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Abstract

A remarkable result in economic theory is that price competition between a small number of sellers producing a homogeneous good may result in the perfectly competitive market outcome. We return to the issue of what prices constitute a pure strategy Bertrand equilibrium when we admit the possibility of coalitional deviations from the market. We consider a market with a finite number of buyers and sellers and standard market primitives. In this context we introduce a new core notion which we term the *Bertrand core*. A trading price is said to be in the Bertrand core if all sellers quoting this price constitutes a pure strategy Bertrand equilibrium *and* no subset of traders, buyers and sellers, can leave the market and improve their outcomes by engaging in Bertrand price competition by themselves. Under standard assumptions we show that the Bertrand core is non-empty. Moreover, we are able to obtain a partial equilibrium analogue of the well-known Debreu-Scarf (1963) result by showing that as the set of market traders is replicated then any price other than the competitive equilibrium can be blocked by some subset of traders provided that the market is replicated sufficiently many times.

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1 Introduction

A central problem in economic theory is to establish under what market conditions we might expect economic outcomes to be close to the equilibrium when agents have no market power. The original model of price competition proposed by Joseph Bertrand (1883) showed that subject to certain technical conditions, such as smoothness of market demand and constant returns to scale costs, price competition between two or more sellers is sufficient to obtain the competitive equilibrium of the market. However, this outcome is well-known to fail under different market conditions such as when sellers have limited capacities or decreasing returns to scale costs.[1] We reconsider the problem of establishing what price a homogeneous good might be traded at in a market where sellers have strictly convex costs and act as strategic price-makers. The difference in this paper is that we introduce the possibility that coalitions of traders may choose to deviate by leaving the market and trading by themselves. To study which prices may result in the market we introduce a new core concept which we term the *Bertrand core*. A trading price is said to be in the Bertrand core if it constitutes a pure strategy Bertrand equilibrium for the whole market *and* no subset of buyers and sellers can improve their outcomes trading by themselves. Mas-Colell et al.(1995, p.655) note that there is a close relationship between Bertrand price competition and the standard Edgeworth core.[2] The seminal result of Debreu and Scarf (1963) showed that as an economy is replicated the only allocations which remain in the core are Walrasian allocations.[3] In this paper we find that there are some deep similarities between the Edgeworth core and the Bertrand core. Whereas Walrasian allocations always belong to the Edgeworth core we show that price-taking equilibria always belong to the Bertrand core. Moreover, we establish a partial equilibrium analogue of the Debreu-Scarf result: as the number of traders in the market is replicated the only price which remains in the Bertrand core is the competitive equilibrium. Remarkably, this result remains valid even when the limit market possesses uncountably many pure strategy Bertrand equilibria. Therefore, we are able to provide a new strategic foundation for price-taking behaviour in large markets.

The Bertrand core is an original combination of the classical ideas of Bertrand and Edgeworth. It is well-known that Edgeworth (1897) criticized Bertrand's model of price competition which resulted in the study of markets with capacity constraints and decreasing returns to scale costs. However, Edgeworth's other seminal insight, that of the core of an economy, analyzed in Edgeworth (1881), has tended to be studied solely in the context of general equilibrium exchange. This paper combines Edgeworth's insight regarding the core with Bertrand price competition. As noted above, the Bertrand core, although a partial equilibrium concept, displays close similarities with the Edgeworth core.

[1] For a succinct summary of the Bertrand model see Vives (1999, Ch.5) or Baye and Kovenock (2008).

[2] At a technical level the models display a number of similarities. Walrasian allocations belong to the Edgeworth core and competitive equilibria belong to the set of Bertrand equilibria (subject to the sharing rule). Moreover, generically the Edgeworth core has uncountably many allocations and there are generically uncountably many Bertrand equilibrium prices.

[3] This result still holds even if traders increase arbitrarily provided that all traders do not vanish as a fraction of the limit economy (Hildenbrand and Kirman, 1988, pp.190-9).

A number of papers have considered strategic price-making foundations of competitive equilibrium. Dixon (1992) analyzed a model where sellerss had symmetric, strictly convex costs and showed that if sellers post prices and can commit to supplying a quantity greater than their competitive supplies, subject to a no-bankruptcy condition, then the only candidate pure strategy equilibrium is the price-taking equilibrium. A sufficient condition was found to be that all but one seller could supply the market demand at the competitive price without incurring a loss. In an influential paper, Dastidar (1995) considered price competition, with a commitment to supply all demand forthcoming, between sellers with strictly convex costs. In a market with symmetric sellers and equal sharing at prices ties it was shown that there are uncountably may pure strategy Bertrand equilibria and the competitive equilibrium belongs to the set (Vives, 1999, p.122). Chowdhury and Sengupta (2004) considered when the refinement of coalition proofness reduces the equilibrium set in standard Bertrand games. It was established that if sellers have symmetric costs then the game admits a unique coalition-proof Bertrand equilibrium. They showed that if one considers sequences of economies then as the number of sellers in the market becomes large the set of coalition-proof equilibria coincides with the competitive equilibrium of the market provided all sellers are active in the limit. Yano (2006a) analyzed a market model with free entry where sellers had u-shaped average costs. Sellers posted prices and a set of quantities they were willing to sell at the posted prices. It was shown that under certain conditions the competitive outcome is a Nash equilibrium of the game despite only a small number of sellers being active in the market. In a related paper, Yano (2006b) showed that the Bertrand paradox and Edgeworth criticism could be obtained as special cases of the game where sellers post prices and quantities.

We follow the tradition of these papers by analyzing price competition between sellers producing a single perfectly homogeneous good. However, unlike most of the previous literature, we model the demand side of the market in an explicit manner by assuming that there is a finite number of buyers. This framework then permits a rich set of trading possibilities as any subset of buyers and sellers could trade by themselves. We also allow for asymmetries between buyers and sellers so the model imposes few restrictions upon buyers' market demands and sellers' cost functions. In the next section we introduce standard mathematical notation used throughout the rest of the paper. In the following section we present the market model, define the Bertrand core, and present the main results. The final section presents some suggestions for future research.

2 Notation

The following notation is used throughout the rest of the paper.
$\Re^n$ denotes n-dimensional Euclidean space.
$\Re^n_+$ is the non-negative orthant of $\Re^n$.
2^X denotes all the subsets of X.
$|X|$ denotes the cardinality of X.
$\setminus$ denotes set theoretic subtraction.

$\emptyset$ denotes the emptyset.
$\mathbb{N}$ denotes the set of natural numbers.
$\mathbb{Q}$ denotes the set of rational numbers.

3 The Bertrand game

Consider the market for a perfectly homogeneous good. In the market there is a finite set of buyers $B = \{1, ..., b\}$, $b \geq 2$, and a finite set of sellers $S = \{1, ..., s\}$, $s \geq 2$. Each seller in the market has a cost function $C_i : \Re_+ \to \Re_+$ which is C^2, strictly convex and satisfies $C_i(0) = 0$ and $C_i'(0) = 0$. Each buyer in the market has a demand function $D_j : \Re_+ \to \Re_+$ which is C^2 and for each $j \in B$ there exist strictly positive finite real numbers $\bar{P}_j, \bar{Q}_j$ such that $D_j(\bar{P}_j) = 0$ and $D_j(0) = \bar{Q}_j$. Also, $D_j'(P) < 0$ and $D_j''(P) < 0$ for all $P \in (0, \bar{P}_j)$. In what follows we shall make frequent use of sellers' competitive supplies. The profit of each seller, as a function of quantity, is $\pi_i(Q) = PQ - C_i(Q)$. The competitive supply of the seller, as a function of price, is $h_i(P) = \arg\max_{Q \in \Re_+} \pi_i(Q)$. As each seller's cost function is strictly convex the function $\pi_i(Q)$ is strictly concave in Q and $h_i(P)$ is well-defined and single-valued. Also let $\pi_i^*(P) = Ph_i(P) - C_i(h_i(P))$ so $\pi_i^*(P)$ is the value function. We shall want to consider a Bertrand price competition game between possible subsets of buyers and sellers so let $\chi^B = \{M : M \in 2^B \setminus \emptyset\}$ and let $\chi^S = \{M : M \in 2^S \setminus \emptyset\}$. The set χ^B is all the non-empty subsets of buyers and χ^S is all the non-empty subsets of sellers. For any $B' \in \chi^B$ and $S' \in \chi^S$ consider a classical Bertrand price game between these buyers and sellers. Each seller simultaneously and independently chooses a $P_i \in \Re_+$ with a commitment to supply all the demand forthcoming from the buyers, B'. If a seller posts the unique minimum price in the market then it serves all the demand forthcoming at that price. If a seller is undercut then it obtains no demand and its profit is zero. If a seller ties with other sellers at the minimum price then a sharing rule describes how the market demand is shared. Throughout we shall assume the market demand is shared according to capacity sharing.[4] Let $\beta_i(P) = S_i(P)/\sum_{j \in A} S_j(P)$ is the share of the market demand which seller i obtains when it ties with $A \setminus \{i\}$ other sellers at minimum price P. Letting $E_i(P_i, P_{-i})$ denote the profit of seller i, P_{-i} denote the prices of the sellers $S' \setminus \{i\}$, and $D(B', P) = \sum_{j \in B'} D_j(P)$ we can summarize the profit as:

$$E_i(P_i, P_{-i}) = \begin{cases} P_i D(B', P_i) - C_i(D(B', P_i)) & \text{if } P_i < P_k \ \forall k \neq i; \\ P_i \beta_i(P_i) D(B', P_i) - C_i(\beta_i(P_i) D(B', P_i)) & i \text{ ties with } A \setminus \{i\} \text{ at min price;} \\ 0 & \text{if } P_i > P_k \text{ for some } k. \end{cases} \tag{1}$$

Then for any $B' \in \chi^B$ and $S' \in \chi^S$ we shall let $G(B', S')$ denote the Bertrand game in which the set of buyers is B' and the set of sellers is S'.

Definition 1. *In a market with $B' \in \chi^B$ buyers and $S' \in \chi^S$ sellers a pure strategy Bertrand*

[4] This sharing rule has been used, amonst others, by Dastidar (1997) and Chowdhury and Sengupta (2004).

equilibrium is a vector of prices $(P_i^*, P_{-i}^*) \subseteq \Re_+^{|S'|}$ *such that* $E_i(P_i^*, P_{-i}^*) \geq E_i(P_i, P_{-i}^*)$ *for all* $P_i \in \Re_+$ *and* $i \in S'$.

We shall let $\mathcal{E}(G(B', S')) \subseteq \Re_+^{|S'|}$ denote the set of pure strategy Bertrand equilibria of the price-setting game with B' buyers and S' sellers. Before proceeding to the equilibrium properties of the game we introduce some lemmas which will help in understanding the results.

Lemma 1. $h_i(0) = 0$ *and* $h_i'(P) > 0$ *for all* $i \in S$.

Proof. As $h_i(P) = \arg\max_{Q \in \Re_+} \pi_i(Q)$ if $P = 0$ then the profit of the seller is $\pi_i(Q) = -C_i(Q)$. Therefore the profit maximizing output is $Q = 0$. To establish the second part of the lemma note that $h_i(P)$ must satisfy the first-order condition for maximization:

$$P - C_i'(h_i(P)) = 0.$$

Differentiating w.r.t. P we obtain:

$$1 - C_i''(h_i(P))h_i'(P) = 0.$$

Rearranging:

$$h_i'(P) = 1/C_i''(h_i(P)).$$

As sellers have strictly convex cost functions $C_i''(\cdot) > 0$ and therefore $h_i'(P) > 0$. ■

Lemma 2. $\pi_i^{*\prime}(P) > 0$ *for all* $P > 0$.

Proof. From the definition $\pi_i^*(P) = Ph_i(P) - C_i(h_i(P))$ and therefore:

$$\pi_i^{*\prime}(P) = h_i(P) + Ph_i'(P) - C_i'(h_i(P))h_i'(P).$$

Factorizing:

$$\pi_i^{*\prime}(P) = h_i(P) + h_i'(P)[P - C_i'(h_i(P))].$$

From the first-order condition $P - C_i'(h_i(P)) = 0$ therefore:

$$\pi_i^{*\prime}(P) = h_i(P).$$

From Lemma 1 we know that $h_i(P) > 0$ for all $P > 0$ which establishes the result. ■

3.1 Price-taking equilibrium and the Bertrand core

In a market where all sellers take prices as given a price-taking, or competitive, equilibrium is a price such that the quantities the sellers are willing to supply to the market is exactly equal to the quantity demanded by the buyers. We state this formally in the next definition.

Definition 2. *A price-taking equilibrium in a market with* $B' \in \chi^B$ *buyers and* $S' \in \chi^S$ *sellers is a price* $P' \in \Re_+$ *such that:*

$$\sum_{i \in S'} h_i(P') = D(B', P'). \tag{2}$$

We shall let $\mathcal{P}(B', S') \subseteq \Re_+$ denote the price-taking equilibria of the market with B' buyers and S' sellers. We now present the following result which shows that a market possesses a unique price-taking equilibrium.

Proposition 1. *For any $B' \in \chi^B$ and $S' \in \chi^S$ $\mathcal{P}(B', S') \neq \emptyset$ and $|\mathcal{P}(B', S')| = 1$.*

Proof. Define the function $f(P) = D(B', P) - \sum_{i \in S'} h_i(P)$. The function $f(P)$ is the excess demand function. From the first-order condition $h_i(P) = C_i'^{-1}(P)$ and as the cost function $C_i(\cdot)$ is C^2 the first derivative is continuous and the inverse of the first derivative is continuous. Therefore $f(P)$ is a is a continuous function of price. Note that $f(0) = \sum_{j \in B'} \bar{Q}_j > 0$ and letting $\bar{P} = \max\{\bar{P}_j : j \in B'\}$ we have $f(\bar{P}) = -\sum_{i \in S'} h_i(\bar{P}) < 0$. As $f(P)$ is continuous, the intermediate value theorem guarantees that $\exists$ a $P' \in (0, \bar{P})$ such that $f(P') = 0$ which implies $D(B', P') = \sum_{i \in S'} h_i(P')$. To see that the price-taking equilibrium is unique note that $f'(P) < 0$. ■

Having established that a market possesses a price-taking equilibrium we now show that this implies that the set of pure strategy Bertrand equilibria is non-empty provided the number of sellers are at least two sellers in the market.[5]

Proposition 2. *For any $B' \in \chi^B$ and $S' \in \chi^S$ if $\mathcal{P}(B', S') = \{P'\}$ then $(P', ..., P') \in \mathcal{E}(G(B', S'))$ provided $|S'| \geq 2$.*

Proof. Suppose we have a market with $B' \in \chi^B$ buyers and $S' \in \chi^S$ sellers. If each seller quotes price P' to the buyers, with $\mathcal{P}(B', S') = \{P'\}$ the profit which the sellers obtain at this price is:

$$P'\beta_i(P')D(B', P') - C_i(\beta_i(P')D(B', P')).$$

As $\beta_i(P') = h_i(P')/\sum_{j \in S'} h_j(P')$ and $\sum_{j \in S'} h_j(P') = D(B', P')$ the profit of each seller simplifies to:

$$P'h_i(P') - C_i(h_i(P') = \pi_i^*(P').$$

Now consider whether any seller could profitably deviate from quoting this price. If a seller were to quote a price $P'' < P'$ then the maximum profit they could obtain is $\pi_i^*(P'')$. Lemma 2 then implies $\pi_i^*(P'') < \pi_i^*(P')$ and this is not a profitable deviation. If a seller increases their price then as $|S'| \geq 2$ they lose all demand and earn zero profit which is not a profitable deviation. Therefore $(P', ..., P') \in \mathcal{E}(G(B', S'))$. ■.

The price-taking outcome can be achieved as the result of sellers setting prices rather than acting as price takers. Therefore this gives a strategic explanation for price-taking behaviour. However, in Bertrand games with sellers as described here there will often be uncountably many Bertrand equilibria. This is certainly the case when the market sellers are symmetric. As a result, the outcomes of Bertrand price competition may be quite different from the competitive equilibrium. We now turn to the question of whether a stronger foundation for price-taking behaviour can be established in the context of this price-setting game. Specifically, we admit the possibility that a group of traders may break away from the market and by engaging in Bertrand price competition by themselves improve their outcomes. Then, a price vector will be said to belong to the *Bertrand core* if it is immune to these coalitional deviations. Formally we introduce this new core concept below.

[5] As far as the author is aware Dastidar (1997) was the first to establish this result.

Definition 3. *A price vector* $(P_1, ..., P_{|S|}) \in \Re_+^{|S|}$ *is in the Bertrand core if* $(P_1, ..., P_{|S|}) \in \mathcal{E}(G(B, S))$ *and* $\nexists\ B' \in \chi^B$, $S' \in \chi^S$ *and* $(P'_1, ..., P'_{|S'|}) \in \mathcal{E}(G(B', S'))$ *such that:*

$$(i) \quad \min\{P'_1, ..., P'_{|S'|}\} < \min\{P_1, ..., P_{|S|}\} \tag{3}$$

$$(ii) \quad \pi_i(P'_i, P'_{-i}) > \pi_i(P_i, P_{-i}) \quad \forall i \in S'. \tag{4}$$

A price vector belongs to the Bertrand core if it constitutes a pure strategy Bertrand equilibrium for the whole market, and there does not exist a subset of buyers and sellers which could leave the market and improve their outcomes by trading by themselves. By an improvement we mean that there exists a pure strategy Bertrand equilibrium for the market formed by the deviating agents in which: (i) buyers are able to obtain the homogeneous good at a lower price (this is expressed in eq.(3)); (ii) the deviating firms obtain higher profits at the new equilibrium price vector (this is expressed in eq.(4)). We shall let $\mathcal{C}(B, S) \subseteq \Re_+^{|S|}$ denote the set of Bertrand core prices. It should be clear that $\mathcal{C}(B, S) \subseteq \mathcal{E}(G(B, S))$. We now show that under the assumptions made here the Bertrand core is non-empty.

Proposition 3. $\mathcal{C}(B, S) \neq \emptyset$.

Proof. We shall show that if $\mathcal{P}(B, S) = \{P^C\}$ then $(P^C, ..., P^C) \in \mathcal{C}(B, S)$. That is, the price-taking equilibrium for the whole market belongs to the Bertrand core. Suppose a set of buyers, $B' \in \chi^B$, and a set of sellers, $S' \in \chi^S$, deviate from the market. The profit which a seller $i \in S'$ earned at the price-taking equilibrium was $\pi_i^*(P^C)$. Suppose that $(P'_1, ..., P'_{|S'|}) \in \mathcal{E}(G(B', S'))$ is the equilibrium price at which trade takes place amongst B' and S'. Let $P'_j = \min\{(P'_1, ..., P'_{|S'|})\}$. If $P'_j < P^C$ then the maximum profit firm j obtains from deviating is $\pi_j^*(P'_j) < \pi_j^*(P^C)$ and deviating is not profitable for firm j. If $P'_j \geq P^C$ then the deviating coalition is not a strict improvement for buyers. Therefore $(P^C, ..., P^C) \in \mathcal{C}(B, S)$. ■

Therefore the market has a non-empty Bertrand core. In a market with a finite number of traders the Bertrand core will typically be a strict subset of the set of pure strategy Bertrand equilibria $\mathcal{C}(B, S) \subset \mathcal{E}(G(B, S))$. The following example shows how the Bertrand core reduces the equilibrium set.

3.2 Example 1

Consider a market with two buyers, $B = \{1, 2\}$, and three sellers, $S = \{1, 2, 3\}$. The market demand of each buyer is given by the piecewise-affine function $D(P) = \max\{0, 5 - \frac{1}{2}P\}$. Each seller's cost function is given by $C(Q) = Q^2$. Standard calculations[6] reveal that the Bertrand equilibrium set for the whole market is $\mathcal{E}(G(B, S)) = \{P \in \Re_+^3 : P_i = P_j, \forall j \neq i, P_i \in [2\frac{1}{2}, 5\frac{5}{7}]\}$. There are a number of different coalitions which could deviate from the market. One possibility is that a single seller leaves the market and trades with a subset of buyers. However, routine calculation shows that the monopoly price which a seller facing a single buyers would charge is $6\frac{2}{3}$. Therefore this coalition would not benefit buyers. Second, a coalition with two sellers and one buyer, $B' = \{1\}$ and $S' = \{1, 2\}$, could form. Routine

[6]See Vives (1999, pp.120-2) or Dastidar (1995).

calculations show that $\mathcal{E}(G(B',S')) = \{P \in \Re^2_+ : P_i = P_j, \forall j \neq i, P_i \in [2, 4\frac{2}{7}]\}$. Of the possible coalition prices it is straightforward to check that all prices in the interval $[2, 3\frac{13}{19})$ represent profitable deviations from the whole market. The final possible coalition is that of two buyers and two sellers, $B'' = \{1,2\}$ and $S'' = \{1,2\}$. The set of equilibria of this market is $\mathcal{E}(G(B'',S'')) = \{P \in \Re^2_+ : P_i = P_j, \forall j \neq i, P_i \in [3\frac{1}{3}, 6]\}$. Of the possible coalition prices the prices in the interval $(4\frac{6}{11}, 5\frac{5}{7}]$ represent profitable deviations from the whole market. Therefore the Bertrand core is $\mathcal{C}(B,S) = \{P \in \Re^3_+ : P_i = P_j, \forall j \neq i, P_i \in [3\frac{13}{19}, 4\frac{6}{11}]\} \subset \mathcal{E}(G(B,S))$. Note that the competitive supply of each seller is $h(P) = \frac{P}{2}$ and the price-taking equilibrium is $P^C = 4$.

3.3 A limit result on the Bertrand core

Given that the Bertrand core includes the price-taking equilibrium it is desirable to know what other properties the core set has. We now turn to this issue. As was shown in the previous example, the Bertrand core of a finite market includes prices that may be quite different from the competitive outcome. However, we shall show that as the number of buyers and sellers in the market becomes large the Bertrand core converges to the price-taking equilibrium. Moreover, this will be the case even when there are uncountably many pure strategy Bertrand equilibria in the large markets. Therefore, admitting coalitional deviations from the market provides a new strategic foundation for price-taking behaviour. To show what happens as the market becomes large we introduce the standard concept of a replicated market. Formally, the $r \in \mathbb{N}$ replication of the market with S sellers and B buyers is the market in which there are r number of each type of buyer and seller. Following the notation used above we shall let $\mathcal{P}_r(B,S) \in \Re_+$ denote the price-taking equilibria of the r-replicated market, $\mathcal{E}_r(G(B,S)) \subseteq \Re^{r|S|}_+$ will denote the set of pure strategy Bertrand equilibria of the r-replicated market, and $\mathcal{C}_r(B,S) \subseteq \Re^{r|S|}_+$ will denote the set of Bertrand core prices of the r-replicated market.

Proposition 4. *$\mathcal{P}_r(B,S) = \mathcal{P}(B,S)$ for all $r \in \mathbb{N}$.*

Proof. Define the excess demand of the replicated market as $f(P,r) = rD(B,P) - \sum_{i \in S} rh_i(P)$. Factorizing gives $f(P,r) = r(D(B,P) - \sum_{i \in S} h_i(P))$. As $r \in \mathbb{N}$, $f(P',r) = 0$ if and only if $f(P') = 0$. ■

Proposition 5. *$\mathcal{C}_r(B,S) \neq \emptyset$ for all $r \in \mathbb{N}$.*

Proof. As $\mathcal{P}_r(B,S) = \mathcal{P}(B,S)$ for all $r \in \mathbb{N}$ the same steps used in the proof of Proposition 3 establish that for $P^C \in \mathcal{P}(B,S)$ the price vector $(P^C, ..., P^C) \in \mathcal{C}_r(B,S)$ for all $r \in \mathbb{N}$. ■

Having established that the Bertrand core is non-empty for each replicated market we now present the main result which proves that the price-taking equilibrium is the only price vector which remains in the Bertrand core as the market becomes large. Before doing so, we introduce the following two lemmas which will be helpful in proving the main result.

Lemma 3. *In a market with $S' \in \chi^S$ symmetric sellers (identical cost functions) and $B' \in \chi^B$ buyers $\exists$ a unique $\tilde{P}$ with $0 < \tilde{P} < \bar{P}$ and $\bar{P} = \max\{\bar{P}_j : j \in B'\}$ such that:*

$$\frac{1}{|S'|}\tilde{P}D(B',\tilde{P}) - C_i(\frac{1}{|S'|}D(B',\tilde{P})) = 0 \quad \forall i \in S'.$$

Moreover, for $P' \in \mathcal{P}(B', S')$, $\tilde{P} < P'$, and provided $|S'| \geq 2$, all sellers quoting $P'' \in [\tilde{P}, P']$ is a pure strategy Bertrand equilibrium. That is, $(P'', ..., P'') \in \mathcal{E}(G(B', S'))$.
Proof. Define the function $g(P)$ as:

$$g(P) = \frac{1}{|S'|} P D(B', P) - C_i(\frac{1}{|S'|} D(B', P)).$$

Routine calculations reveal that $g''(P) < 0$. Therefore $g(P)$ is strictly concave in P. Let $P' \in \mathcal{P}(B', S')$. As all sellers are symmetric we have:

$$g(P') = P' h_i(P') - C_i(h_i(P')) = \pi_i^*(P').$$

As $P' \in (0, \bar{P})$ Lemma 2 implies $h(P') > 0$. Then $h(\bar{P}) = 0$, $h(0) < 0$ and the strict concavity of $h(P)$ imply that $\exists$ a unique $\tilde{P}$ with $0 < \tilde{P} < P'$ such that $h(\tilde{P}) = 0$. For the last part of the lemma, that all sellers quoting $P'' \in [\tilde{P}, P']$ is a pure strategy Bertrand equilibrium the reader is referred to Vives (1999, pp.120-2). ■

The next lemma shows that there is a simple form of equal treatment in the Bertrand core.
Lemma 4. *Suppose $(P'_1, ..., P'_{r|S|}) \in \mathcal{C}_r(B, S)$ and $P'_i = \min\{(P'_1, ..., P'_{r|S|})\}$. Then all sellers of the same type as seller i post price P'_i.*
Proof. If a seller of type i posts the minimum price in the market they must earn non-negative profit. Letting A denote the set of firms tied at the minimum price we have:

$$(\frac{h_i(P'_i) D(A, P'_i)}{\sum_{j \in A} h_j(P'_i)}) P'_i - C_i(\frac{h_i(P'_i) D(A, P'_i)}{\sum_{j \in A} h_j(P'_i)}) \geq 0. \tag{5}$$

If a firm of type i posted a price above this price then the share of the demand they could obtain by joining the minimum price tie is:

$$\frac{h_i(P'_i)}{\sum_{j \in A} h_j(P'_i) + h_i(P'_i)}.$$

The demand shares are such that:

$$0 < \frac{h_i(P'_i)}{\sum_{j \in A} h_j(P'_i) + h_i(P'_i)} < \frac{h_i(P'_i)}{\sum_{j \in A} h_j(P'_i)}.$$

Therefore $\exists$ a $\gamma \in (0, 1)$ such that:

$$\gamma(\frac{h_i(P'_i)}{\sum_{j \in A} h_j(P'_i)}) = \frac{h_i(P'_i)}{\sum_{j \in A} h_j(P'_i) + h_i(P'_i)}. \tag{6}$$

By the convexity of the cost function:

$$\gamma C_i(\frac{h_i(P'_i) D(A, P'_i)}{\sum_{j \in A} h_j(P'_i)}) + (1 - \gamma) C_i(0) > C_i(\frac{h_i(P'_i) D(A, P'_i)}{\sum_{j \in A} h_j(P'_i) + h_i(P'_i)}).$$

As $C_i(0) = 0$ this gives:

$$\gamma C_i(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i')}) > C_i(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i') + h_i(P_i')}). \tag{7}$$

Combining eq.(5) and $\gamma > 0$ gives:

$$\gamma((\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i')})P_i' - C_i(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i')})) \geq 0.$$

$$\gamma(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i')})P_i' - \gamma C_i(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i')}) \geq 0. \tag{8}$$

Eq.(6) and eq.(8) give:

$$(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i') + h_i(P_i')})P_i' - \gamma C_i(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i')}) \geq 0. \tag{9}$$

Then eq.(7) and eq.(9) give:

$$(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i') + h_i(P_i')})P_i' - C_i(\frac{h_i(P_i')D(A,P_i')}{\sum_{j\in A} h_j(P_i') + h_i(P_i')}) > 0.$$

Which means the seller of type i posting a price above the minimum price has a profitable deviation by joining the minimum price tie. ■

The result in Lemma 4 states that if a seller of a given type posts the minimum price in the market then all other sellers of the same type must also post the minimum price in the market. Note that this is a weaker property than the equal treatment property of the Edgeworth core as Lemma 4 imposes no restrictions upon the prices of sellers which are greater than the minimum price. Let $P^{min} = \min\{P_1, ..., P_{r|S|} : (P_1, ..., P_{r|S|}) \in \mathcal{C}_r(B,S)\}$ and $P^{max} = \max\{P_1, ..., P_{r|S|} : (P_1, ..., P_{r|S|}) \in \mathcal{C}_r(B,S)\}$. That is, P^{min} is the minimum price across all price vectors in the r-replicated Bertrand core and therefore the minimum price at which trade takes place. Further, P^{max} is the maximum price of any vector in the r-replicated Bertrand core. We now present our main result which shows that as the market replication becomes large all trade takes place at the price-taking equilibrium as the upper and lower bounds of the Bertrand core converge to price-taking equilibrium.

Proposition 6. *As $r \to \infty$, $P^{min} \to P^C$ and $P^{max} \to P^C$ with $P^C \in \mathcal{P}(B,S)$.*

Proof. The result will be established by showing that if $P^{min} < P^C$ then $\exists\ \bar{r} \in \mathbb{N}$ such that the sellers quoting P^{min} can deviate from the market for any $r \geq \bar{r}$ and if $P^{max} > P^C$ the sellers quoting P^{max} can deviate from the market for all $r \geq \bar{r}$. Therefore any prices different from the price-taking equilibrium can be blocked by some coalition of traders provided the market is replicated sufficiently many times.

Suppose $P^{min} < P^C$. Then there is a seller of some type, say $i \in S$, which in some equilibrium vector posts price P^{min}. The minimum demand which the seller posting P^{min} serves in the r-replicated market is:

$$h_i(P^{min})\frac{rD(B,P^{min})}{r\sum_{j\in S} h_j(P^{min})} = h_i(P^{min})\frac{D(B,P^{min})}{\sum_{j\in S} h_j(P^{min})}.$$

As $P^{min} < P^C$ we know that $D(B, P^{min}) > \sum_{j \in S} h_j(P^{min})$ and:

$$h_i(P^{min}) \frac{D(B, P^{min})}{\sum_{j \in S} h_j(P^{min})} > h_i(P^{min}).$$

Therefore seller i serves greater demand than its competitive supply and the profit which the seller obtains from posting P^{min} must be strictly less than $\pi_i^*(P^{min})$.

We shall show that if the market is replicated sufficiently many times sellers of type i can obtain profit arbitrarily close to $\pi_i^*(P^{min})$. Fix an $\epsilon > 0$ and $j \in B$ with $D_j(P^{min}) > 0$ and we have:

$$\frac{h_i(P^{min})}{D_j(P^{min})} < \frac{h_i(P^{min}) + \epsilon}{D_j(P^{min})}.$$

By the everywhere denseness of the rationals in the real line[7] $\exists\ z \in \mathbb{Q}$ such that:

$$\frac{h_i(P^{min})}{D_j(P^{min})} < z < \frac{h_i(P^{min}) + \epsilon}{D_j(P^{min})}.$$

As $z \in \mathbb{Q}$ we can write $z = x/y$ with $x, y \in \mathbb{N}$ and $y \geq 2$ so that:

$$\frac{h_i(P^{min})}{D_j(P^{min})} < \frac{x}{y} < \frac{h_i(P^{min}) + \epsilon}{D_j(P^{min})}.$$

Now let $\bar{r} = \max\{x, y\}$ and consider the market with x buyers of type j and y sellers of type i. By deviating from the market at price P^{min} they obtain demand:

$$h_i(P^{min}) < \frac{x D_j(P^{min})}{y} < h_i(P^{min}) + \epsilon.$$

Therefore sellers of type i can obtain profit arbitrarily close to $\pi_i^*(P^{min})$. Now suppose that each firm of type i posts a price $P^{min} - \delta$, $\delta > 0$, so that the trade is strictly beneficial for buyers of type j. All that remains to see that this market blocks the price vector with P^{min} from being in the Bertrand core is to show that $(P^{min} - \delta, ..., P^{min} - \delta) \in \mathcal{E}(G(x, y))$.[8] That is, the price vector $(P^{min} - \delta, ..., P^{min} - \delta)$ is a pure strategy Bertrand equilibrium of the market formed by the deviating traders. As:

$$h_i(P^{min}) < \frac{x D_j(P^{min})}{y} < h_i(P^{min}) + \epsilon.$$

This tells us that the price $P^{min} < P'$ with $P' \in \mathcal{P}(x, y)$. That is, P^{min} is below the price-taking equilibrium of the market formed by the deviating traders. Therefore provided $\delta > 0$ and $\epsilon > 0$ are sufficiently small the result in Lemma 3 implies $(P^{min} - \delta, ..., P^{min} - \delta) \in \mathcal{E}(G(x, y))$. We can conclude that $P^{min} \to P^C$ as $r \to \infty$.

[7]See, for example, Rudin (1976, p.9).

[8]We are slightly abusing the notation by letting $\mathcal{E}(G(x, y))$ denote the pure strategy Bertrand equilibria of the market with x buyers and y sellers. However, we hope the reader is aware of what is meant.

Now suppose that $P^{max} > P^C$. There are two possible cases to consider: (i) the seller which posts price P^{max} is undercut by another seller (ii) all sellers post price P^{max} in the market we consider these cases separately.

(i) The seller posting P^{max} is undercut. Suppose a seller of type i posts price P^{max}. From Lemma 4 we know all sellers of type i must post prices higher than P' and therefore all sellers of type i earn zero profit. Suppose that $\hat{P} < P^{max}$ is the price at which trade takes place.

As above fix an $\epsilon > 0$ and $j \in B$ with $D_j(\hat{P}) > 0$ and we have:

$$\frac{h_i(\hat{P})}{D_j(\hat{P})} < \frac{h_i(\hat{P}) + \epsilon}{D_j(\hat{P})}.$$

Following the same argument as above $\exists$ a rational number expressed as x/y with $x, y \in \mathbb{N}$ and $y \geq 2$ such that:

$$\frac{h_i(\hat{P})}{D_j(\hat{P})} < \frac{x}{y} < \frac{h_i(\hat{P}) + \epsilon}{D_j(\hat{P})}.$$

Let $\bar{r} = \max\{x, y\}$ and consider a market composed of x buyers of type j and y sellers of type i. Following the same argument as above this group of buyers and sellers can leave the market and sellers of type i post prices $\hat{P} - \delta$, $\delta > 0$, and provided ϵ, δ are sufficiently small the sellers obtain profit arbitrarily close to $\pi_i^*(P^{\hat{P}}) > 0$ and buyers obtain the good at a lower price $\hat{P} - \delta$. Therefore, for $r \geq \bar{r}$ this price vector does not belong to the Bertrand core.

(ii) Now consider the other case where all firms post price P^{max}. The demand which a seller of type i serves is:

$$h_i(P^{max}) \frac{D(B, P^{max})}{\sum_{j \in S} h_j(P^{max})}.$$

As $P^{max} > P^C$ we know that $D(B, P^{max}) < \sum_{j \in S} h_j(P^{max})$ and:

$$h_i(P^{max}) \frac{D(B, P^{max})}{\sum_{j \in S} h_j(P^{max})} < h_i(P^{max}).$$

Therefore the demand which any sellers serves at this price is strictly less than their competitive supply and the profit which seller i earns at this price is strictly less than have $\pi_i^*(P^{max})$. As above fix an $\epsilon > 0$ and $j \in B$ with $D_j(P^{max}) > 0$ and we have:

$$\frac{h_i(P^{max})}{D_j(P^{max})} < \frac{h_i(P^{max}) + \epsilon}{D_j(P^{max})}.$$

By repeating the previous steps it can then be shown that there is rational number x/y with $x, y \in \mathbb{N}$ such that a market composed of x buyers of type j and y sellers of type i could profitably deviate from the market. Therefore as $r \to \infty$, $P^{max} \to P^C$. $\blacksquare$

3.4 Example 2

Consider the market in Example 1. We found that the Bertrand core was $\mathcal{C}(B,S) = \{P \in \Re^3_+ : P_i = P_j, \forall j \neq i, P_i \in [3\frac{13}{19}, 4\frac{6}{11}]\}$. The set of Bertrand equilibria was $\mathcal{E}(G(B,S)) = \{P \in \Re^3_+ : P_i = P_j, \forall j \neq i, P_i \in [2\frac{1}{2}, 5\frac{5}{7}]\}$ and the price-taking equilibrium was $P^C = 4$. Now consider what happens as the market is replicated. Routine calculation reveal that $\mathcal{E}_r(G(B,S)) = \{P \in \Re^{r3}_+ : P_i = P_j, \forall j \neq i, P_i \in [2\frac{1}{2}, 5\frac{5}{7}]\}$ for all $r \in \mathbb{N}$. Therefore the set of pure strategy Bertrand equilibria is not reduced as the market is replicated. Letting $P^{min} = \min\{P_1, ..., P_{r|S|} : (P_1, ..., P_{r|S|}) \in \mathcal{C}_r(B,S)\}$ and $P^{max} = \max\{P_1, ..., P_{r|S|} : (P_1, ..., P_{r|S|}) \in \mathcal{C}_r(B,S)\}$ we know from Proposition 6 that as $r \to \infty$, $P^{max} \to 4$ and $P^{min} \to 4$. The only price which remains in the Bertrand core as the market becomes large is the price-taking equilibrium.

4 Conclusion

This paper has reconsidered what price a homogeneous good may be traded at when there are a finite number of buyers and sellers in the market. Despite markets possessing uncountably many pure strategy Bertrand equilibria we have shown that if coalitional deviations from the market are permitted then the only price which remains in the Bertrand core as the market becomes large is the price-taking equilibrium. This is a partial equilibrium analogue of the Debreu-Scarf result. Moreover, the ideas presented here suggest there are close connections between the Edgeworth core and the Bertrand core. Given that the Bertrand core is a new concept which permits different trading possibilities to what have usually been considered in price-setting games there are a number of possible extensions for future research. First, Aumann (1964) showed that in markets with an atomless measure space of traders the Edgeworth core is equal to the set of Walrasian allocations. It is possible that a partial equilibrium result of this type regarding the Bertrand core and the set of price-taking equilibria could be established in a market with demand generated by an atomless measure space. Second, a significant amount of research has analyzed the Edgeworth core under asymmetric information (Glycopantis and Yannelis, 2005). It may be possible to model an oligopoly market with asymmetric information and characterize which prices are in the Bertrand core. Finally, in markets with non-convexities a price-taking equilibrium may fail to exist but the set of Bertrand equilibria may still be non-empty.[9] It would be interesting to establish whether the Bertrand core is non-empty in such markets and whether there is convergence to price-taking behaviour as the market becomes large.

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