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Separating Curvature and Elevation: A Parametric Weighting Function

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Abstract. This paper presents a preference foundation for a family of probability weighting functions that involves two parameters which reflect two independent aspects of probability attitude. The first aspect, related to curvature, represents the diminishing effect of optimism and pessimism. The second aspect, related to elevation, represents the strength or dominance of optimism relative to pessimism. These two independent measures allow for comparative analyses between individuals and, e.g., when comparing attitudes towards probabilities of gains and attitudes towards probabilities of losses, a comparative analysis within individuals. Our empirical analysis shows that the new weighting function fits elicited probability weights well, and that it can explain differences in the probabilistic risk attitudes for gain probabilities compared to attitudes for probabilities of losses. We are therefore able to provide a theoretical link between important measures for individual behavior used in the psychology literature and well-established notions of probabilistic risk attitudes used in economics.

Keywords: Curvature, elevation, optimism, pessimism, preference foundation, prospect theory.

Journal of Economic Literature Classification Numbers: D81.

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1 Introduction

A pragmatic approach for the analysis of risk attitudes is offered by the classic expected utility theory (von Neumann and Morgenstern 1944): there is a single measure, the utility function, that captures all aspects of risk attitude. Then, risk averse (prone) choice behavior comes down to concave (convex) utility and the degree of curvature of utility can be used to determine absolute and relative measures of risk (Arrow 1951, Pratt 1964). A similar pragmatic approach is offered by the dual theory (Yaari 1987): there is a single measure, the probability weighting function, that captures risk attitudes. Now risk aversion (proneness) is equivalent to a convex (concave) weighting function over cumulative probabilities.

Extensive empirical research has demonstrated that both the expected utility and the dual theory approaches lack descriptive accuracy, and that theories which combine both measures, thus the joint effect of the outcome sensitivity and the probability sensitivity components of risk attitude, offer sufficient flexibility to account for the description of observed risk behavior. These so-called rank-dependent theories include the rank-dependent utility model of Quiggin (1981, 1982), the Choquet-expected utility model of Schmeidler (1989), the rank- and sign-dependent model of Luce (1991), and the modern prospect theory of Tversky and Kahneman (1992) and Starmer and Sugden (1989). Compared to the former, the two latter theories include a further descriptively important aspect of risk behavior through the distinction of outcomes into gains and losses from a reference point. The extreme sensitivity towards potential losses relative to comparable gains is captured under the notion and interpretation of loss aversion (Kahneman and Tversky 1979, Tversky and Kahneman 1992, Schmidt and Zank 2005, Zank 2008a).

This paper focuses on the role of the probability weighting functions for risk behavior and explores two aspects of probabilistic risk attitude. First, there is the widely documented sensitivity to probabilities of extreme (best and worst) outcomes, revealed through overweighting of small probabilities and underweighting of large probabilities, potentially caused by optimism about obtaining best outcomes and pessimism about obtaining worst outcomes of prospects. The second aspect refers to the observation that this sensitivity is diminishing with changes towards more moderate cumulative, respectively, decumulative probabilities for these extreme outcomes. Together, these two aspects can explain why, in the evaluation of prospects, extreme outcomes receive large decision weights relative to objective probabilities and also comparatively larger decision weights than those of intermediate outcomes with similar objective probabilities.

Recall that, in the dual theory, risk aversion is equivalent to a convex weighting function. In the more general rank-dependent utility theory this is interpreted as probabilistic risk aversion (Wakker 1994). Such a weighting function exhibits little sensitivity towards changes in probabilities away from 0 where it is relatively flat, but exhibits extreme sensitivity towards changes in probabilities away from 1 where it is relatively steep. As a result, relative to the objective probabilities of outcomes, larger decision weights are obtained for equally likely and lower ranked, less good, outcomes. This probabilistic attitude is known as pessimism (Wakker 2001). Similarly, probabilistic risk proneness or optimism, thus a concave weighting function over cumulative probabilities, exhibits strong sensitivity away from 0 but relatively little sensitivity away from 1. The corresponding decision weights decrease relative to the probabilities of the corresponding outcomes, and equally likely lower ranked outcomes receive comparatively less decision weight.

The most common empirical finding is that of strong sensitivity at probabilities close to 0 and similar strong sensitivity towards probabilities close to 1 (see Wakker 2001 for a discussion and review of empirical evidence). This suggests that both optimism about obtaining best outcomes and pessimism about obtaining worst outcomes have significant influence on probabilistic risk behavior.[2] Because the sensitivity diminishes as probabilities are farther away from the boundaries of the probability interval (Tversky and Kahneman 1992), less sensitivity is observed for moderate probabilities. It seems, therefore, plausible to adopt inverse-S shaped probability weighting functions that exhaustively divide the probability interval into a region where the weighting function is concave (small probabilities) and a region where it is convex (moderate and large probabilities). But while such curvature of the weighting function can account for the diminishing sensitivity towards probabilities of outcomes it may not necessarily induce large decision weights for extreme outcomes. An inverse-S shaped weighting function can be completely below the 45-degree line depicting non-transformed probabilities, or it can be completely above it. So, having an inverse-S shaped weighting function alone may not be sufficient to explain all of the empirically observed sensitivity.

Sensitivity for probabilities of extreme outcomes, reflecting the effect of optimism and pessimism, are modeled through large decision weights for those outcomes. They are obtained if the weighting function is above small probabilities, and if it is below large probabilities. In that case we can say that small probabilities are overweighted and large ones are underweighted. While reflecting sensitivity for probabilities of extreme outcomes, overweighting and underweighting does not imply diminishing sensitivity. It ensures, however, that an inverse-S curved weighting function is sufficiently elevated, so that optimism and pessimism are fully accounted for.

The importance of curvature and elevation of weighting functions has been discussed before. Gonzalez and Wu (1999) provided psychological arguments for these separate components.

[2] Other aspects that seem to influence risk behavior are also discussed, e.g., in Viscusi, Magat, and Huber (1987) and the recent review of Birnbaum (2008).

They interpret the curvature of the weighting function as reflecting the ability of an individual to discriminate between probabilities. For example, an expected utility maximizer discriminates equally between probabilities: adding a 1% chance to the cumulative probability of any outcome leads to the same decision weight in the evaluation of prospects. The weighting function under expected utility is linear and continuous. By contrast, an individual who values a 1% probability more if added to a 99% probability of a good outcome than if added to a 10% probability of the same outcome, shows less ability to discriminate.

The interpretation given to elevation is that it reflects how attracted an individual is to the chance domain of prospects. For example, individuals who have a medical profession may be more confident when choosing among prospects involving particular health outcomes (or reveal more optimism about such decisions) compared to decisions among prospects involving business or financial investments.

Gonzalez and Wu (1999, p. 139) argued that curvature and elevation are logically independent aspects and that this should be reflected in two separate measures within the weighting function. Accordingly, they suggest that weighting functions involving two parameters, like the one proposed by Goldstein and Einhorn (1987) and Lattimore, Baker and Witte (1992) and the one proposed by Prelec (1998), where one parameter influences mainly curvature and one mainly elevation, provide a plausible account for discriminability and attractiveness, and hence, for observed probabilistic risk attitude in general. They do remark, however (see Gonzalez and Wu 1999, p. 140), that a completely independent separation of curvature and elevation is not possible for the previous parametric forms.

That such a separate modeling of curvature and elevation is warranted has also been illustrated in Kilka and Weber (2001). They interpreted the parameters of the Goldstein and Einhorn (1987) weighting function as source sensitivity and source preference, respectively (see

Fox and Tversky 1995). They argued for a two-stage decomposition model for decision weights under uncertainty, that is, for decision situations where probabilities of events are not given. According to that model, in the first stage individuals assess probabilities (or beliefs) to the events, and in the second stage they transform those probabilities by a weighting function (see also Tversky and Fox 1995, Fox and Tversky 1998, Wakker 2004, and Abdellaoui, Vossmann and Weber 2005, for further theoretical and empirical analyses). The experimental data collected by Kilka and Weber support the interpretation that decision weights under uncertainty and decision weights under risk are both depending on the source generating uncertainty or risk. They found a significant effect for curvature (source sensitivity), but no significant effects for elevation (source preference), when comparing decision weights generated from investments in known stocks versus decision weights from investments in unknown stocks. Gonzalez and Wu (1999) did interpersonal comparisons and found significant differences in both components. More recently, in Abdellaoui, Vossmann and Weber (2005), no significant differences in the curvature of the weighting functions for probabilities of losses compared to probabilities of gains were found, but they found more elevation for loss probabilities (for a similar finding see Abdellaoui 2000). It should be noted that, by adopting weighting functions which clearly separate between elevation and curvature, a much better accommodation of the current empirical evidence can be obtained. Such a separation may also lead to more advanced measurement instruments for these distinct components of probability weighting.

Our goal in this paper is to review the intuition regarding curvature and elevation by exploring its relation to optimism and pessimism. As a result, we propose a new family of parametric weighting functions in which there is a clear separation between the parameter controlling for curvature and the parameter controlling for elevation. We estimate these parameters using empirical data recently obtained by Abdellaoui, l'Harridon and Paraschiv (2008), and provide an axiomatic preference foundation for the new family of weighting functions by extending the work of Diecidue, Schmidt and Zank (2008).

The results are structured as follows. In Section 2 general notation is presented. In Section 3 we review arguments supporting the two components of probability weighting functions, namely curvature and elevation, and we relate them to optimism and pessimism. In Section 4 we present the new class of parametric weighting functions that capture measures for curvature and elevation which are independent. In Section 5 we provide an empirical analysis leading to estimates for the parameters of elevation and curvature. We provide data at the aggregate and the individual level. Finally, in Section 6 we provide a preference foundation for the proposed parametric inverse-S shaped probability weighting function in the framework of prospect theory. Proofs are deferred to the Appendix.

2 Preliminaries

Let X denote the set of outcomes. For simplicity of exposition, we assume a finite set of outcomes, such that $X = \{x_1, \ldots, x_n\}$. Our results hold for general sets of outcomes if we have at least four distinct outcomes. A *prospect* is a finite probability distribution over the set X. Prospects can be represented by $P = (\tilde{p}_1, x_1; \ldots; \tilde{p}_n, x_n)$ meaning that outcome $x_j \in X$ is obtained with probability $\tilde{p}_j$, for $j = 1, \ldots, n$. Naturally, $\tilde{p}_j \geq 0$ for $j = 1, \ldots, n$ and $\sum_{i=1}^{n} \tilde{p}_i = 1$.

Let L denote the set of all prospects. A preference relation $\succcurlyeq$ is assumed over L, and its restriction to subsets of L (e.g., all degenerate prospects where one of the outcomes is received for sure) is also denoted by $\succcurlyeq$. The symbol $\succ$ denotes strict preference while $\sim$ denotes indifference ($\preccurlyeq$ and $\prec$ denote reversed weak and strict preferences, respectively). To further simplify the exposition, we assume that no two outcomes in X are indifferent, and further, that outcomes are ordered from best to worst, i.e., $x_1 \succ \cdots \succ x_n$.

It will be convenient to use an alternative notation for prospects, following Abdellaoui (2002) and Zank (2008b). In the cumulative probabilities notation $P = (p_1, \ldots, p_{n-1}, 1)$, where $p_j = \sum_{i=1}^{j} \tilde{p}_i$ denotes the probability of obtaining outcome x_j or better, $j = 1, \ldots, n$. Similarly, in the decumulative probabilities notation $P = (1, 1-p_1, \ldots, 1-p_{n-1})$ where entries denote the probability of obtaining outcome x_j or less, $j = 1, \ldots, n$. Note that we have dropped outcomes from the (de)cumulative probability notation for prospects.

Recall, that under *expected utility* (EU) prospects are evaluated by

$$EU(p_1, \ldots, p_{n-1}, 1) = \sum_{j=1}^{n-1} p_j [U(x_j) - U(x_{j+1})] + U(x_n),$$

with a utility function, U, which assigns to each outcome a real number and is monotone (that is, U agrees with the preference ordering over outcomes). Utility is cardinal, i.e., it is unique up to scale and location.

A more general model is *rank-dependent utility* (RDU) where a prospect $P = (p_1, \ldots, p_{n-1}, 1)$ is evaluated by

$$RDU(p_1, \ldots, p_{n-1}, 1) = \sum_{j=1}^{n-1} w(p_j) [U(x_j) - U(x_{j+1})] + U(x_n).$$

Utility is similar to EU, however, RDU involves a probability weighting function, w, which is uniquely determined. Formally, a *weighting function*, w, is a mapping from the probability interval $[0, 1]$ into $[0, 1]$ that is strictly increasing with $w(0) = 0$ and $w(1) = 1$. In this paper the axiomatically derived weighting functions are continuous on $[0, 1]$. There is, however, empirical and theoretical interest in discontinuous weighting functions at 0 and at 1 (Kahneman and Tversky 1979, Birnbaum and Stegner 1981, Bell 1985, Cohen 1992, Chateauneuf, Eichberger and Grant 2007, Webb and Zank 2008). In the next section, we also use linear but discontinuous weighting functions for illustrative purposes.

A weighting function, w, is *convex* if for all probabilities p, q, r such that $p + q + r \leq 1$ we have $w(q+p) - w(q) \leq w(q+r+p) - w(q+r)$. The weighting function is *concave* (*linear*) if for all probabilities p, q, r such that $p+q+r \leq 1$ we have $w(q+p) - w(q) \geq w(q+r+p) - w(q+r)$ ($w(q+p) - w(q) = w(q+r+p) - w(q+r)$). Note that in all cases monotonicity and the respective curvature imply that the weighting function is continuous on $]0,1[$. If w is strictly convex then also continuity at 0 follows, and if w is strictly concave then also continuity at 1 is implied (see also Schmidt and Zank 2008). In this paper there is specific interest in weighting functions that are initially concave, say for probabilities in an interval $[0, \delta]$ for $0 < \delta < 1$, and convex for remaining probabilities, thus, on $[\delta, 1]$. We call these functions *inverse-S shaped* weighting functions, reflecting the shape of the corresponding mapping.

Related to the curvature of weighting functions is the notion of probabilistic risk behavior (see Wakker 1994, 2001, Abdellaoui 2002, Zank 2008b). A convex weighting function characterizes probabilistic risk *aversion* (or *pessimism*) and a concave weighting function characterizes probabilistic risk *proneness* (or *optimism*). A linear weighting function characterizes probabilistic risk *neutrality*. Observe that, as EU is characterized by probabilistic risk neutrality, optimism and pessimism are measuring probabilistic attitudes relative to EU as benchmark. This is similar to the way concave or convex utility measures corresponding risk attitudes towards (monetary) outcomes with the dual theory (Yaari 1987, so RDU with linear utility) as a benchmark. Figure 1 depicts examples of continuous weighting functions of the form $w(p) = p^{\gamma}$ corresponding to the previous notions of (a) optimism ($0 < \gamma < 1$), (b) neutrality ($\gamma = 1$), and (c) pessimism ($\gamma > 1$), respectively.

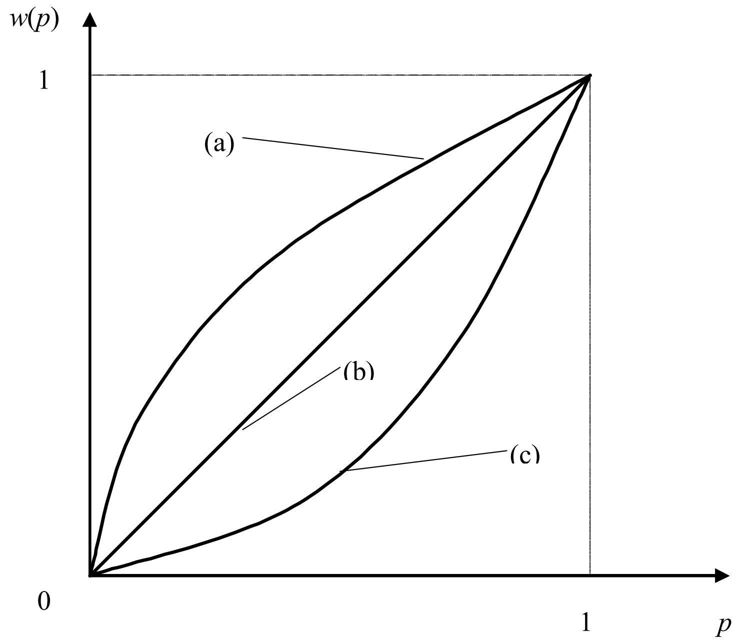

Figure 1: (a) optimistic, (b) pessimistic, (c) neutral probability attitudes.

Observe, that in Figure 1 the concave weighting function is never below the linear weighting function and that the convex one is never above it. This holds in general and is natural because optimism, respectively pessimism, is exhibited for all probabilities, thus, as a global property of probabilistic risk attitude.

In the literature the previous EU and RDU formulae are sometimes displayed using decision weights, hence, a weighted sum over utilities of outcomes of the form:

$$V(P) = \sum_{j=1}^{n} \pi_j U(x_j),$$

with the decision weights, $\pi_j, j = 1, \ldots, n$, explained next.

In the case of expected utility, the decision weights are the probabilities of obtaining the respective outcome, i.e., $\pi_j = \tilde{p}_j, j = 1, \ldots, n$. In the case of RDU, the decision weights are differences in transformed cumulative probabilities as follows: $\pi_1 = w(p_1)$ and $\pi_j = w(p_j) - w(p_{j-1}), j = 2, \ldots, n$. One can infer from Figure 1, that optimism has the implication that the decision weight of the best outcome is larger than the probability of obtaining that outcome, a feature called *overweighting* (i.e., $w(p) > p$ for all $p \in]0,1[$). In the case of pessimism we have *underweighting* (i.e., $w(p) < p$ for all $p \in]0,1[$) of the probability of the worst outcome, which leads to a decision weight of the best outcome which is larger than the probability of obtaining that outcome. Observe that in general optimism and pessimism do not lead to similar implications for decision weights of intermediate outcomes. This follows from the fact that $w(p)$ equals the decision weight of the best outcome, while for intermediate outcomes the decision weights are not necessarily equal to the transformed probabilities of those outcomes.

Overweighting and underweighting of probabilities is intimately related to sensitivity towards small and large probabilities. Recall that probabilistic risk neutrality acts as the benchmark for measuring optimism or pessimism. One can therefore think of a linear weighting function as exhibiting *objective sensitivity.* A weighting function exhibits (subjectively) *increased sensitivity* towards extreme probabilities if $w(p)/p > 1$ for $p \in]0, \varepsilon[$ and $[1 - w(p)]/(1 - p) > 1$ for $p \in]1 - \varepsilon', 1[$ for some $\varepsilon, \varepsilon' > 0$ arbitrary small. It exhibits *reduced sensitivity* if $w(p)/p < 1$ for $p \in]0, \varepsilon[$ and $[1 - w(p)]/(1 - p) < 1$ for $p \in]1 - \varepsilon', 1[$ for some $\varepsilon, \varepsilon' > 0$ arbitrary small. For example the weighting functions proposed by Goldstein and Einhorn (1987), Tversky and Kahneman (1992) and Prelec (1998) exhibit *extreme sensitivity* in the sense that $w(p)/p$ and $[1 - w(p)]/(1 - p)$ are unbounded for p approaching 0 and 1, respectively (see Zank (2008a) for a discussion). The class of weighting functions of Bell (1985), Cohen (1992), and Webb and Zank (2008) too exhibit extreme sensitivity due to discontinuity of those weighting functions at 0 and at 1. It is, however, the combination of increased sensitivity and concavity for small probabilities of a weighting function followed by convexity for moderate and large probabilities which has been most successful in accommodating the empirical findings regarding probabilistic risk attitude (Tversky and Kahneman 1992, Prelec 1998, Abdellaoui 2000, Wakker 2001).

We conclude this section by recalling prospect theory. Here, and elsewhere (e.g., Section 6), it will be convenient to use a notation for prospects that mixes cumulative and decumulative probabilities. Under prospect theory outcomes are interpreted as deviations from a reference point. Assuming that for some $1 < k < n$ the outcome x_k is the reference point, we call $x_1, \ldots, x_{k-1}$ *gains* and $x_{k+1}, \ldots, x_n$ *losses.* We can then write the prospect P as

$$P = (p_1, \ldots, p_k, 1-p_k, \ldots, 1-p_{n-1}).$$

One can think of $p_1, \ldots, p_k$ as being cumulated probabilities for gains and of $1-p_k, \ldots, 1-p_{n-1}$ as being cumulated probabilities for losses.

Under *prospect theory* (PT) probability weighting and the distinction into gains and losses is relevant. There is a weighting functions for probabilities of gains, w^+, and a (possibly different) weighting for probabilities of losses, w^-. A prospect $P = (p_1, \ldots, p_k, 1-p_k, \ldots, 1-p_{n-1})$ is evaluated by

$$PT(p_1, \ldots, p_k, 1-p_k, \ldots, 1-p_{n-1}) = \sum_{j=1}^{k-1} w^+(p_j)[U(x_j)-U(x_{j+1})] + \sum_{j=k}^{n-1} w^-(1-p_j)[U(x_{j+1})-U(x_j)],$$

where we use the convention that $U(x_k) = 0$.

The weighting functions under PT are uniquely determined and, in general, the utility function is cardinal. Here, we have fixed the location of utility for the PT-model to simplify the presentation. So, utility under PT is a ratio scale. However, the results that we derive below do not depend on this restriction but apply more generally to PT with cardinal utility.

Similar to RDU, in prospect theory the decision weights are differences in transformed cumulative probabilities of gains (i.e., $\pi_1 = w^+(p_1)$ and $\pi_j = w^+(p_j) - w^+(p_{j-1}), j = 2, \ldots, k-1$), respectively, differences in transformed cumulative probabilities of losses (i.e., $\pi_j = w^-(1-p_{j-1}) - w^-(1-p_j), j = k+1, \ldots, n-1$ and $\pi_n = w^-(1-p_n)$). Note that PT reduces to RDU if we have duality between the weighting functions, i.e., if $w^+(p) = 1 - w^-(1-p)$ for all $p \in [0, 1]$, or if we have only gains (only losses).

3 Curvature and Elevation

In this section we look in more detail at the shape of probability weighting functions and discuss the relation with the empirically observed probabilistic risk attitudes. For simplicity we assume RDU and note that similar results hold for PT.

A common empirical finding is that of large decision weights for unlikely extreme outcomes (Allais 1953, MacCrimmon and Larsson 1979, Kahneman and Tversky 1979). Such decision weights indicate that individuals are extremely sensitive to changes in cumulative probabilities close to 0 and 1. As a result, the typically estimated inverse-S shaped weighting functions are sufficiently elevated so that overweighting of small probabilities and underweighting of large probabilities is captured (Tversky and Kahneman 1992, Gonzalez and Wu 1999, Abdellaoui 2000, Bleichrodt and Pinto 2000, Abdellaoui, Vossmann and Weber 2005).

In general inverse-S shaped weighting functions need not cross the linear and continuous weighting function, and can be completely above or completely below it (except at 0 and at 1). So, theoretically, we may observe optimistic behavior at 0 (or pessimistic behavior at 1) without overweighting of small (underweighting of large) probabilities, hence without increased sensitivity.

In this section we argue that the increased sensitivity towards extreme probabilities is strongly related to optimism and pessimism about small probability best and worst outcomes, respectively, and that optimism and pessimism are the possible sources of increased sensitivity. So, optimism, here, is interpreted not only as concavity of weighting functions over some interval of probabilities but also as encompassing overweighting of small probabilities of best outcomes. Similarly, pessimism is interpreted more broadly as including underweighting of worst outcomes. To illustrate our motivation for this interpretation we initially take a look at the weighting functions discussed by Bell (1985), Cohen (1992), and, more recently, by Chateauneuf, Eichberger and Grant (2007) and Webb and Zank (2008), which are linear and (possibly) discontinuous at 0 and at 1. Their general form is

$$w(p) = \begin{cases} 0 & \text{for } p = 0, \\ \alpha p + \beta & \text{for } 0 < p < 1, \\ 1 & \text{for } p = 1, \end{cases}$$

with $0 \leq \beta < 1$ and $0 < \alpha \leq 1 - \beta$. Figure 2 depicts such a weighting function in comparison to the linear weighting function under expected utility. Recall that a linear weighting function is seen as the benchmark for measuring optimism and pessimism.

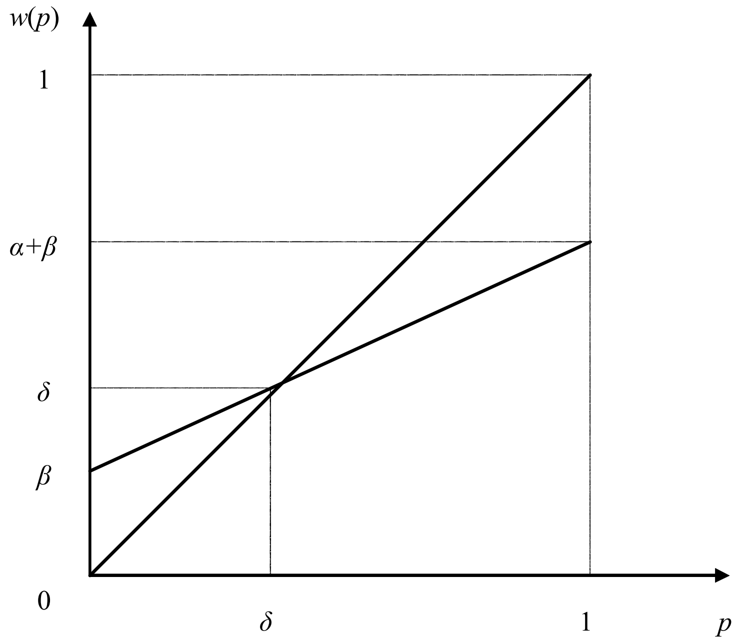

Figure 2: Optimism and pessimism for extreme outcomes.

One observes that the weighting function in Figure 2, which exhibits optimism and pessimism for probabilities of extreme outcomes, has a graph that is rotated around (δ, δ). It exhibits extreme sensitivity and overweighting of small, respectively, underweighting of large probabilities. Note that β measures the deviation from expected utility due to optimism, and can be thought of being an "absolute" index of optimism. Similarly, $1 - \alpha - \beta$ measures the deviation from expected utility due to pessimism. The deviation due to optimism relative to the total deviation from expected utility (i.e., relative to $\beta + (1 - \alpha - \beta)$) is given by the "relative" index of optimism $\delta = \beta/(1 - \alpha)$. Note that at probability δ no deviation from expected utility occurs, so that the two linear weighting functions in Figure 2 intersect. The effects of optimism and pessimism appear to balance each other out at δ. This implies $RDU(\delta, \ldots, \delta, 1) = \delta U(x_1) + (1 - \delta)U(x_n) = EU(\delta, \ldots, \delta, 1)$. Empirical studies, using more general weighting functions, suggest that this intersection is around 1/3 (Prelec 1998, Abdellaoui 2000, Bleichrodt and Pinto 2000, Bleichrodt, Pinto and Wakker 2001, Abdellaoui, Barrios and Wakker 2007).

Obviously, one can compare individuals on the basis of the (maximum) deviations from expected utility caused by optimism and by pessimism, thus using as measures the indexes β and $1-\alpha-\beta$, respectively. Or, as indicated above, one can make a comparison on the basis of the absolute and the relative indexes of optimism, thus using β and δ as measures, respectively.

The relative index of optimism is closely related to the notion of elevation (as discussed elsewhere, e.g., Gonzalez and Wu 1999, Diecidue, Schmidt and Zank 2008). To illustrate this relation, consider two individuals who have the same total deviation from expected utility, $1-\alpha$, but which differ in the absolute index of optimism, say $\beta_1 > \beta_2$. So, compared to individual 2, individual 1 is more optimistic about the best outcome and also less pessimistic about the worst outcome, which leads to an overall more optimistic probabilistic behavior. More precisely, the weighting function of individual 1 results from that of individual 2 by a simultaneous increment of the absolute index of optimism which exactly offsets the decrement of the absolute index of pessimism. The implication of this is depicted in Figure 3.a, which shows that the two weighting functions, w_1, w_2, corresponding to individual 1 and 2, respectively, have parallel graphs. Due to this specific, more optimistic, attitude, w_1 is more elevated than w_2, such that $\delta_1 > \delta_2$.

Figure 3.b depicts weighting functions of two individuals which too have different absolute indexes of optimism but which have the same relative index of optimism δ. Individual 1 is more optimistic and also more pessimistic. More precisely, compared to individual 2, individual 1's absolute index of optimism and 1's total deviation from EU have increased proportionally. As a result, the graph of the weighting function w_1 is more rotated around (δ, δ) than the graph of w_2, indicating proportionally more sensitivity towards probabilities of extreme outcomes.

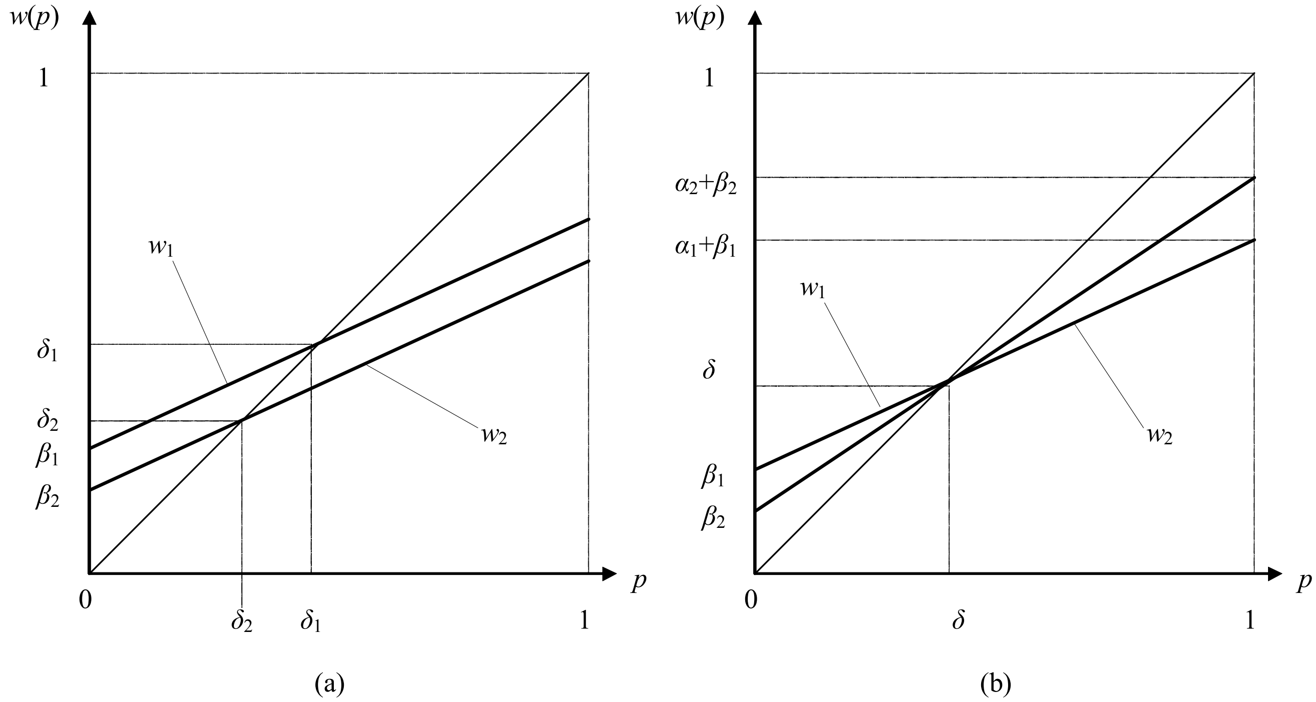

Figure 3: A comparison of elevation (a) and sensitivity (b)

The pragmatism of modeling probability transformations with a linear weighting function is reflected in the fact that the effect of optimism and pessimism is completely absorbed in the decision weights of the extreme outcomes. Remaining decision weights do not capture any of those effects. This can be inferred from observing that, if the probabilities of the extreme outcomes are fixed, then the original preferences restricted to such prospects can be represented by expected utility, the model with probabilistic risk neutrality. This is in contrast to empirical findings which suggest that the effect of optimism and pessimism, even though less pronounced, is observed also for intermediate outcomes (e.g., Wu and Gonzalez 1996). This means that if probabilities of extreme outcomes were fixed, the remaining effect of optimism and pessimism on the probabilities of intermediate outcomes would need to be modeled by a linear weighting function $\tilde{w}$ which results from the original weighting function w by a further rotation of the 45-degree line with fix point δ. This is illustrated in Figure 4, where the probabilities of extreme outcomes are kept fixed (at $\delta/2$ for x_1 and at $(1+\delta)/2$ for x_n) and the remaining effect of optimism and pessimism for the probabilities of second best and second worst outcomes is modeled.

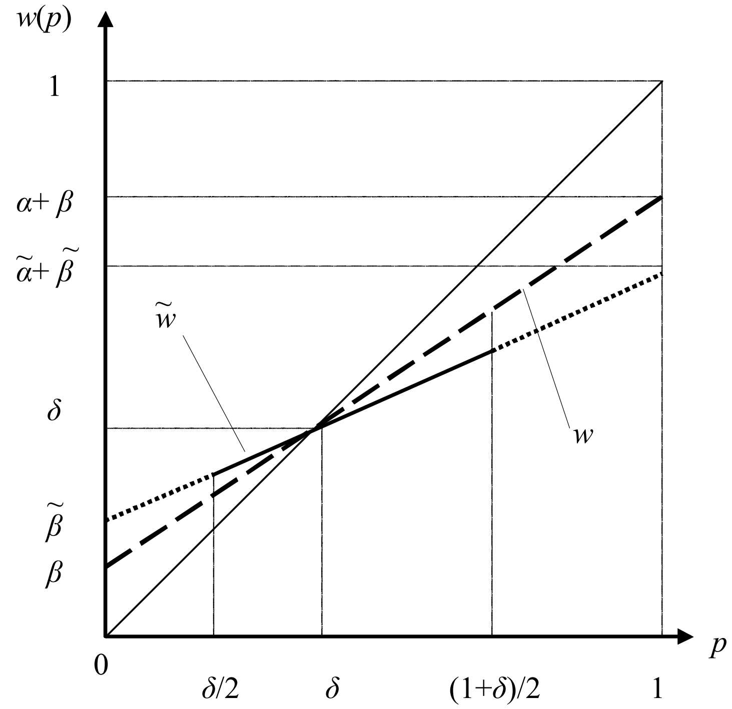

Fig. 4: Adjustment for intermediate outcome probabilities.

The intuition behind Figure 4 is that the probabilistic behavior which lead to deviations from expected utility modeled through w, causes a similar deviation from w, now modeled through $\tilde{w}$. Because these deviations refer to the intrinsic optimistic and pessimistic attitudes of the same individual, it is plausible to assume that the induced effect on decision weights of second best/worst outcomes is proportional to the induced effect on the decision weights of the best/worst outcomes, as if the effects of optimism and pessimism were discounted at the same rate. A consequence of this assumption is that relative optimism (i.e., $\delta = \beta/(1-\alpha) = \tilde{\beta}/(1-\tilde{\alpha})$) is the same for w and $\tilde{w}$, while $\tilde{\alpha}$ and $\tilde{\beta}$ may, in general, depend on the probabilities at which the extreme outcomes were kept fixed. Considering a repetition of this procedure for the decision weights of subsequent intermediate outcomes, now with the probabilities fixed for best/worst outcomes and for second best/worst outcomes, would lead to a further rotation of the 45-degree line around the fix point δ. It is natural to assume that the adjustment for optimism and pessimism that occurs subsequently is even smaller, in accordance with the principle of diminishing sensitivity (Tversky and Kahneman 1992). Hence, because of monotonicity of all involved weighting functions, this process of diminishing adjustment for optimism and pessimism is expected to converge to a linear weighting function with fix point δ, reflecting relative optimism, and slope $0 \leq \gamma < 1$, reflecting sensitivity or, as we argue in the next section, curvature.

4 Constant Relative Sensitivity

In this section we consider a class of weighting functions that models probabilistic risk attitudes by maintaining the intuition of constant relative optimism and (increased) sensitivity as illustrated in the previous section. These weighting functions can be thought of modeling the adjustment for optimism and pessimism in a smoother manner compared to the discrete way illustrated in the example presented in Figure 4 of the previous section. The *constant relative sensitivity* (CRS) weighting functions have the form

$$w(p) = \begin{cases} \delta^{1-\gamma} p^{\gamma}, & \text{if } 0 \leq p \leqslant \delta, \\ 1 - (1-\delta)^{1-\gamma}(1-p)^{\gamma}, & \text{if } \delta < p \leq 1, \end{cases}$$

for $0 \leq \delta \leq 1, 0 < \gamma$. They exhibit the empirically founded inverse-S shape if $0 < \delta < 1$ and $\gamma < 1$, and exhibit the opposite, less frequently found, S-shape if $0 < \delta < 1$ and $\gamma > 1$. The functions are linear if $\gamma = 1$, concave if $\delta = 1$ and $\gamma < 1$ or if $\delta = 0$ and $\gamma > 1$, and convex if $\delta = 1$ and $\gamma > 1$ or if $\delta = 0$ and $\gamma < 1$. It is easily verified that these weighting functions have a fixed point at δ (in addition to 0 and 1), and that their derivative at δ is equal to γ. The latter features establishes a natural connection to the linear discontinuous weighting functions discussed in the previous section, and it reinforces the interpretation of the parameter δ as an index of relative optimism reflecting elevation, and of γ as an index reflecting curvature. Figure 5 depicts an inverse-S shaped CRS weighting function, w, and a linear and discontinuous weighting function that is tangent to w at (δ, δ).

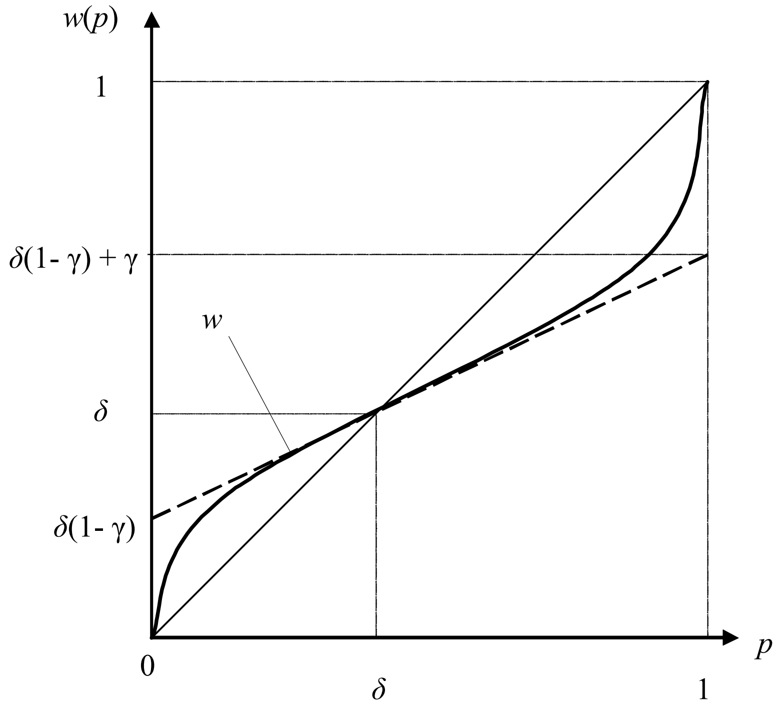

Figure 5: A CRS inverse-S shaped weighting function

The CRS weighting functions are power functions on the interval $[0, \delta]$ and dual power functions on the interval $[\delta, 1]$. This suggests an interpretation for the parameter γ as degree of curvature in much analogy to the measure of relative risk aversion in the case of power utility functions (see

Arrow 1971 and Pratt 1964). The corresponding index of relative sensitivity of the weighting function w is given by $RS(w,p) = \dfrac{-p[\partial^2 w(p)/\partial p^2]}{\partial w(p)/\partial p}$ for $p \in]0,\delta]$, and by (the dual) $RS(w,p) = \dfrac{-(1-p)[\partial^2(1-w(p))/\partial(1-p)^2]}{\partial(1-w(p))/\partial(1-p)}$ for $p \in]\delta,1[$. The index is constant throughout $]0,1[$ and equals $1-\gamma$. This explains the name for this family of weighting functions and indicates the relationship between relative sensitivity and the curvature of these weighting functions.

The CRS weighting functions allow for a comparative analysis based on the index of relative sensitivity, $1-\gamma$, and the index of relative optimism, δ. If $\succcurlyeq_1$ and $\succcurlyeq_2$ describe the preference relations of two individuals with RDU (or PT) preferences and CRS probability weighting functions, then individual 1 exhibits more relative sensitivity than individual 2 iff $1-\gamma_1 > 1-\gamma_2$ (or equivalently, $\gamma_1 < \gamma_2$). Similarly, individual 1 exhibits more relative optimism than individual 2 iff $\delta_1 > \delta_2$.

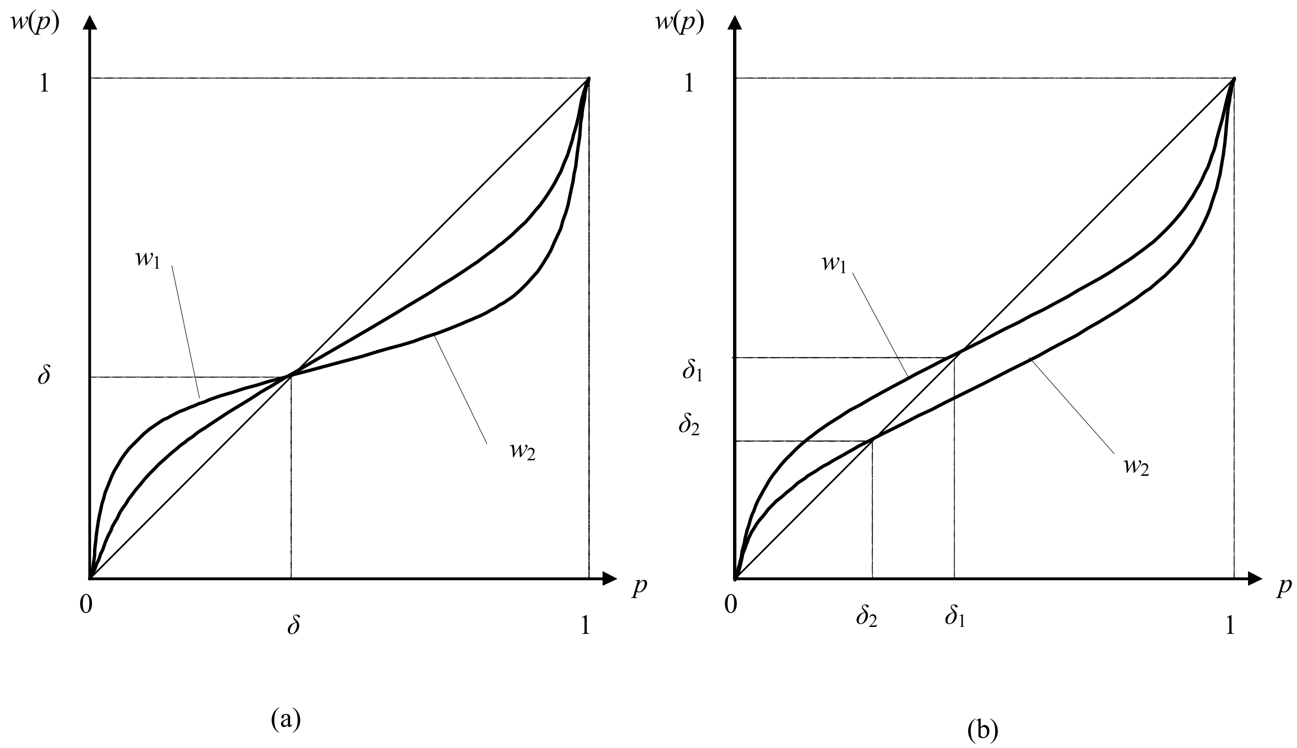

Figure 6: w_1 is more curved than w_2 (a), w_1 is more elevated than w_2 (b).

To establish the relationship with curvature and elevation of the corresponding weighting functions, note that if $\delta_1 = \delta_2$, then w_1 displays more curvature than w_2 iff individual 1 exhibits larger relative sensitivity than individual 2 (see Figure 6.a), and if $\gamma_1 = \gamma_2$ then w_1 is more elevated than w_2 iff individual 1 exhibits more relative optimism than individual 2 (see Figure 6.b).

One observes that, in Figure 6.a, w_1 is more concave than w_2 for probabilities in the interval $[0, \delta]$, hence individual 1 is more optimistic than individual 2 about obtaining best outcomes which have cumulative likelihood below δ. However, for probabilities in $[\delta, 1]$, w_1 is more convex than w_2, and thus individual 1 is more pessimistic about outcomes with cumulative likelihood exceeding δ. Because w_1 is steeper near the endpoints of the probability interval this shows that individual 1 exhibits more increased sensitivity than individual 2.

In Figure 6.b, w_1 is above w_2 for all probabilities except at 0 and 1, hence, w_1 is more elevated than w_2. Because in this case $\gamma_1 = \gamma_2$ and $\delta_1 > \delta_2$ we can write $w_2(p) = (\delta_2/\delta_1)w_1(p)$ for all $p \in [0, 1[$, showing that w_2 is a convex transformation of w_1. This means that overall, individual 2 is more pessimistic than individual 1, or equivalently that individual 1 is more optimistic than individual 2 (see Wakker 1994, 2001), reinforcing the interpretation of higher elevation as indicative for a more optimistic probabilistic attitude.

5 Parameter Estimates

In this section we analyze data obtained from an experiment presented in Abdellaoui, L'Haridon and Paraschiv (2008). This analysis gives some flavor of the empirical potential of the CRS weighting function. Sixty-one subjects took part in the experiment. All subjects were undergraduate students in economics and management at the Institut Universitaire de Technologie of Paris. In order to obtain high quality data with little noise, answers were collected in computer assisted individual sessions. Subjects were told that there were no right or wrong answers, and they were allowed to take a break at any time during the experiment. Their responses to the tasks were entered in the computer by an interviewer so that participants could focus exclusively on the tasks. Subjects were paid 10 euros (approximately 16 US-Dollars at the time of the experiment) for completing the tasks, and a real incentive procedure was implemented to further improve motivation for half of the sample. As discussed in Abdellaoui, L'Haridon and Paraschiv (2008), no significant difference in behavior was observed between the two samples.

The elicitation method for decision weights used in the experiment is based on Abdellaoui, Bleichrodt and L'Haridon (2008). This method consists of two stages. In the first stage utility is elicited for a fixed probability, and then, in the second stage, decision weights associated with the different probabilities are obtained. This two-step elicitation procedure was applied at the individual level for both probabilities of gains and for probabilities of losses. All measurements were based on the elicitation of certainty equivalents for binary prospects. The experiment also included other tasks related to choice under uncertainty, not reported here. The relevant data for the analysis here are five elicited decision weights for probabilities of gains and five decision weights for probabilities of losses, both corresponding to probabilities $p_i = 0.05, 0.25, 0.5, 0.75$ and 0.95.

$w(p_i)$ vs. p_i	$p_1 = 0.05$	$p_2 = 0.25$	$p_3 = 0.5$	$p_4 = 0.75$	$p_5 = 0.95$
$w^+(p_i)$	0.11	0.26	0.42	0.63	0.79
IQR $w^+(p_i)$	$0.05 - 0.16$	$0.17 - 0.32$	$0.35 - 0.50$	$0.52 - 0.75$	$0.63 - 0.89$
$w^-(p_i)$	0.07	0.26	0.44	0.65	0.82
IQR $w^-(p_i)$	$0.03 - 0.13$	$0.19 - 0.34$	$0.36 - 0.54$	$0.54 - 0.73$	$0.74 - 0.90$

Table 1: Median elicited decision weights and corresponding interquartile ranges

Table 1 above, presents the median elicited decision weights for the five different gain, respectively, loss probabilities. Interquartile ranges are presented underneath each median decision weight. The results are consistent with an inverse-S shaped probability transformation for both gain probabilities and loss probabilities: overweighting of small probabilities and underweighting of moderate and large probabilities is observed. Interquartile ranges show considerable variations at the individual level, which is consistent with existing findings in the literature (e.g., Wu and Gonzalez 1996).

Table 2 below presents the results of a parametric fitting of individual decision weights using a linear probability weighting function. Consistent with the data in Table 1, we found both optimism for gain probabilities and pessimism for loss probabilities. The relative index of optimism for gains δ^+ is lower than the relative index of pessimism for losses, δ^-. This result is compatible with the experimental results of Tversky and Kahneman (1992) and of Abdellaoui (2000). However, we did not find significant differences between the relative index of optimism for gain probabilities and the relative index of pessimism for probabilities of losses ($p = 0.45$, one-tail paired t-test). The parameters β^+ and β^- are significantly different from 0, and there is no significant difference between the gain parameter and the loss parameter ($p = 0.07$, one-tail paired t-test).

Linear Weighting Parameters	α	β	$1-\alpha-\beta$	$\delta=\beta/(1-\alpha)$
Gain Probabilities	0.76	0.08	0.18	0.26
IQR	$0.56-0.87$	$0.01-0.14$	$0.07-0.32$	$0.11-0.51$
Loss Probabilities	0.82	0.05	0.14	0.30
IQR	$0.63-0.91$	$-0.01-0.11$	$0.06-0.26$	$0.08-0.55$

Table 2: Median parameter estimates; linear weighting and corresp. interquartile ranges

In Table 3 below we present the parametric fitting results for the CRS probability weighting functions for gain probabilities and for loss probabilities.

CRS-parameters	δ	γ	$RS = 1 - \gamma$
Gain Probabilities	0.28	0.53	0.47
IQR	$0.15 - 0.51$	$0.32 - 0.65$	
Loss Probabilities	0.35	0.58	0.42
IQR	$0.10 - 0.63$	$0.38 - 0.72$	

Table 3: Median parameter estimates; CRS weighting and corresp. interquartile ranges

We observe that subjects exhibited more relative pessimism for probabilities of losses than relative optimism for probabilities of gains ($\delta^- > \delta^+$), but this difference is not significant ($p = 0.37$, one-tailed paired t-test). These results are consistent with findings for the linear probability weighting function. It should be noted that there is considerable variation at the individual level for loss probabilities. The evaluated RS was significantly higher for gain probabilities than for probabilities of losses ($p = 0.03$, one-tail paired t-test). This suggests that subjects were more sensitive to probabilities of gains than to probabilities of losses.

For a better comparison of explanatory power for the different weighting functions, we evaluated the best fits at the individual level. For each individual the best fit corresponds to the parametric form which provided the maximum likelihood for the elicited decision weights. The corresponding findings are given in Table 4 below.

Weighting Function	Linear	CRS
Gain Probabilities	43 (16)	18 (9)
Loss Probabilities	22 (5)	39 (18)

Table 4: Number of best fits weighting function with significantly better fits in parentheses

One observes that most of the decision weighting for gain probabilities in Table 1 is found for intermediate ranged probabilities (i.e., 25%, 50% and 75%). As a results, it is not surprising to see that the linear specification of the probability weighting function is the best parametric form for gain probabilities. For probabilities of losses, however, the opposite finding is obtained: the CRS specification best fitted the data.

In order to get a clearer picture on how the two parametric specifications compare, we also evaluated significant best fits on the basis of a likelihood-ratio test. For gain probabilities, 16 subjects had decision weights significantly better described by the linear specification while only 9 subjects had their decision weights significantly better described by the CRS specification. Note that 36 subjects (59% of the sample) have decision weights compatible with both specifications.

For probabilities of losses, only 5 subjects had decision weights significantly better explained by the linear specification while 18 subjects had decision weights significantly better described by the CRS specification; 38 subjects (62% of the sample) had decision weights compatible with both specifications.

To summarize we note that both the CRS and linear probability weighting function are compatible with a majority of the exhibited probabilistic risk behavior. But, we also found that the CRS probability weighting function seems a better choice for explaining the probabilistic risk behavior for probabilities of losses.

6 A Preference Foundation

This section presents an axiomatic preference foundation for prospect theory preferences with CRS weighting functions. The adopted notation for prospects is that of cumulative probability distributions, with exceptions made for clearly specified contexts. We are interested in conditions for a preference relation, $\succcurlyeq$, on the set of prospects L in order to *represent* the preference relation by a function V. That is, V assigns to each prospect a real value, such that for all $P, Q \in L$,

$$P \succcurlyeq Q \Leftrightarrow V(P) \geq V(Q).$$

A requirement for such a representation V is that $\succcurlyeq$ is a *weak order*, i.e. $\succcurlyeq$ is *complete* ($P \succcurlyeq Q$ or $P \preccurlyeq Q$ for all $P, Q \in L$) and *transitive* ($P \succcurlyeq Q$ and $Q \succcurlyeq R$ implies $P \succcurlyeq R$ for all $P, Q, R \in L$).

Further requirements are those of monotonicity (or first order stochastic dominance) and continuity in probabilities. The preference relation $\succcurlyeq$ satisfies *Jensen-continuity* on the set of prospects L if for all prospects $P \succ Q$ and R there exist $\rho, \mu \in (0, 1)$ such that[3]

$$\rho P + (1 - \rho)R \succ Q \text{ and } P \succ \mu R + (1 - \mu)Q.$$

The preference relation satisfies *monotonicity* (in cumulative probabilities) if $P \succ Q$ whenever $p_j \geq q_j$ for all $j = 1, \ldots, n$ and $P \neq Q$. A monotonic weak order that satisfies Jensen-continuity on L also satisfies the stronger Euclidean-continuity on L (see Abdellaoui 2002, Lemma 18). Further, the three conditions taken together imply the existence of a continuous function $V : L \to I\!R$, strictly increasing in each cumulative probability, that represents $\succcurlyeq$. The latter follows from results of Debreu (1954).

Next, we focus on the additive separability conditions for the representing function V. This requires an independence condition for common cumulative probabilities. To define this property we introduce some useful notation. For $i \in \{1, \ldots, n-1\}$, $P \in L$ and $\sigma \in [0, 1]$, we denote by $\sigma_i P$ the prospect that agrees with P except that p_i is replaced by σ. Whenever this notation is used it is implicitly assumed that $p_{i-1} \leq \sigma \leq p_{i+1}$ (respectively, $\sigma \leq p_{i+1}$ if

[3] The ρ-probability mixture of P with R is the prospect $\rho P + (1-\rho)R = (\rho p_1 + (1-\rho)r_1, \ldots, \rho p_n + (1-\rho)r_n)$. Note that this definition is independent of whether probabilities are cumulative or decumulative. In the case of the cumulative probabilities notation we, obviously, have $\rho p_n + (1 - \rho)r_n = 1$.

$i = 1$ and $p_{i-1} \leq \sigma$ if $i = n-1$) to ensure that $\sigma_i P \in L$. The preference relation $\succcurlyeq$ satisfies *comonotonic independence* if $\sigma_i P \succcurlyeq \sigma_i Q \Leftrightarrow \rho_i P \succcurlyeq \rho_i Q$ for all $\sigma_i P, \sigma_i Q, \rho_i P, \rho_i Q \in L$.

The next lemma follows from results of Wakker (1993).

Lemma 1 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(i) The preference relation $\succcurlyeq$ on L is represented by an additive function

$$V(P) = \sum_{j=1}^{n-1} V_j(p_j),$$

with continuous strictly monotonic functions $V_1, \ldots, V_{n-1} : [0,1] \to I\!R$ which are bounded except maybe V_1 and V_{n-1} which could be unbounded at extreme probabilities (i.e., V_1 may be unbounded at 0 and V_{n-1} may be unbounded at 1).

(ii) The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence.

The functions $V_1, \ldots, V_{n-1}$ are jointly cardinal, that is, they are unique up to location and common scale. $\square$

Next we focus on the condition that, if added to Lemma 1 determines the CRS weighting function. Recall that the CRS weighting functions are either concave for probabilities below some δ and convex for probabilities exceeding δ (or they are convex below δ and concave above δ). In general, the exact value of δ is unknown. But a large body of empirical research over the past two decades suggests that the value of δ is around probability 0.33 for the gain weighting function. For the loss weighting function the value seems, somewhat higher, around 0.4, however, compared to probabilities for gains, there much less empirical evidence for loss probabilities.

As the probability δ separates the probability interval into a range of probabilities with optimistic behavior and a range with pessimistic behavior, it also indicates for which outcomes the corresponding decision weights are mainly influenced by optimism and for which outcomes the decision weights are mainly influenced by pessimism. One can, therefore, distinguish prospects by the maximum number of outcomes for which optimism is determining the decision weights. Accordingly, we denote by L_m the set of prospects for which the m best outcomes $x_1, \dots, x_m$ $(m = 1, \dots, n-1)$ have decision weight reflecting optimistic behavior while the $n - m$ remaining outcomes have decision weights reflecting pessimistic behavior, i.e., $L_m = \{(p_1, \dots, p_{n-1}, 1) \in L : p_m < \delta \leq p_{m+1}\}$. The set $L_0 = \{(p_1, \dots, p_{n-1}, 1) \in L : \delta \leq p_1\}$ contains all prospects where decision weights are determined exclusively by pessimistic behavior.

We now look at the effect on preferences caused by probability mixtures of prospects from L_m with the the prospect that gives x_{m+1} for sure, which we denote by $C_m = (c_1, \dots, c_{m-1}, 1)$ with $c_1 = \cdots = c_m = 0$ and $c_i = 1$ otherwise. The preference relation $\succcurlyeq$ satisfies *proportional invariance* (for changes in probabilities) *away from* δ if

$$P \succcurlyeq Q \Leftrightarrow \sigma P + (1-\sigma)C_m \succcurlyeq \sigma Q + (1-\sigma)C_m,$$

whenever $P, Q, \sigma P + (1-\sigma)C_m, \sigma Q + (1-\sigma)C_m \in L_m$, $m = 0, \dots, n-1$.

The intuition behind this invariance property is revealed when observing that, in the second preference above, the probability of obtaining any of the outcomes in P and Q has been reduced proportionally and the resulting probability mass is given to outcome x_{m+1}. Upon reflection, one observes that the cumulative probabilities of the best m outcomes in the prospects P and Q of the first preference, $p_1, \dots, p_m$, and $q_1, \dots, q_m$, are reduced proportionally to $\sigma p_1, \dots, \sigma p_m, \sigma q_1, \dots, \sigma q_m$, and, in a dual manner, the decumulative probabilities of the remaining $n - m$ outcomes, $1 - p_m, \ldots, 1 - p_{n-1}, 1 - q_m, \ldots, 1 - q_{n-1}$, are similarly reduced proportionally to $\sigma(1 - p_m), \ldots, \sigma(1 - p_{n-1}), \sigma(1 - q_m), \ldots, \sigma(1 - q_{n-1})$, in the second preference. This reflects the idea that the effect of optimism and the opposite effect due to pessimism keep the balance, thereby indicating constant relative optimism. Or, put differently, that the sensitivity towards proportional changes in cumulative probabilities of "good" outcomes is of similar magnitude to the sensitivity towards similar proportional changes in (de)cumulative probabilities of "bad" outcomes.

The next theorem considers the implications of the proportional invariance property when added to Lemma 1. It shows that RDU holds with a weighting functions of the following form

$$w(p) = \begin{cases} ap^{\gamma}, & \text{if } 0 \leq p \leqslant \delta, \\ 1 - b(1-p)^{\gamma}, & \text{if } \delta < p \leq 1, \end{cases},$$

with $b, \gamma > 0$ and (due to continuity at δ) $a = [1 - b(1-\delta)^{\gamma}]/\delta^{\gamma}$. This family of weighting functions is more larger than the CRS-family because it does not require differentiability at δ for $\delta \in]0, 1[$. This aspect is discussed further following the main theorem below.

Before presenting the theorem we note that RDU with such a "generalized CRS" weighting function implies proportional invariance away from $\delta \in [0, 1]$.

Theorem 2 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is represented by RDU with weighting function*

$$w(p) = \begin{cases} ap^{\gamma}, & \textit{if } 0 \leq p \leqslant \delta, \\ 1 - b(1-p)^{\gamma}, & \textit{if } \delta < p \leq 1, \end{cases}$$

with $b, \gamma > 0, 0 \leq \delta \leq 1$ and positive $a = [1 - b(1-\delta)^{\gamma}]/\delta^{\gamma}$.

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and proportional invariance away from $\delta \in [0, 1]$.*

The parameters b, γ and δ are uniquely determined and the utility function U is cardinal. $\square$

It should be noted that Theorem 2 characterizes the class of RDU preferences with weighting functions which can be termed general CRS weighting functions. Our interest was on a specific subfamily which in addition to proportional invariance away from δ also satisfies "constant relative optimism" in the sense that decision weights generated by proportional changes away from δ influenced by optimism (i.e., changes for probabilities in $[0, \delta]$) are proportional to decision weights generated by corresponding dual changes away from δ influenced mainly by pessimism. Formally, we require that for any integer $s = 1, 2, \ldots$ we have the following relation between decision weights:

$$\frac{w(\frac{r\delta}{s}) - w(\frac{(r-1)\delta}{s})}{\delta/s} = \frac{w(1 - \frac{r(1-\delta)}{s}) - w(1 - \frac{(r-1)(1-\delta)}{s})}{(1-\delta)/s},$$

for all $r = 1, \ldots, s$. This condition, which is not determined by specific choice behavior, is inspired from the the idea that the effect of optimism and pessimism observed at extreme probabilities has similar effect on subsequent probabilities due to constant relative optimism, although the effect is discounted due to reduced (or increased) sensitivity away from 0 and 1 modeled through the "rate of decay" parameter γ (see the discussion relating to Figure 4 in Section 3 above). The implications of this condition for the weighting function in Theorem 2 (after substitution and some elementary calculus) is that $b = (1 - \delta)^{\gamma}$, which gives $a = \delta^{1-\gamma}$, and thus the desired CRS weighting function is obtained.

In Theorem 2 we considered RDU. As mentioned before, similar results are obtained in the case of prospect theory (PT). If all outcomes are gains or all outcomes are losses the corresponding preference foundations follow as corollaries from Theorem 2. In the case that we have gains and losses, we can obtain preference foundations by requiring proportional invariance away from δ^+ for probabilities of gains and, separately, proportional invariance away from δ^-

for probabilities of losses. For these preference conditions to take effect we must then require that there are at least 2 gains and at least 2 losses, hence, accounting for the reference point, overall there are at least 5 strictly ranked outcomes. The remaining conditions in Statement (ii) of Theorem 2 remain unaltered, and a preference foundation for PT with CRS weighting functions is obtained.

To conclude this section it is well worth noting that, in the definition of the proportional invariance condition, δ was assumed exogenously given. We have chosen to present the property in this simplified version for convenience of exposition, and it should be noted that the property can be formulated more generally without an exogenously given parameter of relative optimism δ following the ideas presented recently in Diecidue, Schmidt and Zank (2008).

The CRS function provides a clear common link between the behavioral probabilistic risk concepts of optimism and pessimism, which are important for economic analyses and applications, and the concepts of curvature and elevation, which are important behavioral concepts in the psychology literature. Our measure of (relative) sensitivity resembles a clear analogy to the measure of (relative) risk aversion for utility which is central in economics. The results provides a natural link between results and interpretations given in the related economics and psychology literature and, we think, a further opportunity to profit form the dialog between scientists in these different fields of science.

7 Appendix: Proofs

Proof of Theorem 2: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Proportional invariance away from δ follows from substitution of the RDU-functional with a generalized CRS weighting function.

Next we prove that statement (ii) implies statement (i). Obviously statement (ii) in Lemma 1 is satisfied, hence, there exists an additively separable functional representing the preference $\succcurlyeq$. We restrict the attention to the case that $p_1 > 0$ and $p_{n-1} < 1$ to avoid the problem of dealing with unbounded V_1, V_{n-1}. To show that our additive functional in fact is a RDU form with a generalized CRS weighting function we use results presented Diecidue, Schmidt and Zank (2008). If $\delta = 0$ (or $\delta = 1$), then proportional invariance comes down to Diecidue, et al.'s common ratio invariance for cumulative (or decumulative) probabilities, and we apply their Theorem 1 to obtain RDU with power weighting function, $w(p) = p^\gamma$ (or dual power weighting function, $w(p) = 1 - (1-p)^\gamma$).

Next assume that $0 < \delta < 1$. We apply the results of Diecidue, et al. (2008) presented in their Theorem 3. First we observe that proportional invariance implies the common ratio invariance properties used by Diecidue, et al. We thus obtain, from the proof of their Theorem 3, that RDU holds with a weighting function of the form

$$w(p) = \begin{cases} ap^c & \text{if } p < \delta \\ 1 - b(1-p)^d & \text{if } p > \delta \end{cases},$$

with $c, b, d > 0$, and $a = 1/\delta^c - b(1-\delta)^d/\delta^c$. Further, applying proportional invariance away from δ gives that $c = d =: \gamma$.

Uniqueness results follow from Theorem 3 of Diecidue et al. (2008). This completes the proof of Theorem 2. $\square$

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