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On a foundation for Cournot equilibrium

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Abstract

We show in the context of a bilateral oligopoly where all agents are allowed to behave strategically the unexpected result that when the number of buyers becomes large the outcomes in a strategic market game do not converge to those at the Cournot equilibrium. However, convergence to Cournot outcomes is restored if the game is sequential: sellers move simultaneously as do buyers, but the former always move before the latter. This suggests that the ability to commit to supply decisions is an essential feature of Cournot equilibrium.

Keywords: Cournot competition, strategic market game, strategic foundation.
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1 Introduction

Cournot competition has become a widely used model of imperfect competition in modern economic theory. Its premise is that whilst firms behave strategically, buyers act as price takers. Just as in Walrasian models, this price-taking assumption requires a strategic foundation in order to be judged valid. A strategic market game with strategies as quantities is an appropriate framework within which to provide this foundation, in which we need to show that when the number of buyers increases outcomes in the market game tend to those at the Cournot equilibrium. When the sellers move at the same time as the buyers in the market game we show that this convergence does not generally occur. Conversely, in a sequential two-stage game in which sellers move simultaneously as do buyers, but the former move before the latter, convergence to Cournot outcomes is restored. This suggests that in order to provide the Cournot equilibrium concept with a strategic foundation, the sellers must have an opportunity to commit before the buyers make their choices, implying an essential feature of Cournot equilibrium is the ability of firms to commit to supply decisions.

The framework we use is that of bilateral oligopoly in which there are two commodities, the second thought of as money, and agents have a corner endowment. Those agents endowed with the first commodity are called *sellers*, whilst those endowed with the second commodity are called *buyers*. In a strategic market game signals are quantity-based: each agent decides on a proportion of her endowment to send to the market to be exchanged for the other commodity. For sellers we call this an *offer*, whilst buyers make *bids*. In a traditional strategic market game of the type developed by Shapley and Shubik [8] the signals of all agents are placed simultaneously and the rate of exchange of the second commodity for the first (the price) is determined by the ratio of the aggregate bid to offer. In a Cournot market the buyers treat the price as uninfluenced by their actions and the sellers play a quantity-setting game with knowledge of the buyers' choices.

The price-taking hypothesis embedded in the Cournot model should be an accurate representation of behaviour for the buyers when their number is many, for then each should have a negligible influence on the price. With this reasoning, we would expect that as the number of buyers increases the equilibrium outcomes in the strategic market game would converge to those at the Cournot equilibrium. In fact this is not generally the case, and we characterise when convergence does and does not occur. This extends the work of Codognato [2] who was the first to notice this phenomenon in the context of an example.

Busetto, Codognato and Ghosal [1] also analysed this issue, noting that it is the two-stage nature of Cournot competition that likely gives rise to non-convergence. Working in a continuum economy (with some atoms) they show equivalence between the (suitably refined) equilibria in a two-stage market game in which the atoms move first and the Cournot equilibria. Our analysis extends this work to the case of a finite economy where we can investigate precisely the implications when the number of agents is few, and also demonstrate the asymptotic properties as the number of agents increase, gaining intuition as to why the non-convergence emerges. This intuition suggests a two-stage strategic market game is required in order to restore convergence to Cournot outcomes.

In market games where there are two distinct trading stages and all agents of the same type move at the same time, we are able to construct strategic versions of supply and demand and, using these, compare equilibrium outcomes with those at the Cournot equilibrium. We show that as the number of buyers increases without bound the equilibrium outcomes in the game in which the sellers are leaders converge to those at the Cournot equilibrium, whilst when the buyers are leaders outcomes remain distinct even in the limit.

Thus, in order to provide a strategic foundation for Cournot competition we require those agents that are permitted to behave strategically in the Cournot market to move first whilst those that are assumed to be price takers move second. If play is simultaneous or the timing order is reversed, the limit remains distinct from the Cournot equilibrium. This suggests that the essence of a strategically-behaving agent in Cournot competition is their ability to commit to their choices before those agents that behave as price takers. In any market regime where they are not able to make such a commitment, the Cournot equilibrium does not transpire as the natural limit.

The rest of the paper continues as follows. We briefly analyse a strategic market game where moves are simultaneous using the methodology from Dickson and Hartley [3] that exploits the aggregative properties of the game played. This allows us to construct strategic versions of supply and demand, and show that equilibria correspond to their intersection, a technique that is repeated in all the games we analyse. We then characterise the Cournot equilibrium and compare this with the market game equilibrium when the number of buyers increases, showing that generally the limit remains distinct. Next we turn to analyse two-stage strategic market games in which all the sellers move first and all the buyers move second, and the reverse timing structure. This analysis allows us to show that when the sellers are leaders convergence to the Cournot equilibrium always occurs, whilst if the buyers are leaders the equilibrium remains distinct from the Cournot equilibrium even in the many-buyer limit. All proofs are contained in the Appendix.

2 The Economic Framework

We consider the pure exchange economy $\mathcal{E} = \{(e_h, u_h, \mathbb{R}^2_+) : h \in \mathcal{H}\}$. There are two commodities; the first is a standard consumption commodity, the second a commodity money. We partition the set of agents $\mathcal{H}$ into $\mathcal{H}^{\mathrm{S}} \cup \mathcal{H}^{\mathrm{B}}$, $\mathcal{H}^{\mathrm{S}} \cap \mathcal{H}^{\mathrm{B}} = \emptyset$. Those in $\mathcal{H}^{\mathrm{S}}$ are endowed only with the consumption commodity and are termed sellers, those in $\mathcal{H}^{\mathrm{B}}$ are endowed only with the commodity money and we call them buyers. Our ultimate intention is to have a fixed set of sellers and a variable set of buyers whose number may be increased through replication. For this reason we consider replica economies by defining $\mathcal{H}^{\mathrm{S}} = H^{\mathrm{S}} \times \{1, \ldots, m\}$ and $\mathcal{H}^{\mathrm{B}} = H^{\mathrm{B}} \times \{1, \ldots, n\}$, and denote by ${}^{m,n}\mathcal{E}$ the economy in which there are m replicas of each seller in H^{S} and n replicas of each buyer in H^{B}.

In this bilateral oligopoly model we essentially have a partial equilibrium environment where the second commodity can be thought of as representing money used to acquire all other goods in the economy. As the consumption commodity in such circumstances is likely to have a low share in overall expenditure, the most natural assumption is that there are negligible income effects. Indeed, Marshall comments: "When a person buys anything for his own consumption, he generally spends on it a small part of his total resources;...[In such a] case there is no appreciable change in his willingness to part with money." (Marshall [4], pp 335) Zero income effects may be captured by assuming agents have quasi-linear preferences and we make the assumption that buyers are endowed with such preferences: $u_h(x_1, x_2) = v_h(x_1) + x_2 \, \forall h \in \mathcal{H}^{\mathrm{B}}$. In addition, if the sellers are endowed with these preferences they may be thought of as profit-maximising firms in the standard sense, a natural interpretation in our environment[1].

Assumption (Quasi-linearity)**.** *For all $h \in \mathcal{H}$ preferences are representable by the quasi-linear utility function $v_h(x_1) + x_2$ where $v'_h(\cdot) \geq 0$ and $v''_h(\cdot) < 0$.*

3 The Strategic Market Game

In a strategic market game of the type introduced by Shapley and Shubik [8] there is a trading post to which each seller may take a proportion of her endowment of the consumption commodity $q \in [0, e_h]$, which we call an *offer*, to be exchanged for money. Likewise, each buyer may take along a proportion of her endowment of the commodity money $b \in [0, e_h]$, called a *bid*, to be exchanged for the consumption commodity. The trading post then aggregates the offers and bids. Throughout

[1]Whilst we assume quasi-linearity of preferences throughout, many of our results hold for more general utility functions.

we will take market clearing to be at the per-replica level[2]. As such, the rate of exchange of the commodity money for the consumption commodity is determined as

$$p = \frac{\frac{1}{n}\sum_{\mathcal{H}^{\mathrm{B}}} b_h}{\frac{1}{m}\sum_{\mathcal{H}^{\mathrm{S}}} q_h}.$$

Denote by Q the sum of offers of one replica of sellers and by B the sum of bids of one replica of buyers, then the price is simply $p = B/Q$. [If $BQ = 0$ the market is deemed closed and the final allocation is the initial allocation corresponding to the endowments of agents.] The distribution rule in the market game specifies the allocation to each agent is

$$(x_{h1}, x_{h2}) = \begin{cases} (e_h - q, qp) & \text{if } h \in \mathcal{H}^{\mathrm{S}} \text{ or} \\ \left(\frac{b}{p}, e_h - b\right) & \text{if } h \in \mathcal{H}^{\mathrm{B}}. \end{cases}$$

Payoffs are simply the utility forthcoming from this allocation. The equilibrium concept we use is (type-symmetric) Nash equilibrium in pure strategies: a set of strategies that are mutually consistent best responses.

Recently, Dickson and Hartley [3] have conducted an analysis that exploits the aggregative structure of the game in order to characterise the behaviour of buyers and sellers at the aggregate level consistent with Nash equilibrium, and to provide a characterisation of equilibrium. We refer the reader to the original paper for the details of this process, discussing it only briefly here.

By considering the 'partial game' played by each side of the market when the strategies of the other side remain fixed, we can characterise the behaviour of each replica of sellers using what we call *strategic supply*, denoted ${}^m\mathcal{X}_1^{\mathrm{S}}(p)$; for a given price strategic supply gives the sum of the offers of a replica of sellers consistent with a Nash equilibrium at that price. Likewise, the behaviour of each replica of buyers is characterised by *strategic demand*, denoted ${}^n\mathcal{X}_1^{\mathrm{B}}(p)$, that gives the level of demand (equal to the ratio of bid to price) consistent with a Nash equilibrium at a given price.

Strategic supply and demand have several desirable properties, summarised in the following two lemmata.

[2] In studies of many-agent limits market clearing is typically taken to be an aggregate phenomenon. In this case we increase only the number of buyers, so if market clearing was taken at the aggregate level prices would tend to infinity. Thus, an appropriate market clearing requirement is clearing at the average level or, more conveniently for us, at the per-replica level, where the demands of a single replica of buyers are satisfied by the supply of a single replica of sellers. Note that this is consistent with a mixed continuum economy where the sellers are atoms over which we use the counting measure, whilst the buyers are an atomless continuum over which we apply the Lesbegue measure.

Lemma 1. *Strategic supply ${}^m\mathcal{X}_1^{\mathrm{S}}(p)$ is defined for all prices exceeding some lower cutoff ${}^mP^{\mathrm{S}}$ where it is a function that is positive, continuous and non-decreasing in p. ${}^mP^{\mathrm{S}}$ is the price below which no agent would make a positive offer with this price, and is such that*

$$\sum_{H^{\mathrm{S}}} \max\left\{0, m\left(1 - \frac{v_h'(e_h)}{{}^mP^{\mathrm{S}}}\right)\right\} = 1.$$

Lemma 2. *Strategic demand ${}^n\mathcal{X}_1^{\mathrm{B}}(p)$ is defined for all $0 < p < {}^nP^{\mathrm{B}}$ where it is a function that is positive, continuous and strictly decreasing in p. ${}^nP^{\mathrm{B}}$ is the price above which all buyers have zero demand in an equilibrium with this price, and is such that*

$$\sum_{H^{\mathrm{B}}} \max\left\{0, n\left(1 - \frac{{}^nP^{\mathrm{B}}}{v_h'(0)}\right)\right\} = 1.$$

The purpose of constructing these functions is the following fundamental insight: non-autarkic Nash equilibria in the strategic market game are in one-to-one correspondence with intersections of strategic supply and demand.

Proposition 1. *There is a non-autarkic Nash equilibrium in the economy ${}^{m,n}\mathcal{E}$ in which the price is p if and only if*

$${}^m\mathcal{X}_1^{\mathrm{S}}(p) = {}^n\mathcal{X}_1^{\mathrm{B}}(p).$$

This equivalence then gives us a handle on when a non-autarkic Nash equilibrium will exist, and if so whether it is unique[3].

Theorem 1. *In any economy ${}^{m,n}\mathcal{E}$, if ${}^mP^{\mathrm{S}} \geq {}^nP^{\mathrm{B}}$ there is no non-autarkic Nash equilibrium. Conversely, if ${}^mP^{\mathrm{S}} < {}^nP^{\mathrm{B}}$ there is a single non-autarkic Nash equilibrium in which the price is ${}^{m,n}\hat{p}$ such that ${}^m\mathcal{X}_1^{\mathrm{S}}({}^{m,n}\hat{p}) = {}^n\mathcal{X}_1^{\mathrm{B}}({}^{m,n}\hat{p})$.*

Remark 1. *Quasi-linear preferences are sufficient to give us uniqueness. Under much weaker conditions we can get existence, and these are outlined in Dickson and Hartley (2005) [3].*

[3] In addition, there is always an autarkic Nash equilibrium in the simultaneously-played strategic market game. To see this, consider whether the strategies $(\mathbf{0}, \mathbf{0})$ are an equilibrium. When everyone makes a zero offer/bid payoffs are $v_h(e_h)\,\forall h \in \mathcal{H}^{\mathrm{S}}$ and $e_h\,\forall h \in \mathcal{H}^{\mathrm{B}}$. If any seller considers a unilateral deviation to $q > 0$ then her payoff will be $v_h(e_h - q)$ as there is no bid in the market. Clearly this is worse for her. Likewise, if any buyer considers a unilateral deviation to $b > 0$ her payoff will be $e_h - b < e_h$.

4 Cournot Oligopoly

In a Cournot oligopoly the buyers are assumed to behave as price-takers whilst the sellers are permitted to behave strategically in the knowledge of the buyers' behaviour. Each buyer can be thought of as choosing a level of b given that her allocation will be $(x_{h1}, x_{h2}) = (b/p, e_h - b)$ and the fact that she treats the price as a parameter. This corresponds to a standard competitive maximisation problem resulting from which will be a competitive demand schedule for each replica of buyers. The sellers are assumed to know demand, and so know that if their per-replica supply is Q the price will be such that this supply is matched with demand. We denote this price $\tilde{p}(Q)$. Then the payoff to each seller in the ensuing non-cooperatively played quantity-setting game is[4]

$$v_h(e_h - q) + q\tilde{p}\left(\frac{1}{m}((m-1)Q + Q_{-h} + q)\right).$$

Again we consider (type-symmetric) pure strategy Nash equilibria.

Formally, each buyer wishes to maximise her utility from consumption at each price taking the price as given and uninfluenced by her choice of action. In this bilateral oligopoly setting she may be seen as choosing the amount of money b she is willing to forego at each price in exchange for b/p units of the consumption commodity. As such, competitive demand for each replica of buyers will take the form

$$\tilde{\mathcal{X}}_1^{\mathrm{B}}(p) = \sum_{H^{\mathrm{B}}} \frac{\tilde{\mathfrak{b}}_h(p)}{p}$$

where $\tilde{\mathfrak{b}}_h(p)$ is the solution to $\max_{b \in [0, e_h]} v_h(b/p) + e_h - b$.

Competitive demand has several desirable properties.

Lemma 3. *Competitive demand $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ is a continuous function that is zero for all $p \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ whilst for $0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ it is positive and strictly decreasing in the price.*

In equilibrium supply must equate to demand at the per-replica level. If the supply from each replica of sellers as a result of their quantity-setting game is Q this requires the price to be such that the level of demand at that price is the same as this supply. Thus, the price will be of the form $\tilde{p}(Q)$ which is defined such that

$$\tilde{p}(Q) = \{p : Q = \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\}.$$

This price functional has several desirable properties, easily discerned from the preceding lemma.

[4]Each seller sees the per-replica offer as being one mth of the aggregate offer. Thus, $Q = \frac{1}{m}((m-1)Q + Q_{-h} + q)$. In particular, the marginal effect of an individual seller is $\frac{1}{m}\tilde{p}'(Q)$, diminishing as the number of sellers increases.

Lemma 4. *$\tilde{p}(Q)$ is a function that is strictly decreasing in Q with the property that* $\lim_{Q\to 0}\tilde{p}(Q) = \max_{H^{\mathrm{B}}}\{v'_h(0)\}$.

Now, as the sellers are assumed to know the buyers' behaviour before they play their quantity-setting game, they know that if the per-replica offer is Q then the price will be $\tilde{p}(Q)$. As such if seller $h \in \mathcal{H}^{\mathrm{S}}$ uses the offer q her payoff is

$$v_h(e_h - q) + q\tilde{p}\left(\frac{1}{m}((m-1)Q + Q_{-h} + q)\right).$$

A (type-symmetric) Cournot equilibrium is then a set of strategies that are mutual best responses.

The existence of Cournot equilibrium was a prominent research question in the late '70s and early '80s culminating in a body of literature that imposed ever-weaker restrictions on demand and/or cost functions sufficient for the existence of a Cournot equilibrium (see, for example, Novshek [6], Roberts and Sonnenschein [7], Szidarovszky and Yakowicz [9], or Vives [10] for a review). For expositary convenience, but at the expense of generality, we shall make assumptions sufficient to guarantee a *unique* Cournot equilibrium, and these correspond to decreasing average revenue: $\tilde{p}'(Q) + Q\tilde{p}''(Q) \leq 0\,\forall Q > 0$. This condition is satisfied when, for example, inverse demand is a linear function of quantity which can be generated, at least in a homogeneous economy, when buyers' preferences are quadratic i.e. of the form $\alpha_{\mathrm{B}}x_1 - \frac{\gamma_{\mathrm{B}}}{2}x_1^2 + x_2$.

We can characterise the behaviour of each replica of sellers by constructing strategic supply, in this case denoted ${}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ which, for each price, gives the supply of each replica of sellers consistent with a Nash equilibrium at that price (given the behaviour of the buyers). We summarise the properties of strategic supply in the following lemma.

Lemma 5. *Suppose $\tilde{p}'(Q) + Q\tilde{p}''(Q) \leq 0\,\forall Q > 0$. Then strategic supply ${}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ is defined for all $p > \min_{H^{\mathrm{S}}}\{v'_h(e_h)\}$ where it is a function that is positive, continuous and non-decreasing in p.*

Having characterised the (competitive) behaviour of the buyers and the strategic behaviour of the sellers, we are now in a position to discuss the identification of Nash equilibria in the Cournot game. It transpires that there will be a non-autarkic Nash equilibrium if and only if the strategic supply of the sellers intersects with the competitive demand of the buyers.

Proposition 2. *Suppose $\tilde{p}'(Q)+Q\tilde{p}''(Q) \leq 0\,\forall Q > 0$. Then there is a non-autarkic Nash equilibrium in the economy ${}^{m,n}\mathcal{E}$ with price p if and only if*

$${}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(p) = \tilde{\mathcal{X}}_1^{\mathrm{B}}(p).$$

This equivalence between intersections of strategic supply in the Cournot market and competitive demand and non-autarkic Nash equilibria then allows us to determine exactly when a non-autarkic Nash equilibrium will exist, and if so whether it is unique.

Theorem 2. *Suppose* $\tilde{p}'(Q) + Q\tilde{p}''(Q) \leq 0 \, \forall Q > 0$. *Then in any economy* ${}^{m,n}\mathcal{E}$, *if* $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *there is no non-autarkic Nash equilibrium; the only equilibrium is autarky. Conversely, if* $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *there is a single non-autarkic Nash equilibrium (and no autarkic equilibrium) in which the price is* ${}^{m}\hat{p}^{\mathrm{C}}$ *such that* ${}^{m}\tilde{\mathcal{X}}_1^{\mathrm{S}}({}^{m}\hat{p}^{\mathrm{C}}) = \tilde{\mathcal{X}}_1^{\mathrm{B}}({}^{m}\hat{p}^{\mathrm{C}})$.

Notice that the Cournot equilibrium, if it exists, is independent of the number of buyers.

5 Non-Convergence

The aim of this paper is to provide a strategic foundation for Cournot competition. In a Cournot oligopoly the buyers are assumed to behave as price takers, and we expect this model to be valid when the number of buyers is many and each has no market power with which she can manipulate the market price. Such an equilibrium concept uses price-taking assumptions and implicit in the market clearing requirement is the notion of a Walrasian Auctioneer. In order to provide the equilibrium concept with a strategic foundation we would like to show that the price-taking assumption is justified; that is, price-taking behaviour is the natural limit of strategic behaviour in the market game as the number of buyers increases without bound.

In order to address this question we fix the set of sellers (setting $m = 1$ and dropping the ^{m} notation) and consider a sequence of economies $\{{}^{n}\mathcal{E}\}_{n=1}^{\infty}$. We need to show that any sequence of strategic market game equilibria converge to a Cournot equilibrium in the many-buyer limit, and that for any Cournot equilibrium there is a sequence of strategic market game equilibria that approach it in the limit. As we have shown the Cournot equilibrium is unique (and independent of the number of buyers), and indeed that there is only a single strategic market game equilibrium in any economy, it will suffice to show that the sequence of strategic market game equilibria converge to the Cournot equilibrium as the number of buyers increases without bound. We focus on the case in which the Cournot equilibrium is non-autarkic, i.e. $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$.

The allocation structure in both the strategic market game and the Cournot market take the same form; namely

$$(x_{h1}, x_{h2}) = \begin{cases} (e_h - q, qp) & \text{if } h \in \mathcal{H}^{\mathrm{S}} \text{ or} \\ \left(\frac{b}{p}, e_h - b\right) & \text{if } h \in \mathcal{H}^{\mathrm{B}}. \end{cases}$$

Therefore, to demonstrate convergence of equilibrium outcomes it will suffice to demonstrate convergence of the equilibrium price and of individual bids and offers, for then convergence of allocations will follow.

We first show that as the number of buyers increases their strategic demand in the strategic market game converges to their competitive demand for all prices.

Lemma 6. *Strategic demand in the simultaneously-played strategic market game converges to competitive demand as the number of buyers increases without bound, i.e.*

$$^{n}\mathcal{X}_1^{\mathrm{B}}(p) \rightarrow_{n\rightarrow\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\, \forall 0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}.$$

This result implies that at any given price the behaviour of the buyers in the market game gets closer to that were they to behave as price takers as their number increases.

The equilibrium in the strategic market game is identified by the intersection of strategic supply $\mathcal{X}_1^{\mathrm{S}}(p)$ and strategic demand $^{n}\mathcal{X}_1^{\mathrm{B}}(p)$. According to the above lemma, in the many-buyer limit this intersection corresponds to the intersection of strategic supply and competitive demand $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$. The Cournot equilibrium on the other hand is identified by the intersection of strategic supply in the Cournot market $\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ and competitive demand.

As such, the intersection point (and therefore the equilibrium price and quantity of the consumption commodity traded) in the strategic market game will converge to that at the Cournot equilibrium if and only if the strategic supply in the strategic market game is the same as that in the Cournot market, at least in a neighborhood of the Cournot equilibrium.

With this reasoning in mind, the key to our convergence (indeed non-convergence) argument lies in determining exactly when the two strategic supplies (in the strategic market game and in the Cournot oligopoly) are equal, and the following lemma addresses this point.

Lemma 7. *Suppose* $\tilde{p}'(Q) + Q\tilde{p}''(Q) \leq 0\, \forall Q > 0$. *Then*

$$\mathcal{X}_1^{\mathrm{S}}(p) \gtreqless \tilde{\mathcal{X}}_1^{\mathrm{S}}(p) \Leftrightarrow \eta(\tilde{\mathcal{X}}_1^{\mathrm{S}}(p), p) \lesseqgtr 1$$

where $\eta(Q,p) = \left|\frac{p}{Q}\frac{1}{\tilde{p}'(Q)}\right|$.

The function $\eta(Q,p)$ is a measure of the elasticity of competitive demand. [In particular, $\eta(\tilde{\mathcal{X}}_1^{\mathrm{B}}(p), p)$ gives the elasticity of competitive demand when the price is p.] Thus, at any given price strategic supply in the strategic market game will exceed that in the Cournot market if, were there to be an equilibrium at that price (i.e. $\tilde{\mathcal{X}}_1^{\mathrm{S}}(p) = \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$), equilibrium demand would be inelastic. If it is elastic the strategic market game strategic supply will be less than that in the Cournot market. Only when the elasticity is unity will the two be equal. In particular, at the Cournot equilibrium in which the price is $\hat{p}^{\mathrm{C}}$ we have that

$$\mathcal{X}_1^{\mathrm{S}}(\hat{p}^{\mathrm{C}}) \gtreqless \tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p}^{\mathrm{C}}) \Leftrightarrow \eta(\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p}^{\mathrm{C}}), \hat{p}^{\mathrm{C}}) \lesseqgtr 1 :$$

in a neighborhood of the intersection between strategic supply in the Cournot oligopoly and competitive demand, the strategic supply in the strategic market game will be equal to the Cournot strategic supply if and only if the elasticity of competitive demand at the Cournot equilibrium is unity.

As such, the price and aggregate quantity of the consumption commodity traded at the limit strategic market game equilibrium will coincide with those at the Cournot equilibrium if and only if the elasticity of competitive demand at the Cournot equilibrium is unity.

Proposition 3. *Suppose $\tilde{p}'(Q) + Q\tilde{p}''(Q) \leq 0$. Then the price and aggregate amount of the consumption commodity traded in the strategic market game equilibrium converge to those at the Cournot equilibrium as the number of buyers increases without bound if and only if the elasticity of competitive demand at the Cournot equilibrium is unity.*

It only remains to show that in the limit the individual bids and offers are the same in the strategic market game as in the Cournot market if we achieve convergence in the equilibrium price and quantity of the consumption commodity traded (i.e. if the elasticity is one). For the sellers this follows immediately from the realisation that the elasticity must be one if we achieve convergence, which implies shares and, therefore, offers must be the same in the limit (see the proof of Lemma 7). For the buyers, we revealed in the proof of Lemma 6 that for each buyer her bid at any B and p in the strategic market game will converge to her competitive bid for each price, so convergence in the price implies convergence in individual bids.

If competitive demand at the Cournot equilibrium is inelastic, the price in the strategic market game in the limit will be lower and the quantity traded higher than those at the Cournot equilibrium. Conversely, if it is elastic, the price in the limit strategic market game will be higher and the quantity traded lower. In each of these cases the limit outcomes will remain distinct from the Cournot outcome.

To summarise, when the demand elasticity is unity we find that the sequence of equilibrium outcomes in the strategic market game converge to the Cournot equilibrium outcome as the number of buyers increases without bound. Conversely, when competitive demand is not unit elastic at the Cournot oligopoly equilibrium convergence will not (generically at least) occur.

Finally, notice that even though the Cournot equilibrium is non-autarkic (by presumption) it may be the case that the only strategic market game equilibrium is autarky even in the limit. Indeed, since[5] $P^{\mathrm{S}} > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$, even if $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ it is not ruled out that $P^{\mathrm{S}} \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ (so the only limit strategic market game equilibrium is autarky). Moreover, there is also always an autarkic equilibrium in the strategic market game even if a non-autarkic equilibrium exists, and this autarkic equilibrium never converges to the (non-autarkic) Cournot equilibrium.

Our analysis generalises and extends the work of Codognato [2], who showed that convergence occurs in the case of Cobb-Douglas preferences (in which case demand is everywhere unit elastic) whereas when the buyers have preferences quasi-linear in the second commodity the equilibrium remains distinct even in the limit.

6 Two-Stage Strategic Market Games

The fact that Nash equilibria in the strategic market game do not converge to the Cournot equilibrium seems rather paradoxical: one would expect that as the market power of one side of the market diminishes the equilibrium would tend to that which assumes they are price takers. But this only occurs in the special case where the elasticity of competitive demand at the Cournot equilibrium is unity. In any other case the sequence of equilibria remains distinct from the Cournot equilibrium even in the limit. This suggests that the simultaneous-move strategic market game is not an appropriate framework in which to provide a strategic foundation for Cournot equilibrium.

The case in which the competitive demand is unit elastic and convergence is achieved is intriguing. This singular case of convergence presents itself for the following reason: the elasticity of 'demand' in the strategic market game is always unity as well. To see this note that the price is $p = B/Q$, so that $\mathrm{d}p/\mathrm{d}Q = -B/Q^2$ and then we find that the elasticity is

$$\left| \frac{p}{Q} \frac{1}{\frac{\mathrm{d}p}{\mathrm{d}Q}} \right| = \frac{B}{Q^2} \frac{Q^2}{B} = 1.$$

Non-convergence emerges because in the Cournot market the sellers have information about the buyers' choices before they make their quantity decisions whereas in the strategic market game no such *ex ante* inference is possible: sellers make conjectures about demand but these are made simultaneous to their quantity decisions. This is a fundamental difference in the information the sellers have.

[5]To see this, note that $1 = \sum_{H^{\mathrm{S}}} \max\left\{0, 1 - \frac{v_h'(e_h)}{P^{\mathrm{S}}}\right\} \leq |H^{\mathrm{S}}|\left(1 - \frac{\min_{H^{\mathrm{S}}}\{v_h'(e_h)\}}{P^{\mathrm{S}}}\right)$ and this implies $P^{\mathrm{S}} \geq \frac{|H^{\mathrm{S}}|}{|H^{\mathrm{S}}|-1} \min_{H^{\mathrm{S}}}\{v_h'(e_h)\} > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$.

Then even as the number of buyers increases without bound the sellers' behaviour remains inherently different to that were they to infer demand prior to making their decisions, even though the buyers are behaving as if they are price takers in the strategic market game. But in the case where the elasticity of competitive demand is unity the detail about the buyers' demand available to the sellers in the Cournot market gives no more information than is available (by inference) in the strategic market game (as the demand has the same 'structure').

This suggests that in order to align the strategic market game with the Cournot equilibrium in the limit, we need for the sellers to infer, before they play their quantity setting game, the decisions of the buyers. In other words, we need the sellers to commit to their quantity choices before the buyers reveal their demand. One way of doing this is via a market game that has two distinct trading stages in which all the sellers move at the first stage whilst all the buyers move at the second stage.

We now turn to an analysis of two-stage strategic market games in which the order of moves is exogenously specified: we focus on the cases where all agents of the same 'type' move at the same time, beginning with the game in which the sellers move first and the buyers move second, then moving on to an analysis of the reverse timing structure.

6.1 Sellers as Leaders

There are two trading stages. At the first stage the sellers approach the trading post; each seller may offer a proportion of her endowment $q \in [0, e_h]$ to be exchanged for money. At the second stage, having observed the sellers' actions, each buyer then approaches the trading post and offers a proportion of her endowment of money $b \in [0, e_h]$ to be exchanged for the consumption commodity. The rate of exchange of the commodity money for the consumption commodity is determined as $p = B/Q$ in order to achieve market clearing at the per-replica level and allocations are determined in the usual way so that the payoff to each seller $h \in \mathcal{H}^{\mathrm{S}}$ is $v_h(e_h - q) + qp$ whilst that to each buyer $h \in \mathcal{H}^{\mathrm{B}}$ is $v_h(b/p) + e_h - b$. This dynamic game of complete but imperfect information is well defined, as the set of players, their available strategies, their payoffs and the order of moves are all specified.

The equilibrium concept we use is subgame-perfect Nash equilibrium (henceforth SPNE). To identify such an equilibrium we need to fix the set of offers in the first stage (so we specify the subgame) and then compute the optimal actions of the buyers when the offers take said values, and repeat for all possible offer combinations. We then use the optimal (re)actions of the buyers in the sellers' payoff functions (as they infer these actions) and determine a set of mutually consistent best responses from the sellers given the reactions of the buyers.

Let us fix the offers from the sellers and consider the second-stage game played by the buyers. If the offers of each replica of sellers total Q then each buyer will seek to solve the problem

$$\max_{b\in[0,e_h]} v_h\left(\frac{b}{\frac{1}{n}((n-1)B+B_{-h}+b)}Q\right)+e_h-b.$$

This corresponds to the maximisation problem solved in the simultaneous-move game. Our analysis there suggested that in any 'partial game' in which the per-replica offer was fixed there would be a unique set of strategies consistent with equilibrium (as we are able to show that share correspondences, and strategic supply correspondences, were functions).

The nature of this game may formally give us multiplicity of equilibria in the second stage. Although the payoffs to the buyers depend only on the aggregate offer, we have to take into account that each buyer observes the individual offers of the sellers, i.e. the microstructure of the aggregate offer. Then, even though the aggregate offer may remain the same, a change in the microstructure of Q can elicit changes in bids by the buyers. For example, suppose there are two sellers; the first offers 3 whilst the other offers 1, and the response of the first and second buyer is 4 and 2 respectively. It may be the case, if buyer one thinks buyer two cares about the microstructure of Q, that if the offers change to 1 from the first seller and 3 from the second, the equilibrium bids of the buyers may not remain at 4 and 2. Beliefs about other players' behavior can cause a multiplicity of equilibria. We believe, however, that this is unrealistic behavior. Indeed, when there is a 'summary statistic', such as the aggregate offer, the buyers would focus on this and ignore the microstructure. Thus, in order to circumvent the complications of (unrealistic) multiple second-stage solutions we presume that in any subgame with the same per-replica offer, buyers adopt the same (equilibrium) strategies. Busetto, Codognato and Ghosal [1] address this problem by considering the Markov perfection refinement (a concept originally introduced by Maskin and Tirole [5]).

Under this presumption, the buyers' behavior at the per-replica level can be characterized by their consistent bids in response to the per-replica offer, or, indeed, by their strategic demand function ${}^n\mathcal{X}_1^{\mathrm{B}}(p)$.

The sellers, moving at the first stage infer that, if their collective actions result in a per-replica offer of Q then the price will be such that this supply equates to the demand forthcoming at such a price. Thus, the price will take the form ${}^n\dot{p}(Q)$ which is such that

$${}^n\dot{p}(Q)=\{p:Q={}^n\mathcal{X}_1^{\mathrm{B}}(p)\}.$$

From Lemma 2 we know that strategic demand is a function that is continuous and strictly decreasing in the price. This implies that for any Q there will be a single market clearing price, i.e. ${}^n\dot{p}(Q)$ is a function. This is summarised in the following lemma. The proof parallels that of Lemma 4 and so is omitted.

Lemma 8. *${}^n\dot{p}(Q)$ is a function that is continuous and strictly decreasing in Q with the property that* $\lim_{Q\to 0} {}^n\dot{p}(Q) = {}^n P^{\mathrm{B}}$.

Sellers infer that this is the price they face and so the payoff to each seller moving at the first stage is

$$v_h(e_h - q) + q^n\dot{p}\left(\frac{1}{m}((m-1)Q + Q_{-h} + q)\right),$$

and each will choose her level of q to maximise this payoff given the offers of the other sellers in her replica totalling Q_{-h}. In order to ensure this program is concave we require some conditions on the price functional. In particular, sufficient to ensure concavity of the objective is to have $2^n\dot{p}'(Q) + Q^n\dot{p}''(Q) \leq 0\, \forall Q > 0$. A more useful condition, and one we use in the sequel, is ${}^n\dot{p}'(Q) + Q^n\dot{p}''(Q) \leq 0\, \forall Q > 0$ which, since $\dot{p}'(Q) < 0$ implies $2^n\dot{p}'(Q) + Q^n\dot{p}''(Q) \leq 0$. This condition places restrictions on the buyers' preferences in much the same way that the analogous condition used in a Cournot oligopoly does.

When ${}^n\dot{p}'(Q) + Q^n\dot{p}''(Q) \leq 0\, \forall Q > 0$ we can consider the best response of each agent as the solution to the first order condition of the above problem. Simple calculations show that the best response of a typical seller takes the form

$$^{m,n}\dot{\mathrm{BR}}^{\mathrm{S}}_h(Q_{-h}) = \begin{cases} 0 & \text{if } {}^n\dot{p}(Q_{-h}) \leq v_h'(e_h) \text{ or} \\ \min\{^{m,n}\dot{\mathrm{br}}^{\mathrm{S}}_h(Q_{-h}), e_h\} & \text{if } {}^n\dot{p}(Q_{-h}) > v_h'(e_h) \end{cases}$$

where

$$^{m,n}\dot{\mathrm{br}}^{\mathrm{S}}_h(Q_{-h}) = \Bigg\{q : v_h'(e_h - q) = {}^n\dot{p}\left(\frac{1}{m}((m-1)Q + Q_{-h} + q)\right) + \\ q\frac{1}{m}{}^n\dot{p}'\left(\frac{1}{m}((m-1)Q + Q_{-h} + q)\right)\Bigg\}.$$

Instead of analysing this best response directly, we look for offers consistent with a SPNE in which the per-replica offer is Q and the price is p. By replacing Q_{-h} with $Q - q$ and ${}^n\dot{p}(\cdot)$ with p we find the replacement correspondence, and by dividing elements in the replacement correspondence by Q we find the share correspondence of each seller in this two-stage market game. This takes the form

$$^{m,n}\dot{S}^{\mathrm{S}}_h(Q, p) = \begin{cases} 0 & \text{if } p \leq v_h'(e_h) \text{ or} \\ \min\left\{^{m,n}\dot{s}^{\mathrm{S}}_h(Q, p), \frac{e_h}{Q}\right\} & \text{if } p > v_h'(e_h) \end{cases}$$

where

$$^{m,n}\dot{s}^{\mathrm{S}}_h(Q, p) = \left\{s : v_h'(e_h - sQ) = p + sQ\frac{1}{m}{}^n\dot{p}'(Q)\right\}.$$

When multiplied by Q, elements in the share correspondence give the offers of seller h consistent with a Nash equilibrium in which the per-replica offer is Q and the price is p.

At any given price we then look for the consistent per-replica offers; those that generate individual offers that sum to the per-replica offer, or where the sum of the share values of one replica of sellers is equal to one. We denote by ${}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(p)$ the correspondence that contains such offers:

$$ {}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(p) = \{Q : \sum_{H^{\mathrm{S}}} {}^{m,n}\dot{S}_h^{\mathrm{S}}(Q,p) = 1\}, $$

and summarise the properties of this strategic supply in the following lemma.

Lemma 9. *Suppose* ${}^{n}\dot{p}'(Q)+Q^{n}\dot{p}''(Q) \leq 0\,\forall Q > 0$. *Then strategic supply* ${}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(p)$ *is defined for all* $p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ *where it is a function that is positive, continuous and non-decreasing in* p.

We can then use strategic supply at the first stage and strategic demand at the second stage to identify SPNE in this two-stage game in which the sellers are leaders.

Proposition 4. *Suppose* ${}^{n}\dot{p}'(Q) + Q^{n}\dot{p}''(Q) \leq 0\,\forall Q > 0$. *Then there is a non-autarkic SPNE in the two-stage game where the sellers move first and the buyers move second in which the price is* p *if and only if*

$$ {}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(p) = {}^{n}\mathcal{X}_1^{\mathrm{B}}(p). $$

This equivalence between strategic supply and demand then gives us a handle on when a non-autarkic SPNE will exist, and whether it will be unique.

Theorem 3. *Suppose* ${}^{n}\dot{p}'(Q) + Q^{n}\dot{p}''(Q) \leq 0\,\forall Q > 0$. *Then in the economy* ${}^{m,n}\mathcal{E}$, *if* $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} \geq {}^{n}P^{\mathrm{B}}$ *there is no non-autarkic SPNE; only autarky is an equilibrium. Conversely, if* $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < {}^{n}P^{\mathrm{B}}$ *there is a unique non-autarkic SPNE (and no autarkic equilibrium) in which the price is* ${}^{m,n}\hat{p}^{\mathrm{SB}}$ *such that* ${}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}({}^{m,n}\hat{p}^{\mathrm{SB}}) = {}^{n}\mathcal{X}_1^{\mathrm{B}}({}^{m,n}\hat{p}^{\mathrm{SB}})$.

6.2 Buyers as Leaders

We now address the model with the reverse timing structure; where the buyers move at the first stage and the sellers move second. The analysis is analogous to that performed above, so details are kept brief and all proofs, which parallel those of the previous analysis, are omitted.

At the first stage the buyers approach the trading post and make their bid $b \in [0, e_h]$. Having observed the buyers' bids the sellers then move at the second stage making their offers $q \in [0, e_h]$. Bids are aggregated to B and offers to Q at the per-replica level. The price and allocations are then determined in the usual way.

Let us fix the set of bids from the first stage and consider the second-stage game played by the sellers. If the bids of each replica of buyers total B then each seller will solve the problem

$$\max_{q \in [0,e_h]} v_h(e_h - q) + \frac{q}{\frac{1}{m}((m-1)Q + Q_{-h} + q)} B.$$

This is precisely the same as that in the simultaneous-move game, and we know that in any subgame there will be a unique equilibrium among the sellers and if the per-replica bid in any two subgames is the same then the equilibrium in each of these will be the same. The sellers' optimal behaviour can be represented at the per-replica level by their strategic supply ${}^m\mathcal{X}_1^{\mathrm{S}}(p)$. Thus, the buyers infer that if their per-replica bid is B then in order to clear the market the price must be such that $B/p = {}^m\mathcal{X}_1^{\mathrm{S}}(p)$. We collect such prices in the correspondence ${}^m\ddot{p}(B)$, and summarise its properties in the following lemma.

Lemma 10. *${}^m\ddot{p}(B)$ is a function that is continuous and strictly increasing in B with the property that ${}^m\ddot{p}(B) \to_{B\to 0} {}^mP^{\mathrm{S}}$.*

Then we can write the payoff to a typical buyer as

$$v_h\left(\frac{b}{{}^m\ddot{p}(\frac{1}{n}((n-1)B + B_{-h} + b))}\right) + e_h - b$$

which she will maximise over her choice of bid. Sufficient for this problem to be concave is to have $2{}^m\ddot{p}'(B) - B{}^m\ddot{p}''(B) \leq 0$, in which case we can write the best response as

$${}^{m,n}\ddot{\mathrm{BR}}_h^{\mathrm{B}}(B_{-h}) = \begin{cases} 0 & \text{if } {}^m\ddot{p}(B_{-h}) \geq v_h'(0) \text{ or} \\ \min\{{}^{m,n}\ddot{\mathrm{br}}_h^{\mathrm{B}}(B_{-h}), e_h\} & \text{if } {}^m\ddot{p}(B_{-h}) < v_h'(0) \end{cases}$$

where

$${}^{m,n}\ddot{\mathrm{br}}_h^{\mathrm{B}}(B_{-h}) = \Bigg\{ b : v_h'\left(\frac{b}{{}^m\ddot{p}(\frac{1}{n}((n-1)B + B_{-h} + b))}\right) + \frac{{}^m\ddot{p}(\frac{1}{n}((n-1)B + B_{-h} + b))^2}{{}^m\ddot{p}(\frac{1}{n}((n-1)B + B_{-h} + b)) - b\frac{1}{n}{}^m\ddot{p}'(\frac{1}{n}((n-1)B + B_{-h} + b))} \Bigg\}.$$

We construct the share correspondence of each buyer as

$${}^{m,n}\ddot{S}_h^{\mathrm{B}}(B,p) = \begin{cases} 0 & \text{if } p \geq v_h'(0) \text{ or} \\ \min\left\{{}^{m,n}\ddot{s}_h^{\mathrm{B}}(B,p), \frac{e_h}{B}\right\} & \text{if } p < v_h'(0) \end{cases}$$

where

$$ {}^{m,n}\ddot{s}_h^{\mathrm{B}}(B,p) = \left\{ s : v_h'\left(\frac{sB}{p}\right) = \frac{p^2}{p - sB\frac{1}{n}{}^m\ddot{p}'(B)} \right\}. $$

When multiplied by B this share correspondence gives the bids of buyer h consistent with a SPNE in which the aggregate bid of each replica of buyers is B and the price is p. In order to find consistent per-replica bids we seek, for each price, those bids which are such that the sum of the share correspondences of one replica of buyers is equal to one, and we divide the resulting per-replica bid by the price to determine strategic demand which we denote by ${}^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p)$:

$$ {}^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p) = \left\{ \frac{B}{p} : \sum_{H^{\mathrm{B}}} {}^{m,n}\ddot{S}_h^{\mathrm{B}}(B,p) = 1 \right\}. $$

Lemma 11. *Suppose*[6] $2{}^m\ddot{p}'(B) - B{}^m\ddot{p}''(B) \leq 0\,\forall B > 0$. *Then strategic demand* ${}^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p)$ *is defined for all* $0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *where it is a function that is positive, continuous and strictly decreasing in* p.

We can then show that there is a SPNE in the two-stage game in which the buyers move first if and only if the strategic demand of the first-stage buyers is equal to the strategic supply of the second-stage sellers.

Proposition 5. *Suppose* $2{}^m\ddot{p}'(B) - B{}^m\ddot{p}''(B) \leq 0\,\forall B > 0$. *Then there is a non-autarkic SPNE in the two-stage game where the buyers move first and the sellers move second in which the price is* p *if and only if*

$$ {}^m\mathcal{X}_1^{\mathrm{S}}(p) = {}^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p). $$

Again we use this equivalence between intersections of the appropriate strategic supply and demand functions and equilibria to determine exactly when a non-autarkic SPNE will exist.

Theorem 4. *Suppose* $2{}^m\ddot{p}'(B) - B{}^m\ddot{p}''(B) \leq 0\,\forall B > 0$. *Then in the economy* ${}^{m,n}\mathcal{E}$, *if* ${}^mP^{\mathrm{S}} \geq \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *there is no non-autarkic SPNE; the only equilibrium is autarky. Conversely, if* ${}^mP^{\mathrm{S}} < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ *there is a unique non-autarkic SPNE (and no autarkic equilibrium) in which the price is* ${}^{m,n}\hat{p}^{\mathrm{BS}}$ *such that* ${}^m\mathcal{X}_1^{\mathrm{S}}({}^{m,n}\hat{p}^{\mathrm{BS}}) = {}^{m,n}\ddot{\mathcal{X}}_1^{\mathrm{B}}({}^{m,n}\hat{p}^{\mathrm{BS}})$.

[6] We actually need two conditions, the one stated and ${}^m\ddot{p}'(B) + B{}^m\ddot{p}''(B) \geq 0$. However, it can be checked that $2{}^m\ddot{p}'(B) - B{}^m\ddot{p}''(B) \leq 0 \Rightarrow {}^m\ddot{p}'(B) - B{}^m\ddot{p}''(B) \leq 0 \Rightarrow {}^m\ddot{p}'(B) + B{}^m\ddot{p}''(B) \geq 0$.

7 Redressing Non-Convergence

Our previous analysis highlighted the fact that when we increased the number of buyers without bound in the simultaneous-move strategic market game the sequence of outcomes remained distinct from that at the Cournot equilibrium even in the limit. Intuition suggests that it may be the informational aspects of Cournot competition that prevent convergence, and this in turn suggests that a two-stage strategic market game will be more suited to providing a strategic foundation for Cournot competition. In this section we show that when we increase the number of buyers in a two-stage game in which the sellers move first and the buyers move second the sequence of outcomes associated with the sequence of SPNE do indeed converge to the outcome at the Cournot equilibrium in the limit. Conversely, when the timing structure is reversed, so the buyers move first and the sellers move second, we get a non-convergence result that parallels that of the simultaneous-move game. Thus, in order to achieve Cournot outcomes in the limit the trading regime must give the sellers the opportunity to commit to their supply decisions before the buyers make their choices.

Let us first look at the two-stage game in which the sellers are leaders. We fix the number of sellers (and drop the ^{m} notation) and consider the sequence of economies $\{{}^n\mathcal{E}\}_{n=1}^{\infty}$ as the number of buyers increases without bound. Recall that in the Cournot oligopoly the equilibrium is identified by the intersection of strategic supply in the Cournot market $\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ and competitive demand $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$. In the two-stage strategic market game in which the sellers are leaders the equilibrium is identified by the intersection of strategic supply derived for this game ${}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p)$ and strategic demand ${}^n\mathcal{X}_1^{\mathrm{B}}(p)$. Also recall from Lemma 6 that as $n \to \infty$ strategic demand converges to competitive demand: ${}^n\mathcal{X}_1^{\mathrm{B}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$. Thus, to ensure that the equilibrium price and quantity of the consumption commodity traded in the two-stage market game with the sellers as leaders converge to those at the Cournot equilibrium we need only make sure that strategic supply in the market game converges to that in the Cournot oligopoly as the number of buyers increases without bound, and we demonstrate this in the following lemma.

Lemma 12. *Suppose ${}^n\dot{p}'(Q) + Q^n\dot{p}''(Q) \leq 0\,\forall Q > 0$, $\forall n$. Then strategic supply in the two-stage market game in which the sellers are leaders converges to strategic supply in the Cournot market as the number of buyers increases without bound, i.e.*

$$ {}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{S}}(p)\,\forall p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}. $$

As such, in the game in which the sellers are leaders, not only does strategic demand converge to competitive demand, but strategic supply converges to that in the Cournot oligopoly in the many-buyer limit. As such, we know the intersection point of strategic supply and demand in the two-stage game with the sellers as leaders will converge to that at the Cournot equilibrium. This implies that the price and aggregate quantity of the consumption commodity traded at the two-stage market game equilibrium with the sellers as leaders will converge to that at the Cournot equilibrium as the number of buyers increases without bound, and, as the following theorem demonstrates, this is sufficient to guarantee convergence in equilibrium outcomes.

Theorem 5. *Suppose* ${}^{n}\dot{p}'(Q) + Q^{n}\dot{p}''(Q) \leq 0\,\forall Q > 0,\,\forall n$. *Then the allocations and price associated with the SPNE in the two-stage market game in which the sellers are leaders converge to those at the Cournot equilibrium as the number of buyers increases without bound.*

This result implies that the two-stage strategic market game in which the sellers move first is an appropriate fully strategic model in which to provide a foundation for Cournot competition.

We next turn to consider the two-stage strategic market game with the reverse timing structure; that in which the buyers move first and the sellers move second. Recall that the SPNE in this case is identified by the intersection of strategic supply of the second-stage sellers $\mathcal{X}_1^{\mathrm{S}}(p)$ (which is the same as in the simultaneous-move game) and the derived strategic demand ${}^{n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p)$. We recall that in the Cournot oligopoly the equilibrium is identified by the intersection of strategic supply derived in that game $\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ and competitive demand $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$.

We show in the following lemma that strategic demand in the two-stage game in which the buyers are leaders converges to competitive demand as the number of buyers increases without bound.

Lemma 13. *Suppose* $2\ddot{p}'(B) - B\ddot{p}''(B) \leq 0\,\forall B > 0$. *Then strategic demand in the two-stage strategic market game in which the buyers are leaders converges to competitive demand as the number of buyers increases without bound, i.e.*

$${}^{n}\ddot{\mathcal{X}}_1^{\mathrm{B}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}.$$

Given how the equilibria under consideration are identified, this implies that in the many-buyer limit the equilibrium price and quantity of the consumption commodity traded in the two-stage game with the buyers as leaders will be the same as that at the Cournot equilibrium if and only if the strategic supply of the sellers in that game is the same as, or converges to, their strategic supply in the Cournot oligopoly (at least in a neighborhood of the Cournot equilibrium). But in the two-stage game in which the buyers are leaders strategic supply is the same as in the simultaneous-move game, and we demonstrated in Lemma 7 that the two will coincide if and only if the elasticity of competitive demand at the Cournot oligopoly equilibrium is unity. We state this formally in the following proposition.

Proposition 6. *Suppose* $2\breve{p}'(B) - B\breve{p}''(B) \leq 0 \, \forall B > 0$. *Then the price and aggregate quantity of the consumption commodity traded at the equilibrium in the two-stage game in which the buyers are leaders converge to those at the Cournot equilibrium if and only if the elasticity of competitive demand at the Cournot equilibrium is unity.*

Thus, unless we are in the very specific circumstance in which the elasticity of competitive demand is unity, the outcomes in the two-stage game in which the buyers are leaders will remain distinct from those at the Cournot oligopoly equilibrium even in the limit, implying this timing structure is not appropriate in providing a foundation for Cournot oligopoly.

An analogous analysis can be performed in Cournot oligopsony markets where the buyers are few in number and behave strategically. There we would find that in order to achieve Cournot outcomes in the many-seller limit the buyers need to move at the first stage and the sellers move second: unless the elasticity of competitive supply at the Cournot equilibrium is unity a simultaneously-played game, or a game in which the sellers are leaders, will have a limit that remains distinct from the Cournot equilibrium even in the limit. In this case the buyers must commit to their demand decisions before the sellers make their supply decisions.

8 Conclusion

We have exploited the aggregative structure of the market game in both its static and dynamic forms to derive strategic versions of supply and demand and demonstrated that equilibria correspond to intersections of these functions. Using this fact we have been able to compare outcomes in the various strategic market games to those at the Cournot equilibrium, in particular in the many-buyer limit.

In order to provide the Cournot equilibrium concept with a strategic foundation we need the outcomes in a fully strategic trading regime to converge to those at the Cournot equilibrium in the many-buyer limit, and for this we require those agents that are permitted to behave strategically in the Cournot market to move in the first stage whilst those that are assumed to be price takers move in the second stage. This implies that an essential feature of a firm in a Cournot oligopoly is their ability to commit to supply decisions. Indeed, if firms cannot commit before the buyers make their choices the Cournot equilibrium is (in general) not supported in the many-buyer limit.

We recognise, however, that the timing order is exogenously specified. Working in a finite economy, there is no justification to assume that, for example, the sellers have a desire to move in the first stage and the buyers want to move in the second stage. The next phase of our research project involves endogenising the order of moves in an attempt to show that, at least as the number of buyers increases, each seller finds it in her own best interests to commit and move at the first stage whilst buyers find it in their own best interests to delay and move at the second stage.

A Proofs

Proof of Lemma 1. Each seller $h \in \mathcal{H}^{\mathrm{S}}$ may be seen as solving the problem

$$\max_{q \in [0, e_h]} v_h(e_h - q) + \frac{q}{\frac{1}{m}((m-1)Q + Q_{-h} + q)} B.$$

The best response is

$$^m\mathrm{BR}_h^{\mathrm{S}}(Q_{-h}, B) = \begin{cases} 0 & \text{if } v_h'(e_h) \geq \frac{B}{Q_{-h}} \text{ or } Q_{-h} = 0, \text{ or} \\ \min\{^m\mathrm{br}_h^{\mathrm{S}}(Q_{-h}, B), e_h\} & \text{otherwise} \end{cases}$$

where

$$^m\mathrm{br}_h^{\mathrm{S}}(Q_{-h}, B) = \left\{ q : v_h'(e_h - q) = \frac{\frac{1}{m}((m-1)Q + Q_{-h})}{(\frac{1}{m}((m-1)Q + Q_{-h} + q))^2} B \right\}.$$

We seek those offers consistent with a Nash equilibrium in which the per-replica offer is Q and the price is p. Such an offer will be a best response to Q minus itself and $B = pQ$, and they are found by replacing Q_{-h} with $Q - q$ and B/Q with p in the best response correspondence. This gives the replacement correspondence, and by dividing by Q we get shares of the per-replica offer. This share correspondence takes the form

$$^mS_h^{\mathrm{S}}(Q, p) = \begin{cases} 0 & \text{if } p \leq v_h'(e_h) \text{ or} \\ \min\left\{^ms_h^{\mathrm{S}}(Q, p), \frac{e_h}{Q}\right\} & \text{if } p > v_h'(e_h) \end{cases}$$

where

$$^ms_h^{\mathrm{S}}(Q, p) = \left\{ s : v_h'(e_h - sQ) = \left(1 - \frac{1}{m}s\right) p \right\}.$$

When multiplied by Q the share correspondence gives the offers of seller $h \in \mathcal{H}^{\mathrm{S}}$ consistent with a Nash equilibrium in which the per-replica offer is Q and the price is p.

In order to find consistent per-replica offers we must find a per-replica offer that generates individual offers which when summed over a replica of sellers equal this offer, i.e. find where the sum of one replica's share correspondences are equal to one. This defines strategic supply. Thus,

$$^m\mathcal{X}_1^{\mathrm{S}}(p) = \{Q : \sum_{H^{\mathrm{S}}} {}^mS_h^{\mathrm{S}}(Q, p) = 1\}.$$

In determining the properties of strategic supply the properties of each $^mS_h^{\mathrm{S}}(Q, p)$ will be of crucial importance. We show next that this share correspondence is in fact a continuous function that is strictly decreasing in Q and non-decreasing in p. Recall that $^ms_h^{\mathrm{S}}(Q, p)$ is those value of s such that $v_h'(e_h - sQ) = \left(1 - \frac{1}{m}s\right) p$. When $v_h''(\cdot) < 0$, $v_h'(e_h - sQ)$ is increasing in s and so there can be at most one s such that $v_h'(e_h - sQ) = \left(1 - \frac{1}{m}s\right) p$: $^mS_h^{\mathrm{S}}(Q, p)$ is a function. Continuity is implied by continuity of $v_h'(\cdot)$.

We will now show that $^mS_h^{\mathrm{S}}(Q, p)$ is strictly decreasing in Q whenever it is positive. It will suffice to show $^ms_h^{\mathrm{S}}(Q, p)$ is strictly decreasing in Q. Suppose not, so for $Q' > Q$ we have $s' = {}^ms_h^{\mathrm{S}}(Q', p) \geq {}^ms_h^{\mathrm{S}}(Q, p) = s$. Then we would have $e_h - s'Q' < e_h - sQ$ and $\left(1 - \frac{1}{m}s'\right) p \leq \left(1 - \frac{1}{m}s\right) p$. But then concavity of $v_h(\cdot)$ implies

$$\left(1 - \frac{1}{m}s'\right) p = v_h'(e_h - s'Q') > v_h'(e_h - sQ) = \left(1 - \frac{1}{m}s\right) p,$$

a contradiction. Thus, ${}^m S_h^{\mathrm{S}}(Q,p)$ is strictly decreasing in Q. Next we show that ${}^m s_h^{\mathrm{S}}(Q,p)$ is strictly increasing in p implying ${}^m S_h^{\mathrm{S}}(Q,p)$ is non-decreasing in p. Suppose, contrarily, that for $p' > p$ we have $s' = {}^m s_h^{\mathrm{S}}(Q,p') \le {}^m s_h^{\mathrm{S}}(Q,p) = s$. Then we would have $e_h - s'Q \ge e_h - sQ$ and $\left(1 - \frac{1}{m}s'\right)p' > \left(1 - \frac{1}{m}s\right)p$, but then concavity of $v_h(\cdot)$ implies

$$\left(1 - \frac{1}{m}s'\right)p' = v_h'(e_h - s'Q) \le v_h'(e_h - sQ) = \left(1 - \frac{1}{m}s\right)p,$$

a contradiction. Moreover, it is easy to discern from the definition that as $Q \to 0$, ${}^m s_h^{\mathrm{S}}(Q,p) \to m\left(1 - \frac{v_h'(e_h)}{p}\right)$ and so $\lim_{Q\to 0} {}^m S_h^{\mathrm{S}}(Q,p) = \max\left\{0, m\left(1 - \frac{v_h'(e_h)}{p}\right)\right\}$.

We now seek to aggregate the share functions and determine the property of the solution in Q to $\sum_{H^{\mathrm{S}}} {}^m S_h^{\mathrm{S}}(Q,p) = 1$. When $\sum_{H^{\mathrm{S}}} \max\left\{0, m\left(1 - \frac{v_h'(e_h)}{p}\right)\right\} \le 1$ we know that as $Q \to 0$ the per-replica share function approaches something less than one. Since the per-replica share function is also strictly decreasing in Q this implies there is no $Q > 0$ such that the per-replica share function is equal to one. Thus, for all $p \le {}^m P^{\mathrm{S}}$, which is defined such that $\sum_{H^{\mathrm{S}}} \max\left\{0, m\left(1 - \frac{v_h'(e_h)}{{}^m P^{\mathrm{S}}}\right)\right\} = 1$, there is no $Q > 0$ such that $\sum_{H^{\mathrm{S}}} {}^m S_h^{\mathrm{S}}(Q,p) = 1$.

Conversely, when $p > {}^m P^{\mathrm{S}}$ the per-replica share function will exceed one when Q is small, and, when $Q = \sum_{H^{\mathrm{S}}} e_h$ it will be no greater than one as each ${}^m S_h^{\mathrm{S}}(Q,p) \le \max\left\{1, \frac{e_h}{Q}\right\}$ implying $\sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(\sum_{H^{\mathrm{S}}} e_h, p) \le 1$. Since the per-replica share function is also strictly decreasing in Q this implies there is exactly one $Q \in (0, \sum_{H^{\mathrm{S}}} e_h]$ such that $\sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(Q,p) = 1$. Thus, strategic supply is a function.

Higher values of p mean each individual share function will be no lower than before, and so the per-replica share function will be no lower than before. Since it is also strictly decreasing in Q this implies that for higher values of p the value of Q consistent with the per-replica share function being equal to one can be no lower. Thus, strategic supply is non-decreasing in the price. □

Proof of Lemma 2. We derive the share correspondence of each buyer by operations analogous to those performed for the sellers. Written as being dependent on B and p, this takes the form

$${}^n S_h^{\mathrm{B}}(B,p) = \begin{cases} 0 & \text{if } p \ge v_h'(0) \text{ or} \\ \min\left\{{}^n s_h^{\mathrm{B}}(B,p), \frac{e_h}{B}\right\} & \text{if } p < v_h'(0) \end{cases}$$

where

$${}^n s_h^{\mathrm{B}}(B,p) = \left\{s : v_h'\left(\frac{sB}{p}\right) = \frac{1}{1 - \frac{1}{n}s}p\right\}.$$

It is more convenient, however, to write this in terms of the demand for the first commodity $V = B/p$, and the price. In this way, we get the share correspondence as

$${}^n \breve{S}_h^{\mathrm{B}}(V,p) = \begin{cases} 0 & \text{if } p \ge v_h'(0) \text{ or} \\ \min\left\{{}^n \breve{s}_h^{\mathrm{B}}(V,p), \frac{e_h}{Vp}\right\} & \text{if } p < v_h'(0) \end{cases}$$

where

$${}^n \breve{s}_h^{\mathrm{B}}(V,p) = \left\{s : v_h'(sV) = \frac{1}{1 - \frac{1}{n}s}p\right\}.$$

[Note that ${}^n \breve{S}_h^{\mathrm{B}}(V,p)$ is simply ${}^n S_h^{\mathrm{B}}(Vp,p)$.] Strategic demand is the solution in V to $\sum_{H^{\mathrm{B}}} {}^n \breve{S}_h^{\mathrm{B}}(V,p) = 1$, so it is the properties of the share correspondences that will be of crucial importance. In fact, as $v_h'(sV)$ is decreasing in s under our concavity assumption whilst $\frac{1}{1-\frac{1}{n}s}p$ is increasing in s this correspondence will be a function. Continuity is implied by continuity of $v_h'(\cdot)$.

We show next that the share function is strictly decreasing in both V and p wherever it is positive. It is sufficient to show that ${}^n\breve{s}_h^{\mathrm{B}}(V,p)$ is strictly decreasing in both V and p. First for V: suppose, contrarily, that for $V' > V$ we have $s' = {}^n\breve{s}_h^{\mathrm{B}}(V',p) \geq {}^n\breve{s}_h^{\mathrm{B}}(V,p) = s$. Then we would have $s'V' > sV$ and $\frac{1}{1-\frac{1}{n}s'}p \geq \frac{1}{1-\frac{1}{n}s}p$. But then concavity of $v_h(\cdot)$ implies

$$\frac{1}{1-\frac{1}{n}s'}p = v_h'(s'V') < v_h'(sV) = \frac{1}{1-\frac{1}{n}s}p,$$

a contradiction. Now for p: suppose, again to the contrary, that for $p' > p$ we have $s' = {}^n\breve{s}_h^{\mathrm{B}}(V,p') \geq {}^n\breve{s}_h^{\mathrm{B}}(V,p) = s$. Then we would have $s'V \geq sV$ and $\frac{1}{1-\frac{1}{n}s'}p' > \frac{1}{1-\frac{1}{n}s}p$. But then concavity implies

$$\frac{1}{1-\frac{1}{n}s'}p = v_h'(s'V) \leq v_h'(sV) = \frac{1}{1-\frac{1}{n}s}p,$$

a contradiction.

Note, moreover, that when $V \to 0$, ${}^n\breve{s}_h^{\mathrm{B}}(V,p) \to n\left(1-\frac{p}{v_h'(0)}\right)$ and this implies that $\lim_{V\to 0} {}^n\breve{S}_h^{\mathrm{B}}(V,p) = \max\left\{0, n\left(1-\frac{p}{v_h'(0)}\right)\right\}$.

Now, strategic demand ${}^n\mathcal{X}_1^{\mathrm{B}}(p)$ is the solution in V to $\sum_{H^{\mathrm{B}}} {}^n\breve{S}_h^{\mathrm{B}}(V,p) = 1$. When $\sum_{H^{\mathrm{B}}} \max\left\{0, n\left(1-\frac{p}{v_h'(0)}\right)\right\} \leq 1$ we know that when $V \to 0$ the per-replica share function is no greater than one. In addition, since individual share functions are strictly decreasing in V the per-replica share function will inherit this property and so for all $V > 0$ it will be less than one, and strategic demand is undefined in such a case. This occurs for all $p \geq {}^nP^{\mathrm{B}}$ which is defined such that $\sum_{H^{\mathrm{B}}} \max\left\{0, n\left(1-\frac{{}^nP^{\mathrm{B}}}{v_h'(0)}\right)\right\} = 1$.

When $0 < p < {}^nP^{\mathrm{B}}$ the per-replica share function will exceed one when V is close to zero. When $V = \sum_{H^{\mathrm{B}}} e_h/p$ the per-replica share function will be less than one since each individual share function has an upper bound $e_h/\sum_{H^{\mathrm{S}}} e_h$ at this level of V. Since the per-replica share function is strictly decreasing in V this implies there is a single $V \in (0, \sum_{H^{\mathrm{B}}} e_h/p]$ such that $\sum_{H^{\mathrm{B}}} {}^n\breve{S}_h(V,p) = 1$, so strategic demand will be a function for all $0 < p < {}^nP^{\mathrm{B}}$. To show that it is strictly decreasing in p we note that each individual share function is decreasing in p and the per-replica share function inherits this property. This, together with the fact that the per-replica share function is decreasing in V also implies that for higher values of p the V such that $\sum_{H^{\mathrm{B}}} {}^n\breve{S}_h(V,p) = 1$ will be lower, which gives the desired result. $\square$

Proof of Proposition 1. Suppose ${}^m\mathcal{X}_1^{\mathrm{S}}(\hat{p}) = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$, then we must show there is a Nash equilibrium in which the price is $\hat{p}$. Let $\hat{Q} = {}^m\mathcal{X}_1^{\mathrm{S}}(\hat{p})$ and $\hat{B} = \hat{p}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$, and consider the strategies

$$\begin{aligned} \hat{q}_h &= \hat{Q}^m S_h^{\mathrm{S}}(\hat{Q},\hat{p})\,\forall h \in \mathcal{H}^{\mathrm{S}} \text{ and} \\ \hat{b}_h &= \hat{B}^n S_h^{\mathrm{B}}(\hat{B},\hat{p})\,\forall h \in \mathcal{H}^{\mathrm{B}}. \end{aligned}$$

By construction of the share functions we must have

$$\begin{aligned} \hat{q}_h &= {}^m\mathrm{BR}_h^{\mathrm{S}}(\hat{Q}_{-h},\hat{B})\,\forall h \in \mathcal{H}^{\mathrm{S}} \text{ and} \\ \hat{b}_h &= {}^n\mathrm{BR}_h^{\mathrm{B}}(\hat{B}_{-h},\hat{Q})\,\forall h \in \mathcal{H}^{\mathrm{B}} \end{aligned}$$

and so the strategies $(\hat{\mathbf{q}}, \hat{\mathbf{b}})$ form a Nash equilibrium.

Next, suppose the vector of strategies $(\hat{\mathbf{q}}, \hat{\mathbf{b}})$ form a Nash equilibrium in which the price is $\hat{p} = \hat{B}/\hat{Q}$, then we must show that ${}^m\mathcal{X}_1^{\mathrm{S}}(\hat{p}) = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. Since we have a Nash equilibrium the strategies of all agents must be best responses, and then it follows that

$$\begin{aligned} \hat{q}_h &= \hat{Q}^m S_h^{\mathrm{S}}(\hat{Q}, \hat{p}) \, \forall h \in \mathcal{H}^{\mathrm{S}} \text{ and} \\ \hat{b}_h &= \hat{B}^n S_h^{\mathrm{B}}(\hat{B}, \hat{p}) \, \forall h \in \mathcal{H}^{\mathrm{B}}. \end{aligned}$$

But then we have $\sum_{H^{\mathrm{S}}} {}^m S_h^{\mathrm{S}}(\hat{Q}, \hat{p}) = 1 \Leftrightarrow \hat{Q} = {}^m\mathcal{X}_1^{\mathrm{S}}(\hat{p})$ and $\sum_{H^{\mathrm{B}}} {}^n S_h^{\mathrm{B}}(\hat{B}, \hat{p}) = 1 \Leftrightarrow \hat{B}/\hat{p} = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. Then since $\hat{p} = \hat{B}/\hat{Q}$, $\hat{Q} = \hat{B}/\hat{p}$ and it follows that ${}^m\mathcal{X}_1^{\mathrm{S}}(\hat{p}) = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. □

Proof of Theorem 1. We know that under the stated conditions strategic supply is a continuous function defined for all $p > {}^m P^{\mathrm{S}}$ where it is non-decreasing in the price (Lemma 1) and strategic demand is a continuous function defined for all $0 < p < {}^n P^{\mathrm{B}}$ where it is strictly decreasing in the price (Lemma 2). Moreover, non-autarkic Nash equilibria are in one-to-one correspondence with intersections of strategic supply and demand (Proposition 1). When ${}^m P^{\mathrm{S}} \geq {}^n P^{\mathrm{B}}$ there is no price where strategic supply and strategic demand are both defined, so they cannot intersect. In this case, there is no non-autarkic equilibrium: the only equilibrium is autarky, which always exists. When ${}^m P^{\mathrm{S}} < {}^n P^{\mathrm{B}}$ there is an ϵ^{S} such that ${}^m\mathcal{X}_1^{\mathrm{S}}(p) < {}^n\mathcal{X}_1^{\mathrm{B}}(p)$ when $p = {}^m P^{\mathrm{S}} + \epsilon^{\mathrm{S}}$, and an ϵ^{B} such that ${}^m\mathcal{X}_1^{\mathrm{S}}(p) > {}^n\mathcal{X}_1^{\mathrm{B}}(p)$ when $p = {}^n P^{\mathrm{B}} - \epsilon^{\mathrm{B}}$. By continuity, therefore, strategic supply and demand must intersect. Since the former is non-decreasing in the price whilst the latter is strictly decreasing in p, they can intersect only once. Thus, there is a single non-autarkic Nash equilibrium (accompanied, of course, by the autarkic no-trade equilibrium). □

Proof of Lemma 3. Each buyer can be seen as solving the problem

$$\max_{b \in [0, e_h]} v_h \left(\frac{b}{p} \right) + e_h - b$$

the solution to which is

$$\tilde{\mathfrak{b}}_h(p) = \begin{cases} 0 & \text{if } p \geq v_h'(0) \text{ or} \\ \min\{\tilde{b}_h(p), e_h\} & \text{if } p < v_h'(0) \end{cases}$$

where

$$\tilde{b}_h(p) = \left\{ b : v_h' \left(\frac{b}{p} \right) = p \right\}.$$

Competitive demand is simply the summation of individual demands at each price:

$$\tilde{\mathcal{X}}_1^{\mathrm{B}}(p) = \sum_{H^{\mathrm{B}}} \frac{\tilde{\mathfrak{b}}_h(p)}{p}.$$

When $p \geq v_h'(0) \, \forall h \in H^{\mathrm{B}}$, i.e. when $p \geq \max_{H^{\mathrm{B}}} \{v_h'(0)\}$, the demand from each buyer, hence at the per-replica level, will be zero. When $p < \max_{H^{\mathrm{B}}} \{v_h'(0)\}$ there will be some buyers who have positive demand, given by $\tilde{v}_h(p) = \tilde{b}_h(p)/p$. This is the solution in v to $v_h'(v) = p$, which, since $v_h''(\cdot) < 0$, is strictly decreasing in p. Thus, $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ will be strictly decreasing in p. Continuity derives from continuity of $v_h'(\cdot)$. □

Proof of Lemma 4. The price $\tilde{p}(Q)$ is that which is consistent with $Q = \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$. We know from Lemma 3 that $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ is strictly decreasing in p when $v_h''(\cdot) < 0$ and so it directly follows that for higher values of Q the price consistent with $Q = \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ must be lower. Thus, $\tilde{p}(Q)$ is strictly decreasing in Q. The limit is a consequence of continuity of $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p)$ and the easily discernable fact that $\tilde{\mathcal{X}}_1^{\mathrm{B}}(p) \to 0$ as $p \to \max_{H^{\mathrm{B}}}\{v_h'(0)\}$. □

Proof of Lemma 5. Each seller $h \in \mathcal{H}^{\mathrm{S}}$ may be seen as solving the problem $\max_{q\in[0,e_h]} v_h(e_h - q) + q\tilde{p}\left(\frac{1}{m}((m-1)Q + Q_{-h} + q)\right)$. The best response is

$$ {}^m\tilde{\mathrm{BR}}_h^{\mathrm{S}}(Q_{-h}) = \begin{cases} 0 & \text{if } \tilde{p}(Q_{-h}) \le v_h'(e_h) \text{ or} \\ \min\{{}^m\tilde{\mathrm{br}}_h^{\mathrm{S}}(Q_{-h}), e_h\} & \text{if } \tilde{p}(Q_{-h}) > v_h'(e_h) \end{cases} $$

where

$$ {}^m\tilde{\mathrm{br}}_h^{\mathrm{S}}(Q_{-h}) = \left\{ q : v_h'(e_h - q) = \tilde{p}\left(\frac{1}{m}((m-1)Q + Q_{-h} + q)\right) + q\frac{1}{m}\tilde{p}'\left(\frac{1}{m}((m-1)Q + Q_{-h} + q)\right)\right\}. $$

Now consider those offers consistent with a Nash equilibrium in which the per-replica offer and price take certain values. By replacing Q_{-h} with $Q - q$ and $\tilde{p}(Q)$ with p we find such offers, and by considering shares of the per-replica offer we find the share correspondence of each seller in a Cournot oligopoly. This takes the form

$$ {}^m\tilde{S}_h^{\mathrm{S}}(Q,p) = \begin{cases} 0 & \text{if } p \le v_h'(e_h) \text{ or} \\ \min\left\{{}^m\tilde{s}_h^{\mathrm{S}}(Q,p), \frac{e_h}{Q}\right\} & \text{if } p > v_h'(e_h) \end{cases} $$

where

$$ {}^m\tilde{s}_h^{\mathrm{S}}(Q,p) = \left\{ s : v_h'(e_h - sQ) = p + sQ\frac{1}{m}\tilde{p}'(Q)\right\}. $$

When multiplied by Q, this correspondence gives those offers any seller $h \in \mathcal{H}^{\mathrm{S}}$ consistent with a Cournot equilibrium in which the per-replica offer is Q and the price is p.

At any given price we then seek those per-replica offers that are consistent, in that they generate individual offers that sum to it. Alternatively, we look for those values of Q where the sum of one replica's share correspondences is equal to one. Thus, strategic supply is

$$ {}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(p) = \{Q : \sum_{H^{\mathrm{S}}} {}^m\tilde{S}_h^{\mathrm{S}}(Q,p) = 1\}. $$

In order to determine the properties of strategic supply we must first determine the properties of individual share correspondences. Since $v_h''(\cdot) < 0$ we know $v_h'(e_h - sQ)$ is increasing in s. Moreover, since $\tilde{p}'(Q) < 0$ we know $p + sQ\frac{1}{m}\tilde{p}'(Q)$ is decreasing in s. As such, for any p and Q there will be only a single s such that $v_h'(e_h - sQ) = p + sQ\frac{1}{m}\tilde{p}'(Q)$, so ${}^m\tilde{s}_h^{\mathrm{S}}(Q,p)$, hence ${}^m\tilde{S}_h^{\mathrm{S}}(Q,p)$, will be a function.

We show next that it is strictly decreasing in Q and non-decreasing in p. Sufficient is to show that ${}^m\tilde{s}_h^{\mathrm{S}}(Q,p)$ is strictly decreasing in Q and strictly increasing in p. First for Q: suppose, contrarily, that for $Q' > Q$ we have $s' = {}^m\tilde{s}_h^{\mathrm{S}}(Q',p) \ge {}^m\tilde{s}_h^{\mathrm{S}}(Q,p) = s$. Then we would have $e_h - s'Q' < e_h - sQ$ and so concavity of $v_h(\cdot)$ implies

$$ p + s'Q'\frac{1}{m}\tilde{p}'(Q') = v_h'(e_h - s'Q') > v_h'(e_h - sQ) = p + sQ\frac{1}{m}\tilde{p}'(Q). $$

However,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}Q}\{sQ\tilde{p}'(Q)\} &= s(\tilde{p}'(Q)+Q\tilde{p}''(Q))+\frac{\mathrm{d}s}{\mathrm{d}Q}\{Q\tilde{p}'(Q)\}\\ &\leq 0\end{aligned}$$

as $\tilde{p}'(Q)+Q\tilde{p}''(Q)\leq 0$, $\tilde{p}'(Q)<0$ and $\frac{\mathrm{d}s}{\mathrm{d}Q}\geq 0$ by presumption, which is a contradiction as the first inequality implies $s'Q'\frac{1}{m}\tilde{p}'(Q')>sQ\frac{1}{m}\tilde{p}'(Q)$.

Next to show that ${}^m\tilde{s}_h^{\mathrm{S}}(Q,p)$ is strictly increasing in p. In order to demonstrate this we note that

$$\begin{aligned}\frac{\partial^m\tilde{s}_h^{\mathrm{S}}(Q,p)}{\partial p} &= -\frac{\frac{\partial}{\partial p}\{v_h'(e_h-sQ)-p-sQ\frac{1}{m}\tilde{p}'(Q)\}}{\frac{\partial}{\partial s}\{v_h'(e_h-sQ)-p-sQ\frac{1}{m}\tilde{p}'(Q)\}}\\ &= -\frac{1}{Qv_h''(e_h-sQ)+Q\frac{1}{m}\tilde{p}'(Q)}>0\end{aligned}$$

as $v_h''(\cdot)<0$ and $\tilde{p}'(\cdot)<0$, which gives the result.

Now, when $p>v_h'(e_h)$ and Q is small we may have situations in which the share function exceeds one. To avoid this economically meaningless case we restrict the domain of the share function to $Q>\underline{Q}^h(p)$ where $\underline{Q}^h(p)$ is such that $v_h'(e_h-\underline{Q}^h(p))=p+\underline{Q}^h(p)\frac{1}{m}\tilde{p}'(\underline{Q}^h(p))$. By definition, ${}^m\tilde{S}_h^{\mathrm{S}}(\underline{Q}^h(p),p)=1$.

If $p\leq\min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ all sellers' share functions are zero, as is the per-replica share function. When $p>\min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ there is a set of sellers, denoted H_*^{S}, for whom $p>v_h'(e_h)$ and who have positive share functions defined for $Q>\underline{Q}^h(p)$ which are continuous, bounded above by $\min\left\{1,\frac{e_h}{Q}\right\}$, strictly decreasing in Q, non-decreasing in p and such that ${}^m\tilde{S}_h^{\mathrm{S}}(\underline{Q}^h(p),p)=1$. We take the per-replica share function to be defined for all $Q\geq\max_{H_*^{\mathrm{S}}}\{\underline{Q}^h(p)\}$. When $Q=\max_{H_*^{\mathrm{S}}}\{\underline{Q}^h(p)\}$ the per-replica share function is no lower than one. When $Q=\sum_{H^{\mathrm{S}}}e_h$ it is no higher than one. As it is strictly decreasing in Q there is a single $Q\in[\max_{H_*^{\mathrm{S}}}\{\underline{Q}^h(p)\},\sum_{H^{\mathrm{S}}}e_h]$ such that $\sum_{H^{\mathrm{S}}}{}^m\tilde{S}_h^{\mathrm{S}}(Q,p)=1$. Thus, strategic supply is a function. As individual share functions are non-decreasing in p, so is $\sum_{H^{\mathrm{S}}}{}^m\tilde{S}_h^{\mathrm{S}}(Q,p)$ and, since this function is strictly decreasing in Q, the value of Q such that $\sum_{H^{\mathrm{S}}}{}^m\tilde{S}_h^{\mathrm{S}}(Q,p)=1$ can be no lower for higher levels of p: ${}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ is non-decreasing in p. □

Proof of Proposition 2. First we show that if ${}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})=\tilde{\mathcal{X}}_1^{\mathrm{B}}(\hat{p})$ then there must be a Nash equilibrium in which the price is $\hat{p}$. Let $\hat{Q}={}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})$, then by definition $\hat{p}=\tilde{p}(\hat{Q})$. For each seller we know $\hat{q}_h=\hat{Q}{}^m\tilde{S}_h^{\mathrm{S}}(\hat{Q},\hat{p})$ and this implies $\hat{q}_h={}^m\tilde{\mathrm{BR}}_h^{\mathrm{S}}(\hat{Q}_{-h})$, so the vector of strategies $\hat{\mathbf{q}}$ forms a Nash equilibrium.

Next, suppose the strategies $\hat{\mathbf{q}}$ form a Nash equilibrium with per-replica offer $\hat{Q}$ and price $\hat{p}=\tilde{p}(\hat{Q})$. Then the demand from the buyers is $\tilde{\mathcal{X}}_1^{\mathrm{B}}(\hat{p})$. Since the strategies form a Nash equilibrium we know $\hat{q}_h={}^m\tilde{\mathrm{BR}}_h(\hat{Q}_{-h})\,\forall h\in\mathcal{H}^{\mathrm{S}}$ and this implies $\hat{q}_h=\hat{Q}{}^m\tilde{S}_h^{\mathrm{S}}(\hat{Q},\hat{p})\,\forall h\in\mathcal{H}^{\mathrm{S}}$, in turn implying $\sum_{H^{\mathrm{S}}}{}^m\tilde{S}_h^{\mathrm{S}}(\hat{Q},\hat{p})=1$ and so $\hat{Q}={}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})$. But the price is $\hat{p}=\tilde{p}(\hat{Q})$ implying $\hat{Q}=\tilde{\mathcal{X}}_1^{\mathrm{B}}(\hat{p})$, thus it follows that ${}^m\tilde{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})=\tilde{\mathcal{X}}_1^{\mathrm{B}}(\hat{p})$. □

Proof of Theorem 2. We know from Lemma 3 that competitive demand is positive only when $0<p<\max_{H^{\mathrm{B}}}\{v_h'(0)\}$ where it is a continuous strictly decreasing function. From Lemma 5 we know that strategic supply is defined only for $p>\min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ where it is a function that is positive, continuous and non-decreasing in p. Thus, if $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\}\geq\max_{H^{\mathrm{B}}}\{v_h'(0)\}$

strategic supply and demand never intersect at a positive level and the only equilibrium is autarky. Conversely, when $\min_{H^{\mathrm{S}}}\{v'_h(e_h)\} < \max_{H^{\mathrm{B}}}\{v'_h(0)\}$ they intersect once and only once by arguments analogous to those presented in the proof of the former uniqueness theorem. Then applying Proposition 2 we get our result. Unlike in the simultaneously-played strategic market game, autarky is not always an equilibrium in a Cournot market: when a non-autarkic equilibrium exists, it is the only equilibrium. □

Proof of Lemma 6. We will first show that for each $h \in \mathcal{H}^{\mathrm{B}}$

$$B^n S_h^{\mathrm{B}}(B,p) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(p)\, \forall B > 0,\, \forall p.$$

The magnitude $B^n S_h^{\mathrm{B}}(B,p)$ is equivalent to the replacement function ${}^n R_h^{\mathrm{B}}(B,p)$ which gives the bid of a buyer consistent with a Nash equilibrium in which the per-replica bid is B and the price is p. It is found by replacing B_{-h} with $B-b$ and B/Q with p in the best response function, and is such that

$${}^n R_h^{\mathrm{B}}(B,p) = \begin{cases} 0 & \text{if } p \geq v'_h(0) \text{ or} \\ \min\{{}^n r_h^{\mathrm{B}}(B,p), e_h\} & \text{if } p < v'_h(0) \end{cases}$$

where

$${}^n r_h^{\mathrm{B}}(B,p) = \left\{ b : v'_h\left(\frac{b}{p}\right) = \frac{1}{1-\frac{1}{n}\frac{b}{B}}p \right\}.$$

It will suffice to show that ${}^n r_h^{\mathrm{B}}(B,p) \to_{n\to\infty} \tilde{b}_h(p)\, \forall B > 0, \forall p$. This is obviously true as $\frac{1}{1-\frac{1}{n}\frac{b}{B}}p \to_{n\to\infty} p\, \forall B > 0, \forall p$ implying ${}^n r_h^{\mathrm{B}}(B,p)$ tends to the solution in b to $v'_h\left(\frac{b}{p}\right) = p$, which is precisely $\tilde{b}_h(p)$.

Then we have that

$$B \sum_{H^{\mathrm{B}}} {}^n S_h^{\mathrm{B}}(B,p) \to_{n\to\infty} \sum_{H^{\mathrm{B}}} \tilde{\mathfrak{b}}_h(p)\, \forall B > 0,\, \forall p.$$

Then setting $B = p^n \mathcal{X}_1^{\mathrm{B}}(p)$ which is positive when $0 < p < {}^n P^{\mathrm{B}}$ (one can check that $\lim_{n\to\infty} {}^n P^{\mathrm{B}} = \max_{H^{\mathrm{B}}}\{v'_h(0)\}$) and dividing by p we find

$${}^n\mathcal{X}_1^{\mathrm{B}}(p) \sum_{H^{\mathrm{B}}} {}^n S_h^{\mathrm{B}}(p^n \mathcal{X}_1^{\mathrm{B}}(p), p) \to_{n\to\infty} \sum_{H^{\mathrm{B}}} \frac{\tilde{\mathfrak{b}}_h(p)}{p} = \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\, \forall 0 < p < \max_{H^{\mathrm{B}}}\{v'_h(0)\}.$$

But $\sum_{H^{\mathrm{B}}} {}^n S_h^{\mathrm{B}}(p^n \mathcal{X}_1^{\mathrm{B}}(p), p) = 1$ by definition, so

$${}^n\mathcal{X}_1^{\mathrm{B}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\, \forall 0 < p < \max_{H^{\mathrm{B}}}\{v'_h(0)\},$$

which is the desired result. □

Proof of Lemma 7. When $\eta(Q,p) = \left|\frac{p}{Q}\frac{1}{\tilde{p}'(Q)}\right| \lesseqgtr 1$ we have

$$\begin{aligned} -p &\gtreqless Q\tilde{p}'(Q) \Rightarrow \\ (1-s)p &\gtreqless p + sQ\tilde{p}'(Q) \end{aligned}$$

by multiplying the first inequality by s and adding p to each side. Now, $s_h^{\mathrm{S}}(Q,p)$ is that s where $v'_h(e_h - sQ) = (1-s)p$, whilst $\tilde{s}_h^{\mathrm{S}}(Q,p)$ is that s where $v'_h(e_h - sQ) = p + sQ\tilde{p}'(Q)$. Since

$v_h'(e_h - sQ)$ is increasing in s (by concavity of $v_h(\cdot)$) it follows that when $(1-s)p \gtreqless p + sQ\tilde{p}'(Q)$ we have that $s_h^{\mathrm{S}}(Q,p) \gtreqless \tilde{s}_h(Q,p)$. Thus, it follows that

$$\eta(Q,p) \lesseqgtr 1 \Leftrightarrow \sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(Q,p) \gtreqless \sum_{H^{\mathrm{S}}} \tilde{S}_h^{\mathrm{S}}(Q,p).$$

Setting $Q = \tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ we get that

$$\eta(\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p) \lesseqgtr 1 \Leftrightarrow \sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p) \gtreqless \sum_{H^{\mathrm{S}}} \tilde{S}_h^{\mathrm{S}}(\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p) = 1.$$

When $\sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p) \gtreqless 1$ the value of Q that ensures equality with one, which is precisely $\mathcal{X}_1^{\mathrm{S}}(p)$, must be $\gtreqless \tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ since the function $\sum_{H^{\mathrm{S}}} S_h^{\mathrm{S}}(Q,p)$ is strictly decreasing in Q. As such,

$$\eta(\tilde{\mathcal{X}}_1^{\mathrm{S}}(p),p) \lesseqgtr 1 \Leftrightarrow \mathcal{X}_1^{\mathrm{S}}(p) \gtreqless \tilde{\mathcal{X}}_1^{\mathrm{S}}(p).$$

□

Proof of Proposition 3. If the elasticity of competitive demand at the Cournot equilibrium is unity we know that strategic supply in the strategic market game is the same as that in the Cournot market in a neighborhood of the Cournot equilibrium price. Moreover, we know from Lemma 6 that strategic demand converges to competitive demand as $n \to \infty$. This implies that the price and per-replica quantity of the consumption commodity traded in the strategic market game will converge to those at the Cournot equilibrium: ${}^n\hat{p} \to_{n\to\infty} \hat{p}^{\mathrm{C}}$ and ${}^n\hat{Q} \to_{n\to\infty} \hat{Q}^{\mathrm{C}}$. Moreover, we know that when the elasticity of competitive demand is one $S_h^{\mathrm{S}}(Q,p)$ and $\tilde{S}_h^{\mathrm{S}}(Q,p)$ coincide for each $h \in \mathcal{H}^{\mathrm{S}}$ and this implies that at a given price and per-replica offer individual offers will coincide. As the price and per-replica offer converge we thus find ${}^n\hat{q}_h = S_h^{\mathrm{S}}({}^n\hat{Q}, {}^n\hat{p}) \to_{n\to\infty} \tilde{S}_h^{\mathrm{S}}(\hat{Q}^{\mathrm{C}}, \hat{p}^{\mathrm{C}}) = \hat{q}_h^{\mathrm{C}}$. In addition, for the buyers we recall from the proof of Lemma 6 that $B\,{}^nS_h^{\mathrm{B}}(B,p) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(p)\,\forall B > 0$ and this, combined with the fact that the price converges implies individual bids will converge: ${}^n\hat{b}_h \to_{n\to\infty} \hat{b}_h^{\mathrm{C}}\,\forall h \in H^{\mathrm{B}}$. Since the allocation mechanism is the same we thus achieve convergence in allocations and prices in the many-buyer limit.

Conversely, when the elasticity of competitive demand is not unity the equilibrium price and per-replica offer will not converge in the limit, and so generically we will see a discrepancy between offers and bids in the limit, implying allocations and prices will not converge. □

Proof of Proposition ??. We could infer this from our previous analysis (in the proof of Lemma 2) concerning share functions, but for completeness we show it directly. Fix the offers of each replica of sellers at $\{q_h\}_{h\in H^{\mathrm{S}}}$ so we specify the subgame. In this subgame the per-replica offer is Q. As the maximisation problem of each buyer is the same as in the simultaneous-move game we know her best response will be ${}^n\mathrm{BR}_h^{\mathrm{B}}(B_{-h},Q)$. Let us consider her bids consistent with an equilibrium in this subgame in which the per-replica bid is B. Such a bid will be a best response to B minus itself (and Q) and these (or rather their ratio to B) can be represented by the share correspondence

$${}^n\check{S}_h^{\mathrm{B}}(B,Q) = \begin{cases} 0 & \text{if } v_h'(0) \le \frac{B}{Q} \text{ or} \\ \min\left\{{}^n\check{s}_h^{\mathrm{B}}(B,Q), \frac{e_h}{B}\right\} & \text{if } v_h'(0) > \frac{B}{Q} \end{cases}$$

where

$${}^n\check{s}_h^{\mathrm{B}}(B,Q) = \left\{ s : v_h'(sQ) = \frac{1}{1-\frac{1}{n}s}\frac{B}{Q} \right\}.$$

The properties of this share correspondence are outlined in the following lemma.

Lemma 14. *The share correspondence* ${}^{n}\check{S}_h^{\mathrm{B}}(B,Q)$ *is a function that is continuous and, where positive, strictly decreasing in* $B > 0$. *When* $B > \bar{B}^h(Q)$ *(which is equal to* $v_h'(0)Q$*) it is identically zero. When* $0 < B < \bar{B}^h(Q)$ *it is positive, bounded above by* $\min\left\{1, \frac{e_h}{B}\right\}$*, strictly decreasing in* $B > 0$ *and is such that* $\lim_{B\to 0} {}^{n}\check{S}_h^{\mathrm{B}}(B,Q) = 1$.

Proof. That it is a function follows from realising that $v_h'(sQ)$ is strictly decreasing in s (by concavity) whilst $\frac{1}{1-\frac{1}{n}s}\frac{B}{Q}$ is strictly increasing in s so there can be at most one s consistent with equality between the two. For any given $Q > 0$ there will be some cutoff value $\bar{B}^h(Q) = v_h'(0)Q$. When $B \geq \bar{B}^h(Q)$, $v_h'(0) \leq \frac{B}{Q}$ and so ${}^{n}\check{S}_h^{\mathrm{B}}(B,Q) = 0$. When $B < \bar{B}^h(Q)$, $v_h'(0) > \frac{B}{Q}$ and ${}^{n}\check{S}_h^{\mathrm{B}}(B,Q) = \min\left\{{}^{n}\check{s}_h^{\mathrm{B}}(B,Q), \frac{e_h}{B}\right\}$. To show that the function is decreasing in B suppose, to the contrary, that when $B > B'$ we have $s' = {}^{n}\check{s}_h^{\mathrm{B}}(B',Q) \geq {}^{n}\check{s}_h^{\mathrm{B}}(B,Q) = s$. Then $s'Q \geq sQ$ and $\frac{1}{1-\frac{1}{n}s'}\frac{B'}{Q} > \frac{1}{1-\frac{1}{n}s}\frac{B}{Q}$. But concavity of $v_h(\cdot)$ implies

$$\frac{1}{1-\frac{1}{n}s'}\frac{B'}{Q} = v_h'(s'Q) \leq v_h'(sQ) = \frac{1}{1-\frac{1}{n}s}\frac{B}{Q},$$

a contradiction. When $B \to 0$, $\frac{1}{1-\frac{1}{n}s}\frac{B}{Q}$ approaches a $\lrcorner$ shape (where s is on the horizontal axis) with the corner at $(1,0)$. As such, intersection between $v_h'(sQ)$ and $\frac{1}{1-\frac{1}{n}s}\frac{B}{Q}$ tends to occur when $s = 1$. $\square$

When multiplied by B the share function gives the bid of the buyer consistent with an equilibrium in which the per-replica bid is B in the subgame in which the per-replica offer is Q. In order to identify an equilibrium in this subgame we need only find a consistent per-replica bid, i.e. such that the individual bids generated by it add up to the per-replica bid, or where the per-replica share function is equal to one. Indeed, one can check in the routine way that there is a Nash equilibrium in the subgame in which the per-replica offer is Q with per-replica bid B if and only if $\sum_{H^{\mathrm{B}}} {}^{n}\check{S}_h^{\mathrm{B}}(B,Q) = 1$. Now, we know that when B is arbitrarily close to zero, $\sum_{H^{\mathrm{B}}} {}^{n}\check{S}_h^{\mathrm{B}}(B,Q) > 1$, and when $B = \sum_{H^{\mathrm{B}}} e_h$ the per-replica share function will be lower than one due to the upper bound on individual share functions. Since individual share functions are strictly decreasing in B the per-replica share function inherits this property and there will be a unique $B \in (0, \sum_{H^{\mathrm{B}}} e_h]$ such that $\sum_{H^{\mathrm{B}}} {}^{n}\check{S}_h^{\mathrm{B}}(B,Q) = 1$, *ergo* a unique Nash equilibrium in which the strategy of each buyer is $B\,{}^{n}\check{S}_h^{\mathrm{B}}(B,Q)$.

The magnitude of of this per-replica bid, hence the nature of individual strategies, is only dependent on the per-replica offer Q, not its composition. As such, in any subgame in which the aggregate offer is the same, the optimal responses of the buyers will be the same. $\square$

Proof of Lemma 9. This proof exactly parallels that of Lemma 5 but where the strategic supply function ${}^{m}\tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$ is replaced with ${}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(p)$, the share function ${}^{m}\tilde{S}_h^{\mathrm{S}}(Q,p)$ is replaced with ${}^{m,n}\dot{S}_h^{\mathrm{S}}(Q,p)$ and the price functional $\tilde{p}(Q)$ is replaced with ${}^{n}\dot{p}(Q)$. The details are thus omitted. $\square$

Proof of Proposition 4. First we show that if ${}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(\hat{p}) = {}^{n}\mathcal{X}_1^{\mathrm{B}}(\hat{p})$ then there is a SPNE in which the price is $\hat{p}$. Let $\hat{Q} = {}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})$ and $\hat{B} = \hat{p}\,{}^{n}\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. Then for each buyer we know $\hat{b}_h = \hat{B}\,{}^{n}S_h^{\mathrm{B}}(\hat{B},\hat{p})$ and this implies that $\hat{b}_h = {}^{n}\mathrm{BR}_h^{\mathrm{B}}(\hat{B}_{-h},\hat{Q})\,\forall h \in \mathcal{H}^{\mathrm{B}}$. Thus, in the subgame defined by $\hat{Q}$ buyers are playing equilibrium strategies. Moreover, since the price is $\hat{p} = {}^{n}\dot{p}(\hat{Q})$ we know that for each seller the strategy $\hat{q}_h = \hat{Q}\,{}^{m,n}\dot{S}_h^{\mathrm{S}}(\hat{Q},\hat{p})$ is by construction such that $\hat{q}_h = {}^{m,n}\dot{\mathrm{BR}}_h^{\mathrm{S}}(\hat{Q}_{-h})$. Thus, there is a SPNE in which the price is $\hat{p}$.

Next, suppose the strategies $(\hat{\mathbf{q}}, \hat{\mathbf{b}})$ form a SPNE in which the price is $\hat{p}$. For each buyer $\hat{b}_h = {}^n\mathrm{BR}_h^{\mathrm{B}}(\hat{B}_{-h}, \hat{Q})$ and so $\hat{b}_h = \hat{B}\,{}^nS_h^{\mathrm{B}}(\hat{B}, \hat{p})$ by construction. Thus, $\sum_{H^{\mathrm{B}}} {}^nS_h^{\mathrm{B}}(\hat{B}, \hat{p}) = 1$ implying $\hat{B}/\hat{p} = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$, and in addition that $\hat{p} = {}^n\dot{p}(\hat{Q})$. For each seller we have $\hat{q}_h = {}^{m,n}\dot{\mathrm{BR}}_h^{\mathrm{S}}(\hat{Q}_{-h})$, and then it follows, since $\hat{p} = {}^n\dot{p}(\hat{Q})$, that $\hat{q}_h = \hat{Q}\,{}^{m,n}\dot{S}_h^{\mathrm{S}}(\hat{Q}, \hat{p})$. But then $\sum_{H^{\mathrm{S}}} {}^{m,n}\dot{S}_h^{\mathrm{S}}(\hat{Q}, \hat{p}) = 1$ implying $\hat{Q} = {}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(\hat{p})$. Since ${}^n\dot{p}(Q)$ is such that $Q = {}^n\mathcal{X}_1^{\mathrm{B}}(p)$ and $\hat{p} = {}^n\dot{p}(\hat{Q})$ we have that $\hat{Q} = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$ implying ${}^{m,n}\dot{\mathcal{X}}_1^{\mathrm{S}}(\hat{p}) = {}^n\mathcal{X}_1^{\mathrm{B}}(\hat{p})$. □

Proof of Theorem 3. We recall from Lemma 2 that strategic demand is positive only for $0 < p < {}^nP^{\mathrm{B}}$ where it is a function that is continuous and strictly decreasing in p, and from Lemma 9 that strategic supply in the two-stage market game in which the sellers move first is positive only for $p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ where it is a continuous function that is non-decreasing in p. Moreover, Proposition 4 tells us that non-autarkic SPNE are in one-to-one correspondence with intersections of strategic supply and demand. When $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} \geq {}^nP^{\mathrm{B}}$ there are no prices where both strategic supply and demand are defined, so they cannot intersect. Thus, there is no non-autarkic SPNE; the only equilibrium is autarky. Conversely, when $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < {}^nP^{\mathrm{B}}$ strategic supply and demand must intersect, and due to their monotonicity properties they intersect only once. As such, there is a unique non-autarkic SPNE.

In this latter case, contrary to the simultaneous-move market game, autarky is not also an equilibrium. If the per-replica offer from the sellers in the first stage is positive, any buyer, even if she is acting alone, has the incentive to make a positive bid so the per-replica bid in the second stage will generically be strictly positive if the per-replica offer is positive. Given this, individual sellers have an incentive to make a positive offer in an attempt to acquire the whole of this bid if it is individually rational to do so. If $\min_{H^{\mathrm{S}}}\{v_h'(e_h)\} < {}^nP^{\mathrm{B}}$ there will indeed be such an offer and no autarkic equilibrium will exist. □

Proof of Lemma 12. We first show that ${}^n\dot{S}_h^{\mathrm{S}}(Q,p) \to_{n\to\infty} \tilde{S}_h^{\mathrm{S}}(Q,p)\,\forall Q > 0, \forall p$, noting that it will suffice to show ${}^n\dot{s}_h^{\mathrm{S}}(Q,p) \to_{n\to\infty} \tilde{s}_h^{\mathrm{S}}(Q,p)$. The former is that level of s where $v_h'(e_h - sQ) = p + sQ\,{}^n\dot{p}'(Q)$ whilst the latter is where $v_h'(e_h - sQ) = p + sQ\tilde{p}'(Q)$. But we know from Lemma 6 that ${}^n\mathcal{X}_1^{\mathrm{B}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0 < p < \max_{H^{\mathrm{B}}}\{v_h'(0)\}$ and this implies ${}^n\dot{p}(Q) \to_{n\to\infty} \tilde{p}(Q)\,\forall Q > 0$, which gives the desired result.

As a consequence, we know that

$$\sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(Q,p) \to_{n\to\infty} \sum_{H^{\mathrm{S}}} \tilde{S}_h^{\mathrm{S}}(Q,p)\,\forall Q > 0,\, \forall p.$$

Setting $Q = \tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$, which is positive for all $p > \min_{H^{\mathrm{S}}}\{v_h'(0)\}$ we get that

$$\sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(\tilde{\mathcal{X}}_1^{\mathrm{S}}(p), p) \to_{n\to\infty} \sum_{H^{\mathrm{S}}} \tilde{S}_h^{\mathrm{S}}(\tilde{\mathcal{X}}_1^{\mathrm{S}}(p), p) = 1\,\forall p > \min_{H^{\mathrm{S}}}\{v_h'(0)\}.$$

Since $\sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(Q,p)$ is strictly decreasing in Q under the stated conditions this implies that the only value of Q consistent with $\sum_{H^{\mathrm{S}}} {}^n\dot{S}_h^{\mathrm{S}}(Q,p) = 1$ (which is precisely ${}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p)$) in the limit is $Q = \tilde{\mathcal{X}}_1^{\mathrm{S}}(p)$. Thus, ${}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{S}}(p)\,\forall p > \min_{H^{\mathrm{S}}}\{v_h'(0)\}$. □

Proof of Theorem 5. Since ${}^n\mathcal{X}_1^{\mathrm{B}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{B}}(p)\,\forall 0 < p < \min_{H^{\mathrm{B}}}\{v_h'(0)\}$ and ${}^n\dot{\mathcal{X}}_1^{\mathrm{S}}(p) \to_{n\to\infty} \tilde{\mathcal{X}}_1^{\mathrm{S}}(p)\,\forall p > \min_{H^{\mathrm{S}}}\{v_h'(e_h)\}$ we know ${}^n\hat{p}^{\mathrm{SB}} \to_{n\to\infty} \hat{p}^{\mathrm{C}}$ and ${}^n\hat{Q}^{\mathrm{SB}} \to_{n\to\infty} \hat{Q}^{\mathrm{C}}$. It only remains to show that individual bids and offers converge. For the sellers, their individual offers in the two-stage game are ${}^n\hat{Q}^{\mathrm{SB}}\,{}^n\dot{S}_h^{\mathrm{S}}({}^n\hat{Q}^{\mathrm{SB}}, {}^n\hat{p}^{\mathrm{SB}})$ whilst in the Cournot oligopoly they are $Q^{\mathrm{C}}\tilde{S}_h^{\mathrm{S}}(Q^{\mathrm{C}}, p^{\mathrm{C}})$. In the proof of Lemma 12 we showed that for each $h \in H^{\mathrm{S}}$, ${}^n\dot{S}_h^{\mathrm{S}}(Q,p) \to_{n\to\infty}$

$\tilde{S}^{\mathrm{S}}_h(Q,p)\,\forall Q > 0$, $\forall p$. As such, since ${}^n\hat{Q}^{\mathrm{SB}} \to_{n\to\infty} \hat{Q}^{\mathrm{C}}$ and ${}^n\hat{p}^{\mathrm{SB}} \to_{n\to\infty} \hat{p}^{\mathrm{C}}$ it follows that ${}^n\hat{q}^{\mathrm{SB}}_h = {}^n\hat{Q}^{\mathrm{SB}\,n}\dot{S}^{\mathrm{S}}_h({}^n\hat{Q}^{\mathrm{SB}}, {}^n\hat{p}^{\mathrm{SB}}) \to_{n\to\infty} Q^{\mathrm{C}}\tilde{S}^{\mathrm{S}}_h(Q^{\mathrm{C}}, p^{\mathrm{C}}) = \hat{q}^{\mathrm{C}}_h$. The buyers' individual bids in the two-stage game are ${}^n\hat{B}^{\mathrm{SB}\,n}S^{\mathrm{B}}_h({}^n\hat{B}^{\mathrm{SB}}, {}^n\hat{p}^{\mathrm{SB}})$ whilst in the Cournot oligopoly they are $\tilde{\mathfrak{b}}_h(\hat{p}^{\mathrm{C}})$. We showed in the proof of Lemma 6 that $B^nS^{\mathrm{B}}_h(B,p) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(p)\,\forall B > 0$, $\forall p$. As such, since ${}^n\hat{p}^{\mathrm{SB}} \to_{n\to\infty} \hat{p}^{\mathrm{C}}$, it follows that for each $h \in H^{\mathrm{B}}$, ${}^n\hat{b}^{\mathrm{SB}}_h = {}^n\hat{B}^{\mathrm{SB}}S^{\mathrm{B}}_h({}^n\hat{B}^{\mathrm{SB}}, {}^n\hat{p}^{\mathrm{SB}}) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(\hat{p}^{\mathrm{C}}) = \hat{b}^{\mathrm{C}}_h$. □

Proof of Lemma 13. Write the share correspondence of each buyer in terms of the ratio $V = \frac{B}{p}$, in which case it takes the form

$$
{}^n\breve{\ddot{S}}^{\mathrm{B}}_h(V,p) = \begin{cases} 0 & \text{if } p \geq v'_h(0) \text{ or} \\ \min\left\{{}^n\breve{\ddot{s}}^{\mathrm{B}}_h(V,p), \frac{e_h}{Vp}\right\} & \text{if } p < v'_h(0) \end{cases}
$$

where

$$
{}^n\breve{\ddot{s}}^{\mathrm{B}}_h(V,p) = \left\{ s : v'_h(sV) = \frac{p^2}{p - sVp\frac{1}{n}\ddot{p}'(Vp)} \right\}.
$$

The magnitude of a buyer's bid consistent with a SPNE in which the ratio of per-replica bid to price is V and the price is p is given by $Vp^n\breve{\ddot{S}}^{\mathrm{B}}_h(V,p)$, and in order to find the consistent level of V, i.e. strategic demand, we look for that level of V such that $\sum_{H^{\mathrm{B}}} {}^n\breve{\ddot{S}}^{\mathrm{B}}_h(V,p) = 1$. [One can verify in the usual way that ${}^n\breve{\ddot{S}}^{\mathrm{B}}_h(V,p)$ is a function that is strictly decreasing in V.]

Now, the first task is to show $Vp^n\breve{\ddot{S}}^{\mathrm{B}}_h(V,p) \to_{n\to\infty} \tilde{\mathfrak{b}}_h(p)\,\forall V > 0$, $\forall p$. The magnitude $Vp^n\breve{\ddot{S}}^{\mathrm{B}}_h(V,p)$ is equivalent to the replacement value

$$
{}^n\breve{\ddot{R}}^{\mathrm{B}}_h(V,p) = \begin{cases} 0 & \text{if } p \geq v'_h(0)\text{or} \\ \min\{{}^n\breve{\ddot{r}}^{\mathrm{B}}_h(V,p), e_h\} & \text{if } p < v'_h(0) \end{cases}
$$

where

$$
{}^n\breve{\ddot{r}}^{\mathrm{B}}_h(V,p) = \left\{ b : v'_h\left(\frac{b}{p}\right) = \frac{p^2}{p - b\frac{1}{n}\ddot{p}'(Vp)} \right\}
$$

and it will suffice to show ${}^n\breve{\ddot{r}}^{\mathrm{B}}_h(V,p) \to_{n\to\infty} \tilde{b}_h(p)\,\forall V > 0$, $\forall p$. The latter is where $v'_h\left(\frac{b}{p}\right) = p$, so this is clearly true as $\frac{p^2}{p - b\frac{1}{n}\ddot{p}'(Vp)} \to_{n\to\infty} p$, and the desired result follows.

Then we have

$$
Vp\sum_{H^{\mathrm{B}}} {}^n\breve{\ddot{S}}^{\mathrm{B}}_h(V,p) \to_{n\to\infty} \sum_{H^{\mathrm{B}}} \tilde{\mathfrak{b}}_h(p)\,\forall V > 0,\ \forall p.
$$

Setting $V = {}^n\ddot{\mathcal{X}}^{\mathrm{B}}_1(p)$ which is positive for all $0 < p < \max_{H^{\mathrm{B}}}\{v'_h(0)\}$ and dividing by p we get

$$
{}^n\ddot{\mathcal{X}}^{\mathrm{B}}_1(p)\sum_{H^{\mathrm{B}}} {}^n\breve{\ddot{S}}^{\mathrm{B}}_h({}^n\ddot{\mathcal{X}}^{\mathrm{B}}_1(p), p) \to_{n\to\infty} \sum_{H^{\mathrm{B}}} \frac{\tilde{\mathfrak{b}}_h(p)}{p} = \tilde{\mathcal{X}}^{\mathrm{B}}_1(p)\,\forall 0 < p < \max_{H^{\mathrm{B}}}\{v'_h(0)\}.
$$

But $\sum_{H^{\mathrm{S}}} {}^n\breve{\ddot{S}}^{\mathrm{B}}_h({}^n\ddot{\mathcal{X}}^{\mathrm{B}}_1(p), p) = 1$ by definition, which gives the desired result. □

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