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Arbitrage in Stationary Markets

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Abstract

We analyse conditions for arbitrage in financial markets in which asset price vectors change in time as stationary stochastic processes. The main focus of the study is on the case where these vectors are independent and identically distributed. In this case, we find conditions (formulated in terms of the given price distribution) that are necessary and sufficient for the absence of arbitrage opportunities.

1. Introduction. In this note, we examine questions of arbitrage in financial markets where asset price vectors change in time as stationary stochastic processes. The behaviour of self-financing trading strategies in such markets might be at first glance counterintuitive—different from that suggested by the conventional models where asset prices follow a geometric random walk. In particular, Evstigneev and Schenk-Hoppé (2002) showed that fixed-mix (constant proportions) investment strategies in such markets exhibit exponential growth with probability one, provided that the stationary price process satisfies some mild non-degeneracy assumptions. An analogous result in the context of currency markets was obtained by Dempster *et al.* (2003).

In connection with the above results, it is of interest to examine the question of existence of arbitrage opportunities in stationary markets. Specifically, if the price vectors are independent and identically distributed, what conditions guarantee the absence of arbitrage opportunities? Conversely, under what conditions such opportunities exist? In this paper, we provide answers to these questions. The answers are formulated in terms of the given probability distribution of the asset price vectors.

2. The Model. Let $s_0, s_1, ...$ be a discrete-time stochastic process whose value $s_t \in S$ at each time $t = 0, 1, ...$ describes the random "state of the world" at this time. We assume that the state space S is measurable (endowed with a σ-algebra $\mathcal{S}$). Consider a financial market, where $K+1$ assets $k = 0, 1, ..., K$ are traded. At each time $t = 0, 1, 2, ...,$ the prices $p_t^k(s^t)$, $k = 0, ..., K$, of the $K+1$ assets are random variables depending on the history $s^t = (s_0, s_1, ..., s_t)$ of the process $\{s_t\}$. We denote by

$$p_t = p_t(s^t) = (p_t^0(s^t), p_t^1(s^t), ..., p_t^K(s^t))$$

the $K + 1$-dimensioned vector of these prices. The functions $p_t^k(s^t)$ ($k = 0, ..., K$) are assumed to be strictly positive and measurable with respect to the product σ-algebra $\mathcal{S} \times ... \times \mathcal{S}$ on the space $S \times ... \times S$ whose elements are sequences $s^t = (s_0, ..., s_t)$.

Any vector $h_t = (h_t^0, h_t^1, ..., h_t^K)$ represents a *portfolio* of the $K + 1$ assets at time t. A measurable vector function $h_t(s^t) = \left(h_t^0(s^t), ..., h_t^K(s^t)\right)$ (depending on the present and past states of the world $s_0, ..., s_{t-1}, s_t$) is called a *contingent portfolio.* A sequence of contingent portfolios $H = (h_0, ..., h_T)$ is called a *trading strategy* over the time period $0, 1, ..., T$. Those trading strategies $H = (h_0, ..., h_T)$ which satisfy $\langle p_t, h_{t-1}\rangle = \langle p_t, h_t\rangle$ are called *self-financing* (we denote by $\langle\cdot,\cdot\rangle$ the scalar product of two vectors of the same finite dimension). An investor using a self-financing strategy rebalances his/her portfolio from h_{t-1} to h_t, so that the values of h_{t-1} and h_t expressed in terms of the prices p_t^k prevailing at time t coincide: $\sum_{i=1}^K p_t^i h_{t-1}^i = \sum_{i=1}^K p_t^i h_t^i$.

Fix some time period $0, 1, ..., T$. We say that there is an *arbitrage opportunity* over this time period if there exists a self-financing trading strategy $H = (h_0, ..., h_T)$ for which $\langle p_0, h_0\rangle \leq 0$ and $\langle p_T, h_T\rangle \geq 0$ almost surely (a.s.) and $\langle p_T, h_T\rangle > 0$ with strictly positive probability.

The following hypothesis is of fundamental importance in finance.

(NA) There are no arbitrage opportunities in the market.

Our goal in this paper is to examine conditions under which this hypothesis holds in stationary markets. A financial market is called *stationary* if the following two requirements are fulfilled:

(i) the given stochastic process $s_0, s_1, ...$ is stationary[1];

[1] A random process s_t is called *stationary* if for each m and each measurable function ϕ, the distribution of the random variable $\phi(s_t, s_{t+1}, ..., s_{t+m})$ does not depend on t.

(ii) the price vector $p_t(s^t)$ does not explicitly depend on the time variable t and depends only on the current state of the world s_t: $p_t(s^t) = p(s_t)$, where $p(\cdot)$ is a measurable vector function on S.

Remark 1. There are versions of the model studied here (see, e.g., Dempster *et al.* 2003) where the process s_t is given for all $t = 0, \pm 1, \pm 2, ...$ Then stationarity is typically defined by the assumption that $p_t = p(s^t)$, where $s^t = (..., s_{t-1}, s_t)$ is an infinite history. That framework can be reduced to the present one by setting $\sigma_t = s^t$ and regarding σ_t as a new state of the world.

The above cited papers dealt with quite general stationary processes describing states of the world and asset prices. In this article, we concentrate on the case where the structure of the process s_t is as simple as possible: we assume that the random elements $s_0, s_1, ...$ are independent and identically distributed. Further, as it is commonly supposed, we will assume that the 0th asset is *riskless* (numeraire, cash), having a non-random rate of return, which in our stationary context implies that its price p_t^0 is a strictly positive constant, which we will normalize to one. In view of this, we can represent the price vector $p(s_t)$ as $p(s_t) = (1, \gamma(s_t)) = (1, \gamma_t)$, where

$$\gamma_t = \gamma(s_t) = \left(\gamma^1(s_t), ..., \gamma^K(s_t)\right)$$

is the K-dimensional price vector of *risky assets*. Analogously, we can represent any portfolio $h_t = \left(h_t^0, h_t^1, ..., h_t^K\right)$ as $h_t = (h_t^0, \xi_t)$, where $\xi_t = \left(h_t^1, ..., h_t^K\right)$ is the portfolio of risky assets.

In the analysis that follows, we will use another version of the no arbitrage hypothesis which is formulated below (its equivalence to (**NA**) is proved, for example, in Föllmer and Schied (2002), Proposition 5.11.

(NA$'$) For each $t = 0, ..., T-1$, there is no measurable vector function $\xi_t(s^t)$ such that the two conditions

$$\xi_t(s^t)[\gamma(s_{t+1}) - \gamma(s_t)] \geq 0 \quad \text{(a.s.)}, \tag{1}$$

$$P\{\xi_t(s^t)[\gamma(s_{t+1}) - \gamma(s_t)] > 0\} > 0 \tag{2}$$

hold simultaneously.

Here and in what follows, P denotes the underlying probability measure P.

3. The Main Results. In this section we formulate and discuss the main results of the paper. Let π be the probability distribution of the random vector $\gamma(s_t)$. We will always assume that π is non-degenerate, i.e., it is not concentrated at one point. Let W be the support of π and let $V := cl\ co\ W$ be the closure of the convex hull of W. Denote by $\partial_r V$ the relative boundary of V, i.e. the boundary of the convex set V in the smallest linear manifold containing V.

A central result is as follows.

Theorem 1. *The absence of arbitrage opportunities in the market under consideration is equivalent to the condition* $\pi(\partial_r V) = 0$.

Theorem 1 provides a no-arbitrage criterion for the stationary asset market under consideration. This criterion is stated in terms of the closed convex hull V of the support of the distribution π of the random price vector $\gamma(s_t)$. It turns out that if no mass of this distribution is concentrated on the relative boundary of V, then arbitrage opportunities do not exist. Conversely, if $\pi(\partial_r V) > 0$, then arbitrage opportunities exist.

The above result has the following immediate consequences.

Corollary 1. *If the price vector* $\gamma(s_t)$ *takes on a finite number of values, then an arbitrage opportunity exists.*

Corollary 2. *If the distribution of* $\gamma(s_t)$ *is absolutely continuous with respect to the Lebesgue measure, then there are no arbitrage opportunities in the market.*

Thus, the answer to the question of arbitrage depends, roughly speaking, on whether the distribution of the price vector $\gamma(s_t)$ is continuous or discrete. This answer seems somewhat unexpected, and although the result is mathematically simple, we believe that it deserves attention. The question of arbitrage in the stationary context was raised by W. Schachermayer at a conference on Mathematical Finance (Paris, 2003) in the course of a discussion of the paper by Dempster *et al.* (2003). The paper examined the phenomenon of exponential growth of fixed-mix strategies in stationary markets. W. Schachermayer put forward an intuitive explanation of this phenomenon, by linking growth to asymptotic arbitrage (e.g. Klein and Schachermayer 1996). The informal reasoning was based on the fact that the class of stationary processes is in a sense "opposite" to the class of martingales, whereas the no-arbitrage hypothesis requires the existence of an equivalent martingale measure. Our results show that this reasoning cannot be fully formalized. In the context of stationary markets we consider, arbitrage properties over finite time horizons are not determined by stationarity itself. They depend on some (at first glance irrelevant) properties of the probability distribution of the vector of asset prices.

Relations between arbitrage and stationarity were examined in different (deterministic) settings by Cantor and Lipmann (1995) and Adler and Gale (1997).

We conclude this discussion with one more corollary dealing with the case where there is only one risky asset, and so $\gamma(s_t)$ is a scalar-valued random variable.

Corollary 3. *Let $\gamma(s_t)$ be one-dimensional. Then the following assertions are equivalent.*

(a) There is an arbitrage opportunity in the market under consideration.

(b) There is a number r such that $P\{\gamma(s_t) = r\} > 0$ and either $P\{\gamma(s_t) \le r\} = 1$ or $P\{\gamma(s_t) \ge r\} = 1$.

4. Proofs. The proof of Theorem 1 is based on two propositions.

Proposition 1. *If $\pi(\partial_r V) = 0$, then the inequality*

$$\xi\left(s^t\right)\left[\gamma(s_{t+1}) - \gamma(s_t)\right] \ge 0 \text{ (a.s.)} \tag{3}$$

can hold for a measurable vector function $\xi(s^t)$ only if

$$\xi\left(s^t\right)\left[\gamma(s_{t+1}) - \gamma(s_t)\right] = 0 \text{ (a.s.)}.$$

Proof. We will use the following fact:

$(*)$ a point x belongs to $\partial_r V$ if and only if there exists a linear function l on R^n such that

(i) $ly \ge lx$ for all $y \in V$;

(ii) $ly_0 > lx$ for some $y_0 \in V$.

Suppose there is a measurable vector function $\xi(s^t)$ such that inequality (3) holds a.s. and is strict with strictly positive probability. Inequality (3) implies that for almost all s^t, we have

$$\xi\left(s^t\right)\left[w - \gamma(s_t)\right] \ge 0 \tag{4}$$

for all $w \in W$ and hence for all $w \in V$. Indeed, since s^t and s_{t+1} are independent, relation (3) implies that for almost all s^t, the affine function $f(s^t, x) := \xi(s^t)(x - \gamma(s_t))$ is non-negative for π-almost all x and hence it is non-negative for all x in the support W of the measure π, which yields (4) for all $w \in V$. By using assertion $(*)$ with $l := \xi(s^t)$, we conclude that with positive probability, $\gamma(s_t) \in \partial_r V$ (when $\xi(s^t)[\gamma(s_{t+1}) - \gamma(s_t)] > 0$ and $\gamma(s_{t+1}) \in W$), and so $\pi(\partial_r V) > 0$, which is a contradiction. $\square$

Proposition 2. *If $\pi(\partial_r V) > 0$, then there exists a measurable function $\xi(s_t)$ such that*

$$\xi(s_t)\left[\gamma(s_{t+1}) - \gamma(s_t)\right] \geq 0 \text{ (a.s.)} \tag{5}$$

and

$$P\{\xi(s_t)\left[\gamma(s_{t+1}) - \gamma(s_t)\right] > 0\} > 0. \tag{6}$$

Proof. For each $x \in \partial_r V$, consider a linear function $l_x(\cdot)$ on $\mathbb{R}^n$ satisfying conditions (i) and (ii) above. By virtue of Aumann's measurable selection theorem (see, e.g., Arkin and Evstigneev 1987, Appendix I, Section 5) we can find a version of this function which is Borel measurable in x and satisfies (i) and (ii) for π-almost all $x \in \partial_r V$. Define

$$\xi(s_t) = \begin{cases} l_{\gamma(s_t)} \text{ if } \gamma(s_t) \in \partial_r V, \\ 0 \text{ otherwise.} \end{cases}$$

Then the function $\xi(s_t)$ will be measurable as a composition of two measurable functions $\gamma(s_t)$ and l_x. From its definition, we immediately obtain (5). If (6) does not hold, then we can find some $s_t = \tilde{s}_t$ for which $\gamma(\tilde{s}_t) \in \partial_r V$, $l_{\gamma(\tilde{s}_t)}$ satisfies conditions (i) and (ii), and we have $\xi(\tilde{s}_t)\left[\gamma(s_{t+1}) - \gamma(\tilde{s}_t)\right] = 0$ for almost all s_{t+1}. This implies that $\xi(\tilde{s}_t)\left[w - \gamma(\tilde{s}_t)\right] = 0$ for all $w \in W$ and hence for all $w \in V$. This contradicts property (ii) of $\xi(s_t) = l_{\gamma(\tilde{s}_t)}$. □

Proof of Theorem 1. Immediate from Propositions 1, 2 and the equivalence of hypotheses (NA) and (NA′). □

Proof of Corollary 1. In this case, the set V is a convex polyhedron and each vertex of it carries a strictly positive mass of π. Consequently, $\pi(\partial_r V) > 0$, and by virtue of Theorem 1, we conclude that arbitrage opportunities exist. □

Proof of Corollary 2. If the distribution π of $\gamma(\cdot)$ is continuous, we have $\pi(\partial_r V) = 0$ because the Lebesgue measure of the boundary of a closed convex set is zero. □

(a) There is an arbitrage opportunity in the market under consideration.

(b) There is a number r such that $P\{\gamma(s_t) = r\} > 0$ and either $P\{\gamma(s_t) \leq r\} = 1$ or $P\{\gamma(s_t) \geq r\} = 1$.

Proof of Corollary 3. Condition (b) implies (a) because r is on the boundary of the closed convex hull of the support of the distribution π and r is an atom of π. To prove that (a) implies (b), we observe that in the one-dimensional case any closed convex set is either a segment $[a, b]$, or a half line $[a, +\infty)$, or a half line $(-\infty, b]$, or the whole real line $(-\infty, +\infty)$. Since prices are non-negative, the last two cases can be excluded. If arbitrage opportunities exist in the first or the second case, then either a or b (or both) can play the role of the number r described in assertion (b). $\square$

Remark 2. Although in the case of a finite number of values of $\gamma(s_t)$, we always can construct an arbitrage opportunity, this is not necessarily so for a random variable $\gamma(\cdot)$ taking on a countable number of values. The following example illustrates this. Suppose $\gamma(\cdot)$ takes on with strictly positive probabilities each of the following values: $1 + n^{-1}$, $n = 2, 3, ...$; $2 - n^{-1}$, $n = 2, 3, ...$.Then the closed convex hull of the support of this distribution is $[1, 2]$, but this distribution assigns zero mass to its boundary, $\{1\} \cup \{2\}$.

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