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# Capital growth theory and von Neumann-Gale dynamics

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# Capital growth theory and von Neumann-Gale dynamics

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**Abstract**

In the recent paper by Dempster, Evstigneev and Taksar (Annals of Finance 2006) it has been shown that the von Neumann-Gale growth model provides a convenient and natural framework for the analysis of questions of asset pricing and hedging under transaction costs. The present article focuses on a different area of applications of this model in finance. It demonstrates how methods and concepts developed in the context of von Neumann-Gale dynamical systems can be used to build a general theory of optimal financial growth under transaction costs.

*Key words:* Capital growth theory, Transaction costs, Numeraire portfolios, Random dynamical systems, Convex multivalued operators, Von Neumann–Gale model, Rapid paths
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# 1 Introduction

How to invest in order to maximize the asymptotic growth rate of wealth? This question has been in the focus of a large area of research pioneered by Shannon[1] and followed by Kelly [24], Breiman [5], Thorp [32], Algoet and Cover [2] and others—see the survey by Hakansson and Ziemba [7] and references therein. For the most part, results available in the literature on capital growth pertain to markets without transaction costs. Up to now, only some specialized models of markets with "frictions" have been analyzed in this field; see e.g. Taksar, Klass and Assaf [31], Iyengar and Cover [20], and Akian, Sulem and Taksar [1]. The purpose of the present study is to develop a capital growth theory within a general discrete-time framework taking into account transaction costs and trading constraints. Our main tool in this work is one of the most celebrated models in mathematical economics—the von Neumann-Gale growth model.

Generally, the term "von Neumann-Gale model" refers to a special class of multivalued dynamical systems possessing certain properties of convexity and homogeneity. The classical theory of such systems (von Neumann [33], Gale [19] and others) aimed basically at the modeling of economic growth. Originally, this theory was purely deterministic; it did not reflect the influence of random factors on economic growth. The importance of taking these factors into account was realized early on. First attempts of constructing stochastic analogues of the von Neumann–Gale model were undertaken in the 1970s by Dynkin [11, 12, 13], Radner [26, 27] and others. However, the initial attack on the problem left many questions unanswered. Studies in this direction faced serious mathematical difficulties. To overcome these difficulties, new mathematical techniques were required, that were developed only during the last decade—see [15, 17], and [4].

It has recently been observed [9] that stochastic analogues of the von Neumann-Gale model provide a natural and convenient framework for the analysis of some fundamental problems in finance (asset pricing and hedging under transaction costs). This paper focuses on a different area of applications of the model in finance. It demonstrates how methods and concepts

[1]Although Claude Shannon—the famous founder of the mathematical theory of information—did not publish on investment-related issues, his ideas, expressed in his lectures on investment problems, should apparently be regarded as the initial source of that strand of literature which we cite here. For the history of these ideas and the related discussion see Cover [6].

developed in the context of von Neumann-Gale dynamical systems can be used to develop a general theory of optimal financial growth under transaction costs.

The paper is organized as follows. In Section 2 we describe the dynamic securities market model we deal with. Section 3 introduces the basic concepts and results related to the von Neumann-Gale dynamical systems. In Section 4 we apply these results to capital growth theory under transaction costs. Section 5 concludes the paper.

# 2 Dynamic securities market model.

Let $s_0, s_1, ...$ be a stochastic process with values in a measurable space $S$. The process $(s_t)_{t=0}^{+\infty}$ models random factors influencing the market: the random element $s_t$ represents the "state of the world" at date $t = 0, 1, ....$ We denote by

$$s^t := (s_0, s_1, ..., s_t)$$

the history of the process $(s_t)$ up to date $t$.

There are $n$ *assets* traded on the market. A (*contingent*) *portfolio* of assets held by an investor at date $t$ is represented by a vector

$$x_t(s^t) = (x_t^1(s^t), ..., x_t^n(s^t))$$

whose coordinates (portfolio positions) describe the holdings of assets $i = 1, 2, ..., n$. The positions can be described either in terms of "physical units" of assets or in terms of their market values. In the latter case, $x_t^i(s^t)$ is the amount of money invested in asset $i$. A contingent portfolio $x_t(s^t)$ depends generally on the whole history $s^t = (s_0, s_1, ..., s_t)$ of the process $(s_t)$, which means that the investor can select his/her portfolio at date $t$ based on information available by that date. In the applications which we will deal with (capital growth theory), the classical models, e.g. [2], exclude short selling. Following this approach, we will assume that all the contingent portfolios $x_t(s^t)$ are represented by non-negative vector functions. All functions of $s^t$ will be assumed to be measurable and those representing contingent portfolios essentially bounded.

Any sequence of contingent portfolios $x_0(s^0), x_1(s^1), x_2(s^2), ...$ will be called a *trading strategy*. Trading strategies describe possible scenarios of investors' actions at the financial market influenced by random factors. In the model, we are given sets $G_t(s^t) \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^n$ specifying the *self-financing* (*solvency*) constraints. The main focus of the study is on self-financing trading strategies. A strategy $x_0(s^0), x_1(s^1), x_2(s^2), ...$ is called *self-financing* if

$$(x_{t-1}(s^t), x_t(s^t)) \in G_t(s^t) \tag{1}$$

almost surely (a.s.) for all $t \geq 1$. The inclusion $(x_{t-1}(s^t), x_t(s^t)) \in G_t(s^t)$ means that the portfolio $x_{t-1}(s^t)$ can be rebalanced to the portfolio $x_t(s^t)$ at date $t$ in the random situation $s^t$ under transaction costs and trading constraints. The rebalancing of a portfolio excludes inflow of external funds, but it may take into account dividends paid by the assets.

It is assumed that for each $t \geq 1$, the set $G_t(s^t)$ is a *closed convex cone* depending measurably on $s^t$.[2] This assumption means that the model takes into account *proportional* transaction costs. We give examples of the cones $G_t(s^t)$ below.

*Example 1. No transaction costs.* Let $q_t(s^t) = (q_t^1(s^t), ..., q_t^n(s^t)) > 0$ be the vector of the market prices of assets $i = 1, 2, ..., n$ at date $t$. (All inequalities between vectors, strict and non-strict, are understood coordinatewise.) Suppose that portfolio positions are measured in terms of the market values of assets. Define

$$G_t(s^t) := \{(a, b) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^n b^i \leq \sum_{i=1}^n \frac{q_t^i(s^t)}{q_{t-1}^i(s^{t-1})} a^i\}. \tag{2}$$

A portfolio $a = (a^1, ..., a^n)$ can be rebalanced to a portfolio $b = (b^1, ..., b^n)$ (without transaction costs) if and only if $(a, b) \in G_t(s^t)$.

*Example 2. Proportional transaction costs: single currency.* Let $G_t(s^t)$ be the set of those $(a, b) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ for which

$$\sum_{i=1}^n (1 + \lambda_{t,i}^+(s^t))\, (b^i - \frac{q_t^i(s^t)}{q_{t-1}^i(s^{t-1})} a^i)_+ \leq$$

$$\sum_{i=1}^n (1 - \lambda_{t,i}^-(s^t))\, (\frac{q_t^i(s^t)}{q_{t-1}^i(s^{t-1})} a^i - b^i)_+ \, , \tag{3}$$

where $r_+ := \max\{r, 0\}$ for a real number $r$. The transaction cost rates for buying and selling are given by the numbers $\lambda_{t,i}^+(s^t) \geq 0$ and $1 > \lambda_{t,i}^-(s^t) \geq 0$, respectively. A portfolio $a = (a^1, ..., a^n)$ can be rebalanced to a portfolio $b = (b^1, ..., b^n)$ (with transaction costs) if and only if the pair of vectors $(a, b)$ belongs to the cone $G_t(s^t)$. Here, we again assume that the coordinates $a^i$

[2] A closed set $G(s) \subseteq \mathbb{R}^n$ is said to depend measurably on a parameter $s$ if the distance to this set from each point in $\mathbb{R}^n$ is a measurable function of $s$.

and $b^i$ of the portfolio vectors indicate the current market values of the asset holdings. The inequality in (3) expresses the fact that purchases of assets are made only at the expense of sales of other assets.

*Example 3. Proportional transaction costs: several currencies.* Consider an asset market where $n$ currencies are traded. Suppose that for each $t = 1, 2, ...$ a matrix

$$\mu_t^{ij}(s^t) \text{ with } \mu_t^{ij} > 0 \text{ and } \mu_t^{ii} = 1$$

is given, specifying the exchange rates of the currencies $i = 1, 2, ..., n$ (including transaction costs). The number $\mu_t^{ij}(s^t)$ shows how many units of currency $i$ can be obtained by exchanging one unit of currency $j$. A portfolio $a \geq 0$ of currencies can be exchanged to a portfolio $b$ at date $t$ in the random situation $s^t$ if and only if there exists a nonnegative matrix $(d_t^{ji})$ (*exchange matrix*) such that

$$a^i \geqslant \sum_{j=1}^{n} d_t^{ji},\ 0 \leqslant b^i \leqslant \sum_{j=1}^{n} \mu_t^{ij}(s^t) d_t^{ij}.$$

Here, $d_t^{ij}$ $(i \neq j)$ stands for the amount of currency $j$ exchanged into currency $i$. The amount $d_t^{ii}$ of currency $i$ is left unexchanged. The second inequality says that at time $t$ the $i$th position of the portfolio cannot be greater than the sum $\sum_{j=1}^{n} \mu_t^{ij} d_t^{ij}$ obtained as a result of the exchange. The model we deal with here is a version of the multicurrency models considered by Kabanov, Stricker and others (see e.g. [22], [23] and [21]). Note that in this example asset holdings are expressed in terms "physical units" of assets (currencies).

An important class of dynamic securities market models is formed by *stationary model*s. They are defined as follows. A model is called stationary if the stochastic process $(s_t)$ is stationary and the given cones $G_t(s^t)$ (specifying the solvency constraints) are of the following form:

$$G_t(s^t) = G(s^t), \tag{4}$$

where for each $s^t$ the set $G(s^t)$ is a closed convex cone in $\mathbb{R}_+^n \times \mathbb{R}_+^n$ depending measurably on $s^t$. Assumption (4) expresses the fact that the solvency constraints do not explicitly depend on time: their structure depends only on the current and previous states of the world—on the history $s^t$ of the underlying stochastic process. In the stationary context it is convenient to assume that $s_t$ is defined for each $t = 0, \pm 1, \pm 2, ...,$ and in this case the notation $s^t$ refers to the infinite history $s^t = (..., s_{t-1}, s_t)$. This convention will always apply when we shall speak of stationary models.

If the stochastic process $(s_t)$ is stationary, then the models considered in Examples 1 and 2 are stationary if the asset returns $R_t(s^t) := q_t^i(s^t)/q_{t-1}^i(s^{t-1})$ and the transaction cost rates $\lambda_{t,i}^-(s^t)$ and $\lambda_{t,i}^+(s^t)$ do not explicitly depend on $t$:

$$R_t(s^t) = R(s^t),\ \lambda_{t,i}^{\pm}(s^t) = \lambda_i^{\pm}(s^t).$$

The analogue of this assumption in the Example 3 is the condition that the exchange rates do not explicitly depend on $t$: $\mu_t^{ij}(s^t) = \mu^{ij}(s^t)$.

In the analysis of stationary models, we will consider a class of trading strategies called *balanced*. A strategy $x_0, x_1, x_2, ...$ is termed balanced if there exist a vector function $x(s^0) \in \mathbb{R}_+^n$ and scalar function $\alpha(s^0) > 0$ such that

$$x_0(s^t) = x(s^t);\ x_t(s^t) = \alpha(s^t)...\alpha(s^1)x(s^t),\ t \geq 1, \tag{5}$$

and $|x(s^0)| = 1$. (We write $|\cdot|$ for the sum of the absolute values of the coordinates of a vector). According to (5), portfolios $x_t(s^t)$ grow with stationary proportions defined by the random vector process $x(s^0), x(s^1), ...$ and at a stationary rate $\alpha(s^1), \alpha(s^2), ....$ The results of capital growth theory pertaining to stationary models (see Section 3) will be stated in terms of balanced trading strategies.

# 3 Von Neumann–Gale dynamical systems

Von Neumann-Gale dynamical systems are defined in terms of multivalued (set-valued) operators possessing properties of convexity and homogeneity. States of such systems are represented by elements of convex cones $X_t$ $(t = 0, 1, ...)$ in linear spaces. Possible one-step transitions from one state to another are described in terms of given operators $A_t(x)$, assigning to each $x \in X_{t-1}$ a convex subset $A_t(x) \subseteq X_t$. It is assumed that the graphs $Z_t := \{(x, y) \in X_{t-1} \times X_t : y \in A_t(x)\}$ of the operators $A_t(x)$ are convex cones. *Paths* (*trajectories*) of the von Neumann-Gale dynamical system are sequences $x_0 \in X_0, x_1 \in X_1, ...$ such that $x_t \in A_t(x_{t-1})$.

In this work we consider stochastic von Neumann-Gale dynamical systems in which a stochastic process $(s_t)$ and a sequence of random closed convex cones $G_t(s^t) \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^n$ $(t = 1, 2, ...)$ are given. The random elements $s_t$ of a measurable space $S$ are defined either for all non-negative integers $t$ or for all integers $t$. In the former case $s^t := (s_0, ..., s_t)$ and in the latter $s^t := (...s_{t-1}, s_t)$. We denote by $\mathcal{X}_t$ the cone of measurable essentially bounded vector functions $x(s^t)$ with values in $\mathbb{R}_+^n$ and we put

$$Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : (x(s^{t-1}), y(s^t)) \in G_t(s^t) \text{ (a.s.)}\} \tag{6}$$

and

$$A_t(x) := \{y \in \mathcal{X}_t : (x, y) \in Z_t\}. \tag{7}$$

The multivalued operators $x \mapsto A_t(x)$ $(t = 1, 2, ...)$ transforming elements of $\mathcal{X}_{t-1}$ into subsets of $\mathcal{X}_t$ define the von Neumann-Gale dynamical system we deal with. Paths of this system are sequences of vector functions $x_t(s^t)$ such that $x_t \in \mathcal{X}_t$ and $x_t \in A_t(x_{t-1})$. In the applications we have in mind, these paths are self-financing investment strategies in the dynamic securities market model described in the previous section and $G_t(s^t)$ are the solvency cones in this model.

It is assumed that the cone $G_t(s^t)$ depends measurably on $s^t$, and for all $t$ and $s^t$, the following basic conditions hold:
(**G.1**) for any $a \in \mathbb{R}^n_+$, the set $\{b : (a, b) \in G_t(s^t)\}$ is non-empty;

(**G.2**) the set $G_t(s^t)$ is contained in $\{(a, b) : |b| \leq M_t|a|\}$, where $M_t$ is a constant independent of $s^t$;

(**G.3**) there exist a strictly positive constant $\gamma_t > 0$ and a pair of essentially bounded vector functions $(\check{a}_{t-1}(s^t), \check{b}_t(s^t))$ such that $(\check{a}_{t-1}(s^t), \check{b}_t(s^t)) \in G_t(s^t)$ for all $s^t$ and $\check{b}_t(s^t) \geq \gamma_t e$, where $e = (1, ..., 1)$;
(**G.4**) if $(a, b) \in G_t(s^t)$, $a' \geq a$ and $0 \leq b' \leq b$, then $(a, b) \in G_t(s^t)$ ("free disposal hypothesis").

Define

$$G_t^\times(s^t) = \{(c, d) \geq 0 : db - ca \leq 0 \text{ for all } (a, b) \in G_t(s^t)\}, \tag{8}$$

where $ca$ and $db$ denote the scalar products of the vectors. Let $\mathcal{P}_t$ denote the set of measurable vector functions $p(s^t)$ with values in $\mathbb{R}^n_+$ such that $E|p(s^t)| < \infty$. A *dual path* (*dual trajectory*) is a finite or infinite sequence $p_1(s^t), p_2(s^t), ...$ such that $p_t \in \mathcal{P}_t$ $(t \geq 1)$ and

$$(p_t(s^t), E_t p_{t+1}(s^t)) \in G_t^\times(s^t) \text{ (a.s.)} \tag{9}$$

for all $t \geq 1$ for which $p_{t+1}$ is defined. We write $E_t(\cdot) = E(\cdot|s^t)$ for the conditional expectation given $s^t$. By virtue of (8) and (9), $E_t(p_{t+1}y) \leq p_t x$ (a.s.) for any $(x, y) \in Z_t$. This inequality shows that for any path $x_0, x_1, ...$ the sequence of random variables $p_1 x_0, p_2 x_1, ...$ is a supermartingale with respect to the given filtration in the underlying probability space generated by $s^t$.

A dual path $(p_1, ..., p_{N+1})$ is said to *support* a path $(x_0, ..., x_N)$ if

$$p_t x_{t-1} = 1 \text{ (a.s.)} \tag{10}$$

for $t = 1, ..., N+1$ (for infinite paths (10) should hold for all $t \geq 1$). A trajectory is called *rapid* if there exists a dual trajectory supporting it. The term "rapid" is motivated by the fact that

$$\frac{E_t(p_{t+1}y_t)}{p_t y_{t-1}} \leq \frac{E_t(p_{t+1}x_t)}{p_t x_{t-1}} = 1 \text{ (a.s.)}$$

for each path $y_0, y_1, ...$ with $p_t y_{t-1} > 0$ (see (9) and (10)). This means that the path $x_0, x_1, ...$ maximizes the conditional expectation of the *growth rate* at each time $t$, the maximum being equal to 1. Growth rates are measured in terms of the random linear functions $p_t a$. In the applications, where states of the system $x_t$ constituting trajectories represent portfolios of assets, $p_t$ are interpreted as price vectors.

# 4 Capital growth theory and von Neumann-Gale dynamical systems

From the point of view of capital growth, trading strategies growing at an asymptotically optimal rate are of central interest. How can we define asymptotic optimality? In the definition below, we follow essentially Algoet and Cover [2].

Let $x_0, x_1, ...$ be an investment strategy. It is called *asymptotically optimal* if for any other investment strategy $y_0, y_1, ...$ there exists a supermartingale $\xi_t$ such that

$$\frac{|y_t|}{|x_t|} \leq \xi_t, \; t = 0, 1, ... \text{ (a.s.)}.$$

Recall that for a vector $b = (b^1, ..., b^n)$ we write $|b| = |b^1| + ... + |b^n|$. If $b \geq 0$, then $|b| = b^1 + ... + b^n$, and if the vector $b$ represents a portfolio whose positions are measured in terms of the market values of assets, then $|b|$ is the value of this portfolio—the total amount of money invested in all its assets. Note that the above property remains valid if $|b|$ is replaced by any function $\psi_t(s^t, b)$ (possibly random and depending on $t$) , where

$$l|b| \leq \psi_t(s^t, b) \leq L|a| \tag{11}$$

where $0 < l < L$ are non-random constants. As an example of such a function, we can consider the *liquidation value* (also called *net asset value*) of the portfolio

$$\psi_t(s^t, b) = \sum_{i=1}^{n} (1 - \lambda_{t,i}^{-}(s^t)) b^i$$

within the model defined by (3). This is the amount of money the investor gets if he/she decides to liquidate the portfolio (sell all the assets) at date $t$. Clearly condition (11) holds if the random variables $1 - \lambda_{t,i}^-$ are uniformly bounded away from zero.

The strength of the above definition, which might seem not immediately intuitive, is illustrated by the following implications of asymptotic optimality. As long as

$$\frac{|y_t|}{|x_t|} \leq \xi_t, \; t = 0, 1, ... \text{ (a.s.)},$$

where $\xi_t$ is a supermartingale, the following properties hold.

(a) With probability one

$$\sup_t \frac{|y_t|}{|x_t|} < \infty \; ,$$

i.e. for no strategy wealth can grow asymptotically faster than for $x_0, x_1, ...$ (a.s.).

(b) The strategy $x_0, x_1, ...$ a.s. maximizes the exponential growth rate of wealth

$$\limsup_{t \to \infty} \frac{1}{t} \ln |x_t|.$$

(c) We have

$$\sup_t E \frac{|y_t|}{|x_t|} < \infty \;\text{ and }\; \sup_t E \ln \frac{|y_t|}{|x_t|} < \infty.$$

This work aims at obtaining results on optimal growth in models with transaction costs. The main results are concerned with the existence of asymptotically optimal strategies in general (non-stationary) models and the existence of asymptotically optimal balanced strategies in stationary models. Our main tool for analysing the questions of asymptotic optimality is the concept of a *rapid path* in the stochastic von Neumann-Gale system. The results are based on assumption (**G.5**) below.

(**G.5**) There exists an integer $l \geq 1$ such that for every $t \geq 0$ and $i = 1, ..., n$ there is a path $y_{t,i}, ..., y_{t+l,i}$ over the time interval $[t, t+l]$ satisfying

$$y_{t,i} = e_i, ..., y_{t+l,i} \geq \gamma e,$$

where $e_i = (0, 0, ..., 1, ..., 0)$ (the $i$th coordinate is 1 while the others are 0) and $\gamma$ is a strictly positive non-random constant.

**Proposition 1.** *If the constants $M_t$ in condition (**G.2**) do not depend on $t$ and assumption **(G.5)** holds, then any rapid path is asymptotically optimal.*

For a proof of this fact see Evstigneev and Flåm [14], Proposition 2.5. In specific dynamic securities market models, condition (**G.5**) holds typically with $l = 1$. Then it means a possibility of buying some fixed strictly positive amounts of all the assets by selling one unit of any asset $i = 1, ..., n$ (or if portfolio positions are measured in terms of the market values of assets—by selling the amount of asset $i$ worth one dollar).

Thus in order to prove the existence of asymptotically optimal strategies it is sufficient to establish the existence of infinite rapid paths. In the applications to the dynamic securities market model defined in terms of a von Neumann-Gale dynamical system, we will use the terms "paths" and "(self-financing) trading strategies" interchangeably. The main results of this paper are collected in the following theorem.

**Theorem 1.** *(i) Let $x_0(\omega)$ be a vector function in $\mathcal{X}_0$ such that $\delta e \leq x_0(\omega) \leq De$ for some constants $0 < \delta \leq D$. Then there exists an infinite rapid path with initial state $x_0(\omega)$. (ii) If the model is stationary and (**G.5**) holds, then there exists a balanced rapid path. (iii) If the constants $M_t$ in condition (**G.2**) do not depend on $t$ and assumption **(G.5)** holds, then the rapid paths in (i) and (iii) are asymptotically optimal.*

Assertion (iii) is immediate from Proposition 1. Statement (i) of the above theorem is proved in [4], where the existence of infinite rapid paths with the given initial state is established. The proof in [4] is conducted by passing to the limit from finite time horizons, for which the existence of rapid paths is obtained in [14]. The passage to the limit is based on a compactness principle involving Fatou's lemma in several dimensions (Schmeidler [30]).

Assertion (ii) follows from the results of papers [16, 17], where not only the existence of a rapid path is proved, but also it is shown that there exists a balanced rapid path supported by a dual trajectory with the following special structure:

$$p_1(s^1) = p(s^1),\ p_t(s^t) = \frac{p(s^t)}{\alpha(s^{t-1})...\alpha(s^1)},\ t = 2, 3, ..., \tag{12}$$

where $\alpha(s^1) > 0$ and $p(s^1) \geq 0$ are scalar and vector functions such that $E|p(s^1)| < \infty$ (*balanced dual trajectory*). The triplet of functions $\alpha(\cdot)$, $p(\cdot)$, $x(\cdot)$ involved in (5) and (12) is called a *von Neumann equilibrium.* It can be shown that if $\alpha(\cdot)$, $p(\cdot)$, $x(\cdot)$ is a von Neumann equilibrium, then the balanced trajectory defined by (5) maximizes $E \ln \alpha$ among all such trajectories. This means that (5) is a *von Neumann path.* The existence of a von Neumann equilibrium established in [16, 17] is a deep result solving a problem that remained open for more than three decades. It was posed by Eugene Dynkin in the early 1970s in connection with his pioneering studies on stochastic models of economic dynamics (e.g. [11, 12, 13]). In the former of the two papers [16, 17], a version of the existence theorem for a von Neumann equilibrium was obtained which dealt with an extended model defined in terms of *randomized paths*. In the latter paper, the final result was obtained by using the method of elimination of randomization (Dvoretzky, Wald and Wolfowitz [10]).

In the context of the dynamic securities market model, rapid paths may be regarded as analogues of *numeraire portfolios* (Long [25]). The price system $(p_t)$ involved in the definition of a rapid path is such that the value $p_{t+1}x_t$ of the portfolio $x_t$ is always equal to one, while for any other feasible sequence $(y_t)$ of contingent portfolios (feasible trading strategy), the values $p_{t+1}y_t$ form a supermartingale. We arrive at Long's definition of a numeraire portfolio in the classical case when transaction costs are absent by observing that in this case the vectors $p_t$ can be chosen collinear with the market price vectors: $p_t = \lambda_t q_t$, where $\lambda_t^{-1} = q_t x_t$ is the market value of the numeraire portfolio. (Long [25] considered a model with unlimited short selling, and in that context one can speak of martingales rather than supermartingales.)

To use Theorem 1 in specific models, one has to verify assumptions (**G.1**)–(**G.5**) (note that (**G.3**) is a consequence of (**G.5**) with $l = 1$). In Example 1, these conditions follow immediately from the assumption:

(**R**) The asset returns $R_t(s^t) := q_t^i(s^t)/q_{t-1}^i(s^{t-1})$ are uniformly bounded and uniformly bounded away from zero.

Under this condition, (**G.2**) holds with constants $M_t$ independent of $t$.

To obtain (**G.1**)–(**G.5**) in Example 3 it is sufficient to assume that the exchange rates $\mu_t^{ij}(s^t)$ are uniformly bounded away from zero and infinity. In both cases, (**G.2**) holds with constants $M_t$ independent of $t$. All the conditions needed can be obtained in Example 2 if hypothesis (**R**) holds and the following requirement regarding the transaction costs is fulfilled:

(**TC**) The random variables $\lambda_{t,i}^+(s^t)$ are uniformly bounded and the random variables $1 - \lambda_{t,i}^-(s^t) > 0$ are uniformly bounded away from zero.

# 5 Conclusion

This work deals with an important class of multivalued random dynamical systems originally discovered in mathematical economics—von Neumann–Gale dynamical systems. We show how they can be applied to the analysis of some fundamental issues in finance. This is a novel approach allowing us to establish a link with the classical von Neumann and Gale economic growth models, which makes it possible to use concepts, techniques and results from mathematical economics to obtain new theoretical results in finance. Even though one would think that models of economic dynamics are the "next of kin" to dynamic security market models, surprisingly they have not been analysed from this angle for quite a while, and interconnections between those two types of modeling frameworks have not been examined in as much detail as they deserve. In a previous paper [9], this approach was applied to questions of asset pricing and hedging under transaction costs and portfolio constraints. In the present study we show how it can be fruitfully employed to develop a general theory of capital growth under proportional transaction costs.

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