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Economics
Discussion Paper Series
EDP-1318

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September 2013

Economics
School of Social Sciences
The University of Manchester
Manchester M13 9PL

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September 5, 2013

Abstract

We prove new equilibrium existence theorems for games with asymmetric information and discontinuous payoffs. In particular, the seminal work of Reny (1999) is extended to Bayesian and maximin preferences frameworks. We indicate that our results are applicable in situations that recent equilibrium existence theorems in the literature cannot be applied.

Keywords: Discontinuous Game; Asymmetric Information; Bayesian Expected Utility; Maximin Expected Utility

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1 Introduction

Games with discontinuous payoffs have been used to model a variety of important economic problems; for example, Hotelling location games, Bertrand competition, and various auction models. The seminal work by Reny (1999) proposed the "better reply secure" condition and proved the equilibrium existence in quasiconcave compact games with discontinuous payoffs. Since the hypotheses are sufficiently simple and easily verified, the increasing applications of his results has widened significantly in recent years, as evidenced by Jackson and Swinkels (2005) and Monteiro and Page (2008) among others. Recently a number of papers appeared in the topic of discontinuous games and further extensions have been obtained in several directions; see, for example, Lebrun (1996), Bagh and Jofre (2006), Bich (2009), Balder (2011), Carmona (2009, 2011), Prokopovych (2011), de Castro (2011), Reny (2011b, 2013), McLennan, Monteiro and Tourky (2011), Tian (2012) and Barelli and Meneghel (2013).

In this paper, we consider discontinuous games with asymmetric information; i.e., games with a finite set of players and each of whom is characterized by his own private information (which is a partition of an exogenously given state space representing the uncertainty of the world), a strategy set, a state dependent (random) utility function and a prior. This problem arises naturally in situations where privately informed agents behave strategically. Because of its importance, the research trend in this field has been quite active since Harsanyi's seminal work. The Harsanyi approach is based on the Bayesian doctrine; i.e., priors are defined for every state and the strategy of each player is assumed to be measurable with respect to his own private information. The later assumption is needed to capture the information asymmetry (see Remarks 1 and 9). Although several quite general existence of equilibrium theorems for Bayesian games are available in the literature, they are not applicable to situations that the payoffs are discontinuous. One of the main purpose of this paper is to provide new equilibrium existence results for Bayesian games with discontinuous payoffs.

Despite the fact that the Bayesian expected utility theory is widely accepted in economic theory, there are also non-expected utility theories which have obtained great success – recall that the Ellsberg's paradox is solved by using the maximin expected utility (see, for example, Schmeidler (1989) and de Castro and Yannelis (2013)). The maximin expected utility (MEU) approach abandons the Bayesian doctrine as agents need not be able to assign a probability to each state, and priors are defined on the information partition of each player. In this approach although the probability of an event $\{a, b\}$ in the partition of a player is well defined, this is not the case for the state a or b that he cannot distinguish. Thus, the information asymmetry is captured by the MEU itself and the strategies of each player need not be measurable with respect to the private information of each player, contrary to the Bayesian doctrine.

In the current paper, we adopt both approaches, (Bayesian and maximin expected utility) and prove new theorems on the existence of equilibria with discontinuous payoffs.

For the case that all the players have Bayesian preferences, we introduce the notions of finitely/finitely* payoff security and adopt the aggregate upper semicontinuity condition in the ex post games. We show that the (ex ante) Bayesian game is payoff secure and reciprocal upper semicontinuous, and hence Reny's theorem is applicable and a Bayesian equilibrium exists. A key issue here is that the quasiconcavity of the Bayesian game cannot be guaranteed even if all ex post games are quasiconcave. We show by means of counterexamples that the concavity and finitely payoff security conditions of the ex post games are both necessary for the existence of a Bayesian equilibrium.

In the case of more general preferences, we impose a monotonicity condition; i.e., if a player prefers the realizations of a strategy at each state, then he will prefer the strategy in the ex ante game. The maximin expected utility is a concrete example of such monotone preferences. We show that a combination of all ex post equilibrium strategies across all states is also an equilibrium strategy under the MEU in the ex ante game.

There are two main advantages for the MEU approach. Firstly, in Bayesian games, priors are defined at each state and strategies are required to be private information measurable. Thus, the information asymmetry in a Bayesian model is captured by the assumption of private information measurability. However, in the model with maximin expected utility, the information asymmetry is captured by the MEU itself and the restriction of private information measurability can be relaxed. Secondly, by adopting the maximin expected utility, we provide a direct link between ex ante equilibrium and ex post equilibrium which enables us to impose assumptions only on the primitive stage. Thus, no additional assumptions on the interim/ex ante utility functions are needed, and therefore the existing conditions on the deterministic game suffice to prove new equilibrium existence results for the maximin expected utility case.

There is a substantial literature on the existence of equilibrium in discontinuous games with asymmetric information and general action spaces. Based on the modularity idea, the existence of pure strategy monotone equilibrium has been obtained with huge success, see Athey (2001), McAdams (2003) and Reny (2011a) among others. As shown in Section 4.3, there are examples in which an equilibrium may not exist in their setting but it does exist in ours.[1]

The rest of the paper is organized as follows. In Section 2, we introduce a discontinuous game with asymmetric information. Section 3 presents the Bayesian games and proves the existence of a Bayesian equilibrium. In Section 4, we consider more general preferences and prove the existence of equilibrium under maximin expected utility. Some concluding remarks and open questions are collected in Section 5.

2 Asymmetric information game

We consider an **asymmetric information game**

$$G = \{\Omega, (u_i, X_i, \mathcal{F}_i)_{i \in I}\}.$$

- There is a finite set of players, $I = \{1, 2, \cdots, N\}$.
- Ω is a countable state space representing the **uncertainty** of the world, $\mathcal{F}$ is the power set of Ω.
- $\mathcal{F}_i$ is a partition of Ω, denoting the **private information** of player i. $\mathcal{F}_i(\omega)$ denotes the element of $\mathcal{F}_i$ including the state ω.
- Player i's action space X_i is a nonempty, compact, convex subset of a topological vector space, $X = \prod_{i \in I} X_i$.
- For every $i \in I$, $u_i : X \times \Omega \to \mathbb{R}$ is a **random utility function** representing the (ex post) preference of player i.

For player i, a **random strategy** is a function x_i from Ω to X_i and the set of random strategies is denoted by X_i^{Ω}. A game G is called a **compact** game if u_i is bounded for ever $i \in I$; i.e., $\exists M > 0$, $|u_i(x, \omega)| \leq M$ for all $x \in X$ and $\omega \in \Omega$, $1 \leq i \leq N$. A game G is said to be **quasiconcave (resp. concave)** if $u_i(\cdot, x_{-i}, \omega)$ is quasiconcave (resp. concave) for every $i \in I$, $x_{-i} \in X_{-i}$ and $\omega \in \Omega$. For every $\omega \in \Omega$, the **ex post game** is $G_\omega = (u_i(\cdot, \omega), X_i)_{i \in I}$.

[1]Based on a different approach using the communication device, Jackson et al. (2002) also studied discontinuous games with asymmetric information.

3 Existence of equilibrium: Bayesian preferences

3.1 Bayesian preferences

In this section, we assume that all players are Bayesian expected utility maximizers. To this end, suppose that each player has a **private prior** π_i on $\mathcal{F}$ such that $\pi_i(E) > 0$ for any $E \in \mathcal{F}_i$. The **weighted ex post game** is $G'_\omega = (w_i(\cdot,\omega), X_i)_{i\in I}$, where $w_i(\cdot,\omega)$ is a mapping from X to $\mathbb{R}$ and $w_i(\cdot,\omega) = u_i(\cdot,\omega)\pi_i(\omega)$ for each $\omega \in \Omega$.

For every player i, a **strategy** is an $\mathcal{F}_i$-measurable mapping from Ω to X_i. Let

$$L_i = \{f_i : \Omega \to X_i : f_i \text{ is } \mathcal{F}_i\text{-measurable}\},$$

then L_i is convex and compact. $L = \prod_{i\in I} L_i$.

Remark 1. *We adopt the private information measurability requirement for the strategies of all players. If this condition is relaxed, then players behave as they have symmetric information and the information partition does not influence the payoff of each player.*

Given a strategy profile $f \in L$, the **expected utility** of player i is

$$U_i(f) = \sum_{\omega\in\Omega} u_i(f_i(\omega), f_{-i}(\omega), \omega)\pi_i(\omega),$$

then $U_i(\cdot)$ is also bounded by M for every i. Therefore, the **(ex ante) Bayesian game** of G is

$$G_0 = (U_i, L_i)_{1\leq i\leq N},$$

which is also compact and concave if the game G is compact and concave.

Remark 2. *It is well known that quasiconcavity may not be preserved under summation or integration. Thus the Bayesian game G_0 may not be quasiconcave even if G is quasiconcave.*

A strategy profile $f \in L$ is said to be a **Bayesian equilibrium** if for each player i

$$U_i(f) \geq U_i(g_i, f_{-i})$$

for any $g_i \in L_i$.

3.2 Deterministic case

Hereafter, $G_d = (X_i, u_i)_{i=1}^N$ will denote a **deterministic discontinuous game**, i.e., Ω is a singleton.

The following definitions strengthen the payoff security definition of Reny (1999).

Definition 1. *In the game G_d, player i can secure an n-dimensional payoff* $(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ *at* $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$ *if there is* $\overline{x_i} \in X_i$*, such that* $u_i(\overline{x_i}, y_{-i}^k) \geq \alpha_k$ *for all* y_{-i}^k *in some open neighborhood of* x_{-i}^k*,* $1 \leq k \leq n$*.*

Definition 2. *The game G_d is n-**payoff secure** if for every $i \in I$ and* $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$*,* $\forall \epsilon > 0$*, player i can secure an n-dimensional payoff*

$$\left(u_i(x_i, x_{-i}^1) - \epsilon, \cdots, u_i(x_i, x_{-i}^n) - \epsilon\right)$$

at $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$*. The game G_d is said to be **finitely payoff secure** if it is n-payoff secure for any $n \in \mathbb{N}$.*[2]

*If $n = 1$, it is called **payoff secure**.*

Given $x \in X$, let $u(x) = (u_1(x), \cdots, u_N(x))$ be the payoff vector of the game G_d. Define $\Gamma_d = \{(x, u(x)) \in X \times \mathbb{R} : x \in X\}$, i.e., the graph of the payoff vector $u(\cdot)$, then $\overline{\Gamma_d}$ denotes the closure of Γ_d.

The following definition is due to Reny (1999).

Definition 3. *The game G_d is **better-reply secure** if whenever* $(x^*, \alpha^*) \in \overline{\Gamma_d}$ *and x^* is not a Nash equilibrium, some player j can secure a payoff strictly above α_i^* at x^*.*

In their pioneer paper, Dasgupta and Maskin (1986) proposed the following condition which is weaker than the upper semicontinuity condition of the utility functions.

Definition 4. *A game G_d is said to be **aggregate upper semicontinuous** if the summation of the utility functions of all players is upper semicontinuous.*

The following generalization is due to Simon (1987), which is called complimentary discontinuity or reciprocal upper semicontinuity.

Definition 5. *A game G_d is **reciprocal upper semicontinuous** if for any* $(x, \alpha) \in \overline{\Gamma_d} \setminus \Gamma_d$*, there is a player i such that* $u_i(x) > \alpha_i$*.*

Bagh and Jofre (2006) extended the reciprocal upper semicontinuous condition and show that the game G_d is better reply secure if it is payoff secure and weakly reciprocal upper semicontinuous.

[2] It is clear that the uniform payoff security condition of Monteiro and Page (2007) implies our finitely payoff security condition. One can construct counterexamples to show that the converse direction is not true.

Definition 6. *A game* G_d *is* ***weakly reciprocal upper semicontinuous*** *if for any* $(x, \alpha) \in \overline{\Gamma_d} \setminus \Gamma_d$*, there is a player* i *and* $\overline{x_i} \in X_i$ *such that* $u_i(\overline{x_i}, x_{-i}) > \alpha_i$*.*

Theorem 1 (Reny (1999))**.** *Every compact, quasiconcave and better-reply secure deterministic game has a Nash equilibrium.*

We will use this theorem to establish the existence results in Section 3.4 and 4.2. One may easily develop analogous definitions of "n-payoff security" in the framework of many recent papers; for example, Bagh and Jofre (2006), Prokopovych (2011), McLennan, Monteiro and Tourky (2011) and Barelli and Meneghel (2013).

3.3 Two counterexamples

In the literature, the utility function is required to be quasiconcave (see Reny (1999) and Prokopovych (2011) among others). The following example shows that this is not the case if asymmetric information is introduced and Bayesian preferences are adopted. The ex post utility functions in Example 1 below satisfy the conditions of Theorem 1 (indeed they satisfy all the conditions of Theorem 2 in Section 3.4 except concavity), but there is no Bayesian equilibrium.

Example 1 (Importance of concavity)**.**

Consider the following game G*. There are two players* $I = \{1, 2\}$ *competing for a object. The strategy spaces for players* 1 *and* 2 *are respectively* X *and* Y*,* $X = Y = [0, 1]$*. Player* 1 *has only one possible private value* 1*, and player* 2 *has two possible private values* 0 *and* 1*.*

Denote $a = (1, 1)$ *and* $b = (1, 0)$ *(the first component is the private value of player* 1 *and the second component is the private value of player* 2*). The state space is* $\Omega = \{a, b\}$*. The information partitions and priors are as follows:*

$$\mathcal{F}_1 = \{\{a, b\}\}, \pi_1(a) = \pi_1(b) = \frac{1}{2};$$

$$\mathcal{F}_2 = \{\{a\}, \{b\}\}, \pi_2(a) = \pi_2(b) = \frac{1}{2}.$$

For $\omega = a, b$*, the utility function of player* 1 *is*

$$u_1(x, y, \omega) = \begin{cases} 1 - x, & \text{if } x \geq y \\ 0, & \text{otherwise} \end{cases}.$$

Then $u_1(x, y, \cdot)$ *is measurable with respect to* $\mathcal{F}_1$ *for any* $(x, y) \in X \times Y$*.*

The utility function of player 2 is

$$u_2(x, y, a) = \begin{cases} 1 - y, & \text{if } y > x \\ 0, & \text{if } y \leq x \end{cases}$$

and

$$u_2(x, y, b) = \begin{cases} -y, & \text{if } y > x \\ \frac{-y}{2}, & \text{if } y \leq x \end{cases}.$$

1. *At both states, when there is a tie, player 1 will take the good and player 2 gets nothing.*
2. *At state b, the private value of player 2 is 0, bidding for positive price will harm both, thus player 2 will be punished when he bids more than 0 even if he loses the game.*

The ex post games G_a and G_b are 2-payoff secure. Consider the ex post game G_a and player 1. Given $\epsilon > 0$, $x \in X$ and $(y_1, y_2) \in Y \times Y$, player 1 can bid $x + \frac{\epsilon}{2}$ if it is no more than 1 and 0 otherwise. There are two possible cases for $i = 1, 2$.

1. *If $y_i \leq x$, then for any y_i' in a small neighborhood of y_i, $y_i' \leq \min\{x + \frac{\epsilon}{2}, 1\}$, thus the payoff of player 1 is at least $1 - x - \frac{\epsilon}{2}$.*
2. *If $y_i > x$, the payoff of player 1 is 0, which cannot be worse off.*

Similarly, one can show the 2-payoff security of player 2 at state a and b. Therefore, the ex post game is 2-payoff secure at each state. It is easy to see that the summations of ex post utility functions are upper semicontinuous at both states, and the assumptions of quasiconcavity and compactness are satisfied. Thus, there are Nash equilibria for both ex post games. At state a, the unique equilibrium is $(1, 1)$; at state b, the unique equilibrium is $(0, 0)$.

However, there is no Bayesian equilibrium in this game. Suppose (x, y) is an equilibrium, where $y = (y(a), y(b))$. In state b, player 2 will always choose $y(b) = 0$, thus player 1 can guarantee himself a positive payoff by choosing $x = 0$. But if $x < 1$, player 2 has no optimal strategy at state a. Thus, player 1 has to choose $x = 1$ and gets 0 payoff, a contradiction.

Remark 3. *In Example 1, although the ex post utility function is quasiconcave at both states, the expected utility function is not quasiconcave, and hence there is no Bayesian equilibrium.*

The second example shows that the payoff security of every ex post game can not guarantee the payoff security of the Bayesian game.

Example 2 (Ex post payoff security does not imply ex ante payoff security).

Consider the following game: the player space is $I = \{1,2,3\}$*, the state space is* $\Omega = \{a,b\}$*, and the information partitions of all players are* $\mathcal{F}_1 = \mathcal{F}_2 = \{\{a,b\}\}$ *and* $\mathcal{F}_3 = \{\{a\},\{b\}\}$*. Players have common prior* $\pi(a) = \pi(b) = \frac{1}{2}$*. The action space of player* i *is* $X_i = [0,1]$*,* $i = 1,2,3$*. The games* L *and* R *are listed below.*

In both states, players 1 *and* 2 *will play the game* L *if* $x_3 = 0$ *and the game* R *otherwise. Player* 1*'s action is in the left and player* 2*'s action is in the top.*

	0	$(0,1)$	1
0	$(1,6)$	$(0,7)$	$(3,7)$
$(0,1)$	$(5,4)$	$(4,5)$	$(3,7)$
1	$(6,3)$	$(6,3)$	$(4,5)$

L

	0	$(0,1)$	1
0	$(14,14)$	$(16,10)$	$(16,10)$
$(0,1)$	$(13,15)$	$(14,14)$	$(14,14)$
1	$(13,15)$	$(12,15)$	$(12,15)$

R

The utility function of player 3 *is defined as follow:*

$$u_3(x_1,x_2,x_3,\omega) = \begin{cases} 1, & \text{if } x_3 = 0 \text{ at } \omega = a \text{ or } x_3 \in (0,1] \text{ at } \omega = b; \\ 0, & \text{otherwise.} \end{cases}$$

Below we study the ex post game G_a *and show that it is payoff secure but not* 2*-payoff secure. The same result holds for the ex post game* G_b*. However, the Bayesian game is not payoff secure.*

In the game L*, player* 1 *can choose the dominated strategy* $x_1 = 1$ *and player* 2 *can choose the dominated strategy* $x_2 = 1$*, thus the game* L *is payoff secure. In the game* R*, player* 1 *can choose the dominated strategy* $x_1 = 0$ *and player* 2 *can choose the dominated strategy* $x_2 = 0$*, thus the game* R *is payoff secure.*

Suppose state a *realizes. The payoff of player* 3 *is secured since he can always choose* $x_3 = 0$*, which could guarantee his highest payoff. For players* 1 *and* 2*, if player* 3*'s action* $x_3 = 0$*, then players* 1 *and* 2 *will play the game* L *and it is payoff secure since if* x_3 *deviates in a small neighborhood, then players* 1 *and* 2 *will play the game* R *and their payoffs are strictly higher; if* x_3 *stays unchanged and they are still in game* L*, then the payoff security of the game* L *supports our claim. If player* 3*'s action* $x_3 \in (0,1]$*, they will play game* R *and it is payoff secure since a sufficiently small neighborhood of* x_3 *is still included in* $(0,1]$ *and the game* R *itself is payoff secure. Therefore, the ex post game* G_a *is payoff secure.*

However, this game is not 2*-payoff secure. For example, let* $x_1 = 0$*,* $(x_2^1, x_3^1) = (1,0)$ *and* $(x_2^2, x_3^2) = (1,1)$*, there is no action which could guarantee that player* 1 *can secure the* 2 *dimensional payoff vector* $(3,16)$*. Similarly, one could show that the ex post game* G_b *is also payoff secure but not* 2*-payoff secure.*

Finally, we verify our claim that the Bayesian game is not payoff secure. Let *the strategy of player 3 be $x_3 = (x_3(a), x_3(b)) = (1, 0)$, the expected utilities for players 1 and 2 are listed as the following game E. Then player 1 cannot secure*

	0	$(0,1)$	1
0	$(\frac{15}{2}, 10)$	$(8, \frac{17}{2})$	$(\frac{19}{2}, \frac{17}{2})$
$(0,1)$	$(9, \frac{19}{2})$	$(9, \frac{19}{2})$	$(\frac{17}{2}, \frac{21}{2})$
1	$(\frac{19}{2}, 9)$	$(9, 9)$	$(8, 10)$

E

his payoff if $x_1 = 1$ and $x_2 = 0$, and player 2 cannot secure his payoff if $x_1 = 0$ and $x_2 = 0$.

Moreover, this game does not have a Bayesian equilibrium. It is easy to see that player 3 will choose $x_3(a) = 0$ and $x_3(b) \in (0, 1]$. Consequently, the expected payoff matrix of players 1 and 2 is E. However, the game E has no equilibrium.

Remark 4. *The game in Example 2 is obviously compact and satisfies the private information measurability requirement. We will show that the Bayesian game is quasiconcave.*

It is clear that the expected utility of player 3 is quasiconcave. Now we consider players 1 and 2. Given $x_3 = (x_3(a), x_3(b))$. If $x_3 = (0, 0)$, then players 1 and 2 will play the game L in both states. Their expected payoff matrix is L, which is quasiconcave. If $x_3 \in (0, 1] \times (0, 1]$, players 1 and 2 will play the game R in both states, and hence their expected payoff matrix is the quasiconcave game R. Otherwise, players 1 and 2 will play the game L at one state and the game R at the other state. That is, their expected payoff matrix is E, which is also quasiconcave.

In the next section, we provide some positive results for the existence of Bayesian equilibria.

3.4 Existence of Bayesian equilibrium

Propositions 1 and 2 below provide sufficient conditions to guarantee the payoff security of the Bayesian game.

Proposition 1. *If the weighted ex post game G'_ω is finitely payoff secure at every state $\omega \in \Omega$ and $u_i(x, \cdot)$ is $\mathcal{F}_i$-measurable for every $x \in X$ and $i \in I$, then the Bayesian game G_0 is payoff secure.*

Proof. For any $i \in I$, suppose $\mathcal{F}_i = \{E_1, \cdots, E_k, \cdots\}$ is the information partition of player i, M is the bound for u_i. Given any $\epsilon > 0$, there exists a positive integer $K > 0$ such that $\pi_i(\cup_{k=1}^{K} E_k) > 1 - \frac{\epsilon}{3M}$. For $1 \leq k \leq K$, there exists a finite subset $E'_k \subseteq E_k$ such that $\pi_i(E_k \setminus E'_k) < \frac{\epsilon}{3KM}$ and $\pi_i(E'_k) > 0$.

Fix $\omega_k \in E'_k$ such that $\pi_i(\omega_k) > 0$. Given any $f \in L$, because $u_i(x, \cdot)$ and $f_i(\cdot)$ are both $\mathcal{F}_i$-measurable,

$$u_i(f_i(\omega), f_{-i}(\omega), \omega) = u_i(f_i(\omega_k), f_{-i}(\omega), \omega_k)$$

for any $\omega \in E_k$, $1 \leq k \leq K$.

Since G'_{ω_k} is finitely payoff secure, there exists a point $y_i^k \in X_i$, such that

$$w_i(y_i^k, y_{-i}^\omega, \omega_k) \geq w_i(f_i(\omega_k), f_{-i}(\omega), \omega_k) - \frac{\epsilon}{3}\pi_i(\omega_k)$$

for all y_{-i}^ω in some open neighborhood O_ω of $f_{-i}(\omega)$, $\forall \omega \in E'_k$.

Let

$$g_i(\omega) = \begin{cases} y_i^k, & \text{if } \omega \in E_k \text{ for } 1 \leq k \leq K, \\ f_i(\omega), & \text{otherwise.} \end{cases}$$

Then by construction g_i is $\mathcal{F}_i$-measurable.

For any open set O in L_{-i} such that $O \subseteq \left(\prod_{1\leq k\leq K}(\prod_{\omega\in E'_k} O_\omega \times X_{-i}^{E_k \setminus E'_k})\right) \times X_{-i}^{\Omega\setminus\cup_{1\leq k\leq K}E_k}$,

$$\begin{aligned}
U_i(g_i, g'_{-i}) &= \sum_{E\in\mathcal{F}_i}\sum_{\omega\in E} w_i(g_i(\omega), g'_{-i}(\omega), \omega) \\
&\geq \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(g_i(\omega), g'_{-i}(\omega), \omega) \\
&= \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(y_i^k, g'_{-i}(\omega), \omega_k)\frac{\pi_i(\omega)}{\pi_i(\omega_k)} \\
&\geq \sum_{k=1}^{K}\sum_{\omega\in E'_k} [w_i(f_i(\omega_k), f_{-i}(\omega), \omega_k) - \frac{\epsilon}{3}\pi_i(\omega_k)]\frac{\pi_i(\omega)}{\pi_i(\omega_k)} \\
&\geq \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(f_i(\omega_k), f_{-i}(\omega), \omega) - \frac{\epsilon}{3} \\
&\geq \sum_{E\in\mathcal{F}_i}\sum_{\omega\in E} w_i(f_i(\omega_k), f_{-i}(\omega), \omega) - \frac{\epsilon}{3} - M\left(\pi_i(\Omega \setminus (\cup_{k=1}^K E_k)) + \sum_{k=1}^{K}\pi_i(E_k \setminus E'_k)\right) \\
&> U_i(f) - \epsilon
\end{aligned}$$

for every $g'_{-i} \in O$. Thus G_0 is payoff secure. $\square$

Remark 5. *Note that the finitely payoff security of the weighted ex post game $G'_\omega = (w_i(\cdot, \omega), X_i)_{i\in I}$ is slightly weaker than the finitely payoff security of the ex post game $G_\omega = (u_i(\cdot, \omega), X_i)_{i\in I}$, where u_i is the ex post payoff function and*

$w_i(\cdot,\omega) = u_i(\cdot,\omega)\cdot\pi_i(\omega)$ *for every* $i \in I$*. These two conditions will be equivalent if* $\pi_i(\omega) > 0$ *for any* $i \in I$ *and* $\omega \in \Omega$*.*

In Proposition 1, the ex post utility functions are required to be private information measurable. This assumption can be dropped if the finitely payoff security condition is strengthened accordingly.

Definition 7. *An asymmetric information game* G *is* n^****-payoff secure*** *if for every* $i \in I$*, every* $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$ *and every* $(\omega_1, \cdots, \omega_n) \subseteq D$ *for some* $D \in \mathcal{F}_i$*,* $\forall \epsilon > 0$*, there is* $\overline{x_i} \in X_i$*, such that* $u_i(\overline{x_i}, y_{-i}^k, \omega_k) \geq u_i(x_i, x_{-i}^k, \omega_k) - \epsilon$ *for all* y_{-i}^k *in some open neighborhood of* x_{-i}^k*,* $1 \leq k \leq n$*.*

The game G *is said to be* ***finitely**** ***payoff secure*** *if it is* n^**-payoff secure for any* $n \in \mathbb{N}$*.*

Proposition 2. *The Bayesian game* G_0 *is payoff secure if* G *is finitely***-payoff secure.*

Proof. As in the proof of Proposition 1, we could find some positive integer K and finite set E_k' for each $1 \leq k \leq K$ satisfying the same conditions therein.

Given any $f \in L$. Since G is finitely* payoff secure, for each $1 \leq k \leq K$, there exists a point $y_i^k \in X_i$, such that

$$u_i(y_i^k, y_{-i}^\omega, \omega) \geq u_i(f_i(\omega), f_{-i}(\omega), \omega) - \frac{\epsilon}{3}$$

for all y_{-i}^ω in some open neighborhood O_ω of $f_{-i}(\omega)$, $\forall \omega \in E_k'$.

Let

$$g_i(\omega) = \begin{cases} y_i^k, & \text{if } \omega \in E_k \text{ for } 1 \leq k \leq K, \\ f_i(\omega), & \text{otherwise.} \end{cases}$$

Then the rest of the proof proceeds similarly as in the proof of Proposition 1. $\square$

Proposition 3. *In the game* G*, if the weighted ex post game* G_ω' *is aggregate upper semicontinuous at every state* $\omega \in \Omega$*, then the Bayesian game* G_0 *is reciprocal upper semicontinuous.*

Proof. By way of contradiction, suppose that the Bayesian game G_0 is not reciprocal upper semicontinuous. Then there exists a sequence $\{f^n\} \subseteq L$, $f^n \to f$ and $U(f^n) \to \alpha$ as $n \to \infty$, where $U(f) = (U_1(f), \cdots, U_N(f))$ and $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{R}^N$. $U_i(f) \leq \alpha_i$ for $1 \leq i \leq N$ and $U(f) \neq \alpha$.

Denote $\epsilon = \max_{1 \leq i \leq N}(\alpha_i - U_i(f))$, $\epsilon > 0$. Thus,

$$\sum_{i \in I} U_i(f) \leq \sum_{i \in I} \alpha_i - \epsilon.$$

There exists a finite subset $E \subseteq \Omega$ such that $\pi_i(\Omega \setminus E) < \frac{\epsilon}{2NM}$ for every $i \in I$, where M is the bound of u_i for all i.

Then for any $i \in I$, $U_i(f^n)$ can be divided into two parts: $\mu_i^n = \sum_{\omega \in E} w_i(f^n(\omega), \omega)$ and $\nu_i^n = \sum_{\omega \notin E} w_i(f^n(\omega), \omega)$, $U_i(f^n) = \mu_i^n + \nu_i^n$. Let $\mu^n = \{\mu_i^n\}_{i \in I}$, since $\{\mu^n\}_{n \in \mathbb{N}}$ is bounded, there is a subsequence, say itself, which converges to some $\mu \in \mathbb{R}^N$. Since $\nu_i^n \leq M\pi_i(\Omega \setminus E) < \frac{\epsilon}{2N}$ for any $i \in I$ and $n \in \mathbb{N}$, $\mu_i \geq \alpha_i - \frac{\epsilon}{2N}$ for every $i \in I$.

At each state $\omega \in E$ and $i \in I$, since $w_i(f^n(\omega), \omega)$ is bounded, there is a subsequence which converges to some β_i^ω. Since there are only finitely many players and states, we can assume without loss of generality that $w_i(f^n(\omega), \omega) \to \beta_i^\omega$ as $n \to \infty$, then $\sum_{\omega \in E} \beta_i^\omega = \mu_i$.

Since $f^n(\omega) \to f(\omega)$ for every $\omega \in E$ and G'_ω is aggregate upper semicontinuous,

$$\sum_{i \in I} w_i(f(\omega), \omega) \geq \sum_{i \in I} \beta_i^\omega.$$

Thus

$$\sum_{i \in I} U_i(f) \geq \sum_{i \in I} \sum_{\omega \in E} w_i(f(\omega), \omega) \geq \sum_{i \in I} \sum_{\omega \in E} \beta_i^\omega = \sum_{i \in I} \mu_i \geq \sum_{i \in I} \alpha_i - \frac{\epsilon}{2},$$

which is a contradiction. □

By combining Theorem 1, Proposition 1/Proposition 2 and Proposition 3, we obtain the following result which is an extension of Reny (1999) to a framework of asymmetric information.

Theorem 2. *Suppose that an asymmetric information game G is compact, the corresponding Bayesian game G_0 is quasiconcave, and the weighted ex post game G'_ω is aggregate upper semicontinuous at each state ω. Then a Bayesian equilibrium exists if either of the following conditions holds.*

1. *The weighted ex post game G'_ω is finitely payoff secure at every state $\omega \in \Omega$ and $u_i(x, \cdot)$ is $\mathcal{F}_i$-measurable for every $x \in X$ and $i \in I$.*

2. *The game G is finitely* payoff secure.*

Remark 6. *Note that the (ex ante) Bayesian game G_0 is assumed to be quasiconcave. However, Example 1 indicates that the theorem may fail if we only require that G is quasiconcave. To impose conditions in the primitive stage, one possible alternative is to require that G be concave. However, the concavity of the utility function implies that it is continuous on the interior of its domain, and hence the discontinuity only arises on the boundary. This is a rather strong assumption and will deter many possible applications.*

Remark 7. *The existence of Bayesian equilibria with uncountably many states in asymmetric information discontinuous games is an open question. If Ω is a general probability space, the arguments in Propositions 1-3 do not hold even in the case that the finitely payoff security condition is strengthened by the uniform payoff security condition of Monteiro and Page (2007).*

In the next section we consider more general preferences.

4 Existence of equilibrium: general preferences

4.1 Existence

In the previous section we proved the existence of Bayesian equilibria in an asymmetric information discontinuous game by imposing conditions of private information measurability, finitely payoff security and concavity. We now show that all these conditions could be dropped and still prove the existence of an equilibrium.

For any player $i \in I$, a random strategy is a mapping from Ω to X_i and the strategy space is $\mathcal{L}_i = X_i^{\Omega}$. Let $\mathcal{L} = \prod_{i \in I} \mathcal{L}_i$. The ex ante utility of player i is a mapping V_i from $\mathcal{L}$ to $\mathbb{R}$. The **ex ante game** is denoted as $\overline{G_0} = (V_i, \mathcal{L}_i)_{i \in I}$.

The assumption below is self-explanatory.

Assumption. *For player $i \in I$, the ex ante utility V_i is said to be* ***monotone*** *if given any two random strategies f and g such that $u_i(f(\omega), \omega) \geq u_i(g(\omega), \omega)$ for each $\omega \in \Omega$, then $V_i(f) \geq V_i(g)$.*

Remark 8. *Note that the Bayesian expected utilities are not monotone since the strategies of each player are required to be measurable with respect to his private information, and a counterexample can be easily constructed.*

The following lemma shows that the relation between the ex post equilibrium and the ex ante equilibrium is simple by adopting monotone ex ante utilities: if one can find a Nash equilibrium for each ex post game, then a combination of all ex post equilibrium strategies must be an equilibrium strategy in the ex ante game.

Lemma 1. *In an asymmetric information game G with monotone ex ante utilities, if every ex post game G_ω has a Nash equilibrium, then there exists an ex ante equilibrium in $\overline{G_0}$.*[3]

[3]The ex ante equilibrium in the game $\overline{G_0}$ is the equilibrium in the sense of Nash.

Proof. Suppose that $f(\omega)$ is an equilibrium in the ex post game G_ω for every $\omega \in \Omega$. Then f is an equilibrium in G. Indeed, for any $i \in I$ and $g_i \in \mathcal{L}_i$, $u_i(f(\omega), \omega) \geq u_i(g_i(\omega), f_{-i}(\omega), \omega)$ for each $\omega \in \Omega$. By monotonicity we have $V_i(f) \geq V_i(g_i, f_{-i})$, which implies that f is an ex ante equilibrium in $\overline{G_0}$. □

4.2 Maximin expected utility

In this section, we adopt the maximin expected utilities and show that the monotonicity condition is satisfied.

Suppose that players will consider the ex ante utilities based on the worst possible state that can occur and behave as lowest bound utility maximizers. That is, players have maximin preferences. For each player i, the **private prior** π_i is defined on $\mathcal{F}_i$ such that $\pi_i(E) > 0$ for any $E \in \mathcal{F}_i$. Given a strategy profile $f \in \mathcal{L}$, for each player $i \in I$, her **maximin expected utility (MEU)** is

$$V_i(f) = \sum_{E \in \mathcal{F}_i} \inf_{\omega \in E} u_i(f(\omega), \omega)\pi_i(E).$$

Remark 9. *In games with maximin preferences, priors are defined on the information partition of each player (not on each state of nature). Thus, the information asymmetry is captured by the MEU itself, and hence it is natural to relax the restriction of private information measurability. Conversely, the information asymmetry in a Bayesian model is captured by the assumption of private information measurability. As pointed out in Remark 1, players may behave as in a symmetric information game if this assumption is relaxed.*

In Example 1, we provided a game which satisfies all the conditions of Theorem 2 except concavity, and a Bayesian equilibrium fails to exist. This result shows that the concavity is necessary to guarantee the existence of equilibrium when players are Bayesian preference maximizes. Below we show that contrary to the Bayesian framework, if all players are maximin preference maximizes, then an equilibrium exists.

Example 3 (Example 1 under MEU)**.**

Suppose that both players have MEU in Example 1. At state a, the unique ex post Nash equilibrium is $(x, y) = (1, 1)$*; at state b, the unique ex post Nash equilibrium is* $(x, y) = (0, 0)$*. Let* $x_1 = (x_1(a), x_1(b)) = (1, 0)$ *and* $x_2 = (x_2(a), x_2(b)) = (1, 0)$*, then it is easy to see that this is an equilibrium under MEU.*

The lemma below shows that the monotonicity condition is satisfied for maximin expected utilities.

Lemma 2. *The maximin expected utilities are monotone.*

Proof. We will show that for any $i \in I$, given any two random strategies f and g such that $u_i(f(\omega), \omega) \geq u_i(g(\omega), \omega)$ at each state $\omega \in \Omega$, it follows that $V_i(f) \geq V_i(g)$.

Suppose otherwise, i.e., $V_i(f) < V_i(g)$. Then there exists at least one element E in the information partition $\mathcal{F}_i$ such that $\inf_{\omega \in E} u_i(g(\omega), \omega) > \inf_{\omega \in E} u_i(f(\omega), \omega)$, which implies that there exists a state $\omega_1 \in E$ such that

$$u_i(g(\omega_1), \omega_1) \geq \inf_{\omega \in E} u_i(g(\omega), \omega) > u_i(f(\omega_1), \omega_1) \geq \inf_{\omega \in E} u_i(f(\omega), \omega),$$

which is a contradiction. □

Therefore, the existence of equilibria under MEU is immediate.

Theorem 3. *If an asymmetric information game G is compact and quasiconcave, every ex post game G_ω is payoff secure and weakly reciprocal upper semicontinuous at state ω, players are all maximin preference maximizes, then there exists an equilibrium.*

Proof. For any $\omega \in \Omega$, the ex post game G_ω is payoff secure and weakly reciprocal upper semicontinuous, thus it is better reply secure. By Theorem 1 there exists a Nash equilibrium in G_ω. Thus by combining Lemmas 1 and 2, one can complete the proof. □

4.3 Timing games with asymmetric information

We study a class of two-person, non-zero-sum, noisy timing games with asymmetric information. Such games can be used to model behavior in duels as well as in R&D and patent races. In the setup of complete information, these games have been extensively discussed (see, for example, Reny (1999), Bagh and Jofre (2006) and Barelli and Meneghel (2013)). Below we consider the asymmetric information case.

Let G be an asymmetric information timing game. The state space is Ω. For player i, the information partition is denoted as $\mathcal{F}_i$ and the private prior π_i is defined on $\mathcal{F}_i$. The action space for both players is $[0, 1]$. At state ω, the payoff to player i is given by

$$u_i(a_i, a_{-i}, \omega) = \begin{cases} p_i(x_i, \omega), & \text{if } x_i < x_{-i} \\ q_i(x_i, \omega), & \text{if } x_i = x_{-i} \\ h_i(x_{-i}, \omega), & \text{otherwise} \end{cases}.$$

Suppose that the following conditions hold for $i = 1, 2$, every $\omega \in \Omega$ and $x \in [0, 1]$.

1. $p_i(\cdot, \omega)$ and $h_i(\cdot, \omega)$ are both continuous and $p_i(\cdot, \omega)$ is nondecreasing,
2. $q_i(x, \omega) \in \text{co}\{p_i(x, \omega), h_i(x, \omega)\}$,[4]
3. if $q_i(x, \omega) + q_{-i}(x, \omega) < p_i(x, \omega) + h_{-i}(x, \omega)$, then $\text{sgn}(p_i(x, \omega) - q_i(x, \omega)) = \text{sgn}(q_{-i}(x, \omega) - h_{-i}(x, \omega))$.[5]

As shown in Reny (1999), each ex post game is compact, quasiconcave and payoff secure. We claim that each ex post game is reciprocal upper semicontinuous. If $q_i(x, \omega) + q_{-i}(x, \omega) \leq p_i(x, \omega) + h_{-i}(x, \omega)$, then we have that $\text{sgn}(p_i(x, \omega) - q_i(x, \omega)) = \text{sgn}(q_{-i}(x, \omega) - h_{-i}(x, \omega))$. This case has already been discussed by Reny (1999), we only need to consider the case that $q_i(x, \omega) + q_{-i}(x, \omega) > p_i(x, \omega) + h_{-i}(x, \omega)$. Then the reciprocal upper semicontinuity is obvious since there must be some $i \in \{1, 2\}$ such that $q_i(x, \omega) > p_i(x, \omega)$ or $q_i(x, \omega) > h_i(x, \omega)$.

Therefore, if the conditions above hold and the ex ante utility functions of both players are monotone, then the asymmetric information timing game has an ex ante equilibrium due to Lemma 1. In particular, an equilibrium exists if both players are maximin preference maximizers.

However, the following example shows that an asymmetric information timing game may not possess an equilibrium when players have Bayesian preferences. Moreover, a monotone equilibrium does not exist either; i.e., the approach based on modularity ideas (see Athey (2001), McAdams (2003) and Reny (2011a)) is not applicable. However, our Theorem 2 can be used to show the existence of an equilibrium under the MEU as indicated above.

Remark 10 (Nonexistence of Bayesian equilibria)**.**

The state space is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, *where* $\omega_1 = (\frac{1}{2}, \frac{1}{2})$, $\omega_2 = (\frac{1}{2}, 1)$, $\omega_3 = (1, 1)$, $\omega_4 = (1, \frac{1}{2})$. *The information partitions are* $\mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ *and* $\mathcal{F}_2 = \{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\}$. *The ex post utility functions of players at state* $\omega = (t_1, t_2)$ *are given as in the general model, where* $p_i(x, \omega) = x - t_i$, $h_i(x, \omega) \equiv 0$ *and*

$$q_i(x, \omega) = \begin{cases} x - t_i, & \text{if } t_i < t_{-i} \\ \frac{x - t_i}{2}, & \text{if } t_i = t_{-i} \\ 0, & \text{if } t_i > t_{-i} \end{cases}.$$

Players 1 *and* 2 *hold the common prior:* $\pi(\omega_1) = \pi(\omega_2) = \pi(\omega_3) = \frac{1}{3}$ *and* $\pi(\omega_4) = 0$.

It is easy to see that this game satisfies all the above conditions, and hence possesses an equilibrium when both players are maximin preference maximizers. We

[4]The notation co(A) denotes the convex hull of the set A.

[5]Note that this condition is slightly weaker than the corresponding condition in Example 3.1 of Reny (1999). The example in Remark 10 satisfies our condition, but does not satisfy the condition of Reny (1999).

claim that there is no Bayesian equilibrium in this game. By way of contradiction, suppose that (x_1, x_2) *is a Bayesian equilibrium.*

We shall first show that at state $\omega = (t_1, t_2)$, $x_i(\omega) \geq t_i$ *for* $i = 1, 2$. *It is clear that* $x_1(\omega), x_2(\omega) \geq \frac{1}{2}$ *for any* $\omega \in \Omega$, *thus we only need to show that* $x_1(\omega_3) = x_1(\omega_4) = 1$ *and* $x_2(\omega_2) = x_2(\omega_3) = 1$. *Suppose that* $x_1(\omega_3) = x_1(\omega_4) < 1$. *If* $x_2(\omega_3) < x_1(\omega_3)$, *then player* 2 *gets a negative payoff at the event* $\{\omega_2, \omega_3\}$, *he can choose* $x_2(\omega_2) = x_2(\omega_3) = 1$ *to be strictly better off; if* $x_2(\omega_3) \geq x_1(\omega_3)$, *then player* 1 *gets a negative payoff at the event* $\{\omega_3, \omega_4\}$, *he can choose* $x_1(\omega_3) = x_1(\omega_4) = 1$ *to be strictly better off. Thus,* $x_1(\omega_3) = x_1(\omega_4) = 1$. *Then we have that* $x_2(\omega_2) = x_2(\omega_3) = 1$, *otherwise player* 2 *will get a negative payoff at the evert* $\{\omega_2, \omega_3\}$.

Now we consider the choice of player 2 *at state* ω_1.

1. *If* $x_2(\omega_1) = \frac{1}{2}$, *then the best response of player* 1 *at the event* $\{\omega_1, \omega_2\}$ *is to choose the strategy* $x_1(\omega_1) = x_1(\omega_2) = 1$. *However, in this case, there is no best response for player* 2 *at the state* ω_1.
2. *If* $x_2(\omega_1) = 1$, *then there is no best response for player* 1 *at the event* $\{\omega_1, \omega_2\}$.
3. *Suppose that* $x_2(\omega_1) = a \in (\frac{1}{2}, 1)$. *If* $x_1(\omega_1) = x_1(\omega_2) \in [\frac{1}{2}, a)$, *then player* 1 *can always slightly increase his strategy to be strictly better off. If* $x_1(\omega_1) = x_1(\omega_2) = a$, *then player* 1 *can always slightly decrease his strategy to be strictly better off. If* $x_1(\omega_1) = x_1(\omega_2) \in (a, 1]$, *then the best response of player* 1 *must be* $x_1(\omega_1) = x_1(\omega_2) = 1$; *as shown in the first point, there is no best response for player* 2.

Therefore, there is no Bayesian equilibrium. Moreover, the nonexistence of monotone equilibria in this example follows the same argument.

5 Concluding remarks

Our purpose was to impose the same assumptions of Reny (1999) on primitives and prove new equilibrium existence theorems for asymmetric information games. We showed that if players are Bayesians, the conditions of Reny (1999) need to be strengthened. By introducing a new payoff security condition which is a strengthening of the one of Reny (1999), we showed that if our new condition is imposed on the weighted ex post utility functions, then the ex ante expected utility is payoff secure. In view of this result and by assuming that the expected utilities are quasiconcave, we proved an equilibrium existence theorem for discontinuous games with asymmetric information. Also, we pointed out that the concavity assumption plays an important role; specifically, if the ex post utility function of each player is not concave, then a Bayesian equilibrium may not exist

Moreover, we showed that if players are non-expected utility maximizers, in particular, maximin expected utility maximizers, then by introducing a monotonicity condition, we were able to generalize the existence theorem of Reny (1999) to discontinuous games with asymmetric information.

Finally, we showed that our equilibrium result with maximin preferences is applicable to situations that none of the existing theorems in the literature can be applied.

It remains an open question if an equilibrium under the MEU exists in discontinuous games with uncountably many states. Indeed, if Ω is a general state space, weak topologies on the strategy space need to be introduced in order for the MEU to attain a minimum. The existence of an equilibrium in asymmetric information discontinuous games with a continuum of players (with either expected utilities or non-expected utilities) is another interesting open problem.

It seems that the results of this paper can be extended to a social system or abstract economies à la Debreu with asymmetric information and discontinuous expected payoffs. Such results can be applied to concrete economies with asymmetric information and they will enable us to obtain competitive equilibrium/rational expectations equilibrium results with discontinuous expected utility payoffs. We hope to take up those details in subsequent work.

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