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A Heteroskedasticity-Robust F-Test Statistic for Individual Effects

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Abstract

We derive the asymptotic distribution of the standard F-test statistic for fixed effects, in static linear panel data models, under both non-normality and heteroskedasticity of the error terms, when the cross-sections dimension is large but the time series dimension is fixed. It is shown that a simple linear transformation of the F-test statistic yields asymptotically valid inferences and under local fixed (or correlated) individual effects, this heteroskedasticity-robust F-test enjoys higher asymptotic power than a suitably robustified Random Effects test. Wild bootstrap versions of these tests are considered which, in a Monte Carlo study, provide more reliable inference in finite samples.

1 Introduction

In an earlier paper, Orme and Yamagata (2006) added to the already large literature on the analysis of variance testing, by establishing that, in a static linear panel data model, the standard *F-test* for individual effects remains asymptotically valid (large N, fixed T) under non-normality of the error term. Moreover, their (local) asymptotic analysis, supported by Monte Carlo evidence, showed that under (pure) local random effects both the *F-test* and *Random Effects test* (*RE-test*) will have similar power whilst under local fixed effects, or random effects which are correlated with the regressors, the *RE-test* procedure will have lower asymptotic power than the *F-test* procedure.

The key result in the above paper (Proposition 1, p.409) is, essentially, the asymptotic equivalence of the appropriately centred *F-test* statistic and the numerator (test indicator) in the *RE-test* statistic, under homoskedastic, but not necessarily normally distributed, errors. However, it is straightforward to verify (Proposition 1 in Section 3.2 below) that this asymptotic equivalence continues to hold under

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general heteroskedasticity of the errors.[1] The analysis which produces this result also shows that although under quite general, but neglected, heteroskedasticity, the standard (homoskedastic-based) F and RE tests may be asymptotically under, or over, sized, predictions can be made in certain cases. For example, and in addition to the maintained assumption of independent cross-sections, when the linear model error terms are also serially independent, then: (i) if the (unconditional) error variance is constant within a cross section of data, but not across cross-sections, both tests will be asymptotically oversized; (ii) on the other hand, if the (unconditional) error variance is constant across cross-sections, but not through time, both tests will be asymptotically undersized; (iii) furthermore, in the singular case of independently and identically distributed (i.i.d.) data, over *both* the cross-section and time dimensions, then even if the errors are conditionally heteroskedastic, the standard F and RE tests remain asymptotically valid. The assumptions in this paper explicitly allow for independently but not identically distributed data and, therefore, unconditional heteroskedasticity in the errors.

Given the result of Proposition 1, below, Wooldridge's (2010, p.299) heteroskedastic-robust *RE-test* suggests the appropriate transformation required of the standard *F-test* statistic in order to recover its asymptotically validity under general heteroskedasticity of unknown form. This transformation, or correction, involves simple functions of the pooled model's residuals (i.e., the restricted residuals), of which there are a number of asymptotically valid choices. Following the literature on heteroskedasticity robust inference, restricted residuals are employed as advocated, for example, by Davidson and MacKinnon (1985) and Godfrey and Orme (2004), who report reliable sampling performance of tests of linear restrictions in the linear model when employing restricted residuals in the construction of heteroskedasticity robust standard errors.[2]

Importantly, though, the F and RE heteroskedastic-robust tests, so constructed, retain the qualitative properties that were reported by Orme and Yamagata (2006). Specifically: (i) under (pure) local random effects, both tests have the same asymptotic power; and, (ii) under local fixed effects, or random effects which are correlated with the regressors, the *RE-test* procedure will have lower asymptotic power than the *F-test* procedure.

The plan of this paper is as follows. In order to make the current paper self-contained, Section 2 reproduces Orme and Yamagata (2006, Section 2) and introduces the notation and standard test statistics as discussed widely in standard texts; for example Baltagi (2008). Assumptions are introduced in Section 3, justifying the ensuing asymptotic analysis in Section 3.2 which characterises the asymptotic behaviour of the *F-test* statistic, including its relationship with the *RE-test* statistic under the null and local alternatives. All proofs of the main results are relegated to the Appendix. Section 4 illustrates the main findings by reporting the results of a small Monte Carlo study. This also includes an evaluation of a wild bootstrap

[1] Orme and Yamagata (2006) did not cover the case of heteroskedastic errors in the linear model, although their analysis did allow for heteroskedastic individual effects.

[2] As Wooldridge (2010, p.300) points out, standard tests for individual effects essentially test for non-zero correlation in the errors; thus, constructing auto-correlation robust procedures would appear to be counter productive.

procedure scheme, based on Mammen (1993) and Davidson and Flachaire (2008), which might be employed in order to provide closer agreement between the desired nominal and the empirical significance level of the proposed test procedures. Section 5 concludes.

2 The Notation, Model and Test Statistics

We consider the following static linear panel data model

$$\mathbf{y}_i = \alpha_i \boldsymbol{\iota}_T + \mathbf{X}_i \boldsymbol{\beta}_1 + \mathbf{u}_i, \quad i = 1, ..., N \tag{1}$$

where $\mathbf{y}_i = (y_{i1}, ..., y_{iT})'$, $\mathbf{u}_i = (u_{i1}, ..., u_{iT})'$, $\boldsymbol{\iota}_T$ is a $(T \times 1)$ vector of ones, and $\mathbf{X}_i = (\mathbf{x}_{i1}, ..., \mathbf{x}_{iT})'$ a $(T \times K)$ matrix. The innovations, u_{it}, have zero mean and uniformly bounded variances and the α_i are the individual effects. By stacking the N equations of (1), the model for all individuals becomes

$$\mathbf{y} = \mathbf{D}\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta}_1 + \mathbf{u}, \tag{2}$$

where $\mathbf{y} = (\mathbf{y}_1', ..., \mathbf{y}_N')'$ and $\mathbf{u} = (\mathbf{u}_1', ..., \mathbf{u}_N')'$ are both $(NT \times 1)$ vectors, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)'$ is a $(N \times 1)$ vector, $\mathbf{D} = [\mathbf{I}_N \otimes \boldsymbol{\iota}_T]$ is a $(NT \times N)$ matrix, $\mathbf{X} = (\mathbf{X}_1', ..., \mathbf{X}_N')'$ is a $(NT \times K)$ matrix, and $[\mathbf{D}, \mathbf{X}]$ has full column rank. Thus, for the purposes of the current exposition, $\mathbf{x}_{it} = (x_{it1}, ..., x_{itK})'$, $(K \times 1)$, contains no time invariant regressors, in particular a constant term corresponding to an overall intercept. In the context of fixed effects this allows estimation of $\boldsymbol{\beta}_1$, as follows.

In general, define the projection matrices, $\mathbf{P_B} = \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'$ and $\mathbf{M_B} = \mathbf{I}_{NT} - \mathbf{P_B}$, for any $(NT \times S)$ matrix $\mathbf{B}$ of full column rank, with $\tilde{\mathbf{B}} = \mathbf{M_D}\mathbf{B}$ being the residual matrix from a multivariate least squares regression of $\mathbf{B}$ on $\mathbf{D}$ which is, of course, the within transformation. Then the fixed effects (least squares dummy variable) estimator of $\boldsymbol{\beta}_1$ in (2) is given by

$$\tilde{\boldsymbol{\beta}}_1 = (\mathbf{X}'\mathbf{M_D}\mathbf{X})^{-1}\mathbf{X}'\mathbf{M_D}\mathbf{y} = (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}'\tilde{\mathbf{y}}. \tag{3}$$

The null model of no individual effects is the pooled regression model of

$$\begin{aligned} \mathbf{y} &= \beta_0 \boldsymbol{\iota}_{NT} + \mathbf{X}\boldsymbol{\beta}_1 + \mathbf{u}, \\ &= \mathbf{Z}\boldsymbol{\beta} + \mathbf{u}, \end{aligned} \tag{4}$$

where $\mathbf{Z} = [\boldsymbol{\iota}_{NT}, \mathbf{X}] = (\mathbf{Z}_1', ..., \mathbf{Z}_N')'$, where $\mathbf{Z}_i$ has rows $\mathbf{z}_{it}' = (1, x_{it1}, ..., x_{itK}) = \{z_{itj}\}$, $j = 1, ..., K+1$. The (pooled) regression of $\mathbf{y}$ on $\mathbf{Z}$ delivers the Ordinary Least Squares (OLS) estimator $\hat{\boldsymbol{\beta}} = \left(\hat{\beta}_0, \hat{\boldsymbol{\beta}}_1'\right)' = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}$.

The standard *F-test* for fixed effects requires estimation of both (2), treating the α_i as unknown parameters, and (4) whilst the standard *RE-test* only requires estimation of (4). In order to provide a framework in which to investigate the limiting behaviour of the *F-test* and *RE-test* statistics, under both fixed and random effects, the individual effects are assumed to have the form $\boldsymbol{\alpha} = \beta_0 \boldsymbol{\iota}_N + \boldsymbol{\delta}$, $\boldsymbol{\delta} = (\delta_1, ..., \delta_N)'$. Fixed effects correspond to the α_i, $i = 1, ..., N$, being fixed unknown parameters (or, equivalently, $\delta_1 \equiv 0$ with β_0 and δ_i, $i = 2, ..., N$, being the fixed unknown parameters). The case of random effects is accommodated when the δ_i, $i = 1, ..., N$ are random variables. Equations (1) and (2) will be employed to characterise the data generation process, with the restrictions of $H_0 : \boldsymbol{\delta} = \delta_1 \boldsymbol{\iota}_N$ providing the null model of no individual effects (notice that $\boldsymbol{\delta} = \mathbf{0}$ belongs to this set of restrictions). Specifically, when considering the alternative of fixed effects, the $(N-1)$ restrictions placed on (2) are $H_0 : \mathbf{H}\boldsymbol{\alpha} = \mathbf{0}$, where $\mathbf{H} = [\boldsymbol{\iota}_{N-1}, -\mathbf{I}_{N-1}]$, whilst for random effects the null is $H_0 : var(\delta_i) = 0$.

The standard F and RE test statistics are defined as follows:

F-test Statistic

This is constructed as

$$F_N = \frac{(RSS_R - RSS_U)/(N-1)}{RSS_U/(N(T-1)-K)}, \tag{5}$$

where $RSS_R = \hat{\mathbf{u}}'\hat{\mathbf{u}}$ is the restricted sum of squares (from the pooled regression (4)) with $\hat{\mathbf{u}} = \mathbf{M}_{\mathbf{Z}}\mathbf{y}$, and $RSS_U = \tilde{\mathbf{u}}'\tilde{\mathbf{u}}$ is the unrestricted sum of squares (from the fixed effects regression (2)) with $\tilde{\mathbf{u}} = \mathbf{M}_{\tilde{\mathbf{X}}}\tilde{\mathbf{y}}$, the residual vector from regressing $\tilde{\mathbf{y}}$ on $\tilde{\mathbf{X}}$. If normality, homoskedasticity and strong exogeneity were imposed such that, conditional on $\mathbf{X}$, $\mathbf{u}_i \sim N(\mathbf{0}, \sigma^2\mathbf{I}_T)$, $i = 1, ..., N$, then a standard *F-test* would be exact. In the case of non-normal, but homoskedastic, errors Orme and Yamagata (2006) demonstrated that a standard *F-test* would be asymptotically valid.

RE-test Statistic

The usual *RE-test* statistic is[3]

$$R_N = \sqrt{\frac{NT}{2(T-1)}}\left[\frac{\hat{\mathbf{u}}'(\mathbf{I}_N \otimes \mathbf{A})\hat{\mathbf{u}}}{\hat{\mathbf{u}}'\hat{\mathbf{u}}}\right] = \sqrt{\frac{1}{2NT(T-1)}}\left[\frac{\hat{\mathbf{u}}'(\mathbf{I}_N \otimes \mathbf{A})\hat{\mathbf{u}}}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/NT}\right] \tag{6}$$

where $\mathbf{A} = \mathbf{A}' = \boldsymbol{\iota}_T\boldsymbol{\iota}_T' - \mathbf{I}_T$, so that

$$\mathbf{u}'(\mathbf{I}_N \otimes \mathbf{A})\mathbf{u} = \sum_{i=1}^{N} \mathbf{u}_i'\mathbf{A}\mathbf{u}_i = \sum_i \sum_t \sum_{s \neq t} u_{it}u_{is}.$$

R_N has a limit standard normal distribution, as $N \to \infty$, under H_0 and homoskedasticity but not necessarily normality of the errors.

3 Asymptotic Properties of F_N

In this section we describe the properties of F_N, under both local fixed and random effects, by (i) deriving its asymptotic distribution, and (ii) establishing its asymptotic relationship with R_N. In the subsequent analysis asymptotic theory is employed in which $N \to \infty$ and T is fixed. To facilitate this, the next sections details the assumptions that are made, which are of the sort found in, for example, White (2001, p.120):

[3] See, for example, Breusch and Pagan (1980) or Honda (1985).

3.1 Assumptions

A1:

(i) $\{\mathbf{X}_i, \mathbf{u}_i\}_{i=1}^N$ is an independent sequence of $(K+1)$, $(T \times 1)$ vectors;

(ii) $E(u_{it}|\mathbf{X}_i) = 0$, for all i and t;

(iii) $E(u_{it}u_{is}|\mathbf{X}_i) = 0$ for all i and $t \neq s$.

A2:

(i) $E\left(|z_{isj}u_{it}|^{2+\eta}\right) \leq \Delta < \infty$ for some $\eta > 0$, all $s, t = 1, ..., T$, $j = 1, ..., K+1$, and all $i = 1, ..., N$;

(ii) $E\left(|z_{itj}|^{4+\eta}\right) \leq \Delta < \infty$ for some $\eta > 0$, all $t = 1, ..., T$, $j = 1, ..., K+1$, and all $i = 1, ..., N$;

(iii) $E(\mathbf{Z}'\mathbf{Z}/N)$ is uniformly positive definite;

(iv) $E(\tilde{\mathbf{X}}'\tilde{\mathbf{X}}/N)$ is uniformly positive definite;

(v) $\mathbf{V}_N = N^{-1}\sum_{i=1}^N \sum_{t=1}^T E\left(u_{it}^2 \mathbf{z}_{it}\mathbf{z}_{it}'\right)$ is uniformly positive definite;

(vi) $\tilde{\mathbf{V}}_N = N^{-1}\sum_{i=1}^N \sum_{t=1}^T E\left(u_{it}^2 \tilde{\mathbf{x}}_{it}\tilde{\mathbf{x}}_{it}'\right)$ is uniformly positive definite.

Assumption A1 imposes independent sampling of cross-section units and a strong exogeneity assumption on $\mathbf{X}_i$, implying that $E(\tilde{\mathbf{X}}_i'\mathbf{u}_i) = \mathbf{0}$ and thus ruling out (for example) lagged dependent variables. It also constrains the u_{it} to be conditionally serially uncorrelated, and thus serially uncorrelated but not necessarily serially independent. Together with Assumption A2, which allows for heteroskedastic disturbances, we obtain consistency and asymptotic normality of both the pooled and fixed effects least squares regression estimators ($\hat{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\beta}}_1$, respectively), and also consistency of the corresponding heteroskedasticity-robust covariance matrix estimators.[4] These results follow for the fixed effects estimator because Assumption A2(i) and (ii) also imply that $E\left[|\tilde{x}_{isj}u_{it}|^{2+\eta}\right]$ and $E\left[|\tilde{x}_{itj}\tilde{x}_{isl}|^{2+\eta}\right]$ are both uniformly bounded. Thus, in particular, $\frac{1}{\sqrt{N}}\mathbf{Z}'\mathbf{u}$, $\frac{1}{\sqrt{N}}\tilde{\mathbf{X}}'\mathbf{u}$, $\frac{1}{N}\mathbf{Z}'\mathbf{Z}$ and $\frac{1}{N}\tilde{\mathbf{X}}'\tilde{\mathbf{X}}$ are all $O_p(1)$, with $\mathbf{V}_N^{-1/2}\frac{1}{\sqrt{N}}\mathbf{Z}'\mathbf{u} \xrightarrow{d} N(\mathbf{0}, \mathbf{I}_{K+1})$ and $\tilde{\mathbf{V}}_N^{-1/2}\frac{1}{\sqrt{N}}\tilde{\mathbf{X}}'\mathbf{u} \xrightarrow{d} N(\mathbf{0}, \mathbf{I}_K)$, as $N \to \infty$, T fixed. If Assumption A1 (ii) is weakened to $E(\mathbf{X}_i'\mathbf{u}_i) = \mathbf{0}$, or even $E(\mathbf{x}_{it}u_{it}) = \mathbf{0}$ (zero contemporaneous correlation), $\tilde{\boldsymbol{\beta}}_1$ is not guaranteed to be consistent and, when it is inconsistent, the *F-test* is asymptotically invalid anyway, even under normality; for example, in the presence of lagged dependent variables - see the discussion in Wooldridge (2010, Sections 10.5 and 11.6). Note that, although the assumptions constrain $\{u_{it}\}$ to be serially uncorrelated, they allow for rather arbitrary heteroskedasticity, across individuals and/or through time, and do not demand, for

[4] See, for example, White (2001, Exercises 3.14, 5.12 and Chapter 6). Assumption A2(ii) is also required to obtain a heteroskedasticity robust *F-test*.

example, that $\{u_{it}^2\}$ also be serially uncorrelated.[5] Assumptions A1(iii) and A2(v) imply that $\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}E(u_{it}^2) = \frac{1}{N}\sum_{i=1}^{N}E(\sum_{t=1}^{T}u_{it})^2$ is uniformly positive.

For the purposes of this paper, in addition, we assume the following:

A3:

(i) $E\left|u_{it}\right|^{4+\eta} \leq \Delta < \infty$ for some $\eta > 0$, all $t = 1, ..., T$, and all $i = 1, ..., N$;

(ii) $var\left(N^{-1/2}\mathbf{u}'\left(\mathbf{I}_N \otimes \mathbf{A}\right)\mathbf{u}\right) = N^{-1}\sum_{i=1}^{N}E\left(\mathbf{u}_i'\mathbf{A}\mathbf{u}_i\right)^2$ is uniformly positive.

A4:

(i) $\alpha_i = \beta_0 + \dfrac{\delta_i}{N^{1/4}}$, $i = 1, ..., N$;

(ii) the δ_i are independent, satisfying $E\left[u_{it}\delta_i\right] = 0$ and $E\left|\delta_i\right|^{4+\eta} \leq \Delta < \infty$, for all $i = 1, ..., N$.

(iii) $N^{-1}\sum_{i=1}^{N}E\left[\delta_i^2\right]$ is uniformly positive, where $\boldsymbol{\delta}' = (\delta_1, ..., \delta_N)$.

Assumption A3 justifies the limit distribution obtained in Proposition 1 below, and as a consequence also that of R_N. In fact, Assumption A3(i) and Assumption A2(ii) actually imply Assumption A2(i), using the Cauchy-Schwartz inequality. Assumption A4 characterises the alternative data generation process and permits the investigation of asymptotic power, under local individual effects, by restricting the test criteria under consideration to be $O_p(1)$ with well defined limit distributions. Together with Assumptions A3(i) and A2(ii), Assumption A4(ii) implies $E\left|u_{it}\delta_i\right|^{2+\eta} \leq \Delta < \infty$ and $E\left|z_{itj}\delta_i\right|^{2+\eta} \leq \Delta < \infty$, for some $\eta > 0$, and all $i = 1, ..., N$, $t = 1, ..., T$, $j = 1, ..., K+1$. As well as fixed effects (with the δ_i being non-stochastic) it also accommodates local heteroskedastic random effects, but which are uncorrelated with $\mathbf{u}_i$. If the δ_i are also distributed independently of $\mathbf{X}_i$, then we have "pure" random effects whilst if the δ_i are correlated with $\mathbf{X}_i$ then we have "correlated" random effects. (As pointed out by Wooldridge (2010, p.287), in microeconometric applications of panel data models with individual effects, the term fixed effect is generally used to mean correlated random effects, rather than α_i being strictly non-stochastic.)

3.2 The Asymptotic Distribution of F_N

The results concerning the limiting behaviour of both the *F-test* and *RE-test* are driven by the following Lemma, which also substantiates the asymptotic validity of Wooldridge's (2010, p.299) heteroskedasticity-robust test for unobserved effects; see Section 3.2.2.

[5] Note that Assumption A1(iii) is often strengthened to that of u_{it} being independent over $t = 1, ..., T$, conditionally on $\mathbf{X}_i$. However, this would rule out certain forms of conditional heteroskedasticity; such as ARCH or GARCH processes.

Lemma 1 *Define*

$$H_N = \frac{\mathbf{u}'\left(\mathbf{I}_N \otimes \mathbf{A}\right)\mathbf{u}}{\sqrt{NT\left(T-1\right)}} = \frac{1}{\sqrt{NT(T-1)}}\sum_{i=1}^{N}\mathbf{u}_i'\mathbf{A}\mathbf{u}_i$$

and

$$\kappa_N = var\left(H_N\right) = \frac{1}{NT\left(T-1\right)}\sum_{i=1}^{N}E\left\{\mathbf{u}_i'\mathbf{A}\mathbf{u}_i\right\}^2.$$

Then under Assumptions A1(i),(iii) and A3,

$$\kappa_N^{-1/2}H_N \xrightarrow{d} N(0,1),$$

for fixed T, as $N \to \infty$.

Armed with this, the asymptotic distribution of F_N, under non-normality and heteroskedasticity, is given by following proposition:

Proposition 1 *Define* $\bar{\sigma}_N^2 = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}E(u_{it}^2)$.

(i) Under model (2) and Assumptions A1 to A4, $\sqrt{N}\left(F_N - 1\right) = O_p(1)$, *with*

$$\bar{\sigma}_N^2\sqrt{N}\left(F_N-1\right) = \sqrt{\frac{T}{T-1}}H_N + \lambda_N + o_p(1)$$

where H_N is given in Lemma 1 and $\lambda_N = O(1)$ is defined by

$$\begin{aligned}\lambda_N &= E[\boldsymbol{\zeta}_1'\boldsymbol{\zeta}_1/N] = \mu_N - \boldsymbol{\rho}_N'\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\rho}_N \geq 0,\\ \boldsymbol{\zeta}_1 &= \mathbf{D}\boldsymbol{\delta} - \mathbf{Z}\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\rho}_N,\end{aligned}$$

$$\boldsymbol{\Sigma}_N = E\left[\mathbf{Z}'\mathbf{Z}/N\right],\ \boldsymbol{\rho}_N = E\left[\mathbf{Z}'\mathbf{D}\boldsymbol{\delta}/N\right],\ \mu_N = E\left[\boldsymbol{\delta}'\mathbf{D}'\mathbf{D}\boldsymbol{\delta}/N\right].$$

(ii) Furthermore, if $\omega_N = \dfrac{\bar{\sigma}_N^2}{\sqrt{\kappa_N/2}}$, *where κ_N is defined in Lemma 1, then*

$$\omega_N\sqrt{N}\left(F_N-1\right) - \frac{\lambda_N}{\sqrt{\kappa_N/2}} \xrightarrow{d} N\left(0, \frac{2T}{T-1}\right).$$

Given our assumptions, note that both ω_N and λ_N are $O(1)$ satisfying

$$\frac{\frac{1}{NT}\sum_{i=1}^{N}\mathbf{u}_i'\mathbf{u}_i}{\sqrt{\frac{1}{2NT(T-1)}\sum_{i=1}^{N}\left\{\mathbf{u}_i'\mathbf{A}\mathbf{u}_i\right\}^2}} - \omega_N \xrightarrow{p} 0$$

and

$$\frac{\boldsymbol{\delta}'\mathbf{D}'\mathbf{M}_{\mathbf{Z}}\mathbf{D}\boldsymbol{\delta}}{N} - \lambda_N \xrightarrow{p} 0,$$

respectively, with ω_N is uniformly positive by Assumption, although neither ω_N or λ_N need necessarily converge. The special case of no individual effects, with $\boldsymbol{\delta} = \delta_1\boldsymbol{\iota}_N$, yields $\lambda_N \equiv 0$, as it should (this includes the case of $\boldsymbol{\delta} = \mathbf{0}$).

As exploited by Orme and Yamagata (2006), it is easy to show that if ξ_N has an F distribution with $n_1 = N-1$ and $n_2 = N(T-1)-K$ degrees of freedom, then $\xi_N^* = \sqrt{\frac{N(T-1)}{2T}}\,(\xi_N - 1) \sim N(0,1)$, or approximately for large N, $\xi_N \overset{A}{\sim} N\left(1, \frac{2T}{N(T-1)}\right)$. Therefore, by Proposition 1, we can employ the following approximation, under the null,

$$F_\omega \equiv \hat{\omega}_N \{F_N - 1\} + 1 \overset{A}{\sim} F(n_1, n_2), \tag{7}$$

for any choice of $\hat{\omega}_N$ satisfying $\hat{\omega}_N - \omega_N \xrightarrow{p} 0$, implying that F_ω can be used in an asymptotically valid "standard" *F-test* procedure.

Before proceeding to derive a suitable $\hat{\omega}_N$, note that under pure local random effects, with $E[\delta_i|\mathbf{X}_i] = 0$ and $E\left[\delta_i^2|\mathbf{X}_i\right] = \tau^2$, $\boldsymbol{\rho}_N = \frac{T}{N}\sum_{i=1}^N E[\delta_i \bar{\mathbf{z}}_i] = \mathbf{0}$ with $\bar{\mathbf{z}}_i = T^{-1}\sum_{t=1}^T \mathbf{z}_{it}$ so that $\lambda_N = TE\left[\frac{\boldsymbol{\delta}'\boldsymbol{\delta}}{N}\right] = T\tau^2$. In this case we immediately obtain the following Corollary to Proposition 1 (the proof is omitted):

Corollary 1 *Under the alternative of (pure) local random effects, and under the assumptions of Proposition 1,*

$$\hat{\omega}_N \sqrt{N}\,(F_N - 1) - \frac{T\tau^2}{\sqrt{\kappa_N/2}} \xrightarrow{d} N\left(0, \frac{2T}{T-1}\right)$$

for any choice of $\hat{\omega}_N$ *satisfying* $\hat{\omega}_N - \omega_N \xrightarrow{p} 0$.

Therefore, a robust *F-test*, based on F_ω, will have non-trivial asymptotic local power against pure random effects. In fact, and analogous to Orme and Yamagata (2006), a stronger result will be established in Section 3.2.2. There it is shown that, under (pure) local random effects, a robust *F-test* procedure based on F_ω will thus possess the same asymptotic power as a suitably "robustified" *RE-test*, of the sort proposed by Wooldridge (2010, p.299) or Häggström & Laitila (2002). However, under "correlated" local random effects a robust *F-test* will possess higher asymptotic power than a robust *RE-test.*

3.2.1 Asymptotically Valid F-test Statistics

As noted above, an asymptotically valid *F-test* can be constructed if there is a $\hat{\omega}_N$ available satisfying $\hat{\omega}_N - \omega_N \xrightarrow{p} 0$. Using restricted OLS (i.e., pooled) residuals a natural choice for $\hat{\omega}_N$ might be

$$\hat{\omega}_N = \frac{\hat{\sigma}_N^2}{\sqrt{\hat{\kappa}_N/2}}$$

where $\hat{\sigma}_N^2 = \hat{\mathbf{u}}'\hat{\mathbf{u}}/(NT-K-1)$ and

$$\hat{\kappa}_N = \frac{1}{NT(T-1)}\sum_{i=1}^N \{\hat{\mathbf{u}}_i'\mathbf{A}\hat{\mathbf{u}}_i\}^2 = \frac{1}{NT(T-1)}\sum_{i=1}^N \left\{\sum_t \sum_{s\neq t} \hat{u}_{it}\hat{u}_{is}\right\}^2.$$

Indeed, this choice is justified in Proposition 2 below; c.f., Wooldridge (2010, p.299).

However, other choices for $\hat{\kappa}_N$, and thus $\hat{\omega}_N$, emerge if we are willing, or able, to strengthen Assumption A1(iii).[6] To see this, first note that $\sum_t \sum_{s\neq t} u_{it}u_{is} = 2\sum_{t=2}^{T} w_{it}$, where $w_{it} = u_{it}\sum_{s=1}^{t-1} u_{is}$, so that κ_N can equivalently be expressed as

$$\kappa_N = \frac{4}{NT(T-1)}\sum_{i=1}^{N} E\left[\left(\sum\nolimits_{t=2}^{T} w_{it}\right)^2\right]. \tag{8}$$

The first potential strengthening of A1(iii) also strengthens A1(i) and simply states that, conditional on $\mathbf{X}_i$, u_{it} is orthogonal to the entire past history of the errors but without, necessarily, imposing serial independence:

A1(iii)′: $E\left(u_{it}|\mathbf{X}_i, u_{i,t-1}, u_{i,t-2}, ...\right) = 0$, for all i and t.

This is like a martingale difference assumption but is more direct[7] and might be regarded as a mild additional constraint since it still allows, for example, a GARCH process for u_{it}^2. Under this Assumption, $E\left[w_{it}w_{it-m}\right] = 0$, for all $t \geq 3$ and $m = 1, ..., t-1$, so that (8) becomes

$$\kappa_N = \frac{4}{NT(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T} E\left(w_{it}^2\right)$$

where

$$\sum_{t=2}^{T} w_{it}^2 = \sum_{t=2}^{T}\sum_{s=1}^{t-1} u_{it}^2 u_{is}^2 + 2\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{r=3}^{s-1} u_{it}^2 u_{is} u_{ir}. \tag{9}$$

An alternative strengthening of Assumption A1(iii) might be:

A1(iii)″: All distinct pairs (u_{it}, u_{is}) and (u_{ir}, u_{iq}), $(t,s) \neq (r,q)$, are uncorrelated.

In this case, $E\left[w_{it}w_{it-m}\right] = 0$ and $E\left(u_{it}^2 u_{is} u_{ir}\right) = 0$ so that (8) is

$$\begin{aligned}\kappa_N &= \frac{4}{NT(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=1}^{t-1} E\left(u_{it}^2 u_{is}^2\right)\\ &= \frac{2}{NT\left(T-1\right)}\sum_{i=1}^{N}\sum_{t}\sum_{s\neq t} E\left(u_{it}^2 u_{is}^2\right).\end{aligned}$$

This is somewhat stronger than Assumption A1(iii)′, as it rules out an asymmetric GARCH process for u_{it}^2.[8]

In addition to Assumption A1(i), a further strengthening of Assumption A1(iii)′ or A1(iii)″ would be full serial independence:

A1(iii)‴: $\{u_{it}\}_{t=1}^{T}$ is an independent sequence of random variables, for all $i = 1, ...N$.

[6] We shall not, here, consider alternative estimators of $\bar{\sigma}_N^2$, altough this is possible.
[7] See, for example, White (2001, p.54).
[8] See, for example, Goncalves and Killian (2004).

In this case, a GARCH process for u_{it}^2 is ruled out and

$$\kappa_N = \frac{2}{NT\left(T-1\right)} \sum_{i=1}^{N} \sum_{t} \sum_{s \neq t} E\left(u_{it}^2\right) E(u_{is}^2)\,.$$

The preceding discussion suggests differing possible consistent estimators for κ_N, and thus for ω_N, according to the strengthening of Assumption A1(iii). These are described in the following Proposition:

Proposition 2 *Define* $\hat{\sigma}_N^2 = \hat{\mathbf{u}}'\hat{\mathbf{u}}/(NT-K-1)$, $\hat{w}_{it} = \hat{u}_{it} \sum_{s=1}^{t-1} \hat{u}_{is}$ *and*

$$\hat{\kappa}_N^{(1)} = \frac{1}{NT(T-1)} \sum_{i=1}^{N} \left(\sum_{t} \sum_{s \neq t} \hat{u}_{it}^2 \hat{u}_{is}^2 \right)^2 = \frac{4}{NT(T-1)} \sum_{i=1}^{N} \left(\sum_{t=2}^{T} \hat{w}_{it} \right)^2$$

$$\hat{\kappa}_N^{(2)} = \frac{4}{NT(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \hat{w}_{it}^2$$

$$\hat{\kappa}_N^{(3)} = \frac{2}{NT(T-1)} \sum_{i=1}^{N} \sum_{t} \sum_{t \neq s} \hat{u}_{it}^2 \hat{u}_{is}^2 = \frac{4}{NT\left(T-1\right)} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \hat{u}_{it}^2 \hat{u}_{is}^2.$$

Under model (2) and Assumptions A1 to A4:

1. $\hat{\sigma}_N^2 - \bar{\sigma}_N^2 \xrightarrow{p} 0$;
2. $\hat{\kappa}_N^{(1)} - \kappa_N \xrightarrow{p} 0$.

Under Assumptions A1-A4 with A1(iii) strengthened to A1(iii)′, A1(iii)″ or A1(iii)‴ :

3. $\hat{\kappa}_N^{(2)} - \kappa_N \xrightarrow{p} 0$;

Under Assumptions A1-A4 with A1(iii) strengthened to A1(iii)″ or A1(iii)‴ :

4. $\hat{\kappa}_N^{(3)} - \kappa_N \xrightarrow{p} 0$.

From this analysis it follows that asymptotically valid choices for $\hat{\omega}_N$ include the following:

$$\hat{\omega}_N^{(1)} = \frac{\hat{\sigma}_N^2}{\sqrt{\frac{2}{NT(T-1)} \sum_{i=1}^{N} \left(\sum_{t=2}^{T} \hat{w}_{it} \right)^2}}, \tag{10}$$

$$\hat{\omega}_N^{(2)} = \frac{\hat{\sigma}_N^2}{\sqrt{\frac{2}{NT\left(T-1\right)} \sum_{i=1}^{N} \sum_{t=2}^{T} \hat{w}_{it}^2}}, \tag{11}$$

$$\hat{\omega}_N^{(3)} = \frac{\hat{\sigma}_N^2}{\sqrt{\frac{2}{NT\left(T-1\right)} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \hat{u}_{it}^2 \hat{u}_{is}^2}}, \tag{12}$$

depending on assumptions made about the u_{it}, $t = 1, ..., T$. Robust *F-test* statistics can then be constructed as $F_{\omega}^{(m)} = \hat{\omega}_N^{(m)} \left\{F_N - 1\right\} + 1$, $m = 1, 2, 3$, and approximate inferences obtained based on (7).

3.2.2 The Relationship between F_N and R_N

Under the null of no individual effects, it is straightforward to show that

$$\frac{1}{\sqrt{N}}\left[\frac{\hat{\mathbf{u}}'\left(\mathbf{I}_N\otimes\mathbf{A}\right)\hat{\mathbf{u}}}{\hat{\mathbf{u}}'\hat{\mathbf{u}}/NT}\right]=\frac{1}{\sqrt{N}}\frac{\mathbf{u}'\left(\mathbf{I}_N\otimes\mathbf{A}\right)\mathbf{u}}{\bar{\sigma}_N^2}+o_p\left(1\right).$$

From (6), Lemma 1 and Proposition 1, therefore, we can write

$$\begin{aligned}R_N &= \frac{1}{\sqrt{2}}\frac{H_N}{\bar{\sigma}_N^2}+o_p(1)\\ &= \sqrt{\frac{T-1}{2T}}\sqrt{N}\left(F_N-1\right)+o_p(1),\end{aligned}$$

under the null, so that

$$R_\omega\equiv\hat{\omega}_N R_N\xrightarrow{d}N(0,1) \tag{13}$$

for any choice of $\hat{\omega}_N$ satisfying $\hat{\omega}_N-\omega_N\xrightarrow{p}0$. This substantiates Wooldridge's (2010, p.299) suggestion for a heteroskedasticity-robust RE test statistic constructed as $\hat{\omega}_N^{(1)}R_N$; or, under Assumption A1(iii)$''$ or A1(iii)$'''$, $\hat{\omega}_N^{(3)}R_N$ as proposed by Häggström & Laitila (2002).

The following proposition extends this result to the case of local individual effects (fixed or random):

Proposition 3 *Under model (2) and Assumptions A1 to A4,*

$$\hat{\omega}_N R_N=\left\{\sqrt{\frac{(T-1)}{2T}}\right\}\hat{\omega}_N\sqrt{N}\left[F_N-1\right]-\sqrt{\frac{T}{2\left(T-1\right)}}\frac{\gamma_N}{\sqrt{\kappa_N/2}}+o_p\left(1\right),$$

for any choice of $\hat{\omega}_N$ satisfying $\hat{\omega}_N-\omega_N\xrightarrow{p}0$, where $\gamma_N=O(1)$ defined by

$$\begin{aligned}\gamma_N &= E(\boldsymbol{\zeta}_2'\boldsymbol{\zeta}_2/N)=\boldsymbol{\rho}_N'\boldsymbol{\Sigma}_N^{-1}\tilde{\boldsymbol{\Sigma}}_N\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\rho}_N\geq 0,\\ \boldsymbol{\zeta}_2 &= \tilde{\mathbf{Z}}\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\rho}_N,\end{aligned}$$

$\tilde{\boldsymbol{\Sigma}}_N=E(\tilde{\mathbf{Z}}'\tilde{\mathbf{Z}}/N)$, and the limit distribution of $\omega_N\sqrt{N}\left[F_N-1\right]$ is given by Proposition 1.

Again, γ_N need not converge, but it is $O(1)$ and $\gamma_N-\frac{\boldsymbol{\delta}'\mathbf{D}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\tilde{\mathbf{Z}}'\tilde{\mathbf{Z}})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{D}\boldsymbol{\delta}}{N}\xrightarrow{p}0$. As with Proposition 1, $\gamma_N\equiv 0$ obtains under $H_0:\boldsymbol{\delta}=\delta_1\boldsymbol{\iota}_N$, as it should, since $\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}'\mathbf{D}\boldsymbol{\delta}=(\delta_1,\mathbf{0}')'$ and the top-left, $(1,1)$, element of $\tilde{\mathbf{Z}}'\tilde{\mathbf{Z}}$ is 0. As discussed above, under the alternative of (pure) local random effects $\boldsymbol{\rho}_N=\mathbf{0}$, and we obtain the following Corollary, which is immediate from Corollary 1 given Proposition 3:

Corollary 2 *Under the alternative of (pure) local random effects, and under the assumptions of Proposition 1,*

$$\hat{\omega}_N R_N-\left\{\sqrt{\frac{T\left(T-1\right)}{2}}\right\}\frac{\tau^2}{\sqrt{\kappa_N/2}}\xrightarrow{d}N\left(0,1\right)$$

for any choice of $\hat{\omega}_N$ satisfying $\hat{\omega}_N-\omega_N\xrightarrow{p}0$.

Thus, since under (pure) local random effects, $\hat{\omega}_N R_N - \sqrt{\frac{N(T-1)}{2T}}\hat{\omega}_N\left(F_N - 1\right) = o_p(1)$, both the robust *RE* and robust *F-test* procedures, based on (13) and (7), respectively, will have identical asymptotic power functions. However, under local fixed effects or random effects which are correlated with $\mathbf{X}_i$, the robust *F-test* can have greater asymptotic power. In particular, when individual effects are correlated with the mean values of the regressors, $\boldsymbol{\rho}_N \neq \mathbf{0}$ and is $O\left(1\right)$, implying $\gamma_N > 0$ so that a test based on R_N (but suitably robust to heteroskedasticity) should have lower asymptotic local power than one based on F_N. This makes intuitive sense, since F_N is designed to test for individual effects which are correlated with $\bar{\mathbf{z}}_i$, whereas R_N is constructed on the assumption that the individual effects are uncorrelated with all regressor values. The importance of distinguishing between individual effects which are correlated or uncorrelated with regressors, rather than simply labelling them fixed or random, is discussed by Wooldridge (2010, Section 10.2).

3.2.3 Analysis of the Standard F-test and RE-test

Given the analysis above the following conclusions emerge concerning the asymptotic behaviour of both the standard *F-test*, based on F_N, and *RE-test*, based on R_N, in certain special cases and under the null hypothesis. Under Assumption A1(iii)$'$ *and*

$$E\left(u_{it}^2|\mathbf{X}_i, u_{i,t-1}, u_{i,t-2}, ...\right) \equiv E\left(u_{it}^2|\mathbf{X}_i\right), \quad \text{for all } i \text{ and } t, \tag{14}$$

we obtain, without recourse to Assumption A1(iii)$''$, that

$$\kappa_N = \frac{2}{NT\left(T-1\right)} \sum_{i=1}^{N} \sum_{t} \sum_{s \neq t} E(u_{it}^2 u_{is}^2), \tag{15}$$

because, from (9),

$$E\left(u_{it}^2 u_{is} u_{ir}\right) = E\left(E\left(u_{it}^2|\mathbf{X}_i\right) E(u_{is}u_{ir}|\mathbf{X}_i)\right) = 0.$$

In this case, (14) rules out conditional heteroskedastcity of the ARCH/GARCH type.

Therefore:

(i) Under A1(iii)$'$ and (14) but conditional cross-sectional heteroskedasticity only such that $E\left(u_{it}^2|\mathbf{X}_i\right) \equiv h_i > 0$, and $E\left(h_i^2\right) < \infty$, we obtain $\bar{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^{N} E(h_i)$ and $\kappa_N = \frac{2}{N}\sum_{i=1}^{N} E\left(h_i^2\right)$. Thus, $\omega_N < 1$, since

$$\frac{1}{N}\sum_{i=1}^{N} E(h_i^2) - \left\{\frac{1}{N}\sum_{i=1}^{N} E(h_i)\right\}^2 \geq \frac{1}{N}\sum_{i=1}^{N} \sigma_i^4 - \left\{\frac{1}{N}\sum_{i=1}^{N} \sigma_i^2\right\}^2 > 0$$

where $\sigma_i^2 = E(h_i) > 0$. This implies that both the *F-test* based on F_N and *RE-test* based on R_N, without adjustment, will be asymptotically oversized (in that, asymptotically, both will reject a correct null of no individual effects too often for any given nominal significance level).[9]

[9] Indeed, this particular consclusion explains some of the finite sample Monte Carlo results obtained by Häggström & Laitila (2002).

(ii) Under A1(iii)$'$ and (14) but time varying variances such that $E\left(u_{it}^2|\mathbf{X}_i\right) \equiv h_t > 0$, and $E\left(h_t^2\right) < \infty$, we have

$$\omega_N^2 = \frac{\left(\frac{1}{T}\sum_{t=1}^{T}\sigma_t^2\right)^2}{\frac{1}{T(T-1)}\sum_t\sum_{s\neq t}\sigma_t^2\sigma_s^2} > 1,$$

where, here, $\sigma_t^2 = E\left(h_t\right) > 0$, because

$$\left(\frac{1}{T}\sum_{t=1}^{T}\sigma_t^2\right)^2 - \frac{1}{T(T-1)}\sum_t\sum_{s\neq t}\sigma_t^2\sigma_s^2 = \frac{1}{T-1}\left(\frac{1}{T}\sum_{t=1}^{T}\sigma_t^4 - \left(\frac{1}{T}\sum_{t=1}^{T}\sigma_t^2\right)^2\right) > 0.$$

This implies that both test procedures, without adjustment, will be asymptotically undersized.

(iii) Of course, the conclusions in (i) and (ii) must also hold under Assumption A1(ii)$''$ or A1(iii)$'''$. Moreover, if $\{u_{it}, \mathbf{x}_{it}'\}_{t=1}^{T}$ are serially independent with $E(u_{it}^2|\mathbf{x}_{it}) = h(\mathbf{x}_{it}) > 0$ *and* $E\left[h(\mathbf{x}_{it})\right] = \sigma^2 < \infty$, so that the errors are unconditionally homoskedastic; then, $\kappa_N = 2\sigma^4$ and $\omega_N = 1$. In particular, this result is true if the $(u_{it}, \mathbf{x}_{it}')$ are i.i.d., but the u_{it} are conditionally heteroskedastic with $E(u_{it}^2|\mathbf{x}_{it}) = h(\mathbf{x}_{it}) > 0$. This shows that both the *F-test* and *RE-test*, based on F_N and R_N, respectively, remain asymptotically valid without any adjustment.

In order to shed light on the relevance of the preceding asymptotic analysis, the next section reports the results of a small Monte Carlo experiment which illustrates the asymptotic robustness of the *F-test* to non-normality/heteroskedasticity and its power properties relative to the *RE-test*.

4 Monte Carlo Study

The Monte Carlo study investigates the sampling behaviour of the test statistics considered above, (7) and (13), for differing choices of $\hat{\omega}_N$, including $\hat{\omega}_N \equiv 1$. As our analytical results suggest, the tests are justified when $N \to \infty$ with T fixed, we consider $(N, T) = (20, 5)$, $(50, 5)$, $(100, 5)$, $(50, 10)$, $(50, 20)$.

4.1 Monte Carlo Design

The model employed is

$$y_{it} = \alpha_i + \sum_{j=1}^{3} z_{it,j}\beta_j + u_{it},\ u_{it} = \sigma_{it}\varepsilon_{it} \tag{16}$$

where $z_{it,1} = 1$, $z_{it,2}$ is drawn from a uniform distribution on $(1, 31)$ independently for i and t, and $z_{it,3}$ is generated following Nerlove (1971), such that

$$z_{it,3} = 0.1t + 0.5z_{it-1,3} + v_{it},$$

where the value $z_{i0,3}$ is chosen as $5+10\upsilon_{i0}$, and υ_{it} (and υ_{i0}) is drawn from the uniform distribution on $(-0.5, 0.5)$ independently for i and t, in order to avoid any normality in regressors. These regressor values are held fixed over replications. Also, observe that the regression design is not quadratically balanced.[10] Without loss of generality, the coefficients are set as $\beta_j = 1$ for $j = 1, 2, 3$. The i.i.d. standardised errors for ε_{it} are drawn from: the standard normal distribution (SN); the t distribution with five degrees of freedom (t_5); and, the chi-square distribution with six degrees of freedom (χ_6^2).

We consider five specifications for σ_{it} :

1. Homoskedasticity (HET0)
$$\sigma_{it} = \sigma = 1$$

2. Cross-sectional one-break-in-volatility heteroskedasticity (HET1)
$$\begin{aligned} \sigma_{it} &= \sigma_1, \ \ i = 1, ..., N_1, \ \ t = 1, ..., T \\ &= \sigma_2, \ \ i = N_1 + 1, ..., N, \ \ t = 1, ..., T \end{aligned}$$
with $N_1 = \lceil N/2 \rceil$, where $\lceil A \rceil$ is the largest integer not less than A, $\sigma_1 = 0.5$ and $\sigma_2 = 1.5$.

3. Time series one-break-in-volatility heteroskedasticity (HET2)
$$\begin{aligned} \sigma_{it} &= \sigma_1, \ \ i = 1, ..., N, \ \ t = 1, ..., T_1 \\ &= \sigma_2, \ \ i = 1, ..., N, \ \ t = T_1 + 1, ..., T \end{aligned}$$
with $T_1 = \lceil T/2 \rceil$, $\sigma_1 = 0.5$ and $\sigma_2 = 1.5$.

4. Conditional heteroskedasticity depending on a regressor (HET3)
$$\sigma_{it} = \eta_c[(z_{it,2} - 1)/30]/c, \quad i = 1, ..., N, \ \ t = 1, ..., T$$
$\eta_c[\cdot]$ is the inverse of the cumulative distribution function of chi-squared distribution with degrees of freedom c. Since $z_{it,2}$ is drawn from a uniform distribution on $(1, 31)$, σ_{it} has mean 1 and variance $2/c$, so it is easy to control the degree of heteroskedasticity through the choice of c. We employ $c = 1$.

5. Conditional heteroskedasticity, GARCH(1,1) (HET4)
$$u_{it} = \sigma_{it}\varepsilon_{it}, \ \ t = -49, ..., T, i = 1, ..., N$$
where
$$\sigma_{it}^2 = \phi_0 + \phi_1 u_{i,t-1}^2 + \phi_2 \sigma_{i,t-1}^2.$$
The value of parameters are chosen to be $\phi_0 = 0.5$, $\phi_1 = 0.25$ and $\phi_2 = 0.25$, and $u_{i,-50} = 0$ with the first 50 observations being discarded, so that the unconditional variance is $E\left(u_{it}^2\right) = \phi_0 / \left(1 - \phi_1 - \phi_2\right)$.

[10] See the discussion in Orme and Yamagata (2006).

6. Conditional heteroskedasticity, ARCH(1) (HET5)

$$u_{it} = \sigma_{it}\varepsilon_{it},\ t = -49, ..., T, i = 1, ..., N$$

where

$$\sigma_{it}^2 = \phi_0 + \phi_1 u_{i,t-1}^2,$$

with $\phi_0 > 0$ and $0 \le \phi_1 < 1$. In particular, $\phi_0 = 0.5$, $\phi_1 = 0.5$ and $u_{i,50} = 0$, with the first 50 observations being discarded, so that the unconditional variance is $E(u_{it}^2) = \phi_0/(1-\phi_1)$.

For power comparisons, the individual effects are generated according to

$$\alpha_i = \tau_i \left[\sqrt{R^2} g_i(\bar{\mathbf{z}}_i) + \sqrt{1-R^2}\varphi_i\right] \tag{17}$$

where the φ_i are i.i.d. $N(0,1)$, $g_i(\bar{\mathbf{z}}_i) = \boldsymbol{\iota}_3'(\bar{\mathbf{z}}_i - \bar{\bar{\mathbf{z}}})/s$ with $\boldsymbol{\iota}_3 = (1,1,1)'$, $\bar{\bar{\mathbf{z}}}$ being overall average of $\mathbf{z}_{it}$, s being the standard deviation of $\boldsymbol{\iota}_3'\bar{\mathbf{z}}_i$, and the R^2 is from the regression of (17). With this set up, the variance of inside of the square brackets is always unity across designs. We consider two combinations of (τ_i, R^2): (i) $(\tau_i, R^2) = (0,0)$, which is a simple null model specification, with $\alpha_i \equiv 0$, and; (ii) $(\tau_i, R^2) = (v_\alpha, 1)$, which is simple fixed effects specification (given that the z_{it} are fixed over replications).[11] To control the power, we consider $v_\alpha^2 = 0.1$.

4.2 Asymptotic Tests

FOUR versions of the FE and RE test statistics are considered, constructed using $\hat{\omega}_N^{(0)} \equiv 1$ and $\hat{\omega}_N^{(m)}$, $m = 1,2,3$, as defined at (10)-(12), and all are based on the restricted estimator, $\hat{\boldsymbol{\beta}}$.[12] Specifically:

1. *F-test* statistics (denoted F_ω in the Tables)

$$F_\omega^{(m)} = \hat{\omega}_N^{(m)}(F_N - 1) + 1,\ \ m = 0,1,2,3, \tag{18}$$

where

$$F_N = \frac{(RSS_R - RSS_U)/(N-1)}{RSS_U/(N(T-1)-K)} \equiv F_\omega^{(0)},$$

is the standard *F-test* statistic. The corresponding test procedure, for each separate statistic (18), employs critical vales from an F distribution with n_1 and n_2 degrees of freedom, respectively, where $n_1 = N-1$ and $n_2 = N(T-1) - K$. That is, for each $m = 0,1,2,3$, reject H_0 if $F_\omega^{(m)} > c_{N,\alpha}$, where $\Pr(\xi > c_{N,\alpha}) = \alpha$, for chosen α, and $\xi \sim F(n_1, n_2)$

[11] We also considered a pure random effects specification, $\tau_i = v_\alpha$, $R^2 = 0$, and the results show that the power properties of the modified fixed effects test and the modified random effects test are very similar.

[12] The estimator $\tilde{\omega}_N$, based on the unrestricted estimator (i.e., fixed effects estimator), is also considered, but the finite sample performance of the tests considered is monotonically inferior to that based on the estimator of $\hat{\omega}_N$.

2. One sided (positive) *RE-test* statistics (denoted R_ω in the Tables)

$$R_\omega^{(m)} = \hat{\omega}_N^{(m)} R_N, \quad m = 0, 1, 2, 3 \tag{19}$$

where

$$R_N = \sqrt{\frac{NT}{2(T-1)}} \left[\frac{\hat{\mathbf{u}}' (\mathbf{I}_N \otimes \mathbf{A}) \hat{\mathbf{u}}}{\hat{\mathbf{u}}'\hat{\mathbf{u}}} \right] \equiv R_\omega^{(0)}$$

is the one sided (positive) standard RE-test Statistic. The corresponding test procedure, for each separate statistic (19), employs critical values from a $N(0,1)$ distribution. That is, for each $m = 0, 1, 2, 3$, reject H_0 if $R_\omega^{(m)} > z_\alpha$, where $\Pr(Z > z_\alpha) = \alpha$, for chosen α, and $Z \sim N(0,1)$.

4.3 Bootstrap Tests

As is well known, asymptotic theory can provide a poor approximation to actual finite sample behaviour and that bootstrap procedures often lead to more reliable inferences.[13] Therefore, we also consider a simple wild bootstrap procedure scheme, based on Mammen (1993) and Davidson and Flachaire (2008), which might be employed in order to provide closer agreement between the desired nominal and the empirical significance level of the proposed test procedures and which has proved effective in previous studies; see, for example, Godfrey and Orme (2004). The wild bootstrap is implemented using the following steps:

1. Estimate the models (2) and (4) to get $\hat{u}_{it}$, $i = 1, ..., N$, and construct test statistics $F_\omega^{(m)}$ and $R_\omega^{(m)}$, $m = 0, 1, 2, 3$.

2. Repeat the following B times:

   (a) Generate $u_{it}^* = \varepsilon_{it}\hat{u}_{it}$, where the ε_{it} are i.i.d., over i and t, taking the discrete values ± 0.5 with an equal probability of 0.5.

   (b) Construct
   $$y_{it}^* = \mathbf{z}_{it}'\hat{\boldsymbol{\beta}} + \varepsilon_{it}\hat{u}_{it} = \mathbf{z}_{it}'\hat{\boldsymbol{\beta}} + u_{it}^*. \tag{20}$$
   and obtain restricted OLS residuals $\hat{u}_{it}^* = y_{it}^* - \mathbf{z}_{it}'\hat{\boldsymbol{\beta}}^*$, $\hat{\mathbf{u}}^* = \mathbf{y}^* - \mathbf{Z}\hat{\boldsymbol{\beta}}$ and restricted and unrestricted residual sums of squares (RSS_R^* and RSS_U^*, respectively).

   (c) Construct the bootstrap test statistics
   $$F_\omega^{*(m)} = \hat{\omega}_N^{*(m)}(F_N^* - 1) + 1, \quad F_N^* = \frac{(RSS_R^* - RSS_U^*)/(N-1)}{RSS_U^*/(N(T-1)-K)} \equiv F_\omega^{*(0)}$$
   and
   $$R_\omega^{*(m)} = \hat{\omega}_N^{*(m)} R_N^*, \quad R_N^* = \sqrt{\frac{NT}{2(T-1)}} \left[\frac{\hat{\mathbf{u}}^{*\prime} (\mathbf{I}_N \otimes \mathbf{A}) \hat{\mathbf{u}}^*}{\hat{\mathbf{u}}^{*\prime}\hat{\mathbf{u}}^*} \right] \equiv R_\omega^{*(0)}$$
   where $\hat{\omega}_N^{*(m)}$, $m = 1, 2, 3$ is constructed as in (10)-(12) but using $\hat{u}_{it}^*$, and $\hat{\omega}_N^{*(0)} \equiv 1$.

[13]See Godffrey (2009) for an excellent guide to bootstrap test procedures for regression models.

3. Calculate the proportion of bootstrap test statistics, $F_\omega^{*(m)}$ (respectively, $R_\omega^{*(m)}$), from the B repetitions of Step 2c that are at least as large as the actual value of $F_\omega^{(m)}$ (respectively, $R_\omega^{(m)}$). Let this proportion be denoted by $\hat{p}^{(m)}$ and the desired significance level be denoted by α. The asymptotically valid rejection rule, for each m, is that H_0 is rejected if $\hat{p}^{(m)} \leq \alpha$.

The sampling behaviour of all the above the tests are investigated using 5000 replications of sample data and $B = 200$ bootstrap samples, employing a nominal 5% significance level.

4.4 Results

Before looking at the results from the Monte Carlo study, and drawing on the discussion in Godfrey, Orme and Santos-Silva (2006), it is important to define criteria to evaluate the performance of the different tests considered. Given the large number of replications performed, the standard asymptotic test for proportions can be used to test the null hypotheses that the true significance level is equal to its nominal value. In these experiments, this null hypothesis is accepted (at the 5% level) for estimated rejection frequencies in the range 4% to 6%. In practice, however, what is important is not that the significance level of the test is identical to the chosen nominal level, but rather that the true and nominal rejection frequencies stay reasonably close, even when the test is only approximately valid. Following Cochran's (1952) suggestion, we shall regard a test as being robust, relative to a nominal value of 5%, if its actual significance level is between 4.5% and 5.5%. Considering the number of replications used in these experiments, estimated rejection frequencies within the range 3.9% to 6.1% are viewed as providing evidence consistent with the robustness of the test, according to this definition.

Under the null, with homoskedastic standard normal errors (reported in Table 1, $H_0 : \alpha_i = 0$), the rejection frequencies of both the asymptotic $F_\omega^{(0)} \equiv F_N$ and $F_\omega^{(3)}$ tests are close to the nominal significance level of 5%. The asymptotic *F-test* based on $F_\omega^{(2)}$, however, tends to under reject the null when T is relatively large, whilst $F_\omega^{(1)}$ suffers from large size distortion with empirical significance levels being considerably smaller than the nominal 5%. The size properties of the R_ω tests, for different $\hat{\omega}_N$, are qualitatively similar to those of the F_ω tests, but tend to have empirical significance levels that are smaller than those of the corresponding F_ω tests. Turning our attention to the bootstrap tests, all the modified fixed and random effects tests control the empirical significance levels very well. The results are qualitatively similar for t_5 and χ^2_6 errors and, confirming the analysis of Orme and Yamagata (2006), $F_\omega^{(0)} \equiv F_N$ appears quite robust to non-normality, whilst in these cases as well the bootstrap tests provide very close agreement between nominal and empirical significance levels. Given these results, we now just compare the power of the bootstrap tests. All bootstrap F_ω tests have very similar power, as do the bootstrap R_ω tests. However, the power of the bootstrap F_ω tests are uniformly higher than power of the corresponding bootstrap R_ω tests which is as expected given the analysis in Section 3.2.2 because of the correlation between regressors and individual effects.

The above results indicate that, even when the errors are homoskedastic, a wild bootstrap procedure still offers reliable finite sample inference for all variants of the FE and RE tests considered. Now let us look at the results under various heteroskedastic schemes. Table 2 reports the results under cross-sectional one-break-in-volatility scheme (HET1). First, and as predicted by the analysis in Section 3.2.3, both the $F_\omega^{(0)} \equiv F_N$ and $R_\omega^{(0)} \equiv R_N$ tests reject the correct null too often. On the other hand, the empirical significance levels of the other F_ω and R_ω tests are very similar to those presented in homoskedastic case. As before, however, the bootstrap F_ω^* and R_ω^* tests provide close agreement between nominal and empirical significance levels, across all error distributions, so again it is sensible to focus only on the power properties of these tests. In contrast to the power properties under homoskedastic errors, under the HET1 scheme the power of bootstrap F_ω^* tests appear different across different variants. For example, $F_\omega^{*(0)} \equiv F_N^*$ and $F_\omega^{*(3)}$ have similar powers but are slightly lower than that of $F_\omega^{*(2)}$, which is again slightly exceeded by that of $F_\omega^{*(1)}$. This feature is qualitatively similar for the R_ω^* tests, but is less striking. Finally, the results confirm again that F_ω^* has higher power than that of R_ω^*.

Table 3 reports the test results under time-series one-break-in-volatility scheme (HET2). In contrast to the results with HET1 scheme, but still consistent with prediction of Section 3.2.3, both the $F_\omega^{(0)} \equiv F_N$ and $R_\omega^{(0)} \equiv R_N$ tests reject the null too infrequently, especially for $N = 20, 50, 100$ and $T = 5$. As before the bootstrap versions control the size very well, and, interestingly, the power ranking of the bootstrap tests is different than that obtained under HET1. In fact, the $F_\omega^{*(0)} \equiv F_N^*$ and $F_\omega^{*(3)}$ tests (respectively $R_\omega^{*(0)} = R_N$ and $R_\omega^{*(3)}$ tests) still have similar powers but they are now slightly higher than those of the $F_\omega^{*(2)}$ and $F_\omega^{*(1)}$ tests (respectively, $R_\omega^{*(2)}$ and $R_\omega^{*(1)}$ tests), which are in this case comparable.

Based on the analysis in Section 3.2.3 it is possible to derive approximate null rejection frequencies of the $F_\omega^{(0)} \equiv F_N$ test analytically, under the heteroskedastic schemes of HET1 and HET2. Given the "population" value of ω_N, and a nominal significance level of $\alpha \times 100\%$, the rejection frequency of the F_N test is, approximately, $\Pr\left[F_N > c_{\alpha,n1,n2}\right]$, where $\Pr\left[F_{n1,n2} > c_{\alpha,n1,n2}\right] = \alpha$ and $F_{n1,n2} \sim F(n_1, n_2)$. But this is identical to $\Pr\left[F_{n1,n2} > q\right]$, where $q = \omega_N(c_{\alpha,n1,n2} - 1) + 1$. More precisely, consider first the case of HET1 where a little calculation shows that, since N is always even in our experiments, $\omega_N = 0.781$. Using $\alpha = 0.05$, it is then straightforward to obtain q and $\Pr\left[F_{n1,n2} > q\right]$. Similar calculations can be undertaken for the case HET2 but, here, ω_N varies according to whether T is even ($\omega_N = 1.02$) or odd ($\omega_N = 1.13$). From these calculations we obtain the following (approximate) significance levels for our choices of (N, T) :

Approximate Significance Levels of F_N					
	$T = 5$			$N = 50$	
	$N = 20$	$N = 50$	$N = 100$	$T = 10$	$T = 20$
HET1:	8.8%	9.2%	9.4%	9.2%	9.2%
HET2:	3.5%	3.4%	3.3%	4.8%	4.8%

As can be seen, the obtained empirical significance levels, for F_N, are qualitatively very similar to these predicted values.

Table 4 summarises the results under conditional heteroskedasticity depending on a regressor $z_{it,2}$ (HET3), where $\sigma_{it} = \eta_1[(z_{it,2} - 1)/30]$, $i = 1, ..., N$, $t = 1, ..., T$, and $\eta_1[\cdot]$ is the inverse of the cumulative distribution function of the χ_1^2 distribution. Since the $z_{it,2}$ are initially i.i.d. draws from a uniform distribution on $(1, 31)$, the values of $\sigma_{it}(z_{it,2})$ are realisations from a χ_1^2 distribution. This means that even though for a given N (and T) σ_{it} will be held fixed for each replication of data, possibly yielding a realisation of $\omega_N \neq 1$, as N increases a Law of Large Numbers implies that the given realisation of ω_N will converge to unity. For example, when $N = 20$ and $T = 5$, $\omega_N = 1.36$, yielding a predicted (approximate) significance level for F_N of 1.9%, which explains the under-rejection of this test in our experiments. For larger sample sizes, the value of ω_N does, indeed, tend to unity, and the empirical significance level of F_N converges to the nominal level, as expected. Due to the larger average error variance encountered here, than that under other heteroskedastic schemes, the power of the tests are lower although, qualitatively, the results are very similar to those under HET0 but with $F_\omega^{*(0)} = F_N^*$ and $F_\omega^{*(3)}$ (respectively, $R_\omega^{*(0)} = R_N^*$ and $R_\omega^{*(3)}$) enjoying a slight power advantage and the F_ω^* tests being more powerful than their R_ω^* counterparts.

Finally, the results under conditional heteroskedasticity, GARCH(1,1) (HET4) and ARCH(1) (HET5) are reported in Tables 5 and 6, respectively. Similar to the results obtained under HET1, the $F_\omega^{(0)} = F_N$ test rejects a correct null too often but the empirical significance levels of other variants of the F_ω tests are very similar to those presented in homoskedastic case. Again, all the bootstrap F_ω^* tests control the empirical significance levels very well, and the power rankings are, from the lowest, $F_\omega^{*(0)} = F_N^*$ and $F_\omega^{*(3)}$, followed by $F_\omega^{*(2)}$, then $F_\omega^{*(1)}$. The same comments apply to the bootstrap R_ω^* tests, which again exhibit lower power than their F_ω^* counterparts. The results under ARCH(1) (HET5) are are qualitatively similar to those under GARCH(1,1).

5 Conclusions

This paper has provided an asymptotic analysis of the sampling behaviour of the standard *F-test* statistic for fixed effects, in a static linear panel data model, under both non-normality and heteroskedasticity of the error terms, when the number of cross-sections, N, is large and T, the number of time periods, is fixed. First, it has been shown that a linear transformation of the commonly cited F and RE tests (using a simple function of restricted residuals) provides asymptotically valid test procedures, when employed in conjunction with the usual F and *standard normal* critical values (respectively). Although asymptotic theory does not always provide a good approximation to finite sample behaviour, our experiments show that wild bootstrap versions of these tests, employing the resampling scheme advocated by Davidson and Flachaire (2008), yield reliable inferences in the sense of close agreement between nominal and actual significance levels.

Furthermore, it has been established that the asymptotic relationship between the heteroskedastic robust *F-test* and the *RE-test* statistics, carries over from the homoskedastic case. That is, under (pure) local random effects, they share the same asymptotic power, whilst under local fixed (or correlated) individual effects the heteroskedastic robust *F-test* enjoys higher asymptotic power. A final contribution has been to provide qualitative predictions about the approximate true significance levels of the standard F and RE *Tests* in certain special cases. These theoretical findings are supported by Monte Carlo evidence.

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Appendix

In what follows $\|\mathbf{C}\| = \sqrt{tr\left(\mathbf{C}'\mathbf{C}\right)}=\sqrt{\sum_i \sum_j c_{ij}^2}$ denotes the Euclidean norm of a matrix $\mathbf{C} = \{c_{ij}\}$.

Proof of Lemma 1.

Write $W_i = \dfrac{\mathbf{u}_i'\mathbf{A}\mathbf{u}_i}{\sqrt{T\left(T-1\right)}}$, which are independent, so that $H_N = \dfrac{1}{\sqrt{N}}\sum_{i=1}^N W_i$ and $E\left[W_i\right] = 0$, by Assumption A1(iii). Since $\|\mathbf{A}\| = \sqrt{T(T-1)}$, $|W_i| = \dfrac{|\mathbf{u}_i'\mathbf{A}\mathbf{u}_i|}{\sqrt{T\left(T-1\right)}} \leq \dfrac{\|\mathbf{u}_i\|^2 \|\mathbf{A}\|}{\sqrt{T\left(T-1\right)}} = \|\mathbf{u}_i\|^2$. Thus, by Minkowski's inequality and Assumption A3(i), for some $\eta > 0$,

$$E\left|W_i\right|^{2+\eta} \leq E\left|\sum_{t=1}^T u_{it}^2\right|^{2+\eta} \leq \left[\sum_{t=1}^T \left\{E\left|u_{it}^2\right|^{2+\eta}\right\}^{\frac{1}{2+\eta}}\right]^{2+\eta} = O(1),$$

so that $\kappa_N = \frac{1}{2N}\sum_{i=1}^N E\left(W_i^2\right) = O(1)$. With Assumption A3(ii), a standard (Liapounov) Central Limit Theorem yields $\kappa_N^{-1/2} H_N \xrightarrow{d} N(0,1)$. ■

Proof of Proposition 1.

The method of proof is nearly identical to that of Orme and Yamagata (2006, Proposition 1) but where, now, our assumptions allow for heteroskedasticity. Briefly:

(i) Let $S_N = (RSS_R - RSS_U)/(N-1)$ and $\tilde{\sigma}^2 = RSS_U/\left(N\left(T-1\right)-K\right)$, so that

$$\bar{\sigma}_N^2\sqrt{N}(F_N - 1) = \frac{\bar{\sigma}_N^2}{\tilde{\sigma}^2}\sqrt{N}\left(S_N - \tilde{\sigma}^2\right). \tag{21}$$

We first show that $\tilde{\sigma}^2 - \bar{\sigma}_N^2 = o_p\left(1\right)$, so that (since $\bar{\sigma}_N^2$ is uniformly positive by Assumption A2(v)) $\tilde{\sigma}^2/\bar{\sigma}_N^2 \xrightarrow{p} 1$. Following Orme and Yamagata (2006, Proof of Proposition 1), we can write

$$\begin{aligned}\tilde{\sigma}^2 &= \frac{N}{N\left(T-1\right)-K}\frac{\mathbf{u}'\left(\mathbf{M}_{\tilde{\mathbf{X}}} - \mathbf{P}_{\mathbf{D}}\right)\mathbf{u}}{N} \\ &= \frac{N}{N\left(T-1\right)-K}\left\{\frac{\mathbf{u}'\mathbf{u}}{N} - \frac{\mathbf{u}'\mathbf{P}_{\tilde{\mathbf{X}}}\mathbf{u}}{N} - \frac{\mathbf{u}'\mathbf{P}_{\mathbf{D}}\mathbf{u}}{N}\right\} \\ &= \frac{\mathbf{u}'\mathbf{M}_{\mathbf{D}}\mathbf{u}}{N\left(T-1\right)} + O_p(N^{-1})\end{aligned}$$

because $\dfrac{\mathbf{u}'\mathbf{u}}{N}$, $\dfrac{\mathbf{u}'\mathbf{P}_{\mathbf{D}}\mathbf{u}}{N}$ and $\mathbf{u}'\mathbf{P}_{\tilde{\mathbf{X}}}\mathbf{u}$ are all $O_p(1)$ and $\frac{N}{N(T-1)-K} = \frac{1}{T-1} + O(N^{-1})$.

Therefore,

$$\begin{aligned}
\tilde{\sigma}^2 - \bar{\sigma}_N^2 &= \frac{\mathbf{u}'\mathbf{M_D}\mathbf{u}}{N(T-1)} - \frac{T\bar{\sigma}_N^2}{T-1} + \frac{\bar{\sigma}_N^2}{T-1} + O_p(N^{-1}) \\
&= \frac{1}{T-1}\left\{T\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}^2 - \bar{\sigma}_N^2\right) - \left(\frac{1}{NT}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}u_{it}\right)^2 - \bar{\sigma}_N^2\right)\right\} \\
&\quad + O_p(N^{-1}) \\
&= o_p(1),
\end{aligned}$$

because, by Assumption A2(i) and A1(iii), both terms in side the $\{.\}$ above are $o_p(1)$. Thus, provided $\sqrt{N}\left(S_N - \tilde{\sigma}^2\right) = O_p(1)$, (21) yields

$$\bar{\sigma}_N^2\sqrt{N}(F_N - 1) = \sqrt{N}\left(S_N - \tilde{\sigma}^2\right) + o_p(1).$$

But from exactly the same argument employed by Orme and Yamagata (2006, pp.418-419) $\sqrt{N}\left(S_N - \tilde{\sigma}^2\right) = O_p(1)$ with

$$\sqrt{N}\left(S_N - \tilde{\sigma}^2\right) = \frac{1}{(T-1)}\frac{1}{\sqrt{N}}\left[\mathbf{u}'\left(\mathbf{I}_N \otimes \mathbf{A}\right)\mathbf{u}\right] + \lambda_N + o_p(1).$$

Thus, (21) can be expressed as

$$\begin{aligned}
\bar{\sigma}_N^2\sqrt{N}(F_N - 1) &= \frac{1}{\sqrt{N}}\frac{\mathbf{u}'\left(\mathbf{I}_N \otimes \mathbf{A}\right)\mathbf{u}}{T-1} + \lambda_N + o_p(1), \\
&= \sqrt{\frac{T}{T-1}}H_N + \lambda_N + o_p(1),
\end{aligned}$$

(ii) By Lemma 1,

$$\omega_N\sqrt{N}(F_N - 1) - \frac{\lambda_N}{\sqrt{\kappa_N/2}} \xrightarrow{d} N\left(0, \frac{2T}{T-1}\right),$$

and the result follows. This completes the proof. ■

Proof of Proposition 2.

1. By the Triangle Inequality, $\left|\hat{\sigma}^2 - \bar{\sigma}_N^2\right| \le \left|\hat{\sigma}^2 - \frac{\mathbf{u}'\mathbf{u}}{NT}\right| + \left|\frac{\mathbf{u}'\mathbf{u}}{NT} - \bar{\sigma}_N^2\right| = o_p(1)$, since, as previously noted, $\frac{\mathbf{u}'\mathbf{u}}{NT} = \bar{\sigma}_N^2 + o_p(1)$ and $\hat{\sigma}^2 - \frac{\mathbf{u}'\mathbf{u}}{NT} = o_p(1)$ by the arguments of Orme and Yamagata (2006, p.422).

2. From the proof of Lemma 1, we have that

$$\frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{u}_i'\mathbf{A}\mathbf{u}_i\right)^2 - \frac{1}{N}\sum_{i=1}^{N}E\left(\mathbf{u}_i'\mathbf{A}\mathbf{u}_i\right)^2 \xrightarrow{p} 0.$$

Therefore, by the Triangle Inequality, it remains to show that $\frac{1}{N}\sum_{i=1}^{N}(\hat{\mathbf{u}}_i'\mathbf{A}\hat{\mathbf{u}}_i)^2 - \frac{1}{N}\sum_{i=1}^{N}(\mathbf{u}_i'\mathbf{A}\mathbf{u}_i)^2 \xrightarrow{p} 0$. Since, $\hat{\mathbf{u}}_i = \mathbf{u}_i + \hat{\mathbf{v}}_i$, where $\hat{\mathbf{v}}_i = \boldsymbol{\iota}_T\delta_i/N^{1/4} - \mathbf{Z}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$, we can write

$$\begin{aligned}
\hat{\mathbf{u}}_i'\mathbf{A}\hat{\mathbf{u}}_i &= \mathbf{u}_i'\mathbf{A}\mathbf{u}_i + 2\mathbf{u}_i'\mathbf{A}\hat{\mathbf{v}}_i + \hat{\mathbf{v}}_i'\mathbf{A}\hat{\mathbf{v}}_i \\
&= \mathbf{u}_i'\mathbf{A}\mathbf{u}_i + S_i, \quad \text{say,}
\end{aligned}$$

so that

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\mathbf{u}}_i'\mathbf{A}\hat{\mathbf{u}}_i)^2 = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{u}_i'\mathbf{A}\mathbf{u}_i)^2 + \frac{1}{N}\sum_{i=1}^{N}S_i^2 + \frac{1}{N}\sum_{i=1}^{N}\mathbf{u}_i'\mathbf{A}\mathbf{u}_iS_i.$$

Now, $\frac{1}{N}\sum_{i=1}^{N}(\mathbf{u}_i'\mathbf{A}\mathbf{u}_i)^2 = O_p(1)$ and we shall show that $\frac{1}{N}\sum_{i=1}^{N}S_i^2 = o_p(1)$ so that, by Cauchy-Schwartz, $\frac{1}{N}\sum_{i=1}^{N}\mathbf{u}_i'\mathbf{A}\mathbf{u}_iS_i = o_p(1)$; then we are done.
Again by Cauchy-Schwartz, $\frac{1}{N}\sum_{i=1}^{N}S_i^2 = o_p(1)$ if it can be shown that (i) $\frac{1}{N}\sum_{i=1}^{N}(\mathbf{u}_i'\mathbf{A}\hat{\mathbf{v}}_i)^2 = o_p(1)$; and (ii) $\frac{1}{N}\sum_{i=1}^{N}(\hat{\mathbf{v}}_i'\mathbf{A}\hat{\mathbf{v}}_i)^2 = o_p(1)$ and we take each of these in turn:

(i) First, by repeated application of Cauchy-Schwartz, noting that $\|\mathbf{A}\|^2 = T(T-1)$,

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}|\mathbf{u}_i'\mathbf{A}\hat{\mathbf{v}}_i|^2 &\leq \frac{T(T-1)}{N}\sum_{i=1}^{N}\|\mathbf{u}_i\|^2\|\hat{\mathbf{v}}_i\|^2 \\
&\leq T(T-1)\sqrt{\frac{1}{N}\sum_{i=1}^{N}\|\mathbf{u}_i\|^4\frac{1}{N}\sum_{i=1}^{N}\|\hat{\mathbf{v}}_i\|^4}.
\end{aligned}$$

Now, $E\|\mathbf{u}_i\|^4$ is uniformly bounded, by Assumption A3(i), so by Markov's Inequality, $\frac{1}{N}\sum_{i=1}^{N}\|\mathbf{u}_i\|^4 = O_p(1)$ and it suffices to show that $\frac{1}{N}\sum_{i=1}^{N}\|\hat{\mathbf{v}}_i\|^4 = o_p(1)$.
Now,

$$\begin{aligned}
\|\hat{\mathbf{v}}_i\|^2 &= \frac{T\delta_i^2}{\sqrt{N}} - 2\frac{\delta_i}{N^{1/4}}\boldsymbol{\iota}_T'\mathbf{Z}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{Z}_i'\mathbf{Z}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \\
&= S_{i1} + S_{i2} + S_{i3}, \quad \text{say,}
\end{aligned}$$

so that, by Cauchy-Schwartz, $\frac{1}{N}\sum_{i=1}^{N}\|\hat{\mathbf{v}}_i\|^4 = o_p(1)$ if $\frac{1}{N}\sum_{i=1}^{N}S_{im}^2 = o_p(1)$, for $m = 1, 2, 3$. Clearly, $\frac{1}{N}\sum_{i=1}^{N}S_{i1}^2 = \frac{T}{N}\frac{1}{N}\sum_{i=1}^{N}\delta_i^2 = o_p(1)$, by Assumption A4(ii) and, by repeated use of Cauchy-Schwartz,

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}S_{i2}^2 &\leq 4\frac{T}{\sqrt{N}}\left\|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right\|^2\frac{1}{N}\sum_{i=1}^{N}\|\delta_i\mathbf{Z}_i\|^2 \\
&= o_p(1)
\end{aligned}$$

because $\left\|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right\| = o_p(1)$, $\frac{1}{N}\sum_{i=1}^{N}\|\delta_i\mathbf{Z}_i\|^2 = \frac{1}{N}\sum_{i=1}^{N}\sum_t\sum_j|\delta_iz_{itj}|^2 = O_p(1)$, by an application of Markov's Inequality, Cauchy-Schwartz and

Assumptions A2(ii) and A4(ii). Finally,

$$\frac{1}{N}\sum_{i=1}^{N} S_{i3}^2 \leq \left\|\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right\|^4 \frac{1}{N}\sum_{i=1}^{N} \left\|\mathbf{Z}_i'\mathbf{Z}_i\right\|^2$$

where $\left\|\mathbf{Z}_i'\mathbf{Z}_i\right\|^2 = \sum_j \sum_k \left\{\sum_t z_{itj} z_{itk}\right\}^2$ and an application of Markov's Inequality, Minkowski's Inequality, Cauchy-Schwartz and Assumption A2(ii) yields $\frac{1}{N}\sum_{i=1}^{N} \left\|\mathbf{Z}_i'\mathbf{Z}_i\right\|^2 = O_p(1)$ and $\frac{1}{N}\sum_{i=1}^{N} S_{i3}^2 = o_p(1)$.
Thus, $\frac{1}{N}\sum_{i=1}^{N} \left\|\hat{\mathbf{v}}_i\right\|^4 = o_p(1)$.

(ii) It immediately follows that $\frac{1}{N}\sum_{i=1}^{N} \left(\hat{\mathbf{v}}_i'\mathbf{A}\hat{\mathbf{v}}_i\right)^2 \leq T(T-1)\frac{1}{N}\sum_{i=1}^{N} \left\|\hat{\mathbf{v}}_i\right\|^4 = o_p(1)$, and we are done.

3. By Assumption A3(i), and Minkowski's Inequality $E\left|\sum_{t=2}^{T} w_{it}^2\right|^{1+\eta}$ is uniformly bounded so that $\frac{1}{N}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it}^2 - \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} E\left(w_{it}^2\right) \xrightarrow{p} 0$. Thus, by the Triangle Inequality, it remains to show that $\frac{1}{N}\sum_{i=1}^{N}\sum_{t=2}^{T} \hat{w}_{it}^2 - \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} w_{it}^2 \xrightarrow{p} 0$. Since $\hat{u}_{it} = u_{it} + \hat{v}_{it}$, $\hat{v}_{it} = \delta_i/N^{1/4} - \mathbf{z}_{it}'(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})$, we can write

$$\begin{aligned}
\hat{w}_{it} &= w_{it} + \hat{v}_{it}\sum_{s=1}^{t-1} u_{is} + \hat{v}_{it}\sum_{s=1}^{t-1} \hat{v}_{is} + u_{it}\sum_{s=1}^{t-1} \hat{v}_{is} \\
&= w_{it} + \hat{g}_{it}, \quad \text{say.}
\end{aligned}$$

Thus, by Cauchy-Schwartz, it suffices to show that $\frac{1}{N}\sum_{i=1}^{N}\sum_{t=2}^{T} \hat{g}_{it}^2 = o_p(1)$. It will be useful to note that

$$\begin{aligned}
\sum_{t=2}^{T} \hat{g}_{it}^2 &\leq \sum_{t=1}^{T} \left(|\hat{v}_{it}| \sum_{t=1}^{T} |u_{it}| + |\hat{v}_{it}| \sum_{t=1}^{T} |\hat{v}_{it}| + |u_{it}| \sum_{t=1}^{T} |\hat{v}_{it}| \right)^2 \\
&= \sum_{t=1}^{T} \left(S_{it1} + S_{it2} + S_{it3}\right)^2, \quad \text{say,}
\end{aligned}$$

so that, now, it is sufficient to demonstrate that , $\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} S_{itm}^2 = o_p(1)$, $m = 1, 2, 3$.
By Cauchy-Schwartz, we have

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} S_{it1}^2 &= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} \hat{v}_{it}^2 \left(\sum_{t=1}^{T} |u_{it}|\right)^2 \\
&\leq \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T} \hat{v}_{it}^2\right)^2 \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T} |u_{it}|\right)^4}
\end{aligned}$$

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} S_{it2}^2 \leq \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T} \hat{v}_{it}^2\right)^2 \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T} |\hat{v}_{it}|\right)^4}$$

and

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}S_{it3}^2 \le \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}u_{it}^2\right)^2 \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}|\hat{v}_{it}|\right)^4}.$$

Both $\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}|u_{it}|\right)^4$ and $\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}u_{it}^2\right)^2$ are $O_p(1)$, by Markov's Inequality, Minkowski's Inequality and Assumption A3(i). Thus, it suffices to show that $\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\hat{v}_{it}^2\right)^2$ and $\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}|\hat{v}_{it}|\right)^4$ are both $o_p(1)$. The former is identical to $\frac{1}{N}\sum_{i=1}^{N}\|\hat{\mathbf{v}}_i\|^4 = o_p(1)$, by the proof of 2(i), above, and the latter is $o_p(1)$ by Assumption A2(ii), A4(ii) and the consistency of $\hat{\boldsymbol{\beta}}$. This completes the proof of part 3.

4. As in previous proofs, by Assumption A3(i) and the Triangle Inequality it suffices to show that

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{t}\sum_{s\neq t}\hat{u}_{it}^2\hat{u}_{is}^2 - \frac{1}{N}\sum_{i=1}^{N}\sum_{t}\sum_{s\neq t}u_{it}^2u_{is}^2 \xrightarrow{p} 0.$$

Again, since $\hat{u}_{it} = u_{it} + \hat{v}_{it}$, $\hat{v}_{it} = \delta_i/N^{1/4} - \mathbf{z}_{it}'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$, we can write

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\sum_{t}\sum_{s\neq t}\hat{u}_{it}^2\hat{u}_{is}^2 - \frac{1}{N}\sum_{i=1}^{N}\sum_{t}\sum_{s\neq t}u_{it}^2u_{is}^2 &= 2\frac{1}{N}\sum_{i=1}^{N}\sum_{t}\sum_{s\neq t}u_{it}^2V_{is} \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\sum_{t}\sum_{s\neq t}V_{it}V_{is} \\
&= S_{N1} + S_{N2}, \quad \text{say,}
\end{aligned}$$

where $V_{it} = 2u_{it}\hat{v}_{it} + \hat{v}_{it}^2$, and it suffices to show that $S_{Nm} = o_p(1)$, $m = 1, 2$. Now,

$$\begin{aligned}
|S_{N1}| &\le 2\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}^2\sum_{t=1}^{T}|V_{it}| \\
&\le 2\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}u_{it}^2\right)^2 \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}|V_{it}|\right)^2}.
\end{aligned}$$

Thus, since $\frac{1}{N}\sum_{i=1}^{N}\left(\sum_t u_{it}^2\right)^2 = O_p(1)$, it suffices to show that $\frac{1}{N}\sum_{i=1}^{N}\left(\sum_t |V_{it}|\right)^2 = o_p(1)$, or that $\frac{1}{N}\sum_{i=1}^{N}\sum_t V_{it}^2 = o_p(1)$ since $\left(\sum_t |V_{it}|\right)^2 \le T\sum_t V_{it}^2$. But this is true because

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}V_{it}^2 \le \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{v}_{it}^4 + 4\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}\hat{v}_{it}^3 + 4\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}^2\hat{v}_{it}^2.$$

The first term on the right hand side is $o_p(1)$ as are the latter two terms by an application of Cauchy-Schwartz.

Second,

$$|S_{N2}| \leq \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=1}^{T} |V_{it}| \right)^2 = o_p(1)$$

by the preceding result, and this completes the proof. ■

Proof of Proposition 3.

We can write $R_N = \dfrac{1}{\sqrt{2}} \dfrac{\hat{H}_N}{\hat{\sigma}^2}$, where $\hat{\sigma}^2 = \hat{\mathbf{u}}'\hat{\mathbf{u}}/NT$ and

$$\begin{aligned} \hat{H}_N &= \frac{1}{\sqrt{NT(T-1)}} \left[\hat{\mathbf{u}}' \left(\mathbf{I}_N \otimes \mathbf{A} \right) \hat{\mathbf{u}} \right] \\ &= \frac{1}{\sqrt{NT(T-1)}} \left[\mathbf{y}' \mathbf{M}_{\mathbf{Z}} \left(\mathbf{I}_N \otimes \mathbf{A} \right) \mathbf{M}_{\mathbf{Z}} \mathbf{y} \right]. \end{aligned}$$

By Proposition 1, it is sufficient to show that

$$\hat{H}_N = H_N + \sqrt{\frac{T-1}{T}} \lambda_N - \sqrt{\frac{T}{T-1}} \gamma_N + o_p(1)$$

and

$$\hat{\sigma}^2 - \bar{\sigma}_N^2 = o_p(1)$$

and the result follows.

Establishing the former follows exactly the argument as in Orme and Yamagata (2006, Proof of Proposition 2), and $\hat{\sigma}^2 - \bar{\sigma}_N^2 = o_p(1)$, was established above. This completes the proof. ■

Table 1: Rejection frequencies of the asymptotic and wild-bootstrap modified F-tests and modified random effects tests under homoskedastic errors (HET0).

	$H_0 : \alpha_i = 0$								$H_1 : var\,(\alpha_i) = 0.1$, α_i correlated with regressors							
	Asymptotic Tests				Bootstrap Tests				Asymptotic Tests				Bootstrap Tests			
ω	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$
	SN								SN							
N, T	F_ω				F_ω^*				F_ω				F_ω^*			
20, 5	5.8	2.8	4.7	5.9	6.2	5.9	5.9	6.1	29.0	18.0	25.4	29.4	30.4	29.9	29.8	30.4
50, 5	5.2	2.8	4.5	5.2	5.7	5.8	5.8	5.7	46.8	38.6	44.4	47.1	47.9	48.8	48.3	47.8
100, 5	4.7	3.1	4.1	4.8	5.3	5.2	5.3	5.3	72.1	66.6	70.8	72.3	73.0	74.3	73.6	73.0
50, 10	4.3	1.9	3.1	4.3	4.5	4.7	4.4	4.5	94.2	87.9	92.6	94.2	94.4	94.0	94.4	94.4
50, 20	4.7	1.6	3.4	4.8	5.1	5.2	5.0	5.1	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	R_ω				R_ω^*				R_ω				R_ω^*			
20, 5	5.1	1.5	3.7	5.2	6.1	5.9	6.1	6.1	23.9	10.8	18.8	24.4	27.0	25.4	26.1	26.9
50, 5	4.6	2.3	3.9	4.5	5.6	5.6	5.6	5.5	32.5	20.4	29.3	32.7	35.9	35.1	35.3	35.8
100, 5	4.4	2.8	3.8	4.6	5.3	5.2	5.1	5.3	55.8	44.7	52.6	55.9	57.6	57.1	57.5	57.7
50, 10	4.1	1.7	2.9	4.0	4.5	4.8	4.5	4.5	89.7	77.7	87.2	89.8	90.5	89.6	90.3	90.5
50, 20	4.6	1.6	3.4	4.6	5.2	5.3	5.0	5.2	100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0
	t_5								t_5							
N, T	F_ω				F_ω^*				F_ω				F_ω^*			
20, 5	4.8	2.4	3.9	5.3	5.5	5.4	5.5	5.4	30.1	20.7	27.6	31.5	31.9	31.7	31.4	31.7
50, 5	4.6	2.7	4.0	4.9	5.3	5.0	5.3	5.2	47.9	40.3	46.4	49.0	49.2	50.5	50.0	49.1
100, 5	5.3	3.5	4.5	5.3	5.9	5.9	5.9	5.8	72.6	68.2	71.4	72.7	73.3	74.7	74.0	73.3
50, 10	5.2	2.1	4.0	5.3	5.7	5.4	5.6	5.7	93.6	87.1	92.0	93.6	94.0	93.3	93.5	93.9
50, 20	4.8	1.5	3.4	4.8	5.1	5.2	5.1	5.1	100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0
	R_ω				R_ω^*				R_ω				R_ω^*			
20, 5	4.4	1.5	3.2	4.6	5.7	5.6	5.4	5.4	24.2	12.7	21.2	25.8	28.6	26.7	27.7	28.6
50, 5	4.1	2.3	3.6	4.5	5.5	5.1	5.3	5.4	32.9	21.8	30.2	33.9	36.4	35.1	36.1	36.5
100, 5	5.0	3.3	4.3	5.0	5.8	6.1	5.8	5.9	56.6	47.1	55.1	57.7	58.8	58.7	59.2	58.8
50, 10	5.0	2.0	3.8	5.0	5.9	5.4	5.6	5.8	89.3	77.6	87.1	89.7	90.1	88.3	89.9	90.1
50, 20	4.6	1.5	3.4	4.5	5.2	5.2	5.1	5.2	100.0	99.8	100.0	100.0	100.0	100.0	100.0	100.0
	χ_6^2								χ_6^2							
N, T	F_ω				F_ω^*				F_ω				F_ω^*			
20, 5	4.5	2.3	3.6	4.4	4.7	5.1	4.7	4.5	30.1	19.7	27.4	31.0	31.5	31.6	32.2	31.4
50, 5	5.1	2.3	3.6	4.8	4.9	5.0	5.0	4.9	46.3	38.4	44.2	46.6	47.2	49.1	47.6	47.3
100, 5	4.9	3.0	4.0	4.8	5.0	5.3	5.2	5.0	72.8	67.8	72.3	73.6	74.4	75.5	74.9	74.3
50, 10	4.5	1.9	3.3	4.4	5.1	5.4	5.0	5.1	93.0	86.9	91.5	93.3	93.3	92.7	93.4	93.4
50, 20	5.2	1.8	3.6	4.9	5.1	5.5	5.4	5.1	100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0
	R_ω				R_ω^*				R_ω				R_ω^*			
20, 5	4.3	1.4	2.9	3.9	4.7	5.4	4.6	4.4	24.3	11.4	20.3	25.2	27.8	26.8	27.5	27.6
50, 5	4.6	1.8	3.1	4.1	4.9	4.9	4.9	4.8	31.5	20.5	27.8	31.6	34.2	33.3	34.1	34.2
100, 5	4.8	2.8	3.9	4.6	5.1	5.4	5.4	4.9	57.0	45.9	53.9	57.5	59.2	58.2	58.6	59.3
50, 10	4.4	1.7	3.2	4.3	5.1	5.3	5.0	5.1	89.1	76.5	86.7	89.3	89.8	88.1	89.5	89.9
50, 20	5.1	1.7	3.6	4.8	5.2	5.5	5.4	5.2	100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0

Notes: The model employed is $y_{it} = \alpha_i + \sum_{j=1}^{3} z_{it,j}\beta_j + u_{it}$, $u_{it} = \sigma_{it}\varepsilon_{it}$, where $z_{it,1} = 1$, $z_{it,2}$ is drawn from a uniform distribution on $(1, 31)$ independently for i and t, and $z_{it,3}$ is generated following Nerlove (1971), such that $z_{it,3} = 0.1t + 0.5z_{it-1,3} + \upsilon_{it}$,where the value $z_{i0,3}$ is chosen as $5 + 10\upsilon_{i0}$, and υ_{it} (and υ_{i0}) is drawn from the uniform distribution on $(-0.5, 0.5)$ independently for i and t, in order to avoid any normality in regressors. These regressor values are held fixed over replications. $\beta_j = 1$ for $j = 1, 2, 3$. The i.i.d. standardised errors for ε_{it} are drawn from: the standard normal distribution (SN); the t distribution with five degrees of freedom (t_5); and, the chi-square distribution with six degrees of freedom (χ_6^2). For estimating size of the tests, $\alpha_i = 0$ and power is investigated using $\alpha_i = \sqrt{0.1}g(z_i)$ where $g_i(z_i)$ is the standardised value of $\sum_{j=1}^{3}\sum_{t=1}^{T} z_{it,j}$, so that the regressors and α_i are correlated. F_ω is the modified F-test and R_ω is the modified random effects test, and F_ω^* and R_ω^* are their wild bootstrap tests, with different choice of $\hat{\omega}_N^{(m)}$, $m = 0, 1, 2, 3$ with $\hat{\omega}_N^{(0)} \equiv 1$; see section 4.2 and 4.3 Here $\sigma_{it} = 1$. The sampling behaviour of the tests are investigated using 5000 replications of sample data and 200 bootstrap samples, employing a nominal 5% significance level.

Table 2: Rejection frequencies of the asymptotic and wild-bootstrap modified F-tests and modified random effects tests under cross-sectional one-break-in-volatility heteroskedastic scheme (HET1).

	$H_0 : \alpha_i = 0$								$H_1 : var\,(\alpha_i) = 0.1$, α_i correlated with regressors							
	Asymptotic Tests				Bootstrap Tests				Asymptotic Tests				Bootstrap Tests			
ω	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$
					SN								SN			
N,T		F_ω					F^*_ω			F_ω					F^*_ω	
20, 5	9.4	2.7	4.9	6.6	6.1	5.8	5.9	5.9	26.9	14.2	18.4	21.0	20.5	22.8	20.5	20.0
50, 5	9.2	2.8	4.7	5.6	5.8	5.5	5.8	5.7	37.6	21.2	26.3	28.8	28.0	31.6	29.1	27.6
100, 5	9.1	3.2	4.7	5.4	5.5	5.4	5.5	5.5	55.2	40.4	43.3	44.7	44.2	49.5	46.2	44.1
50, 10	9.1	1.8	3.8	5.2	5.2	4.9	5.1	5.2	82.7	68.1	73.3	74.5	74.1	81.5	78.0	74.1
50, 20	8.7	1.2	2.9	4.5	4.8	4.8	4.9	4.8	99.8	98.5	99.5	99.4	99.5	99.7	99.7	99.5
		R_ω					R^*_ω			R_ω					R^*_ω	
20, 5	8.6	1.6	3.7	5.6	6.1	5.6	5.8	5.9	22.5	8.3	13.8	16.6	18.0	19.8	18.2	17.7
50, 5	8.8	2.0	4.1	5.2	5.8	5.6	5.7	5.7	27.9	10.3	16.3	19.0	20.9	21.4	21.0	20.7
100, 5	8.7	2.8	4.3	5.1	5.5	5.3	5.4	5.4	42.7	24.9	29.7	31.9	32.9	36.3	34.0	32.6
50, 10	8.7	1.6	3.5	5.0	5.2	4.9	5.2	5.2	75.5	53.9	63.3	65.7	66.1	73.4	69.3	66.0
50, 20	8.6	1.2	3.0	4.6	4.8	4.7	4.9	4.8	99.7	97.6	99.0	99.1	99.0	99.5	99.5	99.0
					t_5								t_5			
N,T		F_ω					F^*_ω			F_ω					F^*_ω	
20, 5	8.5	2.7	4.6	6.0	5.8	5.6	5.4	5.6	26.6	16.0	20.0	21.9	21.7	23.5	22.4	21.1
50, 5	8.6	2.9	4.3	5.5	5.2	5.3	5.4	5.1	39.3	24.4	28.6	30.9	30.7	33.1	31.5	30.3
100, 5	10.4	3.3	4.8	6.2	5.8	6.0	6.1	5.8	56.9	43.2	46.4	47.4	47.3	52.3	49.5	47.1
50, 10	9.2	1.8	3.9	5.2	5.6	5.2	5.5	5.5	82.1	68.0	73.7	74.6	73.6	80.4	77.1	73.5
50, 20	8.8	1.2	3.0	4.7	5.1	5.3	4.7	5.1	99.7	98.2	99.0	99.3	99.2	99.4	99.4	99.2
		R_ω					R^*_ω			R_ω					R^*_ω	
20, 5	7.6	1.8	3.6	5.1	5.9	5.5	5.6	5.6	22.4	9.3	14.6	17.6	19.2	21.0	19.4	18.4
50, 5	8.1	2.1	3.8	4.9	5.3	5.2	5.2	5.2	28.6	11.7	17.3	20.5	21.9	22.7	22.0	21.6
100, 5	9.9	2.8	4.4	5.7	6.0	6.0	6.0	5.9	44.5	27.2	32.4	35.1	35.6	38.8	37.1	35.5
50, 10	8.8	1.7	3.7	5.1	5.5	5.1	5.6	5.6	75.3	56.3	64.6	66.5	67.1	72.7	70.2	67.0
50, 20	8.7	1.3	3.0	4.7	5.1	5.3	4.8	5.0	99.5	96.9	98.5	99.0	98.9	99.0	99.1	98.9
					χ^2_6								χ^2_6			
N,T		F_ω					F^*_ω			F_ω					F^*_ω	
20, 5	8.4	2.6	4.5	5.5	5.1	5.4	5.0	4.9	26.4	15.1	18.8	20.8	19.9	23.0	21.1	19.3
50, 5	8.4	2.2	3.7	4.5	4.5	4.9	4.7	4.4	36.6	21.2	25.7	27.6	27.5	30.2	28.4	27.1
100, 5	9.5	3.0	4.4	5.2	5.5	5.5	5.3	5.4	57.3	41.0	44.5	45.9	45.7	50.6	47.6	45.6
50, 10	9.1	1.7	3.4	4.7	4.8	5.1	5.0	4.8	81.4	67.6	72.7	74.0	73.9	80.0	76.7	73.9
50, 20	8.5	1.6	3.4	5.1	4.7	5.1	4.7	4.7	99.7	98.4	99.4	99.5	99.4	99.5	99.6	99.4
		R_ω					R^*_ω			R_ω					R^*_ω	
20, 5	7.6	1.4	3.2	4.7	5.3	5.2	5.2	5.0	22.2	8.5	14.1	16.2	17.8	19.7	18.5	17.3
50, 5	7.9	1.7	3.1	4.0	4.5	4.9	4.7	4.4	26.7	10.6	15.5	18.3	19.8	20.2	20.1	19.5
100, 5	9.1	2.5	4.1	4.9	5.4	5.4	5.4	5.4	43.9	25.9	30.2	32.9	33.2	37.1	34.8	33.0
50, 10	8.9	1.6	3.1	4.4	4.9	5.0	5.0	4.8	74.6	53.7	63.2	65.7	66.1	71.5	69.5	66.0
50, 20	8.4	1.6	3.4	5.1	4.8	5.2	4.7	4.8	99.6	97.3	99.0	99.1	99.1	99.3	99.4	99.0

Notes: See notes to Table 1. The DGP is identical to that for Table 1 except $\sigma_{it} = \sigma_1, \;\; i = 1, ..., N_1, \; t = 1, ..., T$ and $\sigma_{it} = \sigma_2, \;\; i = N_1 + 1, ..., N, \; t = 1, ..., T$ with $N_1 = \lceil N/2 \rceil$, where $\lceil A \rceil$ is the largest integer not less than A, $\sigma_1 = 0.5$ and $\sigma_2 = 1.5$.

Table 3: Rejection frequencies of the asymptotic and wild-bootstrap modified F-tests and modified random effects tests under time-series one-break-in-volatility heteroskedastic scheme (HET2).

	$H_0 : \alpha_i = 0$								$H_1 : var(\alpha_i) = 0.1$, α_i correlated with regressors							
	Asymptotic Tests				Bootstrap Tests				Asymptotic Tests				Bootstrap Tests			
ω	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$
	SN								SN							
N,T	F_ω				F^*_ω				F_ω				F^*_ω			
$20,5$	3.1	3.1	4.3	5.2	5.5	5.6	5.7	5.6	27.2	20.4	25.8	32.7	33.3	29.5	29.7	33.5
$50,5$	3.2	3.6	4.5	5.1	5.5	5.9	5.6	5.5	44.9	41.5	46.0	51.5	52.2	49.8	49.2	52.2
$100,5$	2.9	3.7	4.5	4.8	5.5	5.5	5.5	5.5	70.7	68.6	71.0	75.7	76.2	74.3	73.2	76.3
$50,10$	3.8	1.8	3.1	4.5	4.7	4.7	4.8	4.7	85.8	75.2	80.5	87.0	86.8	85.7	83.9	86.9
$50,20$	4.2	1.6	3.3	4.5	5.1	5.2	5.4	5.1	99.9	99.5	99.7	99.9	99.9	99.8	99.9	99.9
	R_ω				R^*_ω				R_ω				R^*_ω			
$20,5$	2.6	2.0	3.6	4.5	5.7	5.4	5.7	5.7	21.3	12.1	19.1	26.5	30.5	25.2	26.1	30.6
$50,5$	2.7	2.7	3.8	4.5	5.6	5.7	5.7	5.7	29.8	23.3	29.7	36.0	39.6	35.2	35.6	39.5
$100,5$	2.7	3.3	4.1	4.4	5.6	5.5	5.4	5.6	53.3	48.6	53.4	59.8	62.3	57.5	57.7	62.3
$50,10$	3.5	1.6	3.0	4.1	4.7	4.7	4.8	4.7	78.9	62.1	71.7	80.2	81.5	78.0	76.9	81.5
$50,20$	4.1	1.6	3.2	4.4	5.1	5.2	5.3	5.1	99.8	99.0	99.5	99.8	99.9	99.7	99.7	99.9
	t_5								t_5							
N,T	F_ω				F^*_ω				F_ω				F^*_ω			
$20,5$	3.0	3.4	4.0	4.9	5.4	5.4	5.4	5.5	29.1	23.3	29.6	35.8	36.2	32.2	33.2	36.3
$50,5$	2.9	3.4	4.0	4.6	5.2	5.2	5.3	5.1	46.5	44.4	48.4	53.8	54.6	51.5	51.1	54.5
$100,5$	3.3	4.4	5.0	5.5	6.0	6.0	6.1	6.0	71.9	70.7	72.9	76.9	77.7	75.1	75.0	77.6
$50,10$	4.4	2.2	4.2	5.0	5.7	5.3	5.7	5.7	85.3	75.9	80.7	86.8	86.9	84.8	83.9	86.9
$50,20$	4.0	1.7	3.2	4.5	5.1	5.2	4.9	5.1	99.9	99.1	99.5	99.9	99.9	99.7	99.6	99.9
	R_ω				R^*_ω				R_ω				R^*_ω			
$20,5$	2.3	1.9	2.8	3.9	5.4	5.4	5.3	5.3	22.9	13.9	21.8	29.3	32.9	27.8	29.4	33.1
$50,5$	2.3	2.7	3.4	4.0	5.3	5.3	5.4	5.3	31.0	25.7	32.4	38.4	41.8	37.1	38.2	41.7
$100,5$	3.0	3.9	4.5	5.0	5.8	6.1	6.1	5.8	55.5	52.6	56.7	63.2	64.8	60.5	61.0	64.9
$50,10$	4.1	2.0	3.9	4.7	5.7	5.4	5.8	5.8	79.2	63.6	72.9	80.7	81.9	78.0	77.5	81.8
$50,20$	3.9	1.6	3.2	4.3	5.0	5.2	5.0	5.0	99.9	98.7	99.3	99.9	99.9	99.5	99.5	99.8
	χ^2_6								χ^2_6							
N,T	F_ω				F^*_ω				F_ω				F^*_ω			
$20,5$	3.2	2.8	3.8	4.3	4.6	5.2	4.7	4.6	28.8	22.7	27.9	35.1	35.9	31.1	32.0	35.9
$50,5$	3.4	3.0	3.8	4.2	4.8	4.6	4.7	4.7	44.2	41.4	45.2	51.0	52.2	49.4	49.0	52.1
$100,5$	3.3	3.7	4.5	4.6	5.1	5.0	5.2	5.0	71.8	70.7	72.9	77.4	77.6	75.6	74.6	77.6
$50,10$	4.4	2.3	3.7	4.9	5.3	5.5	5.4	5.3	84.1	74.7	79.8	85.5	85.8	83.8	82.8	85.8
$50,20$	4.8	2.0	3.5	4.8	4.8	5.2	4.8	4.8	99.9	99.2	99.7	99.9	99.9	99.9	99.9	99.9
	R_ω				R^*_ω				R_ω				R^*_ω			
$20,5$	2.5	1.9	2.7	3.4	4.5	5.1	4.6	4.4	22.1	14.3	21.1	28.1	32.1	26.8	27.8	32.2
$50,5$	2.9	2.5	3.2	3.7	4.7	4.9	4.6	4.6	27.9	22.2	28.1	35.0	38.3	33.9	34.4	38.5
$100,5$	3.1	3.4	4.1	4.4	5.2	5.2	5.2	5.1	54.7	50.2	54.8	62.0	64.1	59.0	59.1	64.2
$50,10$	4.2	1.9	3.4	4.6	5.1	5.5	5.4	5.1	77.8	61.4	71.5	79.7	81.0	76.9	77.0	81.0
$50,20$	4.7	2.0	3.5	4.7	4.7	5.0	4.8	4.7	99.9	98.7	99.4	99.9	99.9	99.7	99.8	99.9

Notes: See notes to Table 1. The DGP is identical to that for Table 1 except $\sigma_{it} = \sigma_1,\ i = 1, ..., N,\ t = 1, ..., T_1$, $\sigma_{it} = \sigma_2,\ \ i = 1, ..., N,\ t = T_1 + 1, ..., T$ with $T_1 = \lceil T/2 \rceil$, $\sigma_1 = 0.5$ and $\sigma_2 = 1.5$.

Table 4: Rejection frequencies of the asymptotic and wild-bootstrap modified F-tests and modified random effects tests under conditional heteroskedasticity depending on a regressor (HET3).

	$H_0 : \alpha_i = 0$								$H_1 : var\,(\alpha_i) = 0.1$, α_i correlated with regressors							
	Asymptotic Tests				Bootstrap Tests				Asymptotic Tests				Bootstrap Tests			
ω	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$
	SN								SN							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	2.1	3.9	4.6	5.3	5.7	5.5	5.8	5.4	9.8	13.1	18.5	20.1	20.8	17.5	20.6	20.5
$50,5$	5.7	3.1	4.1	4.8	5.4	5.1	5.1	5.1	18.5	13.9	16.9	18.9	19.8	19.0	19.6	19.4
$100,5$	5.6	3.9	4.9	5.3	6.0	6.3	5.9	5.8	30.0	25.9	29.7	31.1	32.7	32.7	32.5	32.4
$50,10$	5.5	2.6	4.0	5.1	5.5	5.5	5.6	5.5	47.5	31.4	40.5	45.2	46.3	43.5	45.3	46.1
$50,20$	5.4	2.2	3.9	5.1	5.5	5.4	5.5	5.5	80.6	64.2	75.6	79.6	80.0	77.7	79.7	80.0
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	1.7	2.5	3.4	4.4	5.9	6.0	5.9	6.0	7.2	7.8	13.4	16.0	18.9	15.8	18.6	18.6
$50,5$	5.1	2.5	3.3	4.1	5.6	5.3	5.2	5.3	13.1	7.6	10.8	12.6	15.4	13.8	15.1	14.8
$100,5$	5.3	3.5	4.4	4.9	6.0	6.3	6.0	5.8	21.7	16.1	20.7	22.7	25.2	23.9	24.9	24.9
$50,10$	5.3	2.5	3.7	4.8	5.5	5.4	5.5	5.4	41.0	22.6	33.0	38.5	40.5	36.5	38.8	40.2
$50,20$	5.3	2.2	3.7	5.0	5.4	5.4	5.4	5.4	76.9	57.5	71.6	75.9	77.0	73.8	76.8	76.9
	t_5								t_5							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	1.6	4.1	4.7	5.2	5.4	5.4	5.3	5.4	10.4	15.5	20.8	22.5	23.5	19.8	22.8	23.1
$50,5$	5.6	3.7	4.6	5.3	5.7	5.7	5.4	5.6	18.7	16.6	20.0	21.5	23.1	22.2	23.3	22.9
$100,5$	4.4	3.4	4.2	4.5	5.4	5.4	5.4	5.3	30.9	28.8	33.0	35.1	36.7	35.5	36.4	36.3
$50,10$	6.1	2.9	4.2	5.5	5.9	5.8	5.9	5.8	49.7	35.6	44.8	48.5	49.4	46.8	48.7	49.4
$50,20$	4.6	1.8	3.2	4.6	5.1	5.5	5.0	5.1	79.4	65.1	76.2	79.6	79.8	77.5	79.5	79.7
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	1.3	2.4	3.1	3.8	5.6	5.7	5.3	5.3	7.7	9.2	15.2	17.9	21.2	17.4	20.4	20.8
$50,5$	5.1	2.7	3.7	4.3	5.7	5.6	5.4	5.5	13.3	9.1	12.4	14.9	17.6	15.4	17.0	17.3
$100,5$	4.2	3.1	3.8	4.3	5.3	5.2	5.3	5.2	21.9	18.0	23.3	25.6	28.0	26.4	27.8	27.6
$50,10$	5.7	2.4	3.8	5.2	5.9	5.8	5.9	5.8	43.3	26.7	38.0	42.8	44.4	40.0	43.0	44.2
$50,20$	4.6	1.7	3.2	4.4	5.1	5.5	5.0	5.1	76.3	59.8	72.8	76.5	77.4	73.7	76.8	77.3
	χ_6^2								χ_6^2							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	1.8	4.4	4.9	5.3	5.7	6.0	5.5	5.5	10.6	13.2	17.9	19.3	20.7	17.6	20.3	20.2
$50,5$	5.4	3.4	4.3	4.8	5.4	5.0	5.0	5.0	16.9	13.3	15.8	17.2	18.5	18.1	18.6	18.0
$100,5$	5.4	3.7	4.2	4.7	5.6	5.5	5.5	5.4	27.8	25.1	28.1	29.9	31.4	31.1	31.0	31.0
$50,10$	6.0	2.5	3.7	4.6	5.0	5.5	5.5	4.9	47.9	33.7	41.5	46.1	47.3	46.6	46.7	47.2
$50,20$	4.5	1.5	3.1	3.9	4.2	4.5	4.7	4.2	80.0	66.7	77.0	80.5	80.9	79.4	80.9	80.8
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	1.4	2.7	3.2	4.0	5.7	5.9	5.6	5.6	8.0	8.2	13.1	14.8	18.4	15.3	17.9	18.0
$50,5$	4.8	2.8	3.6	4.2	5.3	5.1	5.1	4.9	12.3	6.9	9.7	11.4	14.4	12.6	13.8	14.1
$100,5$	5.2	3.4	3.9	4.4	5.6	5.4	5.4	5.3	20.5	14.7	19.0	21.2	23.5	21.8	22.8	23.1
$50,10$	5.7	2.3	3.5	4.4	5.1	5.5	5.4	5.0	40.9	23.7	33.2	38.6	40.6	38.6	39.7	40.3
$50,20$	4.4	1.4	3.0	3.8	4.2	4.4	4.7	4.2	76.5	60.5	72.9	77.0	78.0	75.8	77.9	77.9

Notes: See notes to Table 1. The DGP is identical to that for Table 1 except $\sigma_{it} = \eta_c[(z_{it,2} - 1)/30]/c$, $i = 1,...,N$, $t = 1,...,T$, where $\eta_c[\cdot]$ is the inverse of the cumulative distribution function of chi-squared distribution with degrees of freedom c. Since $z_{it,2}$ is drawn from a uniform distribution on $(1,31)$, σ_{it} has mean 1 and variance $2/c$, so it is easy to control the degree of heteroskedasticity through the choice of c. We employ $c = 1$.

Table 5: Rejection frequencies of the asymptotic and wild-bootstrap modified F-tests and modified random effects tests under conditional heteroskedasticity, GARCH(1,1) (HET4).

	$H_0 : \alpha_i = 0$								$H_1 : var(\alpha_i) = 0.1$, α_i correlated with regressors							
	Asymptotic Tests				Bootstrap Tests				Asymptotic Tests				Bootstrap Tests			
ω	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$
	SN								SN							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	7.2	2.3	4.5	6.0	6.0	5.8	5.8	5.8	30.5	17.6	24.3	27.5	27.5	28.8	28.1	27.1
$50,5$	7.9	2.3	4.4	5.5	5.8	5.6	5.4	5.7	47.2	34.9	39.1	41.2	40.9	46.2	42.8	40.7
$100,5$	8.8	2.9	5.1	6.3	6.0	5.8	6.0	6.0	71.1	60.8	62.9	64.4	63.4	70.4	66.2	63.3
$50,10$	6.8	1.8	3.9	5.2	5.7	5.9	5.5	5.7	92.7	85.3	89.9	90.9	90.8	92.6	91.9	90.8
$50,20$	5.6	1.6	3.6	4.9	5.3	5.6	5.3	5.3	100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	6.5	1.3	3.5	5.3	6.0	5.7	5.8	5.8	25.7	10.1	17.9	22.5	24.2	24.6	24.5	23.8
$50,5$	7.4	1.9	3.7	5.1	5.8	5.4	5.5	5.7	33.7	18.4	24.9	28.6	29.2	32.3	30.4	29.0
$100,5$	8.6	2.6	4.5	5.9	6.0	5.8	5.9	6.0	55.6	40.3	45.9	48.1	49.0	53.2	50.5	49.0
$50,10$	6.5	1.6	3.7	5.1	5.6	5.8	5.5	5.4	88.6	74.0	83.2	86.0	85.7	87.4	87.0	85.7
$50,20$	5.5	1.6	3.6	4.8	5.3	5.6	5.3	5.3	100.0	99.8	100.0	99.9	100.0	100.0	100.0	100.0
	t_5								t_5							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	7.9	1.9	4.2	5.7	5.3	5.1	5.1	5.1	32.7	20.8	27.0	29.7	29.0	31.4	29.8	28.5
$50,5$	9.2	2.6	4.4	5.8	5.4	5.3	5.1	5.3	49.7	36.5	41.1	42.7	41.4	47.1	43.6	41.1
$100,5$	11.5	3.5	5.6	6.5	6.3	6.2	6.4	6.3	70.8	59.0	59.9	60.3	59.3	67.3	62.2	59.2
$50,10$	8.2	1.9	4.0	5.5	5.6	5.5	5.3	5.5	91.9	82.8	86.9	87.8	86.8	90.6	88.8	86.8
$50,20$	6.9	1.5	3.8	5.3	5.4	5.5	5.5	5.3	99.9	99.3	99.6	99.4	99.3	99.7	99.7	99.3
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	7.4	1.3	3.2	5.1	5.4	5.1	5.2	5.2	27.7	12.0	19.8	24.0	25.7	26.3	26.2	25.2
$50,5$	8.7	2.0	3.7	5.3	5.4	5.1	5.2	5.3	36.4	20.0	26.3	29.2	30.5	33.3	31.4	30.2
$100,5$	11.1	3.0	5.0	6.2	6.3	6.3	6.4	6.3	56.9	39.5	44.6	46.3	46.6	52.4	48.8	46.4
$50,10$	8.0	1.8	3.6	5.3	5.6	5.4	5.3	5.6	87.7	72.1	80.2	82.3	81.9	85.4	84.4	81.9
$50,20$	6.8	1.5	3.7	5.2	5.4	5.5	5.5	5.3	99.9	99.0	99.4	99.4	99.2	99.7	99.6	99.1
	χ_6^2								χ_6^2							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	6.9	1.9	3.3	4.4	3.7	4.2	3.7	3.4	29.8	17.2	23.2	26.2	25.1	28.0	25.9	24.6
$50,5$	8.1	2.0	3.3	4.7	4.3	4.2	4.1	4.2	46.2	33.1	36.9	38.9	37.6	44.6	40.3	37.3
$100,5$	9.3	1.8	3.0	4.6	3.8	4.4	3.5	3.7	68.3	56.2	57.3	58.7	57.3	66.5	60.3	57.0
$50,10$	7.4	1.2	2.9	4.5	4.3	4.7	4.1	4.3	92.4	84.4	87.6	89.2	88.4	92.3	89.9	88.3
$50,20$	6.4	1.1	2.7	4.5	4.8	4.8	4.4	4.8	100.0	99.8	99.9	99.9	99.8	100.0	99.9	99.8
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	6.1	1.2	2.2	3.9	3.7	4.1	3.9	3.5	24.5	9.7	16.5	20.5	22.3	23.6	22.7	21.7
$50,5$	7.6	1.5	2.5	4.0	4.2	4.4	4.0	4.1	33.6	17.1	22.8	26.1	27.0	30.2	27.7	26.5
$100,5$	9.0	1.5	2.7	4.2	3.8	4.4	3.6	3.8	53.1	35.4	40.1	42.6	43.0	48.8	44.6	42.9
$50,10$	7.2	1.0	2.6	4.4	4.3	4.8	4.2	4.3	87.8	72.4	81.1	84.0	83.3	86.3	85.1	83.2
$50,20$	6.3	1.1	2.7	4.4	4.8	4.8	4.4	4.8	100.0	99.6	99.8	99.9	99.8	100.0	99.9	99.8

Notes: See notes to Table 1. The DGP is identical to that for Table 1 except $u_{it} = \sigma_{it}\varepsilon_{it}$, $t = -49, ..., T, i = 1, ..., N$, where $\sigma_{it}^2 = \phi_0 + \phi_1 u_{i,t-1}^2 + \phi_2 \sigma_{i,t-1}^2$. The value of parameters are chosen to be $\phi_0 = 0.5$, $\phi_1 = 0.25$ and $\phi_2 = 0.25$.

Table 6: Rejection frequencies of the asymptotic and wild-bootstrap modified F-tests and modified random effects tests under conditional heteroskedasticity, ARCH(1) (HET5).

	$H_0 : \alpha_i = 0$								$H_1 : var\,(\alpha_i) = 0.1$, α_i correlated with regressors							
	Asymptotic Tests				Bootstrap Tests				Asymptotic Tests				Bootstrap Tests			
ω	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$	1	$\hat{\omega}_N^{(1)}$	$\hat{\omega}_N^{(2)}$	$\hat{\omega}_N^{(3)}$	1	$\hat{\omega}_N^{*(1)}$	$\hat{\omega}_N^{*(2)}$	$\hat{\omega}_N^{*(3)}$
	SN								SN							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	9.8	2.7	5.4	6.8	6.2	6.0	6.0	6.0	32.7	19.6	25.4	28.1	27.0	30.4	28.0	26.5
$50,5$	10.3	2.5	5.1	6.4	5.7	5.6	5.7	5.6	47.2	33.2	37.2	38.1	36.2	44.4	39.1	35.9
$100,5$	11.6	2.6	4.5	6.2	5.3	5.3	5.3	5.2	68.3	53.4	54.9	55.1	52.4	63.7	57.2	52.1
$50,10$	7.8	1.6	3.3	5.1	4.8	5.1	4.9	4.8	91.6	82.0	85.6	85.7	84.8	90.6	87.7	84.7
$50,20$	6.8	1.4	3.3	5.0	5.0	5.2	5.2	5.0	100.0	99.6	99.7	99.6	99.5	99.8	99.8	99.5
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	9.1	1.3	4.3	6.1	6.3	6.0	5.9	6.1	28.0	11.4	19.0	22.8	24.1	25.8	24.9	23.5
$50,5$	9.6	1.7	4.1	5.6	5.8	5.6	5.8	5.5	35.1	17.6	23.8	26.8	26.9	31.0	28.8	26.7
$100,5$	11.2	2.1	4.0	5.8	5.3	5.5	5.3	5.3	53.3	33.3	38.3	40.1	38.9	47.5	42.9	38.7
$50,10$	7.6	1.4	3.0	4.9	4.9	5.1	5.0	4.8	86.6	70.6	78.9	80.4	79.8	84.9	83.1	79.7
$50,20$	6.7	1.4	3.2	4.9	5.0	5.2	5.2	5.0	100.0	99.4	99.7	99.6	99.4	99.8	99.7	99.4
	t_5								t_5							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	9.2	2.2	4.2	6.0	5.5	5.3	5.2	5.2	36.3	23.7	28.7	31.4	30.6	33.9	31.6	30.0
$50,5$	11.9	2.5	5.0	7.0	5.9	5.7	5.7	5.8	51.6	38.2	40.5	41.3	39.4	47.4	42.4	38.8
$100,5$	15.1	3.1	5.4	7.0	6.0	5.9	5.8	5.9	69.0	55.3	55.3	55.1	52.3	63.9	56.6	52.0
$50,10$	10.9	2.0	4.0	6.0	5.5	5.6	5.4	5.4	90.7	80.4	82.5	82.9	81.5	88.0	84.6	81.3
$50,20$	9.0	1.3	3.5	5.6	5.2	4.8	5.2	5.2	99.5	98.0	98.2	98.2	97.7	99.0	98.5	97.7
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	8.5	1.2	3.0	4.9	5.6	4.9	5.2	5.3	31.5	14.4	22.3	26.2	27.2	29.5	27.6	26.5
$50,5$	11.6	1.8	4.0	6.0	5.8	5.6	5.6	5.6	39.7	20.1	26.8	29.3	29.4	34.1	31.4	29.2
$100,5$	14.8	2.4	4.2	6.3	6.0	5.9	5.7	6.0	57.7	36.7	41.0	41.8	41.3	50.1	45.1	41.0
$50,10$	10.6	1.9	3.7	5.7	5.4	5.6	5.6	5.3	86.6	70.9	77.0	78.0	76.9	83.0	80.4	76.7
$50,20$	9.0	1.3	3.3	5.6	5.2	4.9	5.3	5.2	99.4	97.5	97.9	97.9	97.3	98.8	98.3	97.2
	χ_6^2								χ_6^2							
N,T	F_ω				F_ω^*				F_ω				F_ω^*			
$20,5$	8.3	2.3	4.0	5.2	4.5	4.3	4.2	4.2	33.2	20.2	25.8	27.6	26.7	31.3	28.1	26.0
$50,5$	10.9	1.8	4.0	5.7	4.2	4.0	4.2	4.0	46.2	31.1	34.3	35.3	33.1	41.7	36.2	32.9
$100,5$	11.9	1.8	3.7	5.3	4.4	4.2	4.2	4.3	66.6	51.5	51.3	52.0	49.4	61.2	53.4	49.1
$50,10$	9.0	1.1	2.7	5.0	4.4	4.3	4.0	4.3	89.8	79.9	82.0	83.4	81.8	88.6	84.7	81.6
$50,20$	7.9	1.3	2.3	4.5	4.1	4.3	3.7	4.1	99.8	99.1	98.9	99.0	98.8	99.6	99.1	98.8
	R_ω				R_ω^*				R_ω				R_ω^*			
$20,5$	7.6	1.0	2.5	4.4	4.5	4.1	4.2	4.2	28.2	10.9	18.3	22.3	23.0	25.8	24.4	22.5
$50,5$	10.5	1.0	2.8	4.8	4.2	4.1	4.1	4.0	33.6	15.2	20.3	23.5	23.7	27.9	25.0	23.2
$100,5$	11.8	1.3	2.9	4.7	4.4	4.2	4.2	4.4	52.9	32.2	34.6	37.6	36.6	45.5	39.5	36.3
$50,10$	8.8	1.0	2.4	4.8	4.4	4.2	4.0	4.4	85.4	68.4	74.7	77.7	76.2	82.6	79.0	76.1
$50,20$	7.8	1.3	2.2	4.5	4.1	4.3	3.7	4.1	99.7	98.7	98.7	98.9	98.6	99.5	98.9	98.6

Notes: See notes to Table 1. The DGP is identical to that for Table 1 except $u_{it} = \sigma_{it}\varepsilon_{it}$, $t = -49, ..., T, i = 1, ..., N$, where $\sigma_{it}^2 = \phi_0 + \phi_1 u_{i,t-1}^2$. The value of parameters are chosen to be $\phi_0 = 0.5$ and $\phi_1 = 0.5$.