do not have preferences in the model, properties of rules that concern preferences (e.g., strategy-proofness and efficiency) have not been studied. We fill this gap by working with partial preferences derived directly from opinions. We characterize three families of rules that are nested. The most general is the family of voting-by-committees rules, characterized by strategy-proofness alone. By additionally imposing equal treatment of equals, we identify a condition that committees should satisfy and we call the resulting family the voting-by-*equitable*-committees rules. The consent rules are a special case within this family and we characterize them by strategy-proofness and symmetry. We also show that under strategy-proofness, non-degeneracy is necessary and sufficient for efficiency. This implies that a rule satisfies strategy-proofness, efficiency, and equal treatment of equals if and only if it is a non-degenerate voting-by-equitable-committees rule. This family is new in the literature and contains, in addition to the consent rules, dictatorial and oligarchic rules that may be particularly relevant when agents' opinions need to be weighted differently.

JEL Classification Codes: C70, D70, D71.

Key Words: group identification; strategy-proofness; equal treatment of equals; symmetry; efficiency; voting-by-committees rules; consent rules.

*School of Social Sciences, University of Manchester, Oxford Road, Manchester, M13 9PL, United Kingdom. jo.cho@manchester.ac.uk.

†School of Social Sciences, University of Manchester. alejandro.saporiti@manchester.ac.uk.

1 Introduction

The axiomatic theory of group identification begins with Kasher and Rubinstein (1997), who focus on the problem of identifying members of a given ethnic-religious community. The building blocks of this theory are (i) a group of agents, who seek to identify those with, or without, a certain qualification; (ii) the agents' opinions about each other, including themselves, which are the main input for qualifying or disqualifying an individual as a group member; and (iii) a rule that aggregates those opinions into a (social) decision.[1] From the standpoint of economics, an important disadvantage of this framework is that the lack of information about individual preferences prevents us from analyzing incentive compatibility and efficiency of rules.

One way of circumventing this problem is to ask agents to submit their preferences, define a problem as a profile of such preferences and a rule as a mapping associating with each preference profile a decision. The model so obtained is a non-trivial application of Barberà et al. (1991), who study selecting subsets from an abstract set of alternatives. A drawback of this approach is that the information requirement is very demanding. Preferences are defined over 2^n alternatives (where n is the number of agents); so just with $n = 10$, each agent should evaluate more than 3.6 million alternatives and report his preferences over them. Thus, the approach may not be feasible in real-life applications.

Motivated by this limitation, we derive from each agent's opinion a partial ordering that represents his preferences over decisions. Our preference specification ranks the agent's opinion as his most preferred decision; and it partially orders other decisions by looking at each agent's membership separately from the others' and by comparing it to the opinion. For instance, suppose that agent i views agent j as a member. Consider any pair of decisions that differ only in agent j's membership. Then agent i prefers the decision that qualifies agent j as a member to the decision that disqualifies him. Any complete preferences satisfying "separability" should be consistent with such a partial ordering. This way of extending opinions to preferences resembles the way in which preferences over sure outcomes are extended to preferences over lotteries by first-order stochastic dominance.

With preferences defined, we can now study properties of rules that involve preferences. The first one is an incentive property known as strategy-proofness. This property ensures that no agent ever benefit from misrepresenting his opinions. We find that on a "rich" opinion domain (e.g., the universal domain), strategy-proofness characterizes the family of voting-by-committees rules (Theorem 1).[2] The latter rules are first introduced

[1]Opinions and decisions are represented by profiles of 0's and 1's, with 0 meaning "out" and 1 "in".

[2]Domain richness is given by two properties: connectedness and non-restoration. Both are formally defined in Section 3. Also, Fig. 1 illustrates these concepts.

by Barberà et al. (1991) in their abstract social choice model. A voting-by-committees rule determines each agent's membership using a committee for the agent that consists of winning coalitions (subsets of agents). An agent is qualified as a member if he obtains the approval of a winning coalition in his committee.

Our proof for the characterization of the voting-by-committees rules hinges on two results that are interesting in their own right. First, strategy-proofness is equivalent to a local notion of incentive compatibility, called adjacent strategy-proofness (Proposition 1). Two opinions are adjacent if they differ in only one agent's membership. Adjacent strategy-proofness requires that no agent gain by reporting an opinion that is adjacent to the truth. In general, adjacent strategy-proofness is weaker than strategy-proofness. Yet on rich domains, the two are equivalent. Moreover, this equivalence implies that a strategy-proof rule responds to changes in opinions in an intuitive way: when agent i changes his opinion about agent j while keeping his opinions about all other agents the same, agent j's membership, if it is affected, should change in the same direction while the membership for all other agents should remain unaffected. Conversely, any rule satisfying this property is strategy-proof (Corollary 1).

The second result upon which our characterization relies concern two properties proposed by Samet and Schmeidler (2003): monotonicity and independence. Monotonicity requires that a rule should adopt decisions that change (weakly) in the same direction as opinions do. Independence, on the other hand, requires that each agent's membership should be decided based solely on opinions about him. We show that strategy-proofness is equivalent to these two properties together (Proposition 2). This result provides a preference foundation for monotonicity and independence by Samet and Schmeidler (2003).

In addition to seeking to elicit individual opinions sincerely, we are also interested in rules that treat agents fairly. We consider two properties of fairness. The first property is equal treatment of equals (Kasher and Rubinstein, 1997).[3] Two agents are equals if they are not distinguishable in any way (i.e., they have the same opinion about all agents and all agents have the same opinion about them). Equal treatment of equals says that the membership decisions for two equal agents should be the same. This property is the most basic fairness requirement in our model. This axiom is not well-defined in the abstract model of Barberà et al. (1991), but it is meaningful in our model because of the additional structure that alternatives have in group identification.

We show that a rule satisfies strategy-proofness and equal treatment of equals if and only if it is a voting-by-committees rule whose associated committees satisfy the following

[3] Kasher and Rubinstein (1997) call this property "symmetry". We use a different terminology to distinguish it from the concept of symmetry defined by Samet and Schmeidler (2003), which is stronger than equal treatment of equals and is equivalent to the conjunction of essential anonymity and essential neutrality (Çengelci and Sanver, 2010).

equitable condition: for each pair of agents i and j and for each winning coalition M that is in agent i's committee but not in agent j's, exactly one of the two agents belongs to M, so that whenever the two agents are equals, M plays no role in determining their qualification (Theorem 2). These rules are new in the literature and we call them the voting-by-*equitable*-committees rules.

The family of consent rules proposed by Samet and Schmeidler (2003) is a special case within the class of voting-by-equitable-committees rules. Depending on the choice of the parameters, the consent rules can embody different degrees of democracy and liberalism. We characterize these rules by strategy-proofness and symmetry, a fairness notion stronger than equal treatment of equals (Theorem 3). Symmetry (Samet and Schmeidler, 2003) essentially requires that the names of the agents should not matter ("essentially" because an agent's opinion about himself plays a special role in the committee that deals with his own qualification). Our characterization of the consent rules relies on a characterization by Samet and Schmeidler (2003) and our result that strategy-proofness is equivalent to monotonicity and independence combined.

Finally, we also investigate implications of efficiency. Because the preferences derived from opinions are incomplete, efficiency turns out to be a weak property. In fact, we show that in the presence of strategy-proofness, efficiency is equivalent to non-degeneracy (Proposition 4). Non-degeneracy requires that for no agent, should the membership decision be fixed throughout the opinion domain. Therefore, using our previous results, we conclude that a rule satisfies strategy-proofness, efficiency, and equal treatment of equals if and only if it is a non-degenerate voting-by-equitable-committees rule (Corollary 2).

To summarize, our analysis in this work unveils a class of desirable rules that, paraphrasing Samet and Schmeidler (2003), goes "beyond liberalism and democracy". The family of voting-by-equitable-committees rules includes rules that are non-anonymous, but all of them satisfy strategy-proofness, efficiency, and equal treatment of equals. The subfamily of consent rules are made of essentially anonymous committees, for which only the size of winning coalitions (i.e., quotas) matters, but not the names of their members, except for the agent whose membership a committee deals with. By contrast, the voting-by-equitable-committees rules allow committees to be oligarchic (e.g., UN Security Council) or even dictatorial. We believe that these rules are particularly relevant when agents are heterogenous not only in their opinions but also in other dimensions (e.g., power and expertise), which may justify assigning different weights to their opinions.

The rest of the paper proceeds as follows. We briefly review the related literature in Section 2 and set up the model and notation in Section 3. The results about strategy-proofness are in Section 4. In Sections 5 and 6, we additionally impose fairness and efficiency, respectively. We discuss other incentive properties in Section 7. For expositional convenience, all proofs are in Appendix A.

2 Related Literature

The first model of group identification is due to Kasher and Rubinstein (1997), who relate the problem of collective identity to social choice theory. The paper shows that when the range of a rule is restricted to be a proper subset of the set of agents, it satisfies independence and consensus (efficiency in our model) if and only if it is dictatorial. This result is known in the literature as the "who is a J" impossibility theorem. Saporiti (2012) enhances this result by weakening consensus and provides a proof that exploits the structure of decisive coalitions. Kasher and Rubinstein (1997) also propose equal treatment of equals (they call this property symmetry), a fairness requirement that a rule should not discriminate between two agents on any basis other than that represented by opinions. Finally, they introduce and characterize the liberal rule, according to which an agent is a member if and only if he believes himself to be.

Samet and Schmeidler (2003) propose and characterize the family of consent rules, which contains the liberal rule as a special case. The consent rules are parameterized by the weights given to the individuals viz-a-viz the group for determining their membership, accommodating different levels of social intervention in an agent's status. Samet and Schmeidler (2003) point out the connection between the group identification model and Barberà et al. (1991). In particular, they discuss the possibility of misrepresenting opinions, but they do not characterize strategy-proof rules. In that sense, our work is an extension of their study. Also, Theorem 3 provides an alternative justification for the consent rules by showing that they are the only strategy-proof and symmetric rules.

In the papers cited above, there is a single group whose membership is to be decided. By contrast, Miller (2008) considers the setup where the group under question can vary. He imposes axioms that require decisions to be consistent with respect to conjunction and disjunction of groups. Cho and Ju (2014) study a model where multiple groups are identified simultaneously. They investigate consequences of an independence axiom, similar to Arrow's (1951) independence of irrelevant alternatives, requiring that identification of each group should depend only on opinions about that group. Sung and Dimitrov (2005), Houy (2007), Çengelci and Sanver (2010), and Ju (2013) provide further characterizations of group identification rules.[4]

In a model where agents vote on subsets of abstract alternatives, Barberà et al. (1991) characterize the voting-by-committees rules on the domain of separable preferences by an incentive property and a full range condition (voter sovereignty). Since agents submit

[4]See also Dimitrov (2011) for a recent and comprehensive review of the literature.

their preferences over *all* alternatives, their incentive property is weaker than our notion of strategy-proofness. Theorem 1 in our paper shows that the same family of rules is characterized by strategy-proofness alone when individual preferences are separable but represented by partial orders. Our result holds on rich domains where some decisions may not be feasible. In this regard, our work is also related to Barberà et al. (2005) who study choosing abstract sets of objects with constraints. A major distinction between these two papers and ours, apart from the incomplete nature of our preference specification, is that we consider fairness properties—namely, equal treatment of equals and symmetry—and characterize the structures of committees emerging from them.

Sato (2013), Carroll (2012), and Cho (2014) pursue the question of when a local incentive property is sufficient for the global incentive property. In particular, Sato (2013) identifies a condition on the preference domain that ensures the sufficiency for deterministic social choice rules and Cho (2014) extends it to probabilistic rules. Our result that on rich opinion domains, strategy-proofness and adjacent strategy-proofness are equivalent is related to these papers. Moreover, in the spirit of Cho (2014), we also show that strategy-proofness is equivalent to a stronger incentive property, called lie monotonicity, which says that each agent's welfare weakly decreases as he submits increasingly bigger lies.

3 The Model

We study the problem of determining members of a certain group based on individual opinions. Let $\boldsymbol{N} \equiv \{1, \ldots, n\}$ $(n \geq 2)$ be a finite set of agents. There is a group that these agents seek to identify among themselves. Each agent $i \in N$ has an **opinion** $\boldsymbol{p_i} \equiv (p_{ij})_{j \in N} \in \{0,1\}^N$, where for each $j \in N$, $p_{ij} = 1$ (and $p_{ij} = 0$) if agent i approves (and disapproves, respectively) agent j's membership for the group. It may be that some opinions are not permitted; e.g., we may want to identify a group whose maximum size is $\frac{n}{2}$. Let $\boldsymbol{\mathcal{D}} \subseteq \{0,1\}^N$ be the domain of admissible opinions. We call $\{0,1\}^N$ the universal domain and denote it by $\boldsymbol{\mathcal{U}}$.

A (identification) **problem** is a profile $\boldsymbol{p} \equiv (p_i)_{i \in N}$ of opinions. We treat individual opinions as $1 \times n$ row vectors and problems as $n \times n$ matrices. For each $i \in N$, let $\boldsymbol{p^i}$ be the i-th column of p (the opinions about agent i). Let $\boldsymbol{\mathcal{D}^N}$ be the set of problems. A (social) **decision** is a profile $x \equiv (x_i)_{i \in N} \in \{0,1\}^N$, where for each $i \in N$, $x_i = 1$ (and $x_i = 0$) if agent i is approved as a member (and a non-member, respectively) of the group. Let $\boldsymbol{X} \equiv \{0,1\}^N$ be the set of all social decisions. A (identification) **rule** $\varphi : \mathcal{D}^N \to X$ associates with each problem a decision.

What we have described so far is the standard model of group identification (Kasher and Rubinstein, 1997). However, since the model lacks the information about agents' preferences, we cannot define properties of rules, such as strategy-proofness and efficiency, that pertain to these preferences. Yet there is a way to interpret opinions as preferences. Consider agent i's opinion p_i. It is natural to interpret p_i as agent i's most preferred alternative in $\{0,1\}^N$. We do not know how he ranks the other alternatives in $\{0,1\}^N$. But if we place a restriction on admissible preferences, a "partial" preference relation can be recovered from p_i. We assume that preferences are **separable** in the sense that each agent's membership decision is valued independently of other agents' membership decision.

Formally, given $p_i \in \mathcal{D}$, for each $j \in N$ and each pair $x, y \in X$ such that for each $k \in N\backslash\{j\}$, $x_k = y_k$ and $x_j \neq y_j$, agent i with opinion p_i prefers x to y if and only if $x_j = p_{ij} \neq y_j$. Then each separable preference relation that top-ranks p_i is consistent with the strict preference relation P_i defined as follows: for each pair $x, y \in X$,

$$x \, P_i \, y \iff \text{for each } j \in N, \; p_{ij} \neq x_j \text{ implies } p_{ij} \neq y_j.$$

Thus, under the separability assumption, for each pair $x, y \in X$, if $x \, P_i \, y$, then agent i who most prefers p_i in $\{0,1\}^N$ prefers x to y. Also, if x and y are not comparable according to P_i, then agent i who most prefers p_i may or may not prefer x to y. We treat P_i as the preferences of agent i with opinion p_i and call it the **preference extension of $\boldsymbol{p_i}$**. Denote the preference extension of p'_i and $\hat{p}_i$ by P'_i and $\hat{P}_i$, and so on.

Note that P_i is irreflexive, transitive, and incomplete. For example, suppose that $N = \{1,2\}$. Then the universal domain $\mathcal{U}$ contains four opinions. According to the preference extension P_i of opinion $p_i = (0,1)$, decision $(0,1)$ is most preferred, $(1,0)$ is least preferred, and $(0,0)$ and $(1,1)$ are not comparable. In that regard, our preference extension is similar to extending preferences over sure outcomes to preferences over lotteries defined on those outcomes using first-order stochastic dominance. The latter method has been widely used to design ordinal mechanisms in the context of voting (Gibbard, 1977) and object assignment (Bogomolnaia and Moulin, 2001).

Finally, it is worth mentioning that Barberà et al. (1991) also assume separable preferences in their abstract social choice framework. However, a major distinction between our approach and theirs, apart from the additional structure of group identification, is that in our model each agent only submits his opinion (most preferred alternative) in $\{0,1\}^N$; in Barberà et al. (1991), each agent submits his entire preferences, which is a linear ordering over the subsets of a finite set of alternatives.

Below we use the following notation. For each $p \in \mathcal{D}^N$ and each $i \in N$, let $\boldsymbol{N_0(p,i)} \equiv \{j \in N : p_{ji} = 0\}$ be the set of agents who believe that agent i is not a member. Similarly, let $\boldsymbol{N_1(p,i)} \equiv \{j \in N : p_{ji} = 1\}$ be the set of agents who believe that agent i is a member. For each $p \in \mathcal{D}^N$ and each $i \in N$, let $\boldsymbol{\varphi_i(p)}$ be the i-th entry of the decision $\varphi(p)$. For each $p \in \mathcal{D}^N$, each $i \in N$, and each $p'_i \in \mathcal{D}$, let $\boldsymbol{(p'_i, p_{-i})}$ be the problem where agent i has opinion p'_i and for each $j \in N\backslash\{i\}$, agent j has opinion p_j. Let $\boldsymbol{1_{n\times n}} \in \mathcal{U}^N$ be the problem where all of its entries are 1 and $\boldsymbol{1_{1\times n}} \in X$ the decision where all of its entries are 1. The problem $0_{n\times n}$ and the decision $0_{1\times n}$ are similarly defined.

4 Incentives

With individual preferences defined, we can now consider properties of rules that refer to these preferences. The first of these properties deals with incentive compatibility: the decision an agent gets by truthfully reporting his opinion should be weakly preferred to all other decisions he gets by lying.

Strategy-proofness: For each $p \in \mathcal{D}^N$, each $i \in N$, and each $p'_i \in \mathcal{D}$, either $\varphi(p) = \varphi(p'_i, p_{-i})$ or $\varphi(p)\, P_i\, \varphi(p'_i, p_{-i})$.[5]

According to the definition of strategy-proofness, agents are free to report any opinion in the domain. We can formulate a similar but weaker property by restricting lies to be "close" to the truth. To define such property, we first make precise the meaning of closeness in $\mathcal{D}$. For each pair $p_i, p'_i \in \mathcal{D}$, p_i and p'_i are **adjacent** if there is exactly one $j \in N$ such that $p_{ij} \neq p'_{ij}$. Opinions that are adjacent to the truth are the smallest lies, and we require that agents should not benefit from reporting those lies.

Adjacent strategy-proofness: For each $p \in \mathcal{D}^N$, each $i \in N$, and each $p'_i \in \mathcal{D}$ such that p_i and p'_i are adjacent, either $\varphi(p) = \varphi(p'_i, p_{-i})$ or $\varphi(p)\, P_i\, \varphi(p'_i, p_{-i})$.

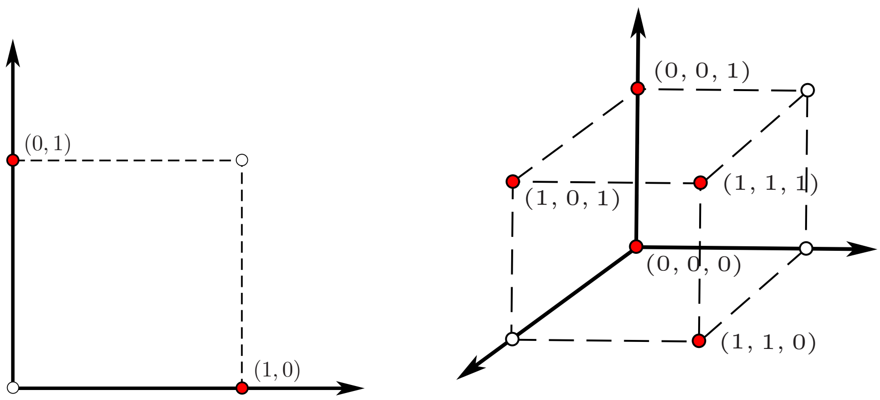
While strategy-proofness has long been studied in the mechanism design literature, adjacent strategy-proofness is a relatively new concept (Carroll, 2012; Sato, 2013). In general, adjacent strategy-proofness is weaker than strategy-proofness. However, the two properties are equivalent on some domains. This equivalence is particularly useful for characterizing the behavior of strategy-proof rules (see Corollary 1 below).

Next, to define domain properties, we adapt Sato's (2013) concepts to our setup. Let $p_i, p'_i \in \mathcal{D}$ be opinions. First, a **path from $\boldsymbol{p_i}$ to $\boldsymbol{p'_i}$ in $\boldsymbol{\mathcal{D}}$** is a sequence of opinions $\{p_i^0, p_i^1, \ldots, p_i^k\}$ in $\mathcal{D}$ such that (i) $p_i^0 = p_i$ and $p_i^k = p'_i$; and (ii) for each $h \in \{0, 1, \ldots, k -$

[5] One can formulate an alternative incentive property as follows: for each $p \in \mathcal{D}^N$ and each $i \in N$, there is no $p'_i \in \mathcal{D}$ such that $\varphi(p'_i, p_{-i})\, P_i\, \varphi(p)$. Since the preference extension only gives an incomplete preference relation, this property is weaker than strategy-proofness. More importantly, strategy-proofness is a more appropriate notion than the above alternative property because we require a rule to be immune to manipulation by agents with *any* separable preferences. A similar issue arises in the context of object assignment where agents compare lotteries based on incomplete preference relations that are obtained by first-order stochastic dominance (Bogomolnaia and Moulin, 2001).

1}, p_i^h and p_i^{h+1} are adjacent. For the path $\{p_i^0, p_i^1, \ldots, p_i^k\}$ from p_i to p_i' in $\mathcal{D}$, we call k the **length of the path**. The domain $\mathcal{D}$ is **connected** if for each pair $p_i, p_i' \in \mathcal{D}$, there is a path from p_i to p_i' in $\mathcal{D}$. Second, the path $\{p_i^0, p_i^1, \ldots, p_i^k\}$ from p_i to p_i' in $\mathcal{D}$ is **without restoration** if for each $h \in \{0, 1, \cdots, k-1\}$ and each $j \in N$, if $p_{ij}^h \neq p_{ij}^{h+1}$, then $p_{ij}^{h+1} = p_{ij}^{h+2} = \ldots = p_{ij}^k$. The domain $\mathcal{D}$ satisfies **non-restoration** if for each pair of connected opinions $p_i, p_i' \in \mathcal{D}$, there is a path from p_i to p_i' in $\mathcal{D}$ without restoration. Finally, the domain $\mathcal{D}$ is **rich** if it satisfies connectedness and non-restoration.

To illustrate the concepts defined above, Fig. 1 exhibits two opinion domains (in red) that are not rich. The domain in Fig. 1a is not connected but trivially satisfies non-restoration. On the contrary, the domain in Fig. 1b is connected but violates non-restoration. Note that the universal domain is rich.



(a) Violation of connectedness

(b) Violation of non-restoration

Figure 1: Opinion domains

As a first step toward understanding strategy-proofness, we explore the logical relation between strategy-proofness and adjacent strategy-proofness on a rich opinion domain. In the standard social choice model where agents report strict preferences over alternatives, these properties are equivalent, regardless of whether the social choice rule is deterministic (Sato, 2013) or probabilistic (Carroll, 2012; Cho, 2014). Our first result shows that a similar equivalence extends to the group identification model.

Proposition 1 *Let $\mathcal{D}$ be a rich opinion domain. A rule is strategy-proof if and only if it is adjacent strategy-proof.*

Proposition 1 plays a key role in unveiling the structure of strategy-proof rules. Indeed, consider a rule φ. Let $p \in \mathcal{D}^N$ and $i \in N$. Suppose that agent i changes his opinion from p_i to an adjacent opinion $p_i' \in \mathcal{D}$ such that for some $j \in N$, $p_{ij} \neq p_{ij}'$. Assume that φ is adjacent strategy-proof. If φ returns different decisions for p and (p_i', p_{-i}), then by adjacent strategy-proofness, the two decisions should differ only in agent j's membership.

Moreover, the change in the two decisions should be the same as the change in opinions p_i and p'_i. Thus, $\varphi_j(p) = p_{ij} \neq p'_{ij} = \varphi_j(p'_i, p_{-i})$ and for each $k \in N \backslash \{j\}$, $\varphi_k(p) = \varphi_k(p'_i, p_{-i})$. It is simple to see that the latter behavior of φ is necessary and sufficient for adjacent strategy-proofness. Now by Proposition 1, it is also necessary and sufficient for strategy-proofness. Thus, we have the following characterization of strategy-proof rules.

Corollary 1 *Let $\mathcal{D}$ be a rich opinion domain. A rule φ is strategy-proof if and only if for each $p \in \mathcal{D}^N$, each $i \in N$, and each $p'_i \in \mathcal{D}$ such that p_i and p'_i are adjacent, with $p_{ij} \neq p'_{ij}$ for some $j \in N$, either* (i) *$\varphi(p) = \varphi(p'_i, p_{-i})$, or* (ii) *$\varphi_j(p) = p_{ij} \neq p'_{ij} = \varphi_j(p'_i, p_{-i})$ and for each $k \in N \backslash \{j\}$, $\varphi_k(p) = \varphi_k(p'_i, p_{-i})$.*

Corollary 1 relates strategy-proofness to two well-known properties in group identification. The first property states that a rule should respond monotonically to changes in problems (Samet and Schmeidler, 2003).

Monotonicity: For each pair $p, p' \in \mathcal{D}^N$ such that $p \geq p'$, $\varphi(p) \geq \varphi(p')$.

The second property requires independence of decisions across agents. That is, to determine agent i's membership for the group, a rule should only consider opinions about him.

Independence: For each $i \in N$ and each pair $p, p' \in \mathcal{D}^N$ such that $p^i = (p')^i$, $\varphi_i(p) = \varphi_i(p')$.

By Corollary 1, it is simple to see that strategy-proofness implies monotonicity and independence. Our next result shows that the converse is also true. To prove this, we use the fact that adjacent strategy-proofness is equivalent to strategy-proofness.

Proposition 2 *Let $\mathcal{D}$ be a rich opinion domain. A rule is strategy-proof if and only if it is monotonic and independent.*

The characterization in Proposition 2 is tight. That is, if either monotonicity or independence is dropped, the equivalence no longer holds. First, it is clear that independence does not imply strategy-proofness. Second, to show that monotonicity does not imply strategy-proofness, define a rule φ on $\mathcal{U}^N$ as follows: for each $p \in \mathcal{U}^N$, (i) if $p_{11} = p_{22} = \ldots = p_{nn} = 1$, let $\varphi(p) = 1_{1\times n}$; and (ii) otherwise, let $\varphi(p) = 0_{1\times n}$. Clearly, φ is monotonic. Take agent $1 \in N$. Let $p_1 \equiv (0, 1, \ldots, 1), p'_1 \equiv 1_{1\times n} \in \mathcal{U}$, and for each $i \in N \backslash \{1\}$, $p_i \equiv 1_{1\times n} \in \mathcal{U}$. Then $\varphi(p) = 0_{1\times n}$ and $\varphi(p'_i, p_{-i}) = 1_{1\times n}$. However, it is not the case that $\varphi(p) = \varphi(p'_i, p_{-i})$ or $\varphi(p)\, P_1\, \varphi(p'_i, p_{-i})$. Thus, φ is not strategy-proof.

Except for the preference extension we defined, the group identification model can be seen as an application of Le Breton and Sen (1999), which extends Barberà et al. (1991) by considering an abstract social choice model where alternatives are drawn from a product set of issues and preferences are separable on that space. Le Breton and Sen (1999) show that a social choice function is strategy-proof if and only if it can be decomposed into a collection of "marginal" social choice functions. This decomposability requires that for each issue, a rule focus only on the agents' marginal preferences over that issue. The axiom of independence (Samet and Schmeidler, 2003) has the same spirit in our model. However, as shown by Proposition 2, independence is not sufficient for strategy-proofness. This is due to the incompleteness of the preferences derived from the agents' opinions, which in turn implies that our notion of strategy-proofness is stronger than that of Le Breton and Sen (1999) and Baberà et al. (1991).

Çengelci and Sanver (2010) characterize the family of rules satisfying monotonicity and independence.[6] By Proposition 2, this family is also the family of strategy-proof rules. To describe these rules, we borrow the following concepts from Barberà et al. (1991). A **coalition** is a subset of the set of agents N (a coalition is allowed to be empty or N). A **committee** for agent i is a collection $\mathcal{W}_i$ of coalitions satisfying the following condition: for each pair $M, M' \subseteq N$, if $M \in \mathcal{W}_i$ and $M \subseteq M'$, then $M' \in \mathcal{W}_i$. Elements of $\mathcal{W}_i$ are **winning coalitions with respect to $\boldsymbol{\mathcal{W}_i}$**.

Let $\mathcal{W} \equiv (\mathcal{W}_i)_{i \in N}$ be a profile of committees. The **voting-by-committees rule with respect to $\boldsymbol{\mathcal{W}}$**, denoted $\boldsymbol{\varphi^{\mathcal{W}}}$, is defined as follows: for each $p \in \mathcal{D}^N$ and each $i \in N$, $\varphi_i^{\mathcal{W}}(p) = 1$ if and only if $N_1(p,i) \in \mathcal{W}_i$. In words, a voting-by-committees rule is one that qualifies each agent i if and only if the agents who consider agent i to be a member of the group form a winning coalition with respect to $\mathcal{W}_i$.[7] Our next result shows that the voting-by-committees rules are the only strategy-proof rules on a rich domain.

Theorem 1 *Let $\mathcal{D}$ be a rich opinion domain. A rule is strategy-proof if and only if it is a voting-by-committees rule.*

As we said earlier, the group identification model is related to Barberà et al. (1991). Let us rename the incentive property in Barberà et al. (1991) to BSZ-strategy-proofness. Barberà et al. (1991) show that when agents have complete and separable preferences, the voting-by-committees rules are characterized by BSZ-strategy-proofness and a full range condition termed voter sovereignty.[8] In contrast, Theorem 1 says that with our preference extension, the same family of rules is characterized by a stronger notion of

[6]The notion of monotonicity in Çengelci and Sanver (2010) is slightly weaker than our notion of monotonicity. However, in the presence of independence, the two are equivalent.

[7]Notice that a committee is not required to be proper (or strong, respectively). Hence, the case in which each coalition is winning, i.e., $\mathcal{W}_i = 2^N$ (or the case in which no coalition is winning, i.e., $\mathcal{W}_i = \emptyset$, respectively), is allowed. As a result, a voting-by-committees rule can be degenerate for an agent, in the sense that for all problems, the rule returns the same decision for him.

[8]Voter sovereignty requires that for each alternative, there exists a preference profile for which the rule chooses that alternative.

strategy-proofness alone, without voter sovereignty. Clearly, Theorem 1 can also be extended to the setup of Barberà et al. (1991), where alternatives are abstract objects.

5 Fairness

In this section, we define two fairness properties. The first property concerns how a rule treats "equal" agents. Given a problem $p \in \mathcal{D}^N$, we say that agents i and j, say, are **equal** if they have the same opinion (i.e., $p_i = p_j$) and all agents have the same opinion about them (i.e., $p^i = p^j$). A reasonable request is that the membership decisions for two equal agents should be the same (Kasher and Rubinstein, 1997).

Equal treatment of equals: For each pair $i, j \in N$ and each $p \in \mathcal{D}^N$ such that $p_i = p_j$ and $p^i = p^j$, $\varphi_i(p) = \varphi_j(p)$.

We seek to identify the family of rules satisfying strategy-proofness and equal treatment of equals. By Theorem 1, these rules are voting-by-committees rules. To see properties that the committees $\mathcal{W} = (\mathcal{W}_i)_{i \in N}$ should have in order for the rule $\varphi^{\mathcal{W}}$ to satisfy equal treatment of equals, take any two agents $i, j \in N$. If $\mathcal{W}_i = \mathcal{W}_j$, then whenever agents i and j are equal, their membership decisions are the same. Thus, any potential violation of equal treatment of equals is due to the winning coalitions in $\mathcal{W}_i \backslash \mathcal{W}_j$ and $\mathcal{W}_j \backslash \mathcal{W}_i$.

Without loss of generality, let $M \in \mathcal{W}_i \backslash \mathcal{W}_j$. If $M \cap \{i, j\} = \emptyset$ or $\{i, j\}$, we can construct a problem $p \in \mathcal{D}^N$ such that agents i and j are equals in p and M is the set of agents who view agent i (and j) as a member. Because $M \in \mathcal{W}_i \backslash \mathcal{W}_j$, for such p, $\varphi^{\mathcal{W}}$ decides that agent i is a member and agent j is not, violating equal treatment of equals. Therefore, equal treatment of equals requires that for each $M \in \mathcal{W}_i \backslash \mathcal{W}_j$, $|M \cap \{i, j\}| = 1$. In fact, this condition is also sufficient for equal treatment of equals. Now say that a profile of committees $\mathcal{W} \equiv (\mathcal{W}_i)_{i \in N}$ is **equitable** if for each pair $i, j \in N$ and each $M \in \mathcal{W}_i \backslash \mathcal{W}_j$, $|M \cap \{i, j\}| = 1$. A **voting-by-equitable-committees rule** is a voting-by-committees rule whose associated committees are equitable.

Theorem 2 *Let $\mathcal{D}$ be the universal domain. A rule satisfies strategy-proofness and equal treatment of equals if and only if it is a voting-by-equitable-committees rule.*

Unlike our other results, Theorem 2 is proved on the universal domain $\{0, 1\}^N$, not an arbitrary rich domain. Richness is not enough for Theorem 2 because the necessity part of the proof involves constructing an opinion profile that may not be admissible unless the domain is sufficiently diverse.[9]

[9] A case in point takes place when $n = 3$ and $\mathcal{D} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)\}$, for which the

Next is a stronger fairness property. Following Samet and Schmeidler (2003), we may require that the names of the agents should not matter for group identification. This idea is expressed by permutations of the set of agents, which represent name changes. A permutation is a one-to-one function $\pi : N \to N$. For each $i \in N$, $\pi(i)$ is agent i's old name. Given $p \in \mathcal{D}^N$, let $\boldsymbol{p_\pi} \equiv (p_{\pi(i),\pi(j)})_{i,j \in N}$ be the problem in the new names, and $\boldsymbol{\varphi_\pi(p)} \equiv \big(\varphi_{\pi(i)}(p)\big)_{i \in N}$ the decision in the new names. Then for each permutation π, the two decisions $\varphi_\pi(p)$ and $\varphi(p_\pi)$ should be the same.

Symmetry: For each $p \in \mathcal{D}^N$ and each permutation $\pi : N \to N$, $\varphi_\pi(p) = \varphi(p_\pi)$.

By imposing symmetry, we obtain a further subfamily of rules, namely, the consent rules (Samet and Schmeidler, 2003). Let $s, t \in \{1, \ldots, n\}$ be such that $s + t \leq n + 2$. The **consent rule with (s, t)**, denoted $\boldsymbol{\varphi^{st}}$, is defined as follows: for each $p \in \mathcal{D}^N$ and each $i \in N$, (i) if $p_{ii} = 1$, then $[\varphi_i^{st}(p) = 1$ if and only if $N_1(p, i) \geq s]$; and (ii) if $p_{ii} = 0$, then $[\varphi_i^{st}(p) = 0$ if and only if $N_0(p, i) \geq t]$. The consent rules embody various degrees of liberalism and democracy: with $s = t = 1$, we have the liberal rule; with $s = t = \lceil \frac{n+1}{2} \rceil$, the simple majority rule; and with $s = t = n$, the unanimity rule.[10] By Samet and Schmeidler (2003), a rule is monotonic, independent, and symmetric if and only if it is a consent rule. Since strategy-proofness is equivalent to monotonicity and independence (Proposition 2), we obtain another characterization of the consent rules on rich domains.

Theorem 3 *Let $\mathcal{D}$ be a rich opinion domain. A rule is strategy-proof and symmetric if and only if it is a consent rule.*

Figure 2 illustrates the three characterizations obtained so far. The voting-by-equitable-committees rules are a superset of the consent rules and a subset of the voting-by-committees rules. Both inclusion relations are strict. For the former inclusion, let $n = 3$ and consider the committees $\mathcal{W}_1 = \mathcal{W}_2 = \{\{1, 2\}, N\}$ and $\mathcal{W}_3 = \{N\}$. Then $\varphi^{\mathcal{W}}$ satisfies equal treatment of equals but not symmetry. For the latter inclusion, redefine the previous committees in such a way that $\mathcal{W}_1 = \mathcal{W}_2 = \{\{1, 3\}, N\}$ and $\mathcal{W}_3 = \{N\}$. This defines a voting-by-committees rule $\varphi^{\mathcal{W}}$ that satisfies strategy-proofness but not equal treatment of equals.

6 Efficiency

Let us now define efficiency based on our preference extension and study its implications. Let $p \in \mathcal{D}^N$. For each pair $x, y \in X$, ***x* Pareto dominates *y* for *p*** if for each $i \in N$,

committees $\mathcal{W}_1 = \mathcal{W}_3 = \{\{1, 2\}, N\}$ and $\mathcal{W}_2 = \{N\}$ define a voting-by-committees rule that satisfies strategy-proofness and equal treatment of equals, but the committees $(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3)$ are not equitable.

[10] For each $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer no less than x.

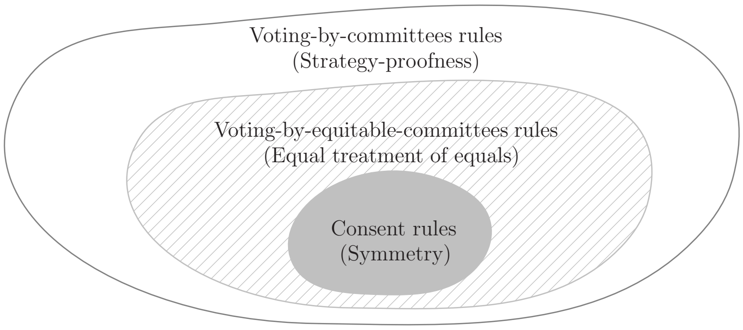


Figure 2: Main characterizations

$x\, P_i\, y$.[11] For each $x \in X$, x is **efficient for $\boldsymbol{p}$** if there is no $y \in X$ such that y Pareto dominates x for p.

Efficiency: For each $p \in \mathcal{D}^N$, $\varphi(p)$ is efficient for p.

Recall that the preference relation P_i recovered from the opinion p_i is coarse. This means that our notion of Pareto dominance is strong and consequently, our notion of efficiency is weak.[12] However, when all separable preferences over $\{0,1\}^N$ are allowed, this is the strongest notion of efficiency that can be considered.[13] Our next result shows that efficiency has an implication only for those agents about whom all agents have the same opinion. That is, a decision is efficient if and only if each agent whom all agents view as a member (and a non-member) is a member (and a non-member, respectively).

Proposition 3 *Let $p \in \mathcal{D}^N$ and $x \in X$. Then x is efficient for p if and only if for each $i \in N$ such that $p_{1i} = p_{2i} = \ldots = p_{ni}$, $x_i = p_{1i}$.*

Kasher and Rubinstein (1997) study implications of "consensus". A rule φ satisfies **consensus** if for each $i \in N$, (i) for each $p \in \mathcal{D}^N$ with $p^i = 1_{n\times 1}$, $\varphi_i(p) = 1$; and (ii) for each $p \in \mathcal{D}^N$ with $p^i = 0_{n\times 1}$, $\varphi_i(p) = 0$. Proposition 3 shows that efficiency is equivalent to consensus and provides a preference foundation for consensus. However, in the presence of strategy-proofness, efficiency reduces to a property weaker than consensus, which is now defined.

[11] The usual definition of Pareto dominance requires that (i) for each $i \in N$, either $x\, P_i\, y$ or $x = y$; and (ii) for some $j \in N$, $x\, P_j\, y$. Clearly, (i) and (ii) are equivalent to our definition of Pareto dominance.

[12] We can formulate another notion of efficiency for the setup where agents report preferences over $\{0,1\}^N$. This alternative notion is stronger than our definition. To see this, let $N = \{1,2\}$ and $p = ((1,0),(0,1))$. Consider two decisions $(0,0)$ and $(1,1)$. According to our definition, $(0,0)$ is efficient. However, if each agent prefers $(1,1)$ to $(0,0)$, $(1,1)$ Pareto dominates $(0,0)$.

[13] A similar comment applies to "ordinal efficiency" (Bogomolnaia and Moulin, 2001), a notion based on (partial) preferences over lotteries that are obtained from preferences over sure outcomes by applying first-order stochastic dominance.

A rule is **degenerate for an agent** if it always returns the same decision for him, regardless of the problems under consideration. The following property requires that for no agent should the rule be degenerate.

Non-degeneracy: For each $i \in N$, there are $p, p' \in \mathcal{D}^N$ such that $\varphi_i(p) \neq \varphi_i(p')$.

By Proposition 2, strategy-proofness implies monotonicity and independence, which in turn ensure that non-degeneracy implies efficiency.

Proposition 4 *Let φ be a strategy-proof rule on a rich opinion domain. Then φ is efficient if and only if it is non-degenerate.*

Combined with Theorem 1, this proposition allows us to characterize the family of rules satisfying strategy-proofness and efficiency.

Theorem 4 *Let $\mathcal{D}$ be a rich opinion domain. A rule is strategy-proof and efficient if and only if it is a non-degenerate voting-by-committees rule.*

It is simple to see that a voting-by-committees rule $\varphi^{\mathcal{W}}$ is non-degenerate if and only if for each $i \in N$, $\mathcal{W}_i \neq \emptyset$ and $\mathcal{W}_i \neq 2^N$

Combining Theorems 2 and 4, we can characterize the rules satisfying strategy-proofness, equal treatment of equals, and efficiency.

Corollary 2 *Let $\mathcal{D}$ be the universal domain. A rule satisfies strategy-proofness, equal treatment of equals, and efficiency if and only if it is a non-degenerate voting-by-equitable-committees rule.*

7 Discussion

7.1 A weaker incentive property

In Section 4, we introduce an incentive property where decisions are compared according to the preference extension of opinions. We use the preference extension there because we require that a rule should not be manipulated by agents with *any* separable preferences. With a smaller set of admissible preferences, one can formulate a weaker incentive property. As a simple case, we may assume that when comparing decisions, each agent takes a weighted sum of the differences between his opinion and decisions. We now explore the logical relation between strategy-proofness and the incentive property associated with this type of preferences.

Let $i \in N$. Let $w_i \equiv (w_{ij})_{j\in N} \in \mathbb{R}^N_{++}$. For each pair $x, y \in \{0,1\}^N$, let $||x-y||_{w_i} \equiv \sum_{j\in N} w_{ij} \cdot |x_j - y_j|$ be the weighted difference between x and y. We assume that for each $i \in N$, each $p_i \in \{0,1\}^N$, and each pair $x, y \in \{0,1\}^N$, agent i with opinion p_i weakly prefers x to y if and only if $||p_i - x||_{w_i} \leq ||p_i - y||_{w_i}$. In contrast with the preference extension in Section 3, these preferences are complete (and separable). When $w_{i1} = \cdots = w_{in}$, agent i's preferences simply minimize the number of different entries in his opinion and a decision.

Let $w \equiv (w_i)_{i \in N}$. The following property requires that when each agent i has the above preferences induced by w_i, no agent gain by lying.

$\boldsymbol{w}$-strategy-proofness: For each $p \in \mathcal{D}^N$, each $i \in N$, and each $p'_i \in \mathcal{D}$, $||p_i - \varphi(p)||_{w_i} \leq ||p_i - \varphi(p'_i, p_{-i})||_{w_i}$.

It is clear that w-strategy-proofness is weaker than strategy-proofness. However, once independence is imposed, the two properties are equivalent.

Proposition 5 *Let $\mathcal{D}$ be a rich opinion domain and φ an independent rule. Then φ is w-strategy-proof if and only if it is strategy-proof.*

Several corollaries follow from this proposition. First, by Proposition 2, w-strategy-proofness and independence together imply monotonicity. By Theorem 1, the family of voting-by-committees rules is characterized by w-strategy-proofness and independence. Finally, by Theorem 3, the consent rules are the only rules that satisfy w-strategy-proofness, independence, and symmetry on a rich opinion domain.

7.2 A stronger incentive property

How is individual welfare affected when an agent submits a lie to a strategy-proof rule? In spite the fact that the rule is strategy-proof, this question may be relevant for two reasons. First, the true opinion of an agent may not be in the domain $\mathcal{D}$ and he has no choice but to lie. In this case, he would seek to identify an (untruthful) opinion that maximizes his welfare. Second, if strategy-proofness has an additional implication that constrains the response of rules to lies, we can successfully design a strategy-proof rule only if that implication is taken into account.

To deal with the question posed above, Cho (2014) considers an incentive property, called "lie monotonicity," that strengthens strategy-proofness as follows: as an agent reports increasingly bigger lies, his welfare must weakly decrease. He finds that lie monotonicity is equivalent to strategy-proofness. Below we show that the equivalence extends to group identification problems.

Let $p_i \in \mathcal{D}$. Define an (asymmetric) order $\boldsymbol{>_{p_i}}$ over $\mathcal{D}$ as follows: for all distinct $p'_i, p''_i \in \mathcal{D}$, $p'_i >_{p_i} p''_i$ if there is a path from p_i to p''_i in $\mathcal{D}$ without restoration containing p'_i. The order $>_{p_i}$ is irreflexive, transitive, and incomplete. Also, it measures the degree of lying: for all distinct $p'_i, p''_i \in \mathcal{D}$ such that $p'_i >_{p_i} p''_i$, if we take p_i as the true opinion, p'_i is a smaller lie than p''_i. Now let $p \in \mathcal{D}^N$ and $i \in N$. Then $\varphi_i(\cdot, p_{-i})$ is a function of agent i's report, with the domain $\mathcal{D}$ ordered by $>_{p_i}$ and the co-domain X ordered by P_i. The next property requires that this function be monotonic, so that an agent is worse off announcing a larger lie.

Lie monotonicity: For each $p \in \mathcal{D}^N$ and each $i \in N$, the function $\varphi_i(\cdot, p_{-i}) : (\mathcal{D}, >_{p_i}) \to (X, P_i)$ is monotonic; i.e., for all distinct $p'_i, p''_i \in \mathcal{D}$ such that $p'_i >_{p_i} p''_i$, either $\varphi(p'_i, p_{-i}) = \varphi(p''_i, p_{-i})$ or $\varphi(p'_i, p_{-i}) \, P_i \, \varphi(p''_i, p_{-i})$.

Clearly, lie monotonicity is sufficient for strategy-proofness. However, our next result shows that it is also necessary.

Proposition 6 *Let $\mathcal{D}$ be a rich opinion domain. A rule is strategy-proof if and only if it is lie monotonic.*

As in Cho (2014), the equivalence of strategy-proofness and lie monotonicity follows as a corollary to the equivalence of strategy-proofness and adjacent strategy-proofness. The proof of Proposition 6 rests on that result and the characterization of strategy-proof rules in Corollary 1.

A Appendix: Proofs

A.1 Proof of Proposition 1

First, we define a metric $d(\cdot, \cdot)$ on $\mathcal{D}$ as follows: for each pair $p_i, p'_i \in \mathcal{D}$, let $d(p_i, p'_i)$ be the length of the shortest path from p_i to p'_i in $\mathcal{D}$.[14] Next, we define a family of auxiliary axioms, which is parametrized by $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, we require that no agent benefit from reporting an opinion whose distance from the truth according to $d(\cdot, \cdot)$ is at most m.

Within-m strategy-proofness: For each $p \in \mathcal{D}^N$, each $i \in N$, and each $p'_i \in \mathcal{D}$ such that $d(p_i, p'_i) \leq m$, either $\varphi(p) = \varphi(p'_i, p_{-i})$ or $\varphi(p) \, P_i \, \varphi(p'_i, p_{-i})$.

To prove Proposition 1, it suffices to show that if $\mathcal{D}$ is a rich domain, then for each $m \in \mathbb{N}$, within-m strategy-proofness implies within-$(m+1)$ strategy-proofness. Let φ be a within-m strategy-proof rule. To show that φ is within-$(m+1)$ strategy-proof, let $p \in \mathcal{D}^N$, $i \in N$, and $p'_i \in \mathcal{D}$ be such that $d(p_i, p'_i) = m + 1$. By the richness condition

[14] If $\mathcal{D} = \{0,1\}^N$, then $d(\cdot, \cdot)$ is a special case of the Hamming metric, but in general, the two are different.

on $\mathcal{D}$, there is a path $\{p_i^0, p_i^1, \cdots, p_i^m, p_i^{m+1}\}$ from p_i to p'_i in $\mathcal{D}$ without restoration. Let $\hat{p}_i \equiv p_i^m$. Let $x \equiv \varphi(p)$, $\hat{x} \equiv \varphi(\hat{p}_i, p_{-i})$, and $x' \equiv \varphi(p'_i, p_{-i})$.

If $\hat{x} = x'$, then by within-m strategy-proofness, either $x = \hat{x} = x'$ or $x \, P_i \, \hat{x} = x'$. Thus, assume, henceforth, that $\hat{x} \neq x'$. Because $\hat{p}_i$ and p'_i are adjacent, there is exactly one $j \in N$ such that $\hat{p}_{ij} \neq p'_{ij}$.

Now we show that for each $k \in N \backslash \{j\}$, $\hat{x}_k = x'_k$. Suppose, by contradiction, that for some $k \in N \backslash \{j\}$, $\hat{x}_k \neq x'_k$. Note that $\hat{p}_{ik} = p'_{ik}$. First, suppose that $\hat{x}_k = \hat{p}_{ik} = p'_{ik} \neq x'_k$. By within-$m$ strategy-proofness, $x' \, P'_i \, \hat{x}$. By the definition of the preference extension, $p'_{ik} \neq x'_k$ implies $p'_{ik} \neq \hat{x}_k$, a contradiction. Second, suppose that $\hat{x}_k \neq \hat{p}_{ik} = p'_{ik} = x'_k$. By within-$m$ strategy-proofness, $\hat{x} \, \hat{P}_i \, x'$. By the definition of the preference extension, $\hat{p}_{ik} \neq \hat{x}_k$ implies $\hat{p}_{ik} \neq x'_k$, a contradiction.

Next, we show that $\hat{p}_{ij} = \hat{x}_j$. Suppose not. By within-m strategy-proofness, $\hat{x} \, \hat{P}_i \, x'$. By the definition of the preference extension, $\hat{p}_{ij} \neq \hat{x}_j$ implies $\hat{p}_{ij} \neq x'_j$. Thus, $\hat{x}_j = x'_j$. However, since $\hat{x} \neq x'$, the argument in the previous paragraph shows that $\hat{x}_j \neq x'_j$, a contradiction.

Finally, we show that $\hat{x} \, P_i \, x'$. Recall that the path $\{p_i^0, p_i^1, \ldots, p_i^m, p_i^{m+1}\}$ from p_i to p'_i in $\mathcal{D}$ is without restoration. Since $\hat{p}_{ij} \neq p'_{ij}$, this implies that $p_{ij} = \hat{p}_{ij} \neq p'_{ij}$. Combined with the arguments in the previous two paragraphs, it follows that (i) $p_{ij} = \hat{p}_{ij} = \hat{x}_j \neq x'_j$; and (ii) for each $k \in N \backslash \{j\}$, $\hat{x}_k = x'_k$. Thus, $\hat{x} \, P_i \, x'$.

A.2 Proof of Proposition 2

(*Sufficiency*) Let φ be a monotonic and independent rule. By Proposition 1, it is enough to show that φ is adjacent strategy-proof. Let $p \in \mathcal{D}^N$, $i \in N$, and $p'_i \in \mathcal{D}$ such that p_i and p'_i are adjacent. Let $x \equiv \varphi(p)$ and $x' \equiv \varphi(p'_i, p_{-i})$. We may assume that $x \neq x'$ (otherwise, the proof is completed). Since p_i and p'_i are adjacent, there is exactly one $j \in N$ such that $p_{ij} \neq p'_{ij}$. By independence, for each $k \in N \backslash \{j\}$, $x_k = x'_k$. If $p_{ij} < p'_{ij}$, then by monotonicity and $x \neq x'$, it follows that $x_j < x'_j$. Thus, $x \, P_i \, x'$. A similar argument applies to the case $p_{ij} > p'_{ij}$.

(*Necessity*) Let φ be a strategy-proof rule. First, we show that φ is monotonic. Let $p, p' \in \mathcal{D}^N$ be such that $p \leq p'$. By appealing to an induction argument, we may assume that there is exactly one $(i, j) \in N \times N$ such that $p_{ij} \neq p'_{ij}$. Since $p \leq p'$, $p_{ij} = 0 \neq 1 = p'_{ij}$. Now by Corollary 1, either (i) $\varphi(p) = \varphi(p')$; or (ii) $\varphi_j(p) = p_{ij} = 0 \neq 1 = p'_{ij} = \varphi_j(p')$ and for each $k \in N \backslash \{j\}$, $\varphi_k(p) = \varphi_k(p')$. Thus, $\varphi(p) \leq \varphi(p')$.

Second, we show that φ is independent. Let $p, p' \in \mathcal{D}^N$ and $i \in N$ be such that $p^i = (p')^i$. By appealing to an induction argument, we may assume that there is exactly one $(k, j) \in N \times (N\backslash\{i\})$ such that $p_{kj} \neq p'_{kj}$. Then by Corollary 1, for each $\ell \in N\backslash\{j\}$, $\varphi_\ell(p) = \varphi_\ell(p')$. In particular, $\varphi_i(p) = \varphi_i(p')$.

A.3 Proof of Theorem 1

(*Sufficiency*) Each voting-by-committees rule is monotonic and independent. Thus, by Proposition 2, the rule is strategy-proof.

(*Necessity*) Let φ be a strategy-proof rule. Recall that φ is a voting-by-committees rule if there is a profile of committees $(\mathcal{W}_i)_{i\in N}$ such that for each $p \in \mathcal{D}^N$ and each $i \in N$,

$$\varphi_i(p) = 1 \iff \{j \in N : p_{ji} = 1\} \in \mathcal{W}_i. \tag{1}$$

Let $i \in N$. By Proposition 2, for each $p \in \mathcal{D}^N$, $\varphi_i(p)$ depends only on p^i. Thus, with a slight abuse of notation, $\varphi_i(\cdot)$ can be seen as a function that maps the i-th columns of all problems $p \in \mathcal{D}^N$ into $\{0, 1\}$. Let $\mathcal{D}_i \equiv \{p_{ji} \in \{0, 1\} : p_j \in \mathcal{D}\}$ denote the restriction of $\mathcal{D}$ to the i-th coordinate of individual opinions. Below we construct a collection $\mathcal{W}_i$ of subsets of N for agent i and show that $\mathcal{W}_i$ is a committee.

Case 1. Assume that $\mathcal{D}_i = \{0\}$ (the case $\mathcal{D}_i = \{1\}$ is similar). The domain of φ_i is a singleton $\{0_{n\times1}\}$. If $\varphi_i(0_{n\times1}) = 0$, then set $\mathcal{W}_i = \emptyset$. If $\varphi_i(0_{n\times1}) = 1$, then set $\mathcal{W}_i = 2^N$. It is simple to verify that this specification of $\mathcal{W}_i$ satisfies (1).

Case 2. Assume that $\mathcal{D}_i = \{0, 1\}$. The domain of φ_i is $\{0, 1\}^N$. If φ_i is constant on $\{0, 1\}^N$, then define $\mathcal{W}_i$ accordingly as in Case 1. Assume, henceforth, that φ_i is not constant on $\{0, 1\}^N$. Define $\mathcal{W}_i$ as follows: for each $M \subseteq N$, $M \in \mathcal{W}_i$ if and only if there is $p^i \in \{0, 1\}^N$ such that $\varphi_i(p^i) = 1$ and $\{j \in N : p_{ji} = 1\} = M$. Then monotonicity implies: (i) $\varphi_i(0_{n\times1}) = 0$[15], so that $\emptyset \notin \mathcal{W}_i$; (ii) $\varphi_i(1_{n\times1}) = 1$, so that $N \in \mathcal{W}_i$ and $\mathcal{W}_i \neq \emptyset$; and (iii) $\mathcal{W}_i$ is a committee (i.e., for each pair $M, M' \subseteq N$, if $M \in \mathcal{W}_i$ and $M \subseteq M'$, then $M' \in \mathcal{W}_i$). Now it is clear that $\mathcal{W}_i$ satisfies (1).

Applying the above argument for all $i \in N$, we obtain the profile of committees $(\mathcal{W}_i)_{i\in N}$ satisfying (1). Therefore, φ is a voting-by-committees rule.

A.4 Proof of Theorem 2

(*Sufficiency*) Let $\mathcal{W} \equiv (\mathcal{W}_i)_{i\in N}$ be a profile of equitable committees. By Theorem 1, the voting-by-committees rule $\varphi^{\mathcal{W}}$ is strategy-proof. Now we show that $\varphi^{\mathcal{W}}$ satisfies equal treatment of equals. Let $p \in \mathcal{D}^N$ and $i, j \in N$ be such that $p_i = p_j$ and $p^i = p^j$. Let

[15]If $\varphi_i(0_{n\times n}) = 1$, then by monotonicity, $\varphi_i(\cdot)$ is constant on $\{0, 1\}^N$, a contradiction.

$M \equiv \{k \in N : p_{ki} = 1\}$ $(= \{k \in N : p_{kj} = 1\})$. Because $p_i = p_j$ and $p^i = p^j$, $p_{ii} = p_{ij} = p_{jj} = p_{ji}$. First, if $M \in \mathcal{W}_i \cap \mathcal{W}_j$, then $\varphi_i^{\mathcal{W}}(p) = \varphi_j^{\mathcal{W}}(p) = 1$. Second, if $M \in (\mathcal{W}_i \cup \mathcal{W}_j)^c$, then $\varphi_i^{\mathcal{W}}(p) = \varphi_j^{\mathcal{W}}(p) = 0$. Third, suppose, without loss of generality, that $M \in \mathcal{W}_i \backslash \mathcal{W}_j$. Since $p_{ii} = p_{ij} = p_{jj} = p_{ji}$, $M \cap \{i, j\} = \{i, j\}$ or $\emptyset$, a contradiction.

(*Necessity*) Let φ be a rule satisfying strategy-proofness and equal treatment of equals. By Theorem 1, there is a profile of committees $\mathcal{W} \equiv (\mathcal{W}_i)_{i \in N}$ such that $\varphi = \varphi^{\mathcal{W}}$. Now we show that $\mathcal{W}$ is equitable. Suppose, by contradiction, that there are $i, j \in N$ and $M \in \mathcal{W}_i \backslash \mathcal{W}_j$ such that $|M \cap \{i, j\}| \neq 1$. Then either $M \supseteq \{i, j\}$ or $M \cap \{i, j\} = \emptyset$. Let $p \in \mathcal{U}^N$ be such that $p_i = p_j$, $p^i = p^j$, and $\{k \in N : p_{ki} = 1\} = M$ (such p cannot be constructed if $|M \cap \{i, j\}| = 1$). Since $M \in \mathcal{W}_i \backslash \mathcal{W}_j$, $\varphi_i(p) = \varphi_i^{\mathcal{W}}(p) = 1$ and $\varphi_j(p) = \varphi_j^{\mathcal{W}}(p) = 0$, a contradiction.

A.5 Proof of Proposition 3

(*Sufficiency*) Let $p \in \mathcal{D}^N$ and $x \in X$. Assume that for each $i \in N$ such that $p_{1i} = p_{2i} = \ldots = p_{ni}$, $x_i = p_{1i}$. Suppose, by contradiction, that there is $y \in X$ such that y Pareto dominate x for p. Let $N^* \equiv \{i \in N : p_{1i} = p_{2i} = \ldots = p_{ni}\}$. Clearly, $N^* \neq N$, and since $y \neq x$, there is $i \in N \backslash N^*$ such that $y_i \neq x_i$. Since $i \in N \backslash N^*$, there is $j \in N$ such that $p_{ji} = x_i$. Then this contradicts that $y \, P_j \, x$.

(*Necessity*) Let $p \in \mathcal{D}^N$ and $x \in X$. Assume that x is efficient for p. Suppose, by contradiction, that for some $i \in N$, $p_{1i} = p_{2i} = \ldots = p_{ni}$ and $x_i \neq p_{1i}$. Let $y \in X$ be such that $y_i = p_{1i}$ and $y_{-i} = x_{-i}$. Then y Pareto dominates x for p, a contradiction.

A.6 Proof of Proposition 4

Let φ be a strategy-proof rule. We only show that if φ is non-degenerate, then it is efficient (the other direction is clear). Suppose, by contradiction, that there are $p \in \mathcal{D}^N$ and $y \in X$ such that y Pareto dominates $\varphi(p)$ for p. Let $x \equiv \varphi(p)$. There is $j \in N$ such that $x_j \neq y_j$. For each $i \in N$, since $y \, P_i \, x$, $y_j = p_{ij} \neq x_j$. That is, $y_j = p_{1j} = \ldots = p_{nj} \neq x_j$.

By Proposition 2, φ is monotonic and independent. Monotonicity, together with non-degeneracy, implies that $\varphi(0_{n \times n}) = 0_{1 \times n}$ and $\varphi(1_{n \times n}) = 1_{1 \times n}$. If $y_j = 0$, then $p^j = 0_{n \times 1}$. By independence, $x_j = \varphi_j(p) = \varphi_j(0_{n \times n}) = 0$, a contradiction. Similarly, if $y_j = 1$, then $p^j = 1_{n \times 1}$. Again, by independence, $x_j = \varphi_j(p) = \varphi_j(1_{n \times n}) = 1$, a contradiction.

A.7 Proof of Proposition 5

Let φ be an independent rule. We only prove the necessity. Assume that φ is w-strategy-proof. By Proposition 2, it suffices to show that φ is monotonic. Let $p, p' \in \mathcal{D}^N$ be such that $p \leq p'$. By appealing to an induction argument, we may assume that there is exactly one $(i,j) \in N \times N$ such that $p_{ij} \neq p'_{ij}$. Let $x \equiv \varphi(p)$ and $x' \equiv \varphi(p')$. Since $p \leq p'$, $p_{ij} = 0$ and $p'_{ij} = 1$. For each $k \in N \backslash \{j\}$, $p^k = (p')^k$, so that by independence, $x_k = x'_k$. We may assume that $x_j \neq x'_j$ (otherwise, the proof is complete). Now applying w-strategy-proofness to agent i with true opinion p_i, $||p_i - x||_{w_i} \leq ||p_i - x'||_{w_i}$. It is simple to check that if $x_j = 1$ and $x'_j = 0$, the latter inequality is violated. Thus, $x_j = 0$ and $x'_j = 1$, so that $x \leq x'$.

A.8 Proof of Proposition 6

We only prove the necessity. Let φ be a strategy-proof rule. Let $p \in \mathcal{D}^N$ and $i \in N$. Let $p'_i, p''_i \in \mathcal{D}$ be two distinct opinions such that $p'_i >_{p_i} p''_i$. Since $\mathcal{D}$ is rich, there is a path $\{p_i^0, p_i^1, \ldots, p_i^\ell, \ldots, p_i^k\}$ from p_i to p''_i in $\mathcal{D}$ without restoration such that $p_i^\ell = p'_i$. For each $\tilde{\ell} \in \{0, \ldots, k\}$, let $x^{\tilde{\ell}} \equiv \varphi(p_i^{\tilde{\ell}}, p_{-i})$. Now we show that either $x^\ell = x^{\ell+1}$ or $x^\ell \, P_i \, x^{\ell+1}$. Since p_i^ℓ and $p_i^{\ell+1}$ is adjacent, there is exactly one $j \in N$ such that $p_{ij}^\ell \neq p_{ij}^{\ell+1}$. Since the path $\{p_i^0, p_i^1, \ldots, p_i^\ell, \ldots, p_i^k\}$ is without restoration, $p_{ij} = p_{ij}^\ell \neq p_{ij}^{\ell+1} = p''_{ij}$. Since $\mathcal{D}$ is rich, Corollary 1 applies. Thus, either (i) $x^\ell = x^{\ell+1}$; or (ii) $x_j^\ell = p_{ij}^\ell \neq p_{ij}^{\ell+1} = x_j^{\ell+1}$ and for each $h \in N \backslash \{j\}$, $x_h^\ell = x_h^{\ell+1}$. Combining this with $p_{ij} = p_{ij}^\ell \neq p_{ij}^{\ell+1} = p''_{ij}$, we obtain that either $x^\ell = x^{\ell+1}$ or $x^\ell \, P_i \, x^{\ell+1}$.

In fact, we can use the above argument to show that for each $\tilde{\ell} \in \{\ell, \ell+1, \ldots, k-1\}$, $x^{\tilde{\ell}} = x^{\tilde{\ell}+1}$ or $x^{\tilde{\ell}} \, P_i \, x^{\tilde{\ell}+1}$. Thus, $x^\ell = x^k$ or $x^\ell \, P_i \, x^k$.

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