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Economics
Discussion Paper Series
EDP-1117

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August 2011

Economics
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November 10, 2011

Abstract

In the analysis of group identification, Kasher and Rubinstein (1997), *Logique Analyse* 160, 385-395, have shown that any method to aggregate the opinions of a group of agents about the individuals in the group that posses a specific attribute, such as race, nationality, profession, etc., must be *dictatorial* or, otherwise, it must violate either *consensus* or *independence*. This result is known in the literature as ‘Who is a J’ impossibility theorem. This note offers a *direct* proof of the theorem based on the structure of the family of decisive coalitions.

JEL Classification: D72.

Key words: Self-determination, Arrow’s theorem, social choice.

1 Introduction

Consider a group of agents $N = \{1, \dots, n\}$, with $n > 2$. Denote by A a specific attribute that each member of N might possess. For instance, N could be a set of countries and A might denote the property of being a democracy, a EU’s member, a free market economy, a world’s trading nation, a nuclear nonproliferation country, etc. Alternatively, if we interpret the set of agents merely as individuals, then A could represent for instance the attribute of being a Jewish, hence the name of the theorem (‘Who is a J’) given by Kasher and Rubinstein (1997). For the sake of concreteness, here we stick with the interpretation that N denotes a set of countries and that A is the attribute of being a democracy. We believe this stresses the importance of the result for international organizations such as the United Nations, the European Union and the like.

Let $A_i \subset N$ be the set of countries that, according with country i ’s views, should be recognized as democratic nations in the international community N . We assume that neither $A_i = \emptyset$ nor $A_i = N$. Otherwise, the attribute as a criterium of group identification would be vacuous for country i . This research focuses instead on the more interesting case where each country thinks that some of the members of N have the corresponding attribute, but certainly not all of them. The question addressed in this work is how these (potentially) conflicting opinions about countries in the world that are democracies can be aggregated into a single view valid for the whole international community.

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An aggregation rule F assigns a proper subset of N to each profile $A = (A_i)_{i \in N}$. The outcome $F(A)$ denotes the set of all members of N who are considered to be democracies according with the aggregation rule in place and the views of the group. As a passing remark, notice that the aggregation rule in this paper aggregates each profile of subsets of possible elements (countries) into a subset of those possible elements, whereas in the Arrovian framework the social welfare functional aggregates each profile of *ordered* sets of *all* elements (rankings) into an *ordered* set of those same elements (ranking).¹ Thus, these two exercises are related, but they are not the same.

Consider the following two properties we might wish F exhibits. The first one is **consensus** (CO). An aggregation rule satisfies consensus if any agreement among *all* countries that a certain state is a democracy (resp., a non-democracy) is respected by the rule. Formally,

Consensus: An aggregation rule F satisfies consensus if for every profile $A = (A_i)_{i \in N}$ and any agent $j \in N$, (i) $j \in \bigcap_{i \in N} A_i \Rightarrow j \in F(A)$, and (ii) $j \in \bigcap_{i \in N} (N \setminus A_i) \Rightarrow j \notin F(A)$.

The second property we may be interesting in is **independence** (IN). An aggregation rule satisfies independence if each country is judged on its own merits, independently of how other countries are assessed. That is, country j is judged to be a democracy on the basis of how the community views j individually, and not on how the group assesses other countries different from j . Formally,

Independence: An aggregation rule F satisfies independence if for any two profiles $A = (A_i)_{i \in N}$ and $A' = (A'_i)_{i \in N}$ with the property that for any $j \in N$, and all $i \in N$, $j \in A_i \Leftrightarrow j \in A'_i$, it follows that $j \in F(A) \Leftrightarrow j \in F(A')$.

The main result of Kasher and Rubinstein (1997) is the following theorem:

Theorem 1 (Kasher and Rubinstein, 1997) *An aggregation rule F satisfies consensus and independence if and only if there exists an agent $i^* \in N$ such that $F(A) = A_{i^*}$ for each profile $A = (A_i)_{i \in N}$.*

The next section offers a *direct* proof of Theorem 1 based on the structure of the family of decisive coalitions that resembles Mas-Colell et al's (1995) proof of Arrow's impossibility theorem. We believe this simplifies the existing general proof given by Rubinstein and Fishburn (1986), which is based on algebraic aggregation theory.

2 Proof of Theorem 1

The following concepts will be useful along the proof. A coalition $L \subseteq N$ is said to be **semi-decisive for agent** $i \in N$, denoted by $SD_L|i$, if for any profile $A = (A_j)_{j \in N}$ the following two conditions are satisfied:

$$[\forall j \in L, i \in A_j \ \& \ \forall j \notin L, i \notin A_j] \Rightarrow i \in F(A); \quad (1)$$

¹More generally, the domain of the social welfare functional is the set of complete and transitive binary relations over the set of possible alternatives (which are rankings when indifference is not permitted).

and

$$[\forall j \in L, i \notin A_j \ \& \ \forall j \notin L, i \in A_j] \Rightarrow i \notin F(A). \quad (2)$$

A coalition $L \subseteq N$ is said to be **semi-decisive**, noted $SD_L|N$, if it is semi-decisive for all $i \in N$. Finally, a coalition $L \subseteq N$ is said to be **decisive for agent** $i \in N$, denoted by $D_L|i$, if for any profile $A = (A_j)_{j \in N}$ the following two conditions are satisfied:

$$[\forall j \in L, i \in A_j] \Rightarrow i \in F(A);$$

and

$$[\forall j \in L, i \notin A_j] \Rightarrow i \notin F(A).$$

For the sake of simplicity, in what follows we restrict attention to the three agents case $N = \{1, 2, 3\}$. The argument generalizes easily to $n > 3$. Additionally, for expositional convenience, we organize the proof of Theorem 1 in a series of lemmas. The first one shows that, under the hypotheses of the theorem, there exists a group of nations $L \subset N$ that is semi-decisive for some country $i \in N$.

Lemma 1 *If F satisfies CO and IN, then there exists a coalition $L \subset N$ and an agent $i \in N$ such that L is semi-decisive for i .*

Proof Consider the profile $A = (\{2\}, \{1\}, \{3\})$. By hypothesis, $F(A) \subset N$. First, suppose $F(A) = \{j, k\}$, for some $j \neq k$. Without loss of generality, let $F(A) = \{1, 2\}$. Take the profile $A^* = (\{2, 3\}, \{1, 3\}, \{3\})$. By IN, $\{1, 2\} \subseteq F(A^*)$. By CO, $3 \in F(A^*)$. Hence, $F(A^*) = N$, a contradiction. Thus, $F(A) = \{i\}$, for some $i \in N$. Without loss of generality, suppose $F(A) = \{2\}$. By IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [2 \in \hat{A}_1 \ \& \ 2 \notin \hat{A}_k \ \forall k = 2, 3] \Rightarrow 2 \in F(\hat{A}). \quad (3)$$

That is, (1) holds for $L = \{1\}$ and $i = 2$. If (2) also holds, then $SD_{\{1\}}|2$. Otherwise, consider the profile $A' = (\{1\}, \{2\}, \{2\})$, and suppose, by contradiction, $2 \in F(A')$.

- ▷ If $1 \in F(A')$, consider the profile $A'' = (\{1, 3\}, \{2, 3\}, \{2, 3\})$. By IN, $\{1, 2\} \subseteq F(A'')$. By CO, $3 \in F(A'')$. Hence, $F(A'') = N$, a contradiction. Thus, $1 \notin F(A')$;
- ▷ If $3 \in F(A')$, consider the profile $A'' = (\{1\}, \{2, 1\}, \{2, 1\})$. By IN, $\{2, 3\} \subseteq F(A'')$. By CO, $1 \in F(A'')$. Hence, $F(A'') = N$, a contradiction. Thus, $3 \notin F(A')$.

Therefore, under the above hypothesis, $F(A') = \{2\}$. Moreover, by IN, $\forall \hat{A} = (\hat{A}_j)_{j \in N}$ such that $2 \notin \hat{A}_1$ and $2 \in \hat{A}_k$ for all $k = 2, 3$, we have that $2 \in F(\hat{A})$. Consider next the profile $\tilde{A} = (\{3\}, \{1\}, \{1\})$:

- ▷ If $2 \in F(\tilde{A})$, then $F(\{3, 1\}, \{1, 3\}, \{1, 3\}) = N$, a contradiction;
- ▷ If $3 \in F(\tilde{A})$, then $F(\{3, 1\}, \{1, 2\}, \{1, 2\}) = N$, a contradiction.

Therefore, $F(\tilde{A}) = \{1\}$. Moreover, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [1 \notin \hat{A}_1 \ \& \ 1 \in \hat{A}_k \ \forall k = 2, 3] \Rightarrow 1 \in F(\hat{A}). \quad (4)$$

Finally, consider $\bar{A} = (\{2, 3\}, \{3, 1\}, \{3, 1\})$. By (3) and (4), $\{2, 1\} \subseteq F(\bar{A})$. By CO, $3 \in F(\bar{A})$. Hence, $F(\bar{A}) = N$, which provides the desired contradiction. Thus, (2) also holds for $L = \{1\}$ and $i = 2$, implying that $SD_{\{1\}}|2$. ■

The next lemma shows that if there exists a group of nations that is semi-decisive in assessing the democratic status of a country, then the same group is semi-decisive for all countries.

Lemma 2 *If F satisfies CO and IN and there is a coalition $L \subset N$ with the property that L is semi-decisive for some agent $i \in N$, then L is semi-decisive.*

Proof Without loss of generality, assume that $SD_{\{1\}}|2$. Let $A = (\{3\}, \{1\}, \{1\})$. On one hand, if $2 \in F(A)$, then $F(\{3, 1\}, \{3, 1\}, \{3, 1\}) = N$, contradicting that $F(\{3, 1\}, \{3, 1\}, \{3, 1\})$ is a proper subset of N . On the other hand, if $F(A) = \{1\}$, then by IN, CO and $SD_{\{1\}}|2$, we would have that $F(\{3, 2\}, \{1, 3\}, \{1, 3\}) = N$, a contradiction. Therefore, $3 \in F(A)$ and, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [3 \in \hat{A}_1 \ \& \ 3 \notin \hat{A}_k \ \forall k = 2, 3] \Rightarrow 3 \in F(\hat{A}).$$

That is, (1) holds for $L = \{1\}$ and $i = 3$. If (2) also holds, then $SD_{\{1\}}|3$. Otherwise, consider the profile $A' = (\{1\}, \{3\}, \{3\})$. If $3 \in F(A')$, then IN, CO and $SD_{\{1\}}|2$ implies that $F(\{1, 2\}, \{1, 3\}, \{1, 3\}) = N$, a contradiction. Thus, $3 \notin F(A)$ and, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [3 \notin \hat{A}_1 \ \& \ 3 \in \hat{A}_k \ \forall k = 2, 3] \Rightarrow 3 \notin F(\hat{A}).$$

That is, $SD_{\{1\}}|3$; and repeating the argument once again, it follows that $SD_{\{1\}}|1$. Therefore, by definition, $SD_{\{1\}}|N$. ■

The next lemma shows that if there exist two semi-decisive groups, then they have some agents in common which are by themselves semi-decisive.

Lemma 3 *If F satisfies CO and IN and there exist two semi-decisive coalitions $L \subset N$ and $L' \subset N$, then $L \cap L'$ is semi-decisive.*

Proof First we show that $L \cap L' \neq \emptyset$. Suppose not. Without loss of generality, let $L = \{1\}$ and $L' = \{2, 3\}$. Consider the profile $A = (\{1\}, \{2\}, \{2\})$. Then, $\{1, 2\} \subseteq F(A)$ and, by IN and CO, $F(\{1, 3\}, \{2, 3\}, \{2, 3\}) = N$, a contradiction. Hence, $L \cap L' \neq \emptyset$.

Second, we prove $L \cap L'$ is semi-decisive. Without loss of generality, let $L = \{1, 3\}$ and $L' = \{1, 2\}$. We wish to show that $L \cap L' = \{1\}$ is semi-decisive. Consider the profile $A = (\{1\}, \{3\}, \{2\})$, and assume, by contradiction, $1 \notin F(A)$. If $2 \in F(A)$, then by IN, CO and $SD_{L'}|N$, $F(\{1, 3\}, \{3, 1\}, \{2, 3\}) = N$, a contradiction. Alternatively, if $3 \in F(A)$, then consider the profile $A' = (\{1, 2\}, \{3, 1\}, \{2\})$. By IN, $3 \in F(A')$. By $SD_L|N$, $2 \in F(A')$. By $SD_{L'}|N$, $1 \in F(A')$. Thus, $F(A') = N$, a contradiction. Therefore, $1 \in F(A)$, and IN implies that

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [1 \in \hat{A}_1 \ \& \ 1 \notin \hat{A}_k \ \forall k = 2, 3] \Rightarrow 1 \in F(\hat{A}). \quad (5)$$

Next, consider the profile $A = (\{3\}, \{1\}, \{1\})$. If $1 \in F(A)$, then it follows from IN, $SD_L|N$ and $SD_{L'}|N$ that $F(\{3, 2\}, \{1, 2\}, \{1, 3\}) = N$, a contradiction. Hence, $1 \notin F(A)$ and, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [1 \notin \hat{A}_1 \ \& \ 1 \in \hat{A}_k \ \forall k = 2, 3] \Rightarrow 1 \notin F(\hat{A}). \quad (6)$$

Thus, by (5) and (6), $SD_{\{1\}}|1$; and, by Lemma 2, $SD_{\{1\}}|N$. ■

The next statement shows that for any coalition of countries, either the coalition is semi-decisive or otherwise those in the complement constitute a semi-decisive group.

Lemma 4 *If F satisfies CO and IN, then for any coalition $L \subseteq N$, either L is semi-decisive or $N \setminus L$ is semi-decisive.*

Proof Note that by CO, $SD_N | N$. Without loss of generality, fix $L = \{1, 2\}$ and suppose, by way of contradiction, that $\{1, 2\}$ is not semi-decisive. Then, there must exist a profile, say $A = (A_j)_{j \in N}$, and individual, say $i \in N$, such that either,

$$\forall j = 1, 2, i \in A_j, i \notin A_3, \ \& \ i \notin F(A);$$

or

$$\forall j = 1, 2, i \notin A_j, i \in A_3, \ \& \ i \in F(A). \quad (7)$$

Without loss of generality, suppose (7) holds, and let $i = 1$. By IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [1 \in \hat{A}_3 \ \& \ 1 \notin \hat{A}_k \ \forall k = 1, 2] \Rightarrow 1 \in F(\hat{A}). \quad (8)$$

We wish to prove $N \setminus L = \{3\}$ is semi-decisive for agent 1. To do that, it remains to be shown that

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [1 \notin \hat{A}_3 \ \& \ 1 \in \hat{A}_k \ \forall k = 1, 2] \Rightarrow 1 \notin F(\hat{A}). \quad (9)$$

Consider the profile $A' = (\{1\}, \{1\}, \{2\})$. If $1 \notin F(A')$, then (9) follows from IN. Instead, if $1 \in F(A')$, then we proceed as follows. First, notice that if $2 \in F(A')$, then $F(\{1, 3\}, \{1, 3\}, \{2, 3\}) = N$. Similarly, if $3 \in F(A')$, then $F(\{1, 2\}, \{1, 2\}, \{2\}) = N$. Therefore, it must be that $F(A') = \{1\}$, and by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N} : [1 \notin \hat{A}_3 \ \& \ 1 \in \hat{A}_k \ \forall k = 1, 2] \Rightarrow 1 \in F(\hat{A}). \quad (10)$$

Second, consider the profile $\tilde{A} = (\{3\}, \{3\}, \{2\})$. Repeating the argument, if $1 \in F(\tilde{A})$, then $F(\{3, 2\}, \{3, 2\}, \{2, 3\}) = N$. Alternatively, if $2 \in F(\tilde{A})$, then by (10), CO and IN, $F(\{3, 1\}, \{3, 1\}, \{2, 3\}) = N$. Thus, it has to be that $F(\tilde{A}) = \{3\}$. By IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N} : [3 \notin \hat{A}_3 \ \& \ 3 \in \hat{A}_k \ \forall k = 1, 2] \Rightarrow 3 \in F(\hat{A}). \quad (11)$$

But then, CO, (8) and (11) imply that $F(\{2, 3\}, \{2, 3\}, \{1, 2\}) = N$, a contradiction.

Hence, (9) holds and, together with (8), imply that $N \setminus L = \{3\}$ is semi-decisive for agent 1. Finally, by Lemma 2, we get the desired result, i.e., $SD_{\{3\}} | N$. ■

Next we show that adding countries to a semi-decisive coalition does not erode its power to influence the social outcome.

Lemma 5 *If F satisfies CO and IN and there is a semi-decisive coalition $L \subset N$, then the supra-coalition $L' \supset L$ is also semi-decisive.*

Proof Fix any $L \subset L' \subseteq N$, and suppose L is semi-decisive. Assume, by way of contradiction, that L' is not semi-decisive. By Lemma 4, $N \setminus L'$ is semi-decisive. Since $L \subset L'$, $(N \setminus L') \cap L = \emptyset$, which stands in contradiction with Lemma 3. Hence, $SD_{L'} | N$. ■

Next we show the family of semi-decisive coalitions has a ‘nested property,’ in the sense that smaller nonempty subsets of a semi-decisive coalition are themselves semi-decisive.

Lemma 6 *If F satisfies CO and IN and there is a semi-decisive coalition $L \subseteq N$, with $|L| > 1$, then there exists a sub-coalition $L' \subset L$ such that L' is semi-decisive.*

Proof Take any $h \in L$. If $L \setminus \{h\}$ is semi-decisive, we have proved the desired result. Otherwise, Lemma 4 implies that $N \setminus (L \setminus \{h\}) \equiv N \setminus L \cup \{h\}$ is semi-decisive. By Lemma 3, $N \setminus L \cup \{h\} \cap L = \{h\}$ is semi-decisive; and since $\{h\} \subset L$, this proves the lemma. ■

The next lemma exploits the nested property alluded above and it shows that, under the conditions of Theorem 1, one country in the international community has semi-decisive power.

Lemma 7 *If F satisfies CO and IN, then there exists an agent $h \in N$ such that $\{h\}$ is semi-decisive.*

Proof By CO, N is semi-decisive. By Lemma 6, there exists $L' \subset N$ such that $N \setminus L'$ is semi-decisive. Using Lemma 6 once again, there must exist L'' such that $(N \setminus L') \setminus L''$ is semi-decisive; and since N is finite, repeated applications of Lemma 6 yield that there exists $h \in N$ such that $SD_{\{h\}} | N$. ■

Lemma 8 *If F satisfies CO and IN and there is a semi-decisive coalition $L \subseteq N$, then L is decisive for all $i \in N$.*

Proof Fix any semi-decisive coalition $L \subset N$. By Lemma 7, there exists $h \in L$ such that $SD_{\{h\}} | N$. Without loss of generality, assume $h = 1$. Take any $i \in N$ and suppose, by way of contradiction, that $\{1\}$ is not decisive for agent i . To simplify, let $i = 2$. Then, there must exist a profile $A = (A_j)_{j \in N}$ such that either (a) $2 \in A_1$ and $2 \notin F(A)$; or (b) $2 \notin A_1$ and $2 \in F(A)$. Suppose the former. The other case is similar. Since by hypothesis $\{1\}$ is semi-decisive for agent 2, there has to be a $j \neq 1$ such that $2 \in A_j$. Moreover, there must also exist an agent $k \in N \setminus \{1, j\}$ such that $2 \notin A_k$. Otherwise, by CO, we would get $2 \in F(A)$. Without loss of generality, consider the case where $A = (\{2\}, \{3\}, \{2\})$. (Bear in mind that we have assumed $2 \notin F(A)$.)

If $1 \in F(A)$, then we would have that $F(\{2, 3\}, \{3, 2\}, \{2, 3\}) = N$, a contradiction. Thus, suppose $3 \in F(A)$. By IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [3 \in \hat{A}_2 \ \& \ 3 \notin \hat{A}_k \ \forall k = 1, 3] \Rightarrow 3 \in F(\hat{A}). \quad (12)$$

Consider next the profile $A' = (\{3\}, \{1\}, \{2, 3\})$. If $3 \notin F(A')$, then by IN

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [3 \notin \hat{A}_2 \ \& \ 3 \in \hat{A}_k \ \forall k = 1, 3] \Rightarrow 3 \notin F(\hat{A}). \quad (13)$$

By (12) and (13), $SD_{\{2\}} | 3$ and, by Lemma 2, it follows that $SD_{\{2\}} | N$. However, $\{1\} \cap \{2\} = \emptyset$, which contradicts Lemma 3. Therefore, $3 \in F(A')$. Moreover, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [3 \in \hat{A}_1, 3 \notin \hat{A}_2 \ \& \ 3 \in \hat{A}_3] \Rightarrow 3 \in F(\hat{A}).$$

Since $SD_{\{1\}}|3$, we also know that

$$\forall \hat{A} = (\hat{A}_j)_{j \in N}, [3 \in \hat{A}_1, 3 \notin \hat{A}_2 \ \& \ 3 \notin \hat{A}_3] \Rightarrow 3 \in F(\hat{A}).$$

Therefore, if for all $\hat{A} = (\hat{A}_j)_{j \in N}$, $[3 \in \hat{A}_1, 3 \in \hat{A}_2 \ \& \ 3 \notin \hat{A}_3] \Rightarrow 3 \in F(\hat{A})$, we would get the desired result: i.e., $\{1\}$ would be decisive for agent 3. Otherwise, we can repeat the previous argument and show that $SD_{\{3\}}|N$, which again would contradict Lemma 3 because $\{1\} \cap \{3\} = \emptyset$. Hence, $D_{\{1\}}|3$; and, since $1 \in L$, $\forall \hat{A} = (\hat{A}_j)_{j \in N}$, $[3 \in \hat{A}_j, \forall j \in L] \Rightarrow 3 \in F(\hat{A})$. That is, $D_L|3$. Finally, a reasoning similar to the above shows that $D_L|i$ for all $i \in N$. ■

We are now ready to complete the proof of Theorem 1. We do that by showing the ‘only if’ (necessity) part. The ‘if’ (sufficiency) part follows immediately. Under the hypotheses of the theorem, namely CO and IN, we know from Lemma 7 that there exists an agent $h \in N$ such that $\{h\}$ is semi-decisive. By Lemma 8, $\{h\}$ is also decisive for all $i \in N$. Hence, by definition, for all $A = (A_i)_{i \in N}$, $i \in A_h \Rightarrow i \in F(A)$; and $i \notin A_h \Rightarrow i \notin F(A)$. Therefore, $F(A) = A_h$ for all $A = (A_i)_{i \in N}$.

3 Final Remarks

We conclude this note showing that the two axioms invoked in Theorem 1 are independent of each other. To see that, consider the following examples. Firstly, any aggregation rule that allocates the same proper subset of countries to every profile satisfies IN, but it clearly violates CO.

Secondly, for any agent $j \in N$ and any profile $A = (A_i)_{i \in N}$, let $n_j(A) = |\{A_i : j \in A_i\}|$ be the number of countries at A that think j is a democracy. Denote by $W(A) = \{j \in N : n_j(A) \geq n_k(A) \ \forall k \in N\}$ the set of ‘the most well-recognized’ democracies. For any subset $\hat{N} \subset N$, define the aggregation rule F as follows:

$$F(A) = \begin{cases} W(A) & \text{if } W(A) \neq N, \\ \hat{N} & \text{otherwise.} \end{cases} \quad (14)$$

Let’s show that (14) satisfies consensus. Fix any country $j \in N$, and consider any profile A such that everybody agrees j is not a democracy. By hypothesis, $\forall i \in N$, $A_i \neq \emptyset$; hence there must exist $k \in N$ such that $n_k(A) > 0$. Since $n_j(A) = 0$, we have that $j \notin W(A)$ and, consequently, $W(A) \neq N$. Moreover, using the definition of F given in (14), it follows that $F(A) = W(A)$. Therefore, $j \notin F(A)$, as required by CO.

Proceeding in a similar way, consider any profile A where everybody agrees j is a democracy, so that $n_j(A) = n$. Clearly, $j \in W(A)$. If $W(A) = N$, then $\forall i \in N$, $n_i(A) = n$, contradicting the hypothesis that $A_\ell \subset N$ for all $\ell \in N$. Hence, $W(A) \neq N$ and by (14), $F(A) = W(A)$, implying that $j \in F(A)$. Thus, F satisfies CO.

Finally, it is easy to see that (14) violates IN. Indeed, let $N = \{1, 2, 3, 4, 5\}$. Consider the profile $A = (\{1\}, \{1, 2\}, \{3\}, \{4\}, \{1, 5\})$. Thus, $F(A) = 1$. Next, consider the profile $A' = (\{1, 3\}, \{1, 2\}, \{3\}, \{4, 3\}, \{1, 5, 3\})$. Notice that, $\forall i \in N$, $1 \in A_i \Leftrightarrow 1 \in A'_i$. By IN, we should have that $1 \in F(A')$. However, by definition, $F(A') = \{3\}$.

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