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Abstract

This paper extends Dastidar's [3] analysis of Bertrand equilibria to industries with increasing and strictly convex variable costs and fixed costs. Focusing on a symmetric duopoly, we show first that a price-taking equilibrium (PTE) is sufficient, but not necessary, for the existence of a pure strategy Bertrand equilibrium (PSBE), and that there could exist multiple PSBE even if a PTE does not exist. Then, we prove that a necessary and sufficient condition for the existence of a Bertrand equilibrium in pure strategies where both firms are active is that the total cost function is superadditive at the output corresponding to the duopoly break-even price. Finally, we characterize the set of PSBE, showing that it is a closed subset of the diagonal of the product of the strategy sets, and that it includes (if it exists) the profile where each firm posts the PTE price.

JEL codes: D43, L13.

Keywords: Bertrand equilibrium; price-taking equilibrium; fixed costs; superadditivity.

1 Introduction

This paper analyzes necessary and sufficient conditions for the existence of pure strategy Bertrand equilibria (PSBE) in a symmetric and homogeneous product duopoly with fixed costs.

In an influential work, Dastidar [3] has recently shown that, under strictly decreasing returns to scale, with no fixed costs, and (possibly) a sunk cost, oligopolies with homogeneous products and symmetric firms typically have multiple PSBE, being the price-taking equilibrium (PTE) price an

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element of the set of Bertrand equilibrium prices. He also proved in a later paper that, under certain conditions, this set may even include the perfectly collusive outcome (Dastidar [4]).

On the contrary, in industries with fixed costs, like in the duopoly proposed here, the total cost function is no longer convex around the origin. Under increasing and strictly convex variable costs, the market exhibits variable returns to scale, and neither a PTE nor a PSBE are anymore guaranteed.[1]

In this paper, we deal with this problem of equilibrium existence. First, we show in Lemma 1 that the existence of a price-taking equilibrium is sufficient to guarantee that the market has a Bertrand equilibrium in pure strategies where all firms are active. Furthermore, as our example in Section 3 illustrates, this condition is not necessary, and the duopoly could exhibit multiple PSBE even if a PTE does not exist.

Then, in Proposition 1, we formulate and proof the main result of the paper, namely that a symmetric and homogenous product duopoly has a Bertrand equilibrium in pure strategies where all firms supply positive output if and only if the total cost function is superadditive at the output corresponding to the duopoly break-even price. Finally, in Proposition 2, we also characterize the set of PSBE, showing that it is a closed subset of the diagonal of the product of the strategy sets, and that it includes (if it exists) the profile where each firm posts the PTE price (Corollary 1).

2 The model

Consider a market with a single homogenous good, produced and supplied by two *identical* firms, indexed by $i = 1, 2$. Suppose firms compete in prices à la Bertrand. Let $A_i = [0, \overline{P}] \subset \Re_+$ be firm i's strategy set, with generic element p_i, and $q_i = q_i(p_i, p_j)$ its output supply at $(p_i, p_j) \in A_i \times A_j$, $j \neq i$.

The following assumptions define our model.

Assumption A1 *The market demand $D(P)$ is continuous and twice differentiable in the market price P, with $D'(P) < 0$ and $D''(P) \geq 0$ for all $P \in [0, \overline{P}]$, $D(\overline{P}) = 0$ and $D(0) = K$ for some $K \in \Re_{++}$.*

Assumption A2 *The variable cost $VC(q_i)$ is continuous and twice differentiable in q_i, with $VC(0) = 0$, $VC' > 0$ and $VC'' > 0$. In addition, each firm faces a fixed cost $F > 0$, so that the total production cost is $C_i(q_i) = VC(q_i) + F$ for all $q_i > 0$, and $C_i(0) = 0$ otherwise.*

[1] A related result is provided by Telser [8], who showed that markets with nondecreasing returns to scale may fail to have a nonempty core and, therefore, that a PTE may not exist. However, his framework is different, and there is no reference to whether Bertrand equilibria exist in such industries.

Assumption A3 *Firm i's profits are* $\pi_i(p_i, p_j) = p_i\, d_i(p_i, p_j) - C_i(d_i(p_i, p_j))$, *where* $d_i(p_i, p_j)$ *is i's individual demand at* (p_i, p_j), *and*

$$d_i(p_i, p_j) = \begin{cases} 0 & \text{if } p_i > p_j, \\ \frac{1}{2}\, D(p_i) & \text{if } p_i = p_j, \\ D(p_i) & \text{if } p_i < p_j. \end{cases} \tag{1}$$

In words, (1) simply says that the firm that charges the lowest price gets the whole market, and that the market is equally split between the firms if they set the same price.[2]

Let $p_i^+ \in \arg\max_{p_i \in A_i} \left\{ p_i \frac{D(p_i)}{2} - VC\left(\frac{D(p_i)}{2}\right) - F \right\}$. By A1-A2, p_i^+ exists and it is unique.

Assumption A4 $p_i^+ \frac{D(p_i^+)}{2} - VC\left(\frac{D(p_i^+)}{2}\right) - F \geq 0$.

We denote $G = (A_i,\, \pi_i)_{i=1,2}$ a symmetric and homogenous good price competition game where A_i and π_i satisfy A1-A4. Compared with Dastidar [3], the main difference is that in our model F represents a fixed cost, that can be avoided producing zero output. On the contrary, in Dastidar [3] only sunk costs are allowed, although it is not explicitly stated in that way. Apart from this, the two frameworks are similar.[3]

As we show later, the nature of F has important implications for the existence of pure strategy Bertrand equilibria. If F is a sunk cost, then the total cost $C_i(q_i)$ is strictly convex on $[0, K]$. The market exhibits strictly decreasing returns to scale. Therefore, there is always a price-taking equilibrium, and a pure strategy Bertrand equilibrium where all firms are *active*. In fact, as Lemma 1 shows, the former result implies the latter.

On the contrary, if F is a fixed cost that firms can avoid producing zero output, then the total cost function $C_i(q_i)$ is no longer convex around the origin. The total average cost is U-shaped, and has a unique minimum where it crosses the marginal cost. Hence, the market's returns to scale are variable, and neither a PTE nor a PSBE are anymore guaranteed.

The main goal here is precisely to find conditions under which the existence of a PSBE is ensured. Before doing that, however, we need to provide the formal definition of this equilibrium concept.

Definition 1 (PSBE) *A pure strategy Bertrand equilibrium of G is a strategy profile* $(p_1^*, p_2^*) \in A_1 \times A_2$ *such that, for each* $i \neq j$,

(E1) $\pi_i(p_i^*, p_j^*) \geq \pi_i(\hat{p}_i, p_j^*)$ *for all* $\hat{p}_i \in A_i$,

[2] For Bertrand games under alternative sharing rules, see Hoernig [7] and Dastidar [5].

[3] As Tirole [9] (p. 307-308) wrote, the difference between fixed and sunk costs is that "fixed costs are sunk only in the short run." Instead, "(S)unk costs are those investment costs that produce a stream of benefits over a long horizon but can never be recouped."

(E2) $\pi_i(p_i^*, p_j^*) \geq 0$, *and*

(E3) $q_i(p_i^*, p_j^*) = d_i(p_i^*, p_j^*)$.

Note that Bertand equilibria requires that firms meet all the demand at the prices they post. The only strategic choice of the firms, therefore, is the price that they charge and not the quantity that they sell.[4] In rest of the paper, we focus on the subset of PSBE where both firms are active, in the sense that $d_i(p_i^*, p_j^*) > 0$ for all $i = 1, 2$. We denote this set $\mathcal{B}(G)$. Clearly, if there exists a Bertrand equilibrium $(p_1^*, p_2^*) \in \mathcal{B}(G)$, then $p_1^* = p_2^* = p^*$. Therefore, E1 can be rewritten as

$$p^* \frac{D(p^*)}{2} - VC\left(\frac{D(p^*)}{2}\right) \geq \hat{p}_i D(\hat{p}_i) - VC(D(\hat{p}_i)), \qquad (E4)$$

for all $\hat{p}_i < p^*$, while E2 simply requires that

$$p^* \frac{D(p^*)}{2} - VC\left(\frac{D(p^*)}{2}\right) - F \geq 0. \qquad (E5)$$

Assume now for a while that the price in the industry is given, so that firms are *price-takers* instead of *price-makers*, and it makes sense to talk about price-taking equilibria.

Definition 2 (PTE) *Given a price $P^c \in (0, \overline{P})$, a price-taking equilibrium in the homogenous product duopoly defined by A1-A4 is a pair of outputs $(q_1^c, q_2^c) \in \Re_+^2$ such that, for each $i = 1, 2$,*

(C1) $q_i^c \in \arg\max_{q_i \in \Re_+} \{P^c q_i - C_i(q_i)\}$,

(C2) $P^c q_i^c - C_i(q_i^c) \geq 0$, *and*

(C3) $q_1^c + q_2^c = D(P^c)$.

Note that C3, together with the "equal sharing" rule implicit in (1), imply that, if (q_1^c, q_2^c) is a PTE for a given $P^c \in (0, \overline{P})$, then $q_1^c = q_2^c = q^c \equiv D(P^c)/2$. Hence, in what follows we denote a PTE as (P^c, q^c), understanding that this means that (q^c, q^c) satisfies C1-C3 under P^c.

Given A1-A2, a unique pair (P^c, q^c) satisfying C1 and C2 always exists, because $P^c q_i - C_i(q_i)$ is strictly concave on $\Re_{++}$, and $P^c 0 - C_i(0) = 0$. However, as the example in the next section illustrates, if F is sufficiently large, the three conditions need not be simultaneously satisfied, meaning that a PTE does not necessarily exist in our model.

By contrast, if F represents a sunk cost, like in Dastidar [3], then a PTE always exists. This is because C1, together with A1 and A2, imply that $q^c > 0$ and $P^c > \frac{VC(q^c)}{q^c}$. Thus, $P^c q_i^c - C_i(q_i^c) \geq -F$.

[4] For more on Bertrand and Bertrand-Edgeworth equilibria, where firms face capacity constraints, see Vives [10], Chapter 5.

3 Example

In order to motivate the results derived in Sections 4 and 5, consider first the following numerical example. Let $D(P) = 10 - P$, and suppose $C_i(q_i) = 1/2q_i^2 + F$ if $q_i > 0$, and $C_i(0) = 0$ otherwise. Then, if a PTE exists, C1 and C3 imply that $P^c = 10/3$ and $q^c = 10/3$. However, C2 is satisfied at $(P^c, q^c) = (10/3, 10/3)$ only if $F \leq 50/9$. Thus, for $F > 50/9$ this exercise does not possess a PTE.

Regarding Bertrand equilibria, note that E4 can be written as $1/2p^*(10 - p^*) - 3/8(10 - p^*)^2 \leq 0$, and it is satisfied for all $p^* \leq 30/7$. Similarly, using E5, it follows that $p^* \geq 6 - \sqrt{16 - 8/5\,F}$. Thus, a price p^* that simultaneously satisfies both inequalities exists if and only if $F \leq 400/49$. For instance, if $F = 6$, then $p^* = 4$ is a solution of E4 and E5. Hence, $(4, 4) \in \mathcal{B}(G)$. If $F \leq 50/9$, the set of Bertrand equilibria includes the PTE; i.e. $(10/3, 10/3) \in \mathcal{B}(G)$. However, if $50/9 < F < 400/49$, a PTE does not exist, but the game possesses multiple PSBE. In effect, as Figure 1 below illustrates, if for example $F = 6$, any price between the lower bound $p_L = 6 - \sqrt{32/5}$ and the upper bound $p_H = 30/7$ satisfies conditions E1-E3 and, therefore, it is part of a PSBE.

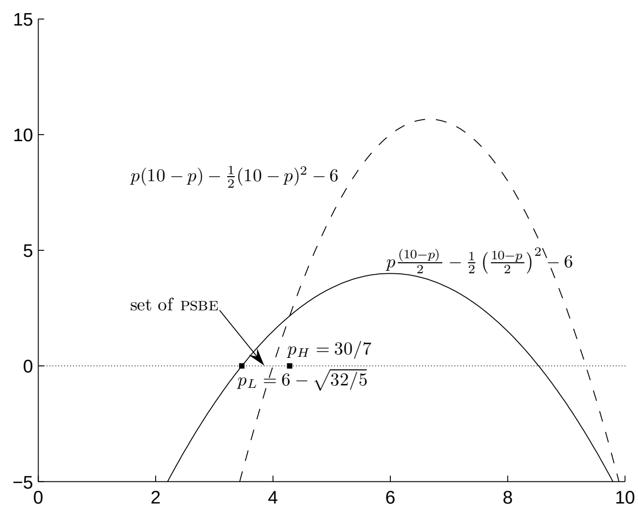

Figure 1:

In addition, it also comes out from the previous analysis that, if $F > 400/49$, then our example does not admit a pure strategy Bertrand equilibrium. For $F = 9$, this is illustrated in Figure 2, where it can be easily seen that, for any strategy profile $(\tilde{p}, \tilde{p})$, $p_L \leq \tilde{p} \leq p'$, that verifies E5, there exists

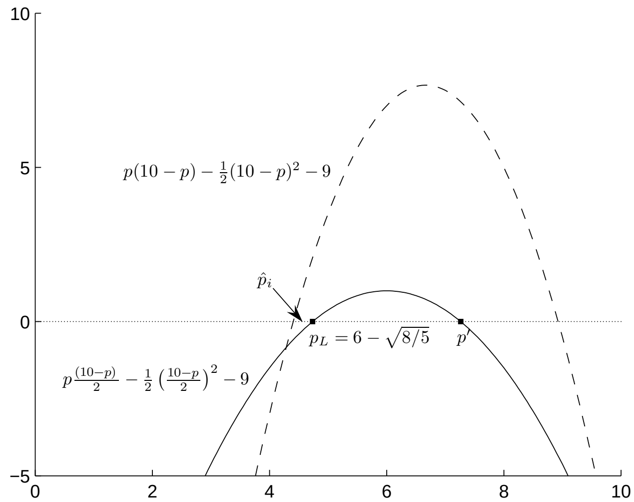

Figure 2:

at least one firm and an individual deviation such that E4 is contradicted at $(\tilde{p}, \tilde{p})$. Effectively, if for instance j is playing p_L, the diagram shows that firm i has an strategy $\hat{p}_i < p_L$ that dominates p_L, in the sense that $\pi_i(\hat{p}_i, p_L) = \hat{p}_i\,(10-\hat{p}_i) - \frac{1}{2}\,(10-\hat{p}_i)^2 - 9 > p_L\,\frac{(10-p_L)}{2} - \frac{1}{2}\left(\frac{10-p_L}{2}\right)^2 - 9 = \pi_i(p_L, p_L)$. Hence, $(p_L, p_L) \notin \mathcal{B}(G)$.

Interestingly, notice that the cost function $C_i(q_i)$ is superadditive at the bound $Q = 10 - [6 - \sqrt{16 - 8/5\,F}]$, (see (3) in Section 4 for the definition of superadditivity), if $1/2(4+\sqrt{16-8/5\,F})^2 \geq 1/4(4+\sqrt{16-8/5\,F})^2 + F$, which is equivalent to $4+\sqrt{16-8/5\,F}-2\sqrt{F} \geq 0$. Moreover, this inequality is satisfied if $F \leq 400/49$, which is precisely the critical level of fixed costs above which the set of pure strategy Bertrand equilibria is empty. As we argue in the next section, this is not accidental. Rather, we will see that there exists a closed relation between superadditivity and the existence of a PSBE. We now move to the existence and characterization analysis.

4 Existence

We begin this section showing that the existence of a PTE is sufficient for a PSBE to exist. Remarkably, the proof does not depend on neither the number of firms in the industry nor the nature of F. Thus, it generalizes to any homogenous product oligopoly, under similar demand and cost conditions that our model, and a finite number of symmetric firms.

Lemma 1 *If (P^c, q^c) is a* PTE*, then $(P^c, P^c) \in \mathcal{B}(G)$.*

Proof: Assume, by contradiction, $(P^c, P^c) \notin \mathcal{B}(G)$. First, note that, if (P^c, q^c) is a PTE, then C2 and C3 imply that E5 and E3 are satisfied at (P^c, P^c). Hence, there must exist i, and $\hat{p}_i \in A_i$, such that $\pi_i(\hat{p}_i, P^c) > \pi_i(P^c, P^c) \geq 0$. That is, $\hat{p}_i\, d_i(\hat{p}_i, P^c) - C(d_i(\hat{p}_i, P^c)) > 0$. Thus, $\hat{p}_i < P^c$. Let $\hat{q}_i = d_i(\hat{p}_i, P^c)$. Totally differentiating $\hat{p}_i\, q_i - VC(q_i) - F$ in a neighborhood of $\hat{q}_i$, we have

$$\hat{p}_i > \frac{\partial VC(\hat{q}_i)}{\partial q_i}. \tag{2}$$

By C1, $P^c = \frac{\partial VC(q^c)}{\partial q_i}$. Moreover, since $\hat{p}_i < P^c$, $D' < 0$ and $VC'' > 0$, it follows that $\frac{\partial VC(q^c)}{\partial q_i} < \frac{\partial VC(\hat{q}_i)}{\partial q_i}$. Therefore, (2) implies that $\hat{p}_i > P^c$: Contradiction. Hence, $(P^c, P^c) \in \mathcal{B}(G)$. □

The previous lemma shows that the existence of a PTE is sufficient to ensure the existence of a Bertrand equilibrium in pure strategies where both firms are active. If F represents a sunk cost, an immediate corollary is therefore that $\mathcal{B}(G)$ is always nonempty, because in that case a PTE always exists. And, of course, the converse of Lemma 1 also applies; that is, if $\mathcal{B}(G) \neq \emptyset$, then a PTE exists.

On the contrary, as the example in the previous section illustrates, if F is a fixed cost, a PTE is not necessary for the existence of a PSBE.[5] Instead, in the rest of this section we show that a necessary and sufficient condition is that the total cost function is superadditive at the output corresponding to the duopoly break-even price.

To show this, we now introduce the definition of subadditivity:

Definition 3 (subadditivity) *A function $f : \Re \to \Re$ is subadditive at $z \in \Re$ if and only if $f(z) < f(x) + f(y)$, for all $x, y \in \Re$ such that $x + y = z$.*

It is immediate to see that, if f is continuous and twice differentiable, $\min_{x,y}\ f(x) + f(y)$ subject to $x + y = z$ (for a given $z \in \Re$), has a unique interior solution where $x = y = z/2$, (provided that, of course, $f'' > 0$). Thus, under these conditions, Definition 3 can be restated as follows: f is subadditive at z only if $f(z) < 2f(z/2)$. Or, alternatively, we will say that f is *superadditive* at z if $f(z) \geq 2f(z/2)$.

Applying superadditivity in our framework, it follows that the cost function $C_i(q_i)$ is superadditive at $Q \in \Re_{++}$ if

$$VC(Q) \geq 2\, VC\left(\frac{Q}{2}\right) + F. \tag{3}$$

[5] A similar result appears in Grossman [6], but associated to a different equilibrium concept. Using a model similar to ours, Grossman has shown that the PTE, if it exists, is a "supply function equilibrium", and that the latter may exist even if a PTE does not.

In words, $C_i(q_i)$ is superadditive at Q if the total cost of producing Q with only one firm is greater than or equal to the sum of the costs of producing it separately with two or more identical firms. Subadditivity is therefore a necessary and sufficient condition for a natural monopoly (Baumol [1]).[6]

The following preliminary results will be useful to prove our main conjecture. Let $H : [0, \overline{P}] \to \Re$ and $J : [0, \overline{P}] \to \Re$ be such that, for all $x \in [0, \overline{P}]$,

$$H(x) = x \, \frac{D(x)}{2} - VC\left(\frac{D(x)}{2}\right) - F,$$

and

$$J(x) = x \, D(x) - VC\left(D(x)\right) - F.$$

By A1-A2, H and J are continuous and strictly concave on $[0, \overline{P}]$.

Lemma 2 *There exists $x' \in (0, \overline{P})$ such that $H(x') = 0$.*

Proof: By A1-A2, $H(0) = -VC(K/2) - F < 0$. By A4, $H(p^+) \geq 0$. Since H is strictly concave, $p^+ \in (0, \overline{P})$. Thus, by the intermediate value theorem, there exists $x' \in (0, p^+]$ such that $H(x') = 0$. $\square$

Lemma 3 *If there exists $\widetilde{x} \in (0, \overline{P})$ such that $J(\widetilde{x}) > H(\widetilde{x}) \geq 0$, then $\sup\{x'' \in (0, \overline{P}) : J(x'') = 0\} > \sup\{x' \in (0, \overline{P}) : H(x') = 0\}$.*

Proof: By Lemma 2, $\sup\{x' \in (0, \overline{P}) : H(x') = 0\}$ exits. By A1-A2, $J(0) = -VC(K) - F < 0$. By hypothesis, $J(\widetilde{x}) > 0$. Hence, $\sup\{x'' \in (0, \overline{P}) : J(x'') = 0\}$ also exists. On the other hand, notice that:

(a) $x \, D(x) > x \, \frac{D(x)}{2}$ for all $x \in (0, \overline{P})$;
(b) $0 \, D(0) = 0 \, \frac{D(0)}{2} = 0$;
(c) $\overline{P} \, D(\overline{P}) = \overline{P} \, \frac{D(\overline{P})}{2} = 0$;
(d) $x \, D(x)$ and $x \, \frac{D(x)}{2}$ are strictly concave on $[0, \overline{P}]$;
(e) $\arg\max_x \, x \, D(x) = \arg\max_x \, x \, \frac{D(x)}{2}$;
(f) $\left|(x \, D(x))'\right| > \left|\left(x \, \frac{D(x)}{2}\right)'\right| \quad \forall \, x \in (0, \overline{P})$;
(g) $VC(D(x)) + F > VC\left(\frac{D(x)}{2}\right) + F \ \ \forall \, x \in [0, \overline{P})$;
(h) $VC(D(0)) + F = VC(K) + F > VC(K/2) + F = VC\left(\frac{D(0)}{2}\right) + F$;
(i) $VC(D(\overline{P})) + F = VC\left(\frac{D(\overline{P})}{2}\right) + F = F$;
(j) $VC(D(x)) + F$ and $VC\left(\frac{D(x)}{2}\right) + F$ are strictly convex on $[0, \overline{P}]$;
(k) $\arg\min_x \, VC(D(x)) + F = \arg\min_x \, VC\left(\frac{D(x)}{2}\right) + F = \overline{P}$;

[6] Of course, it is always possible that the market behaves as a natural monopoly for a certain output, and not for others.

(l) $\left|(VC(D(x))+F)'\right| > \left|\left(VC\left(\frac{D(x)}{2}\right)+F\right)'\right| \quad \forall\, x \in (0,\overline{P})$.

Figure 3 below illustrates $x\,D(x)$, $x\,\frac{D(x)}{2}$, $VC(D(x))+F$ and $VC\left(\frac{D(x)}{2}\right)+F$. As it is clear from the graph, (a)-(l) imply that $\sup\{x'' \in (0,\overline{P}) : J(x'')=0\} > \sup\{x' \in (0,\overline{P}) : H(x')=0\}$. $\square$

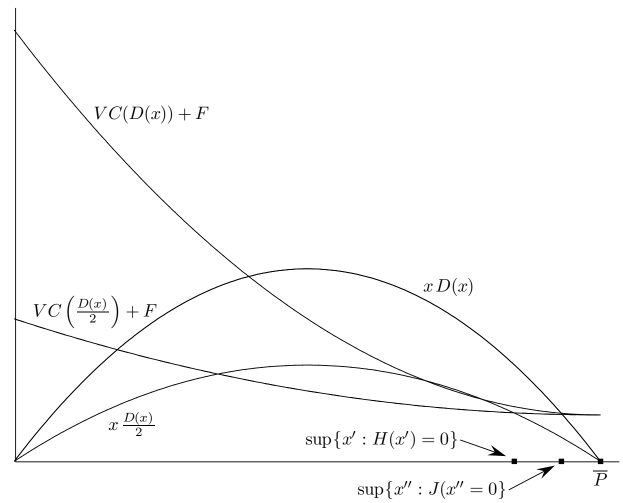

Figure 3:

Lemma 4 $\arg\max_{x\in[0,\overline{P}]} J(x) > \arg\max_{x\in[0,\overline{P}]} H(x)$.

Proof: Totally differentiating H with respect to x, and evaluating it at p^+, we have that

$$D(p^+) + \left[p^+ - VC'\left(\frac{D(p^+)}{2}\right)\right] D'(p^+) = 0. \tag{4}$$

On the other hand, A1-A2 imply that

$$D(p^+) + \left[p^+ - VC'\left(D(p^+)\right)\right] D'(p^+) > 0 \tag{5}$$

But, $\frac{\partial J(x)}{\partial x} = D(x) + [x - VC'(D(x))]\, D'(x)$. Thus, (5) implies $\frac{\partial J(p^+)}{\partial x} > 0$, meaning that $\arg\max_{x\in[0,\overline{P}]} J(x) > p^+ = \arg\max_{x\in[0,\overline{P}]} H(x)$. $\square$

Lemma 5 *There exists a unique $\beta \in [0,\overline{P})$ such that $J(\beta) = H(\beta)$.*

Proof: Consider the function $J(x) - H(x)$ on $[0, \overline{P}]$. Simple algebraic manipulation shows that $J(x) - H(x) = 1/2\, x\, D(x) - VC(D(x)) + VC(D(x)/2)$. Note that $VC(D(x)/2)$ is positive and strictly convex on $[0, \overline{P})$, with $VC(D(0)/2) = VC(K/2)$ and $VC(D(\overline{P})/2) = 0$. On the other hand, $1/2\, x\, D(x) - VC(D(x))$ is strictly concave, it is equal to $-VC(K)$ at zero, it intersects the horizontal axis at a certain $\alpha \in (0, \overline{P})$, (where $VC(D(x))$ crosses from above $1/2\, x\, D(x)$), and it is zero at $\overline{P}$.

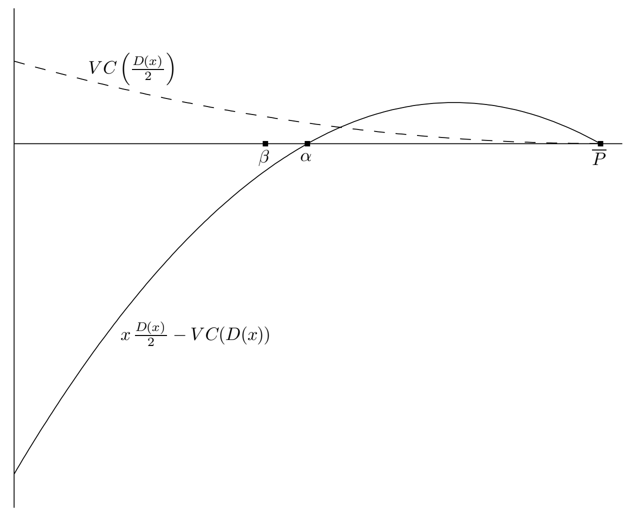

Figure 4:

Hence, as Figure 4 above illustrates, there exists a unique $\beta < \alpha$ on $[0, \overline{P})$ such that $J(\beta) - H(\beta) = 0$. $\square$

Let $p_L = \inf\{x' \in (0, \overline{P}) : H(x') = 0\}$. By Lemma 2, p_L is well defined.

Lemma 6 *If $\mathcal{B}(G) \neq \emptyset$, then $(p_L, p_L) \in \mathcal{B}(G)$.*

Proof: Assume, by contradiction, $(p_L, p_L) \notin \mathcal{B}(G)$. Since (p_L, p_L) fulfills E3 and E5, $\exists\, i$, and $\hat{p}_i \in A_i$, such that $\pi_i(\hat{p}_i, p_L) > \pi_i(p_L, p_L) = 0$, where the last equality follows from the definition of p_L. Hence, $\hat{p}_i < p_L$, and $\pi_i(\hat{p}_i, p_L) = J(\hat{p}_i) > 0$. By hypothesis, $\exists\, p^* \in [0, \overline{P}]$ such that $(p^*, p^*) \in \mathcal{B}(G)$. By E5, $H(p^*) \geq 0 \Rightarrow p^* > p_L$. By E4, $\pi_i(p^*, p^*) \geq J(\hat{p}_i)$ (recall that $\hat{p}_i < p_L < p^*$). Thus, $H(p^*) > 0$. Using Lemmas 2-4, the graphical representation of H and J is as it appears in Figure 5.

Note that, since $H(p^*) \geq J(x)$ for all $x \leq p^*$, $J(\hat{p}_i) > H(p_L)$, and $J(0) < H(0)$, there must exist $\beta^1 \in (0, p_L)$ and $\beta^2 \in (p_L, p^*)$ such that

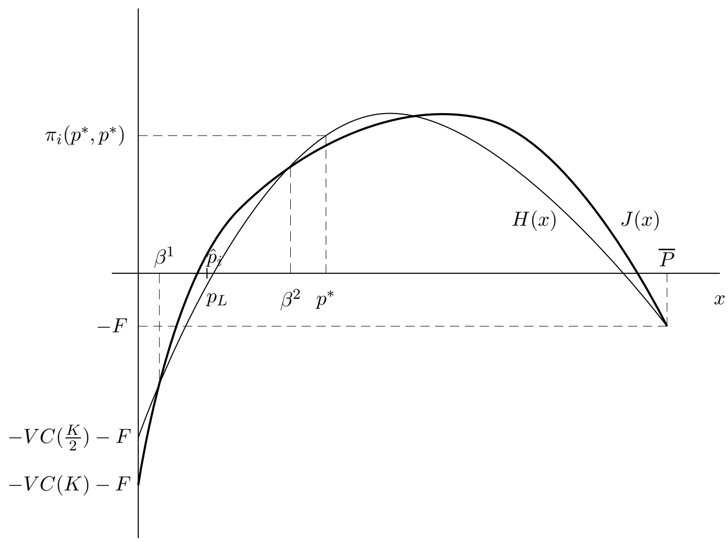

Figure 5:

$H(\beta^k) = J(\beta^k)$ for all $k = 1, 2$. However, this stands in contradiction with Lemma 5. Therefore, $(p_L, p_L) \in \mathcal{B}(G)$. $\square$

Proposition 1 *$\mathcal{B}(G) \neq \emptyset$ if and only if $C_i(q_i)$ is superadditive at $D(p_L)$.*

Proof: (Sufficiency) First, we prove that, if $C_i(q_i)$ is superadditive at $D(p_L)$, then $(p_L, p_L) \in \mathcal{B}(G)$. By definition, p_L is the minimum price that simultaneously satisfies both, $q_i(p_L, p_L) = d_i(p_L, p_L) = \frac{D(p_L)}{2}\ \forall\, i = 1, 2$, and

$$p_L \frac{D(p_L)}{2} - VC\left(\frac{D(p_L)}{2}\right) - F = 0. \tag{6}$$

Suppose, by contradiction, $\exists\, i$, and $\hat{p}_i < p_L$ such that

$$\hat{p}_i D(\hat{p}_i) - VC(D(\hat{p}_i)) > p_L \frac{D(p_L)}{2} - VC\left(\frac{D(p_L)}{2}\right).$$

By (6),

$$\hat{p}_i D(\hat{p}_i) - F > VC(D(\hat{p}_i)). \tag{7}$$

If $C_i(q_i)$ is superadditive at $D(\hat{p}_i)$, we are done: Using (3) and (7), we have that $\hat{p}_i D(\hat{p}_i) - F > 2\, VC\left(\frac{D(\hat{p}_i)}{2}\right) + F$, which is equivalent to $H(\hat{p}_i) > 0$. Applying the intermediate value theorem, it follows that $\exists\, \tilde{p} \in (0, \hat{p}_i)$ such that $H(\tilde{p}) = 0$, which contradicts the definition of p_L.

Instead, if $C_i(q_i)$ is not superadditive at $D(\hat{p}_i)$, then

$$VC(D(\hat{p}_i)) < 2\, VC\left(\frac{D(\hat{p}_i)}{2}\right) + F.$$

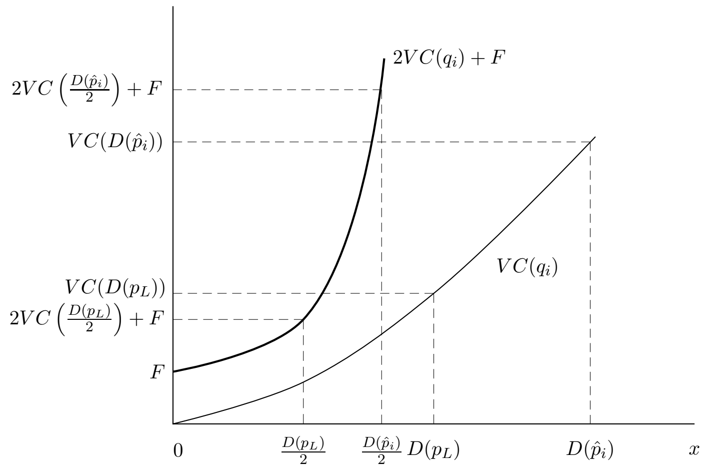

Figure 6:

Notice that $\hat{p}_i < p_L$, $D' < 0$ and $VC' > 0$ imply that $VC(D(\hat{p}_i)) > VC(D(p_L)) \geq 2\,VC\left(\frac{D(p_L)}{2}\right) + F$, where the last inequality follows from the fact that, by hypothesis, $C_i(q_i)$ is superadditive at $D(p_L)$. Thus, combining all these expressions, we have that

$$2\,VC\left(\frac{D(p_L)}{2}\right) + F < VC(D(\hat{p}_i)) < 2\,VC\left(\frac{D(\hat{p}_i)}{2}\right) + F. \tag{8}$$

However, as Figure 6 above illustrates, (8) implies that $2VC(q_i) + F$ and $VC(q_i)$ diverge, which contradicts that by A2, (i.e. because VC is continuous, increasing and strictly convex), $|(2VC(q_i) + F) - VC(q_i)| \to 0$ as $q_i \to \infty$. Hence, $(p_L, p_L) \in \mathcal{B}(G)$.

(Necessity) We want to prove that, if $\mathcal{B}(G) \neq \emptyset$, then $C_i(q_i)$ is superadditive at $D(p_L)$. First, notice that, by Lemma 6, $(p_L, p_L) \in \mathcal{B}(G)$. Now, assume that $C_i(q_i)$ is not superadditive at $D(p_L)$. This means

$$VC(D(p_L)) < 2\,VC\left(\frac{D(p_L)}{2}\right) + F. \tag{9}$$

By definition, $H(p_L) = 0$ can be rewritten as

$$p_L\,D(p_L) - VC(D(p_L)) - F = 2\,VC\left(\frac{D(p_L)}{2}\right) + F - VC(D(p_L)).$$

Therefore, (9) implies that $p_L\,D(p_L) - VC(D(p_L)) - F > 0$. Moreover, since $J(x) = x\,D(x) - VC(D(x)) - F$ is continuous, $\exists\,\epsilon > 0$, and $\tilde{p} < p_L$

closed enough to p_L, such that $0 < J(p_L) - \epsilon < J(\tilde{p})$. However, this contradicts the fact that, by E4 and the definition of p_L, $\hat{p}_i\, D(\hat{p}_i) - VC(D(\hat{p}_i)) - F \leq 0$ for all $\hat{p}_i < p_L$. Hence, $C_i(q_i)$ is superadditive at $D(p_L)$. $\square$

5 Characterization

Having proved the existence of a PSBE, we now provide a full characterization of Bertrand equilibria of G:

Proposition 2 *If $\mathcal{B}(G) \neq \emptyset$, $\exists\, p_H \in (0, \overline{P})$ such that $\mathcal{B}(G) = \{(p^*, p^*) \in A_i \times A_j : p^* \in [p_L, p_H]\}$.*

Proof: If $|\mathcal{B}(G)| = 1$, the claim is trivially true (just let $p_H = p_L$). Thus, suppose $|\mathcal{B}(G)| > 1$. That is, assume there exist $p' < p''$ such that (p', p'), $(p'', p'') \in \mathcal{B}(G)$. Define the linear combination $p^\lambda = \lambda\, p' + (1 - \lambda)\, p''$, $\lambda \in (0, 1)$. We want to prove that $(p^\lambda, p^\lambda) \in \mathcal{B}(G)$ for all $\lambda \in (0, 1)$. Suppose not. Note that, by definition, (p^λ, p^λ) satisfies E3. Moreover, since H is strictly concave and $H(p') \geq 0$ and $H(p'') \geq 0$, it follows that $H(p^\lambda) \geq 0 \Rightarrow$ E5 is satisfied at (p^λ, p^λ). Thus, there must exist i, and $\hat{p}_i \in A_i$ such that $\pi_i(\hat{p}_i, p^\lambda) > \pi_i(p^\lambda, p^\lambda) \Rightarrow \hat{p}_i < p^\lambda$. Hence, $J(\hat{p}_i) > H(p^\lambda)$.

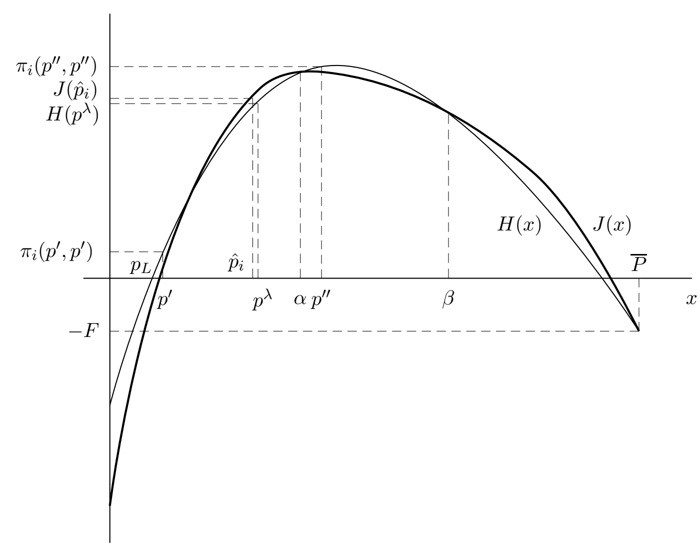

Figure 7:

Notice that, since $(p'', p'') \in \mathcal{B}(G)$, $\exists\, \alpha \in (p^\lambda, p'')$ such that $J(\alpha) = H(\alpha)$. Furthermore, by Lemma 3, $\exists \beta > \alpha$ such that $J(\beta) = H(\beta)$. However, this contradicts Lemma 5. Therefore, $(p^\lambda, p^\lambda) \in \mathcal{B}(G)$. $\square$

Corollary 1 *If (P^c, q^c) is a* PTE*, then $P^c \in [p_L, p_H]$.*

Proof: Immediate from Lemma 1 and Proposition 2. □

6 Conclusion

This paper extends Dastidar's [3] analysis of Bertrand equilibria to industries with increasing and strictly convex variable costs and fixed costs.

Focusing on a symmetric duopoly, we show first that a price-taking equilibrium is sufficient, but not necessary, for the existence of a pure strategy Bertrand equilibrium (Lemma 1), and that there could exist multiple PSBE even if a PTE does not exist.

Then, we prove that a necessary and sufficient condition for the existence of a Bertrand equilibrium in pure strategies where both firms are active is that the total cost function is superadditive at the output corresponding to the duopoly break-even price (Proposition 1).

Finally, we characterize the set of PSBE, showing that it is a closed subset of the diagonal of the product of the strategy sets (Proposition 2), and that it includes (if it exists) the profile where each firm posts the PTE price (Corollary 1).

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