The University of Manchester

Economics
Discussion Paper Series
EDP-0617

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First Published July 2006
Revised October 2007

Economics
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October 2007[‡]

Abstract

This paper analyzes collective choices in a society with strategic voters and single-crossing preferences. It shows that, in addition to single-peakedness, single-crossingness is another meaningful domain over the real line that guarantees the existence of non-manipulable social choice functions. A social choice function is shown to be anonymous, unanimous and strategy-proof on single-crossing domains if and only if it is an extended median rule with $n-1$ parameters distributed on the end points of the feasible set of alternatives. Such rules are known as *positional dictators*, and they include the median choice rule as a particular case. As a by-product, the paper also provides an strategic foundation for the so called "single-crossing version" of the Median Voter Theorem, by showing that the median ideal point can be implemented in dominant strategies through a simple mechanism in which each agent honestly reveals his preferences.

JEL codes: D70, D71.

Keywords: Strategy-proofness; single-crossing; median voter; positional dictators.

1 Introduction

It is well known in economic theory that majority rule and other voting rules may fail to produce acyclic social preferences if neither, the set of

*I warmly thank Nick Baigent, John Duggan, Christian List, Antoine Loeper, Paola Manzini, Jordi Massó, Marco Mariotti, Bernardo Moreno, Herve Moulin, Alejandro Neme, Fernando Tohmé, seminar audiences at LSE, Manchester, Osnabrück, Queen Mary, and Warwick, and participants at the 2005 World Congress of the Econometric Society for their useful comments and suggestions. All remaining errors are my own responsibility.

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‡First Draft: January 2005.

alternatives, nor individual preferences are suitably restricted. It is also known that any voting method defined for all rational preferences over a set of three or more alternatives may be subject to the misrepresentation of individual preferences (Gibbard [18] and Satterthwaite [34]).

To study the validity of these results in more specific economic and political environments, it is common in social choice theory to appropriately restrict the set of individual preferences. If alternatives can be placed over the real line, as for instance when different levels of a public good or different tax rates are the subject of a collective choice, a natural preference restriction is *single-crossingness* (SC). The other one is, of course, single-peakedness.

Single-crossingness makes sense in many political-economic settings. It is technically useful, because it accommodates non-convexities that arise in important applications of majority voting. And it has been extensively used in the literature on political economy in areas such as income taxation and redistribution (Roberts [28], Meltzer and Richard [23], Gans and Smart [17]), local public goods and stratification (Westhoof [36], Epple et al. [13], Epple and Platt [14], Epple et al. [15], Calabrese et al. [7]), coalition stability (Demange [10], Kung [20]) and, more recently, to study policies in the market for higher education (Epple et al. [16]) and the citizen candidate model under uncertainty (Eguia [11]).

In words, a society possesses single-crossing preferences if, given any two policies, one of them more to the right than the other, the more rightist the individual is with respect to the other agents, the more he will be willing to support the right-wing policy over the left-wing one. Thus, for example, if alternatives represent income tax rates, and individuals are *ordered* according to their incomes, this restriction simply means that, the richer the individual is, the lower the tax rate he will be willing to support.

Like other domain conditions, single-crossingness establishes restrictions across individual preferences, i.e. on the character of voters' heterogeneity. However, it does not impose any restriction on the shape of each individual preference relation. The main idea behind SC is that, in some cases, individual preferences are determined by a single parameter, or *type*, such as productivity, income, intertemporal preferences, ideology, etc. Then, agents' preferences are restricted in such a way that, for any pair of alternatives, say x and y, whenever two types, say θ' and θ'', agree to prefer x to y, so do all agents with types *in between*, so that the set of types preferring one of the alternatives all lie to one side of those who prefer the other.

Technically, SC not only guarantees the existence of majority voting equilibria, but it also provides a simple characterization of the core of the majority rule.[1] In effect, the core is simply the ideal point of the median type agent, where the latter is defined over the ordering of individual types that makes preferences single-crossing.[2] This result appeared first in the seminal works of Roberts [28] and Grandmont [19] and, more recently, in Rothstein [30], Gans and Smart [17] and Austen-Smith and Banks [1]. It is sometimes referred to as the *Representative Voter Theorem* (RVT) or, alternatively, as "the second version" of the *Median Voter Theorem* (MVT).

The problem with this result is that, unlike the MVT over single-peaked preferences, whose non-cooperative foundation was provided by Black [5], first, and then by Moulin [24], the RVT is based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere* voting. Hence, a natural question about its legitimacy arises when individual values are private information and voters can behave strategically.

This issue has been recently addressed by Saporiti and Tohmé [33]. In that paper, we showed that SC is sufficient to ensure the existence of non-manipulable social choice rules. In particular, this is true for the median choice rule, which is strategy-proof and group strategy-proof over the full set of alternatives and over every possible policy agenda.

Taking that work as the starting point, in this paper we characterize the family of *anonymous* (A), *unanimous* (U) and *strategy-proof* (SP) social choice functions on single-crossing domains. This family coincides with the class of *positional dictators*, which are extended median rules with $n-1$ parameters distributed on the end points of the feasible set of alternatives. It includes the median choice rule as a particular case.

Although the word "dictator" may initially generate a negative feeling toward our characterization, it is worth noting that the result is far from being a negative one. Anonymity and unanimity are very weak conditions, and strategy-proofness is a desirable incentive compatibility property that is frequently demanded in social choice. On the other hand, as will be clear in Section 2, a positional dictator is an *anonymous* social choice function that only considers the ordering of the announced most preferred alternatives, and always chooses one at a specified rank (e.g., the first ideal point, the second, the median, etc.). The preselected position is a "dictator". But, since in different profiles different individuals may locate at that position,

[1] The core of a preference aggregation rule at any profile of individual preferences is the set of top ranked alternatives of the social preference relation (Austen-Smith and Banks [1], p. 99).

[2] By contrast, under single-peakedness, the core of the majority rule is given by the median ideal point over the ordering of alternatives that makes preferences single-peaked.

there is no such a thing as a dictator, as it is usually understood in social choice.

In our model, positional dictators refer to the simple majority rule and other qualified majorities. Hence, the main message coming out from the analysis is that single-crossing is another simple example, besides single-peakedness, where majority voting works with "maximal" incentives properties. The article explains the root of this good property of single-crossing domains, and how far we can go in changing the majority rule.

To summarize the contribution of this article and to compare it with other important results on the real line, namely, with Moulin's [24] seminal work, we draw a diagram below that shows the family of A, U and SP social choice functions on single-peaked and single-crossing domains.[3] As the figure illustrates, since SC allows any shape in individual preferences, it leads to a smaller (but still large) family of strategy-proof social choice rules. Incidentally, the picture also points out that the class of non-manipulable rules in the intersection of these two domains (whenever nonempty) is still an open question. To the best of the author's knowledge, this subdomain, which contains preferences such as the Euclidean one, has not received enough attention, and a full characterization is still missing.

In the rest of the paper we proceed as follows. In Section 2, we present the model, the notation and definitions. We also restrict the domain of admissible preferences, by introducing the formal definition of single-crossingness. We briefly discuss its relation with single-peakedness, and we prove in Proposition 1 an important property of single-crossing domains, which basically means that, in theory, every agent has the possibility of misrepresenting his preferences, so that strategy-proofness in this framework is not vacuous.

In Section 3, we reproduce, for completeness, the nonstrategic version of the Representative Voter Theorem (Theorem 1), which is, until now, the most important social choice result on single-crossing domains. Then, in Section 4 we present the main results of the paper. We start by proving that every positional dictator is *group strategy-proof* (GSP) on single-crossingness (Proposition 2). Then, in Theorem 3, we show that, although single-crossing does not satisfy Weymark's [37] regularity, U and SP imply tops-onliness (TO). Finally, using anonymity and unanimity as auxiliary conditions, we prove that every strategy-proof social choice function is a positional dictator

[3] Moulin's [24] original characterization on single-peaked preferences over the real line has been extended in several directions by many authors. Some important references within this literature are Border and Jordan [6], Zhou [38], Barberà et al. [2], Barberà and Jackson [3], Ching [9], Berga [4], Schummer and Vohra [35], and Ehlers et al. [12], but this list is by no means exhaustive.

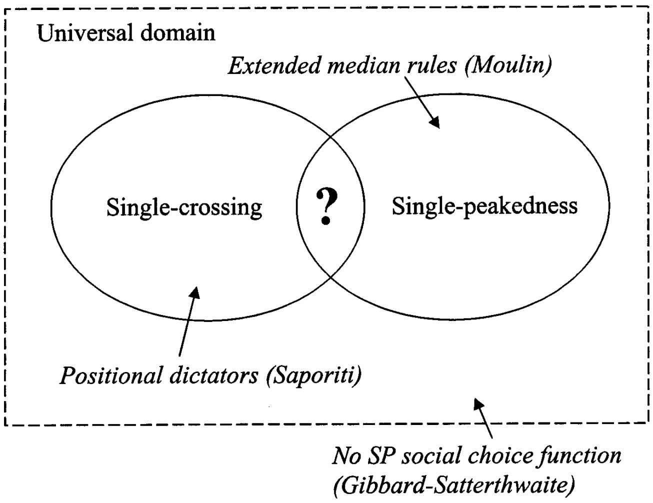

Figure 1:

(Theorem 2), with the natural corollary that in our framework U, A and SP imply Pareto efficiency (Corollary 1). Final remarks and the main contributions of the current research appear in Section 5. The Appendix at the end of the paper contains missing proofs and auxiliary results.

2 Preliminaries

Consider a society $I = \{1, 2, \ldots, n\}$ with a finite number $n \geq 2$ of individuals, who must choose an alternative (e.g. a tax rate) from a finite set $X = \{x, y, \ldots\}$, $|X| > 2$, of the nonnegative real line $\Re_+$.[4]

Let $\mathcal{P}$ be the set of complete, transitive and antisymmetric binary relations over X. Assume each agent $i \in I$ is endowed with a type $\theta_i \in \Theta \subset \Re$, $|\Theta| \geq |\mathcal{P}|$, which completely characterizes his preferences over X.[5] That is, suppose there exists a function $p : \Theta \to \mathcal{P}$ that assigns a unique binary rela-

[4]For every set A, $|A|$ stands for the cardinality of the set.

[5]Abstract models of single-crossing and order-restricted preferences, such as Rothstein [29] and [30], Gans and Smart [17] and Austen-Smith and Banks [1], do not usually distinguish between individuals and types. Here, this simplification is not convenient, because it can lead to the wrong conclusion that the only social choice function that is not *manipulable* on single-crossing domains is the dictatorial one.

tion $p(\theta) \in \mathcal{P}$ to each type $\theta \in \Theta$. We interpret $p(\theta_i)$ as agent i's preferences over X, and we call $\tau(\theta_i)$ his most preferred alternative on X according with $p(\theta_i)$. The next example, taken from Persson and Tabellini [27], illustrates how our abstract setup may naturally emerge in political economy.

Example 1 *Consider the following version of Roberts' [28] model on redistributive linear tax schemes. Suppose each agent i has preferences $u(c_i, l_i) = c_i + v(l_i)$, where c_i denotes private consumption, l_i leisure time, and $v(l_i)$ a continuous and concave function. Let $c_i \leq (1-t)h_i + f$ be the individual budget constraint, where $t \in (0,1)$ is an income tax rate, h_i the individual labor supply, and $f = (\sum_{i \in I} t\, h_i)/n$ a lump-sum transfer.*[6] *Assume each agent is endowed with productivity $\theta_i \in \Re$, and let $l_i + h_i \leq 1 - \theta_i$ be his effective time constraint. If we solve the constrained maximization problem of each individual and substitute the solution into his utility function, then the indirect utility associated to a tax rate t is given by $w(t, \theta_i) = u(c_i^*(t, \theta_i), l_i^*(t, \theta_i)) = h(t) + v[1 - h(t) - \bar{\theta}] - (1-t)(\theta_i - \bar{\theta})$, where $h(t) = 1 - \bar{\theta} - v_l^{-1}(1-t)$ is the average labor supply, v_l the first derivative of $v(l_i)$, and $\bar{\theta}$ the mean productivity. Hence, if there are for example three possible tax rates, say t^1, t^2 and t^3, and $w(t^2, \theta_i) > w(t^1, \theta_i) > w(t^3, \theta_i)$, then agent i's induced preference relation $p(\theta_i) : t^2 t^1 t^3$ over tax rates is fully determined by his productivity θ_i.* $\square$

Like in the previous example, in our framework p is assumed to be equal across agents. Hence, when there is no confusion, each preference relation $p(\theta_i)$ and the profile of society's preferences $(p(\theta_1), \ldots, p(\theta_n))$ are directly represented through their associated types. Abusing the notation, θ will denote a single element of Θ, or a profile of types. As usual, $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$; for each $\hat{\theta}_i$, $(\hat{\theta}_i, \theta_{-i}) = (\theta_1, \ldots, \theta_{i-1}, \hat{\theta}_i, \theta_{i+1}, \ldots, \theta_n)$; and, for every set $S \subseteq I$, $(\theta_S, \theta_{\bar{S}}) = (\{\theta_i\}_{i \in S}, \{\theta_j\}_{j \in \bar{S}})$, where $\bar{S} = I \backslash S$ is the complement of S.

Denote $p(\Theta) \subseteq \mathcal{P}$ the set of all binary relations over X generated by p. If individual preferences were not restricted, then p would be onto, and $p(\Theta)$ would coincide with $\mathcal{P}$. Instead, motivated by Roberts [28] and the rest of the references listed in the Introduction, here we restrict society's admissible preferences to a subset of $\mathcal{P}^n$. Specifically, we focus on a mapping p on Θ that imposes the *single-crossing property* on preference profiles, but no particular shape on each preference relation. Formally,

[6] The real wage is exogenous and normalized at 1.

Definition 1 (SC) *A function $p : \Theta \to \mathcal{P}$ generates single-crossing preference profiles $(p(\theta_1), \ldots, p(\theta_n))$ on X if for all $x, y \in X$, and all $i, j \in I$, if $y > x$, $\theta_j > \theta_i$, and $y\, p(\theta_i)\, x$, then $y\, p(\theta_j)\, x$.*[7]

We call $\mathcal{SC}$ the set of all single-crossing preference profiles. Notice that, when p is held fixed and is equal across agents, and individuals share the same set Θ of possible types, this preference domain becomes a Cartesian product of the form $\mathcal{SC} = p(\Theta)^n$, (namely, the set of preference profiles that can be built with preferences out of $p(\Theta)$).[8] This, of course, is no longer true if the previous conditions are not satisfied. For more details, see Saporiti and Tohmé [33] and, especially, Campbell and Kelly [8], who analyze (a weaker version of) strategy-proofness in preference domains where a Condorcet winner always exists, but that are not necessarily a product set.

The recent interest in single-crossingness is due to the fact that, like single-peakedness, this domain restriction is sufficient to guarantee the existence of majority voting equilibria (see Theorem 1 below). However, apart from this similarity, it should be clear that both conditions are totally independent, in the sense that neither property is logically implied by the other.[9] In Example 1, for instance, the profile of induced preferences is single-crossing on the interval $(0, 1)$, because for any two policies t', $t'' \in (0, 1)$, such that $t' > t''$, the difference $w(t', \theta) - w(t'', \theta)$ is strictly increasing in θ. Instead, for $h(t)$ sufficiently convex, it might violate single-peakedness. Examples 2 and 3 below provide other cases that also illustrate this point.[10]

Example 2 *Assume that individual preferences are as in Table 1, where x, y, $z \in \Re_{++}$, $x < y < z$, and $\theta_1 < \theta_2 < \theta_3$. This profile is single-crossing on $\{x, y, z\}$. However, for any ordering of the alternatives, it violates single-peakedness, because every alternative is ranked bottom in one individual ordering.* $\square$

[7] Other restrictions related with single-crossing are *hierarchical adherence*, *intermediateness*, *order-restriction* and *unidimensional alignment*. For more details, see Roberts [28], Grandmont [19], Rothstein [29] and [30], Gans and Smart [17], Myerson [26], Austen-Smith and Banks [1], List [21] and Saporiti and Tohmé [33].

[8] In Saporiti [32], we discuss how single-crossing and order-restriction are intimately related with the possibility of linearly ordering the set $\mathcal{P}$ of preference relations.

[9] As Gans and Smart [17] showed, single-crossingness is equivalent to Rothstein's [29] and [30] order-restriction (OR), and OR (on triples) is strictly weaker than single-peakedness and single-cavedness, but strictly stronger than Sen's value-restriction, (see Theorems 2 and 3 in Rothstein [29]).

[10] The interesting difference between single-crossing and single-peakness is that the latter is a unique domain once alternatives are ordered, whereas there are still many different SC domains compatible with a given ordering of X. On the other hand, unlike single-peaked preferences, their union covers all preferences on X.

Example 3 *Consider the profile displayed in Table 2, where* w, x, y, $z \in \Re_+$. *These preferences are single-peaked with respect to* $w < x < y < z$. *On the contrary, for every ordering of the types, they violate single-crossing. Moreover, they violate SC not only for* $w < x < y < z$, *but for every ordering of them.* □

Table 1: Single-crossingness

$p(\theta_1) : xyz$
$p(\theta_2) : xzy$
$p(\theta_3) : zyx$

Table 2: Single-peakedness

$p(\theta_1) : xyzw$
$p(\theta_2) : zyxw$
$p(\theta_3) : yxwz$

Since we are interested in social choice functions that are not manipulable over $\mathcal{SC}$, in what follows we restrict our attention to *maximal* domains of single-crossing preferences, in the sense that it would be impossible to add another type in Θ and an associated preference relation in $p(\Theta)$ such that every profile of the enlarged domain $p(\Theta)^n$ still satisfies Definition 1. These domains contain the largest number of possible deviations. Therefore, they are the appropriate framework to analyze incentive compatibility.

In order to make social choices, individual preferences must be aggregated. In this work, we suppose that p is commonly known, but that individual types are private information. Thus, the input for the aggregation process is the set of individuals' *reports* about their preferences. These declarations are intended to provide information about the profile of true types, although agents' sincerity cannot be ensured.

The aggregation process is represented by a social choice function. A social choice function is a single-valued mapping $f : \mathcal{SC} \to X$ that associates to each profile $\theta \in \mathcal{SC}$ a unique outcome $f(\theta) \in X$. Denote $r_f = \{x \in X : \exists\, \theta \in \mathcal{SC}$ such that $f(\theta) = x\}$ the range of f. Given f, $S \subset I$ and $\theta_{\bar{S}} \in p(\Theta)^{|\bar{S}|}$, we call $O^f_S(\theta_{\bar{S}}) = \{x \in X : \exists\, \theta_S \in p(\Theta)^{|S|}$ such that $f(\theta_S, \theta_{\bar{S}}) = x\}$ the *option set* of S, given that the rest, i.e. individuals in $\bar{S} = I \backslash S$, have reported $\theta_{\bar{S}}$.

We are interested in social choice functions that satisfy the following properties on $\mathcal{SC}$. The main one is that agents, acting individually or in groups, never have incentives to misrepresent their preferences. These properties are now formally stated in Definitions 2 and 3, respectively.

Definition 2 (SP) *A social choice function* f *is strategy-proof on* $\mathcal{SC}$ *if for all* $i \in I$, *and all* $(\theta_i, \theta_{-i}) \in \mathcal{SC}$, *there does not exist* $\hat{\theta}_i \in p(\Theta)$ *such that* $f(\hat{\theta}_i, \theta_{-i})\, p(\theta_i)\, f(\theta_i, \theta_{-i})$.

In words, a social choice function f is SP on $\mathcal{SC}$ if for *any* possible report $\theta_{-i} \in p(\Theta)^{n-1}$ that the rest of the agents could make, no individual $i \in I$ would find profitable to make a declaration $\hat{\theta}_i \in p(\Theta)$ different from his own type θ_i. On the contrary, if f is not strategy-proof, then there must exist at least one agent who would be strictly better off misrepresenting his preferences. Therefore, we say that f is *manipulable* by this individual.

Proceeding in a similar way, we can also define *group strategy-proofness*, to study the possibility of group deviations.

Definition 3 (GSP) *A social choice function f is group strategy-proof on $\mathcal{SC}$ if for all $S \subseteq I$, and all $(\theta_S, \theta_{\bar{S}}) \in \mathcal{SC}$, there does not exist $\hat{\theta}_S \in p(\Theta)^{|S|}$ such that $f(\hat{\theta}_S, \theta_{\bar{S}})\, p(\theta_i)\, f(\theta_S, \theta_{\bar{S}})$ for all $i \in S$.*

Another property that we may seek in a social choice function is *unanimity.* This property ensures that, if all agents have the same most preferred alternative, then that alternative is socially selected.

Definition 4 (U) *A social choice function f is unanimous on $\mathcal{SC}$ if for all $x \in X$, and all $\theta \in \mathcal{SC}$ such that $\tau(\theta_i) = x$ for all $i \in I$, $f(\theta) = x$.*

Let $\sigma : I \to I$ be a permutation of the set of individuals. A profile $\theta \in \mathcal{SC}$ is a σ-permutation of another profile $\theta^* \in \mathcal{SC}$ if for every individual $i \in I$, $\theta_i = \theta^*_{\sigma(i)}$. That is, θ is a σ-permutation of θ^* if the lists of preferences under θ and θ^* are identical up to a renaming of agents. We refer to such a pair (θ, θ^*) as a σ-permutation.

Definition 5 (A) *A social choice function f is anonymous on $\mathcal{SC}$ if for each σ-permutation (θ, θ^*), $f(\theta) = f(\theta^*)$.*

In words, a social choice function is anonymous if the names of the individuals holding particular preferences are immaterial in deriving social choices.

One last property that a social choice function may satisfy is *tops-onliness*. We say that f is *tops-only* on $\mathcal{SC}$ if for any preference profile, the social choice is exclusively determined by individuals' most preferred alternatives on the range of the social choice function.

Definition 6 (TO) *A social choice function f is tops-only on $\mathcal{SC}$ if, for all θ, $\hat{\theta} \in \mathcal{SC}$ such that $\tau|_{r_f}(\theta_i) = \tau|_{r_f}(\hat{\theta}_i)$ for all $i \in I$, $f(\theta) = f(\hat{\theta})$.*

Tops-onliness dramatically constrains the scope for manipulation. No agent can expect to be able to affect the social outcome without modifying the peak on r_f of his reported ordering. However, as we show later in Theorem 3, this condition is closely related to SP, in the sense that every U and SP social choice function on single-crossing domains is also TO.

Now we define a class of social choice functions that plays a crucial role in the characterization given in Section 4. To do that we introduce the following notation. For any nonempty subset $V \subseteq \Re$, and any odd positive integer k, we say that $m^k : V^k \to V$ is the k-*median* function on V^k if for each $v = (v_1, \ldots, v_k) \in V^k$, $|\{v_i : m^k(v) \geq v_i\}| \geq \frac{(k+1)}{2}$, and $|\{v_j : v_j \geq m^k(v)\}| \geq \frac{(k+1)}{2}$. Since k is odd, $m^k(v)$ is always well defined.

Definition 7 (EMR) *A social choice function f is an extended median rule on $\mathcal{SC}$ if there exist $n+1$ parameters $\alpha_i \in X$, $i = 1, 2, \ldots, n+1$, also called fixed ballots or phantom voters, such that for all $\theta \in \mathcal{SC}$, $f(\theta) = m^{2n+1}(\tau(\theta_1), \ldots, \tau(\theta_n), \alpha_1, \ldots, \alpha_{n+1})$.*

We denote by f^e a social choice function that satisfies Definition 7, and by $EMR = \{f^e : (\alpha_1, \ldots, \alpha_{n+1}) \in X^{n+1}\}$ the family of all such functions, obtained by reallocating the parameters $\alpha_1, \ldots, \alpha_{n+1}$ in X^{n+1}. A particular case of interest within this family is the well known *median choice rule*, noted f^m, which is obtained from f^e by assigning $(n+1)/2$ fixed ballots at $\underline{X} \equiv \min X$ and the rest at $\overline{X} \equiv \max X$, if n is odd, and $n/2$ at $\underline{X}$ and $n/2+1$ at $\overline{X}$ if n is even.

Proceeding in a similar way, we can derive other rules from EMR, by restricting each α_i to a particular value of X. For example, if $\alpha_i = \alpha$ for all $i = 1, 2, \ldots, n+1$, then f^e is completely insensitive to the preferences reported by the individuals. We might want to exclude such undesirable rules and, in particular, require Pareto efficiency.[11] To do that, we eliminate the possibility of inefficiency by setting $\alpha_n = \underline{X}$ and $\alpha_{n+1} = \overline{X}$. Then, we obtain a social choice rule, noted f^*, with the property that for all $\theta \in \mathcal{SC}$, $f^*(\theta) = m^{2n-1}(\tau(\theta_1), \ldots, \tau(\theta_n), \alpha_1, \ldots, \alpha_{n-1})$. This rule is called the *efficient extended median rule*, and it is characterized by $n-1$ parameters distributed on X^{n-1}. The set of all such rules is denoted $EMR^* = \{f^* : (\alpha_1, \ldots, \alpha_{n-1}) \in X^{n-1}\}$.

Finally, we can also restrict each α_i to take its value at either $\underline{X}$ or $\overline{X}$, so that each phantom voter is either a *leftist* or a *rightist*. The family of

[11] A social choice function f is Pareto efficient on $\mathcal{SC}$ if for all $\theta \in \mathcal{SC}$, there does not exist $y \in X$ such that $y\, p(\theta_i)\, f(\theta)$ for all $i \in I$.

social choice functions obtained in that way was first introduced by Moulin [25], and it is known as *positional dictators*.

These rules select the j-th peak among the tops of the reported preference orderings, for some $j \in \{1, \ldots, n\}$. For example, if $j = 1$, we have the *leftist rule*, which always chooses the smallest reported peak. The median choice rule f^m is also a particular case. We denote by f^j the positional dictator that selects, for all $\theta \in \mathcal{SC}$, the alternative of the sequence $\tau(\theta_1), \ldots, \tau(\theta_n)$ placed at the j-th position according with the order of X. This rule is obtained from f^* by distributing $n - j$ fixed ballots at $\underline{X}$ and the remaining $j - 1$ at $\overline{X}$. The family of all such rules is denoted $PD = \{f^j; j = 1, \ldots, n\}$.

In Section 4, we study how well these choice rules perform, according with the manipulation criteria given above, on single-crossing domains. Before doing that, however, we prove an important property of single-crossingness, which says that (maximal) preference domains satisfying SC are such that, in theory, every agent has the possibility of misrepresenting his preferences, so that incentive compatibility in this framework is not vacuous.

Proposition 1 *For every agent $i \in I$, and each profile $(p(\theta_i), p(\theta)_{-i}) \in \mathcal{SC}$, there exist at least two alternative orderings $p(\theta_i')$ and $p(\theta_i'')$ such that the resulting preference profiles $(p(\theta_i'), p(\theta)_{-i})$ and $(p(\theta_i''), p(\theta)_{-i})$ belong to $\mathcal{SC}$.*

Proof: Fix an agent $i \in I$ and a profile $p(\theta) = (p(\theta_1), \ldots, p(\theta_i), \ldots, p(\theta_n)) \in \mathcal{SC}$. Let $T(p(\theta)) = \{\theta_1, \ldots, \theta_i, \ldots, \theta_n\}$ be the set of *actual* types associated with $p(\theta)$, and $T^*(p(\theta)) \subseteq T(p(\theta))$ the subset associated with different binary relations, in the sense that for all $\theta_j, \theta_k \in T^*(p(\theta))$, $p(\theta_j) \neq p(\theta_k)$. If $|T^*(p(\theta))| > 2$, then the result is immediately obtained.

On the other hand, if $|T^*(p(\theta))| = 2$, assume without loss of generality that $T^*(p(\theta)) = \{\theta^-, \theta^+\}$, where $\theta^- < \theta^+$. Suppose $p(\theta^-)$ and $p(\theta^+)$ differ in the pair (z, w), where $w > z$. By SC, $z\, p(\theta^-)\, w$ and $w\, p(\theta^+)\, z$. Consider an ordering $p(\theta')$ that coincides with $p(\theta^-)$ for every pair of alternatives, except for (z, w), and rank $w\, p(\theta')\, z$. This ordering exists because $p(\Theta)$ is maximal. Moreover, $\theta^- < \theta' \leq \theta^+$, because p generates SC preferences on X. If $p(\theta') \neq p(\theta^+)$, we are done. Otherwise, if $p(\theta') = p(\theta^+)$, then $p(\theta^-)$ and $p(\theta^+)$ differ *only* in the pair (z, w). Consider $(x, y) \subset X$, such that $x \neq w$ or $y \neq z$. This pair exists because $|X| > 2$. Note that either $x\, p(\theta)\, y$, or $y\, p(\theta)\, x$, for all $\theta = \theta^-, \theta^+$. Then, the desired ordering is given by a preference $p(\theta')$ equal to $p(\theta^-)$ (or equal to $p(\theta^+)$), but that ranks y over x, if $x\, p(\theta^-)\, y$, and x over y otherwise. Thus, if $\theta_i = \theta^-$, types θ^+ and θ' are enough to prove the claim. Otherwise, we use θ^- and θ'.

Finally, if $|T^*(p(\theta))| = 1$, the desired result is obtained following the same reasoning of the previous paragraph. $\square$

3 Representative voter theorem

The domain of single-crossing preferences has two useful properties for collective decision making analysis. Firstly, as was already mentioned in other parts of this paper, it guarantees the existence of majority voting equilibria. Secondly, it offers a simple characterization of the core of the majority rule.

More precisely, let $A(X) = \{Y : Y \in 2^X \backslash \emptyset\}$ be the set of all possible nonempty subsets, or *agendas*, of X. If society possesses single-crossing preferences on X, then for all $Y \in A(X)$, the agent endowed with the *median type* on Θ is *decisive* for every pairwise majority contest among alternatives of Y. This result is sometimes referred to as the Representative Voter Theorem (RVT) or, alternatively, as the "second version" of the Median Voter Theorem.

To formally obtain the RVT, let us now introduce the following additional notation. Given $i \in I$, $\theta_i \in p(\Theta)$ and $Y \in A(X)$, let $p|_Y(\theta_i)$ be agent i's *induced* preferences over Y if, for all x, $y \in Y$, $x\, p|_Y(\theta_i)\, y$ if and only if $x\, p(\theta_i)\, y$. Denote $\tau|_Y(\theta_i)$ agent i's most preferred alternative on Y according with $p|_Y(\theta_i)$. Notice that SC is preserved in the induced preferences. That is, if $(p(\theta_i))_{i \in I} \in \mathcal{SC}$, then for all $Y \in A(X)$, $(p|_Y(\theta_i))_{i \in I} \in \mathcal{SC}_Y$, where $\mathcal{SC}_Y$ represents the set of single-crossing profiles on Y. Abusing the notation, let $f : \mathcal{SC}_Y \to Y$ be the restriction of the social choice function f on Y. The nonstrategic version of the RVT is as follows:[12]

Theorem 1 *For each profile $\theta \in \mathcal{SC}$ and every agenda $Y \in A(X)$, $f^m(p|_Y(\theta)) = \tau|_Y(\theta_r)$, where $\theta_r = m^n(\theta_1, \ldots, \theta_n)$.*

Proof: Fix $p(\theta) \in \mathcal{SC}$ and $Y \in A(X)$. Since induced preferences inherit the single-crossing property, for all $i, j \in I$, $\theta_j > \theta_i \Rightarrow \tau|_Y(\theta_j) \geq \tau|_Y(\theta_i)$. Suppose not. That is, assume there exist $i, j \in I$ such that $\theta_j > \theta_i$ and $\tau|_Y(\theta_i) > \tau|_Y(\theta_j)$. By SC, $\tau|_Y(\theta_i)\, p|_Y(\theta_i)\, \tau|_Y(\theta_j) \Rightarrow \tau|_Y(\theta_i)\, p|_Y(\theta_j)\, \tau|_Y(\theta_j)$: contradiction. Hence, $m^n(\tau|_Y(\theta_1), \ldots, \tau|_Y(\theta_n)) = \tau|_Y(\theta_r)$, where $\theta_r = m^n(\theta_1, \ldots, \theta_n)$. And, by definition, $f^m(p|_Y(\theta)) = \tau|_Y(\theta_r)$. $\square$

[12] Notice that we derive a different version of the RVT, since individual preferences are strict, and we focus on choices instead of social orderings. For a complementary analysis, see Rothstein [30], Gans and Smart [17] and Austen-Smith and Banks [1].

In words, Theorem 1 predicts that, for every agenda $Y \in A(X)$, the alternative chosen through pairwise majority voting by a society with single-crossing preferences on X will coincide with the median type agent's most preferred alternative on Y.[13] This result is, of course, very useful in applications, because it allows to treat the electorate as a single representative voter. However, is it robust to individual and group manipulation? That is, can we expect that society will end up choosing in the way predicted by Theorem 1 if voters behave strategically?

The RVT is a result derived under the assumption that individuals honestly reveal their preferences. This is obviously very strong. However, as we will see in the next section, if we relax this assumption, admitting strategic voting and private information about individual values, it turns out that the RVT still holds. As we explain, the reason for this is that each voter has a dominant strategy on single-crossing domains, which is to honestly report his true preferences. Now we derive this formally.

4 Strategic voting

In this section, we prove that positional dictators is the only family of social choice functions that satisfies U, A and SP on single-crossing domains. At the end, we also show that this is a tight characterization, in the sense that relaxing any of the previous axioms enlarges the family of social choice functions.

We start by proving that every positional dictator is GSP.

Proposition 2 *Each positional dictator f^j is group strategy-proof on $\mathcal{SC}$.*

Proof: Fix $f^j \in PD$. Suppose, by contradiction, there exist a coalition $S \subseteq I$, a profile $(\theta_S, \theta_{\bar{S}}) \in \mathcal{SC}$, and a joint deviation $\hat{\theta}_S \in p(\Theta)^{|S|}$ such that $f^j(\hat{\theta}_S, \theta_{\bar{S}})\, p(\theta_i)\, f^j(\theta_S, \theta_{\bar{S}})$ for all $i \in S$. To simplify, denote $f^j(\theta_S, \theta_{\bar{S}}) \equiv \tau$ and $f^j(\hat{\theta}_S, \theta_{\bar{S}}) \equiv \hat{\tau}$, and let $\tau < \hat{\tau}$.

Note that $f^j \in PD \Rightarrow \alpha_i \in \{\underline{X}, \overline{X}\}$ for all $i = 1, 2, \ldots, n-1$. Hence, τ and $\hat{\tau}$ must coincide with the tops reported by two *real* voters. Denote these agents k and k', and their types θ_k and $\theta_{k'}$, respectively. Then, for all $i \in S$, $\tau(\theta_i) > \tau$. Suppose not. That is, assume $\tau(\theta_i) \leq \tau$ for some agent $i \in S$. If $\tau(\theta_i) = \tau$, then $\tau\, p(\theta_i)\, \hat{\tau}$, which contradicts our initial hypothesis. Instead,

[13] Rothstein [30] also showed that, when preferences are antisymmetric and the number of voters is odd, the majority preference relation coincides with the median type agent's ranking. Hence, it inherits all its properties, including transitivity. Gans and Smart [17] found a similar result for non-strict preferences, but under *strict* single-crossingness.

suppose $\tau(\theta_i) < \tau$. Since $\hat{\tau}\, p(\theta_i)\, \tau$ and $(\theta_S, \theta_{\bar{S}}) \in \mathcal{SC}$, we have that $\hat{\tau}\, p(\theta)\, \tau$ for all $\theta \geq \theta_i$. Then, $\theta_k < \theta_i$. And again, since $(\theta_S, \theta_{\bar{S}}) \in \mathcal{SC}$, $\tau\, p(\theta_k)\, \tau(\theta_i)$ implies $\tau\, p(\theta_i)\, \tau(\theta_i)$: contradiction. Hence, $\tau(\theta_i) > \tau$ for all $i \in S$.

Recall that, by definition, $\tau = m^{2n-1}(\{\tau(\theta_i)\}_{i\in S}, \{\tau(\theta_j)\}_{j\in\bar{S}}, \alpha_1, \ldots, \alpha_{n-1})$ and $\hat{\tau} = m^{2n-1}(\{\tau(\hat{\theta}_i)\}_{i\in S}, \{\tau(\theta_j)\}_{j\in\bar{S}}, \alpha_1, \ldots, \alpha_{n-1})$. Thus, there must exist $i \in S$ such that $\tau(\hat{\theta}_i) < \tau$. Otherwise, if $\tau(\hat{\theta}_i) \geq \tau$ for all $i \in S$, we would have that $\hat{\tau} = \tau$. Therefore, if we rename $(\{\tau(\hat{\theta}_i)\}_{i\in S}, \{\tau(\theta_j)\}_{j\in\bar{S}}, \alpha_1, \ldots, \alpha_{n-1})$ as $(y_1, \ldots, y_{2n-1})$, it follows that $|\{j \in \{1, \ldots, (2n-1)\} : y_j \leq \tau\}| \geq n$. But then $m^{2n-1}(y_1, \ldots, y_{2n-1}) \leq \tau$. That is, $f^j(\hat{\theta}_S, \theta_{\bar{S}}) \leq f^j(\theta_S, \theta_{\bar{S}})$, contradicting that $\tau < \hat{\tau}$. Hence, f^j is GSP on $\mathcal{SC}$. □

Falling short of Moulin's [24] results, Proposition 2 shows that every extended median rule is GSP (and, consequently, SP) on single-crossing domains, provided that each fixed ballot is placed at the end points of X, (i.e., at either $\underline{X}$ or $\overline{X}$).[14] Instead, all other extended median rules, which allow the collective outcome to be the top of a *fictitious* voter, are not guaranteed to be SP on $\mathcal{SC}$.

To see this, consider the profile of Table 1, and a rule $f \in EMR^*$, such that $\alpha_1 = y$ and $\alpha_2 = z$. Note that α_1 coincides with neither voters' most preferred alternatives nor the end points of $X = \{x, y, z\}$, (recall that $\underline{X} = x$ and $\overline{X} = z$). Furthermore, $f(\theta) = m^5(x, x, z, \alpha_1, \alpha_2) = y$. But, since y is agent 2's worst outcome on X, he could report $\hat{\theta}_2 = \theta_3$, and generate the outcome $m^5(x, z, z, \alpha_1, \alpha_2) = z$. Agent 2's deviation would be profitable, because $z\, p(\theta_2)\, y$. Hence, individual manipulation cannot be excluded.[15]

As the example illustrates, SP is not ensured for extended median rules other than positional dictators because the latter are the only one within the class of anonymous social choice functions that guarantee that the social choice always coincides with a type's most preferred alternative. However, as we showed in the proof of Proposition 2, without this information manipulation on single-crossing domains cannot be ruled out, because the argument exploits precisely the correlation among individual preferences together with the fact that the outcome *is* the ideal point reported by a real voter.

[14] Note that placing some parameters of f^* at peaks of actual types, in addition to at $\underline{X}$ or $\overline{X}$, yields the same result. However, we ruled out this to ensure that the social choice function is independent of the particular preference profile under consideration.

[15] Interestingly, in the example, agent 2 would prefer to misrepresent his type even if the other agents are reporting their true preferences. That means extended median rules other than positional dictators not only fail to be SP over $\mathcal{SC}$, but also Nash implementable.

The point is that SC does not restrict the shape of individual preferences. Instead, it allows orderings that do not decrease monotonically to both sides of the ideal point. In fact, this is one of the main reasons why SC is an attractive restriction in certain problems of political economy (such as majority voting on distortionary income tax rates). The price for this flexibility, however, is that in general it is impossible to ensure that no agent could be better off misrepresenting his values.

In Figure 2, for instance, $f(\hat{\theta}_i, \theta_{-i})\, p(\theta_i)\, f(\theta_i, \theta_{-i})$, so that in principle agent i would like to manipulate f at (θ_i, θ_{-i}) via $\hat{\theta}_i$. However, this is not possible if f is a positional dictator. In that case, SC is sufficient to rule out any attempt of individual and group manipulation. For example, suppose that $f(\theta_i, \theta_{-i})$ is j's most preferred alternative. If $f(\hat{\theta}_i, \theta_{-i}) p(\theta_i) f(\theta_i, \theta_{-i})$, like in Figure 2, SC would imply $f(\hat{\theta}_i, \theta_{-i})\, p(\theta_k)\, f(\theta_i, \theta_{-i})$ for all $\theta_k \geq \theta_i$. Thus, $f(\theta_i, \theta_{-i}) = \tau(\theta_j) \Rightarrow \theta_j < \theta_i$. But then agent i's preferences cannot exhibit the shape displayed in the figure. Otherwise, $(\theta_i, \theta_{-i}) \in \mathcal{SC}$, $f(\theta_i, \theta_{-i})\, p(\theta_j)\, \tau(\theta_i)$ and $\theta_j < \theta_i$ would imply $f(\theta_i, \theta_{-i})\, p(\theta_i)\, \tau(\theta_i)$, contradicting that $\tau(\theta_i)$ is agent i's ideal point.

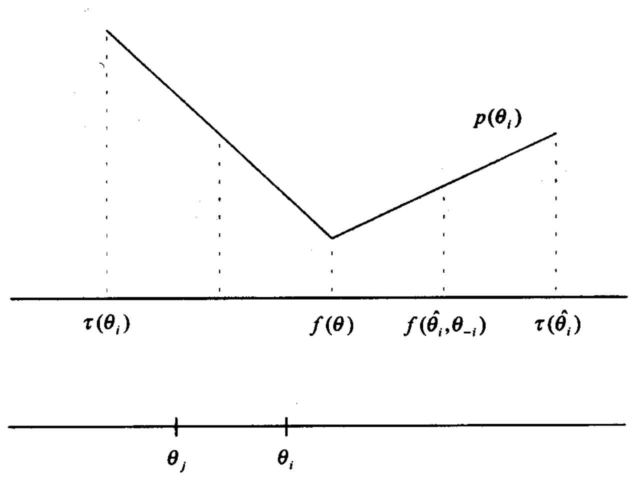

Figure 2:

So, when the social choice function associates to each preference profile an individual's peak, like in the case of positional dictators, the ordering of that agent together with the relation among preferences in single-crossing domains is sufficient to reject any incentive for manipulation. Remarkably, *no additional information about the shape of each preference relation is necessary.*

On the contrary, if social choices are not individual tops, we might think that individuals' preferences can still be inferred from the correlation with other agents' rankings. However, there are profiles on single-crossingness where the way in which one agent orders alternatives bears no relation with other orderings. In those cases, it is impossible to guarantee that all individuals will have the *right* incentives, (i.e., no one will hold an ordering like Figure 2). So, manipulation cannot be excluded.

This conjecture stands in sharp contrast with the main result on single-peaked domains, where extended median rules have been shown to be strategy-proof without any restriction on the distribution of phantom voters. Moreover, it suggests that the family of SP social choice functions on $\mathcal{SC}$ is strictly smaller than the corresponding class on single-peakedness. This is now formally stated and proved in Theorem 2.

Theorem 2 *A social choice function f is unanimous, anonymous, and strategy-proof on $\mathcal{SC}$ if and only if f is a positional dictator.*

Proof: See the Appendix.

Corollary 1 *If a social choice function f is unanimous, anonymous and strategy-proof on $\mathcal{SC}$, then it is Pareto efficient.*

Proof: Suppose, by contradiction, that there exists a social choice function f that satisfies all the hypotheses of Corollary 1, but that f is not Pareto efficient on $\mathcal{SC}$. Then, there must exist $\theta \in \mathcal{SC}$, and a pair $x, y \in X$, $x \neq y$, such that $f(\theta) = x$, while $y\, p(\theta_i)\, x$ for all $i \in I$. Thus, for all $i = 1, \ldots n$, $f(\theta) \neq \tau(\theta_i)$, contradicting that, by Theorem 2, $f \in PD$. □

The proof of Theorem 2, carried out in the Appendix for expositional convenience, rests on two main results. The first one, summarized in Theorem 3 below, shows that on single-crossing domains tops-onlyness is implied by strategy-proofness and unanimity. This result, which is the most important step in the current analysis, is consistent with other results in the literature on strategy-proofness, and captures the intuitive idea that social choice functions that use too much information from society are easier to manipulate.

In our framework, the relationship between SP, U and TO is also interesting because it highlights two important features of single-crossingness.

The first one is that the peak of a preference relation on the entire set X does not necessarily determine its peak on the range of f. This is illustrated in Figure 3, where orderings θ_i and θ_i'' share the same peak $\tau(\theta_i) = \tau(\theta_i'') = w$ over $X = \{x, y, z, w\}$; but if, for example, $r_f = \{z, x, y\}$, then $\tau|_{r_f}(\theta_i) = z \neq y = \tau|_{r_f}(\theta_i'')$.[16]

The second one is that single-crossing is not a *regular* domain. A social choice function f has a regular domain if for any alternative x in the closure of the range of f there is a continuous preference in the individual preference domain that is uniquely maximized on $cl(r_f)$ at x (Weymark [37]). As Figure 3 illustrates, single-crossingness does not satisfy this property. In effect, suppose that there exists $x, y, z \in r_f$, and assume like in the picture that $z < x < y$ and $z\, p(\theta_i)\, x$ and $y\, p(\theta_i)\, x$ for some $i \in I$ and $\theta_i \in p(\Theta)$. Then, it is easy to see that there is no preference relation $p(\theta_i')$ on $cl(r_f)$ such that $\tau|_{r_f}(\theta_i') = x$. On the contrary, suppose that such an ordering exists. If $\theta_i' < \theta_i$, then SC implies $z\, p(\theta_i')\, x$, contradicting that $\tau|_{r_f}(\theta_i') = x$. On the other hand, if $\theta_i' > \theta_i$, then again, by SC, $y\, p(\theta_i')\, x$, and we get the same contradiction.

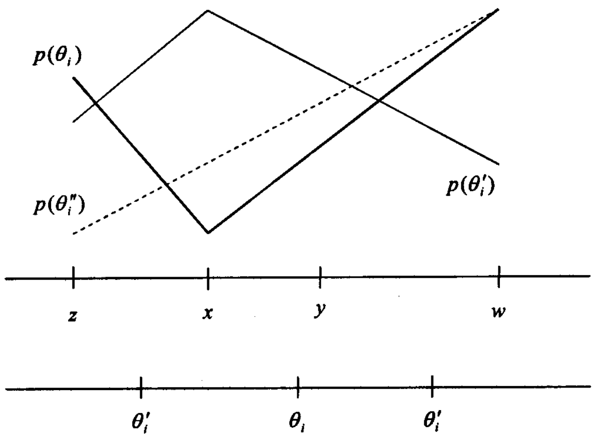

Figure 3:

For the main purpose of this article, the most important implication of the previous two comments is that, in the case of single-crossing, the relationship between strategy-proofness and tops-onliness cannot be examined

[16]Clearly, this cannot happen when preferences are single-peaked and $|r_f| > 2$.

using the general approach proposed by Weymark [37]. That methodology is applicable only if the preference domain is regular and individual ideal points uniquely determine the most preferred alternatives over the range of the social choice function.

Instead, we circumvent this difficulty by following a different approach, which encompasses three main steps: (i) Firstly, we directly prove that, in a two-person society, SP and U implies TO (Proposition 5); (ii) Secondly, using the previous result, we extend the tops-only property to the option sets generated by a unanimous and strategy-proof social choice function (Proposition 3); and, (iii) Finally, invoking (ii), we show that (i) also holds in an n-person society (Theorem 3). To simplify the presentation, we state below only Proposition 3, Theorem 3 and the proof of Theorem 3. The Appendix contains the rest of the analysis.

Proposition 3 *If f is unanimous and strategy-proof on $\mathcal{SC}$, then $\forall\, S \subset I$, and $\forall\, \theta', \theta'' \in \mathcal{SC}$ such that $\tau(\theta'_i) = \tau(\theta''_i)\ \forall\, i \in S$, $O^f_{\bar{S}}(\theta'_S) = O^f_{\bar{S}}(\theta''_S)$.*

Proof: See the Appendix.

Theorem 3 *A social choice function f is unanimous and strategy-proof on $\mathcal{SC}$ only if f is tops-only on $\mathcal{SC}$.*

Proof: Suppose, by contradiction, there exists $(\theta'_i, \theta'_{-i}) \in \mathcal{SC}$, and $\theta''_i \in p(\Theta)$ such that $\tau(\theta'_i) = \tau(\theta''_i)$ and $f(\theta'_i, \theta'_{-i}) = x \neq y = f(\theta''_i, \theta'_{-i})$. Fix $j \neq i$. Since preferences are strict, $x \neq y$ implies that either $x\, p(\theta'_j)\, y$ or $y\, p(\theta'_j)\, x$. Without loss of generality, assume that $y\, p(\theta'_j)\, x$. By Proposition 3, $O^f_j(\theta'_i, \theta'_{-\{i,j\}}) = O^f_j(\theta''_i, \theta'_{-\{i,j\}}) \Rightarrow y \in O^f_j(\theta'_i, \theta'_{-\{i,j\}})$. That is, there exists $\widehat{\theta}_j \in p(\Theta)$ such that $f(\widehat{\theta}_j, \theta'_i, \theta'_{-\{i,j\}}) = y$. However, since $y\, p(\theta'_j)\, x$, this means that j would like to manipulate f at $(\theta'_i, \theta'_{-i})$ via $\widehat{\theta}_j$: contradiction. Hence, f is TO on $\mathcal{SC}$. $\Box$

As we said above, the proof of Theorem 2 rests on Theorem 3 and on another crucial result, summarized in Lemma 1 below. This lemma, which also appears in the context of single-peaked preferences, points out that if a social choice function is SP and U (and therefore TO), then no individual must be able to profit by reporting extreme ideal points, unless such extreme preferences constitute the individual's true ordering. This "median property" at the individual level must simultaneously hold for *every* agent.

To present this more formally, in the sequel we use $p(\underline{\theta}_i)$ (respectively, $p(\overline{\theta}_i)$) to denote agent i's most leftist (respectively, rightist) preference relation on X, so that for all $x, y \in X$, $x\, p(\underline{\theta}_i)\, y$ (respectively, $y\, p(\overline{\theta}_i)\, x$) if and only if $x < y$. Clearly, $\tau(\underline{\theta}_i) = \underline{X}$ and $\tau(\overline{\theta}_i) = \overline{X}$. Moreover, it is easy to check that, for any mapping p on Θ that generates (maximal) single-crossing domains, these rankings always belong to $p(\Theta)$.

Lemma 1 *A social choice function f is unanimous and strategy-proof on $\mathcal{SC}$ only if, for all $i \in I$, and all $\theta \in \mathcal{SC}$,*

$$f(\theta) = m^3(\tau(\theta_i), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i})).$$

Proof: Let f be U and SP on $\mathcal{SC}$.[17] By Theorem 3, f is TO on $\mathcal{SC}$. Fix a profile $\theta \in \mathcal{SC}$ and an agent $i \in I$. If $f(\underline{\theta}_i, \theta_{-i}) > f(\overline{\theta}_i, \theta_{-i})$, then $f(\underline{\theta}_i, \theta_{-i})\, p(\overline{\theta}_i)\, f(\overline{\theta}_i, \theta_{-i})$. Thus, i would like to manipulate f at $(\overline{\theta}_i, \theta_{-i})$ via $\underline{\theta}_i$: contradiction. Hence, $f(\underline{\theta}_i, \theta_{-i}) \leq f(\overline{\theta}_i, \theta_{-i})$. Two cases are possible:

Case 1: $f(\underline{\theta}_i, \theta_{-i}) < \tau(\theta_i) < f(\overline{\theta}_i, \theta_{-i})$. Then, $m^3(\tau(\theta_i), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i})) = \tau(\theta_i)$. Assume, by contradiction, $f(\theta) \neq \tau(\theta_i)$. Without loss of generality, suppose $f(\theta) < \tau(\theta_i) \Rightarrow f(\theta) < f(\overline{\theta}_i, \theta_{-i})$. By SP, $f(\theta_i, \theta_{-i})\, p(\theta_i)\, f(\overline{\theta}_i, \theta_{-i})$ and $f(\overline{\theta}_i, \theta_{-i})\, p(\overline{\theta}_i)\, f(\theta_i, \theta_{-i})$. Define a preference relation $p(\theta'_i)$ such that (i) $\tau(\theta'_i) = \tau(\theta_i)$, and (ii) $f(\overline{\theta}_i, \theta_{-i})\, p(\theta'_i)\, f(\theta_i, \theta_{-i})$ (see Figure 4 below). Since $p(\theta'_i)$ is *between* $p(\theta_i)$ and $p(\overline{\theta}_i)$, $p(\theta'_i) \in p(\Theta)$ and $\theta_i < \theta'_i < \overline{\theta}_i$. By TO, $f(\theta'_i, \theta_{-i}) = f(\theta_i, \theta_{-i})$. Thus, $f(\overline{\theta}_i, \theta_{-i})\, p(\theta'_i)\, f(\theta'_i, \theta_{-i})$: contradiction.

Case 2: $\tau(\theta_i) \leq f(\underline{\theta}_i, \theta_{-i})$.[18] Then, $m^3(\tau(\theta_i), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i})) = f(\underline{\theta}_i, \theta_{-i})$. Assume, by contradiction, $f(\theta) \neq f(\underline{\theta}_i, \theta_{-i})$. First, suppose that $f(\theta) < f(\underline{\theta}_i, \theta_{-i})$. Then, $\theta_i \leq \underline{\theta}_i$. Otherwise, if $\theta_i > \underline{\theta}_i$, SC would imply that $f(\underline{\theta}_i, \theta_{-i})\, p(\theta_i)\, f(\theta_i, \theta_{-i})$, which contradicts SP. However, since $p(\underline{\theta}_i)$ is agent i's most leftist preference relation, $\theta_i \leq \underline{\theta}_i$ implies $\tau(\theta_i) = \tau(\underline{\theta}_i) = \underline{X}$. Hence, by TO, $f(\theta_i, \theta_{-i}) = f(\underline{\theta}_i, \theta_{-i})$: contradiction. Thus, $f(\theta) > f(\underline{\theta}_i, \theta_{-i}) \Rightarrow \tau(\theta_i) \leq f(\underline{\theta}_i, \theta_{-i}) < f(\theta_i, \theta_{-i})$. Note that $\tau(\theta_i) \neq f(\underline{\theta}_i, \theta_{-i})$. Otherwise, if $\tau(\theta_i) = f(\underline{\theta}_i, \theta_{-i})$, then $f(\theta_i, \theta_{-i}) \neq f(\underline{\theta}_i, \theta_{-i})$ would imply that i would like to manipulate f at (θ_i, θ_{-i}) via $\underline{\theta}_i$. On the other hand, SP $\Rightarrow f(\theta_i, \theta_{-i})\, p(\theta_i)\, f(\underline{\theta}_i, \theta_{-i})$. And, $f(\underline{\theta}_i, \theta_{-i}) \neq \tau(\underline{\theta}_i)$, because $\underline{X} = \tau(\underline{\theta}_i) \leq \tau(\theta_i) < f(\underline{\theta}_i, \theta_{-i})$.

In fact, as it can be inferred from Figure 5 below, $f(\underline{\theta}_i, \theta_{-i}) \neq \tau(\theta_j)$ for all $j \neq i$. Otherwise, if $f(\underline{\theta}_i, \theta_{-i}) = \tau(\theta_j)$ for some $j \in I$, $j \neq i$, then $\theta_j > \theta_i$, because $\tau(\theta_i) < f(\underline{\theta}_i, \theta_{-i})$. However, by SC, $\theta_j > \theta_i$, $f(\underline{\theta}_i, \theta_{-i}) < f(\theta_i, \theta_{-i})$,

[17] Notice that U implies that individual peaks on X determine the top on r_f.

[18] The remaining case where $\tau(\theta_i) \geq f(\overline{\theta}_i, \theta_{-i})$ is similar.

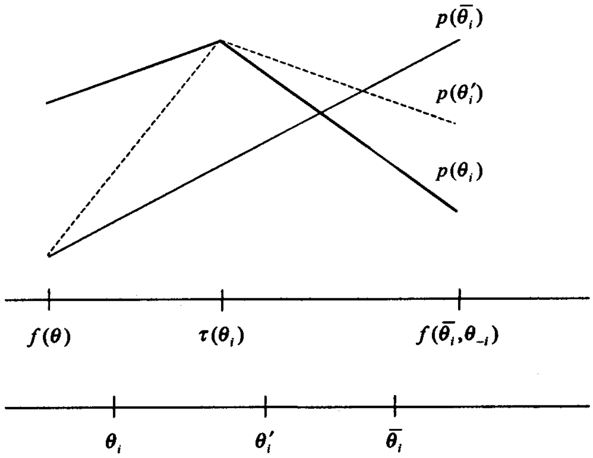

Figure 4:

and $f(\underline{\theta}_i, \theta_{-i})\, p(\theta_j)\, f(\theta_i, \theta_{-i})$ would imply $f(\underline{\theta}_i, \theta_{-i})\, p(\theta_i)\, f(\theta_i, \theta_{-i})$: contradiction. Hence, there exists a type $\theta'_i \in p(\Theta)$ such that (i) $\tau(\theta'_i) = \tau(\theta_i)$, and (ii) $f(\underline{\theta}_i, \theta_{-i})\, p(\theta'_i)\, f(\theta_i, \theta_{-i})$. By TO, $f(\theta'_i, \theta_{-i}) = f(\theta_i, \theta_{-i})$. Therefore, $f(\underline{\theta}_i, \theta_{-i})\, p(\theta'_i)\, f(\theta'_i, \theta_{-i})$: contradiction.

Thus, since $\theta \in \mathcal{SC}$ and $i \in I$ were arbitrarily chosen, Cases 1 and 2 prove the claim. $\square$

We close this section showing the independence of the axioms used in Theorem 2. First, consider the consequence of relaxing SP. As we explained before, any efficient extended median rule that it is not a positional dictator may be subject to individual manipulation on single-crossing domains. However, all of them are U and A. Thus, the family that satisfies U and A on $\mathcal{SC}$ is larger than PD.

Next consider the consequences of relaxing U. Define a social choice function f on $\mathcal{SC}$ in such a way that, for each $\theta \in \mathcal{SC}$,

$$f(\theta) = \begin{cases} a & \text{if } |\{\theta_i : a\, p(\theta_i)\, b\}| > |\{\theta_i : b\, p(\theta_i)\, a\}|, \\ b & \text{otherwise.} \end{cases}$$

where $a, b \in X$, and $a < b$. It is clear that f is A, and that it violates U, since $r_f = \{a, b\}$. Hence, $f \notin PD$. Let us prove that f is SP on $\mathcal{SC}$.

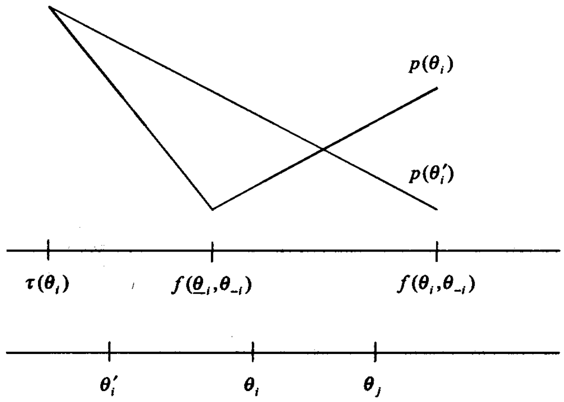

Figure 5:

First, given any profile $\theta \in \mathcal{SC}$, denote $\theta_r = m^n(\theta_1, \ldots, \theta_n)$ the median type in θ. Notice that, if $b\,p(\theta_r)\,a$, then by SC for all $\theta_k \geq \theta_r$, $b\,p(\theta_k)\,a$. Thus, $f(\theta) = b$ if and only if $\theta_r \in \{\theta_k : b\,p(\theta_k)\,a\}$, and $f(\theta) = a$ otherwise.

Second, assume by contradiction that f is manipulable on $\mathcal{SC}$. That is, suppose there exist $i \in I$ and $\theta \in \mathcal{SC}$ such that $f(\theta_i', \theta_{-i})\,p(\theta_i)\,f(\theta_i, \theta_{-i})$, for some $\theta_i' \in p(\Theta)$. Without loss of generality, let $f(\theta_i, \theta_{-i}) = a$ and $f(\theta_i', \theta_{-i}) = b$. The other case is similar. Hence, $\theta_i \neq \theta_r$, since by hypothesis $b\,p(\theta_i)\,a$ and $f(\theta_i, \theta_{-i}) = a$. There are two cases to consider.

If $\theta_i < \theta_r$, then $a < b$ and $a\,p(\theta_r)\,b$ implies that $a\,p(\theta_i)\,b$: contradiction. Thus, $\theta_i > \theta_r$. But then $\theta_i' < \theta_r$. Otherwise, if $\theta_i' \geq \theta_r$, $m^n(\theta_i', \theta_{-i}) = m^n(\theta_i, \theta_{-i}) = \theta_r$ and, therefore, we would have $f(\theta_i', \theta_{-i}) = a$. However, since $\theta_i' \in p(\Theta)$, $\theta_i' < \theta_r$, $a < b$ and $a\,p(\theta_r)\,b$, it follows that $a\,p(\theta_i')\,b$. That is, the number of agents supporting a against b doesn't decrease in going from (θ_i, θ_{-i}) to (θ_i', θ_{-i}). Therefore, by the way in which the social choice function has been defined, we would have that $f(\theta_i', \theta_{-i}) = a$: contradiction. Hence, f is SP on $\mathcal{SC}$.

Finally, regarding anonymity, note that replacing it by, for instance, *non-dictatorship* enlarges indeed the set of social choice functions satisfying U and SP.[19] More precisely, suppose $|I| > 2$. For a given coalition $S \subset I$, $|S| \geq 2$, define a social choice function $\widehat{f}$ on $\mathcal{SC}$ in such a way that, for

[19] A social choice function f is non-dictatorial (ND) on $\mathcal{SC}$ if for each $i \in I$, there exists $\theta \in \mathcal{SC}$, such that $f(\theta) \neq \tau|_{r_f}(\theta_i)$.

all $\theta \in \mathcal{SC}$, $\widehat{f}(\theta) = m^{2|S|-1}(\{\tau(\theta_i)\}_{i\in S}, \alpha_1, \ldots, \alpha_{|S|-1})$. It is immediate to see that $\widehat{f}$ is ND and U. Moreover, following a reasoning analogous to the proof of Proposition 2, it is also easy to prove that $\widehat{f}$ is SP on $\mathcal{SC}$, provided that for all $i = 1, \ldots, |S|-1$, $\alpha_i \in \{\underline{X}, \overline{X}\}$. However, $\widehat{f}$ violates A, since the preferences of all agents in the set $\bar{S} = I \backslash S$ are ignored to make social choices.

5 Final remarks

This paper studies collective choices in a society with strategic voters and single-crossing preferences. While this preference domain ensures that the core of the majority rule is nonempty, this result has been derived assuming sincere voting. This naturally raises the issue of potential individual and group manipulation, motivating the current research.

The main contributions of the paper are the following. First of all, it shows that, in addition to single-peakedness, single-crossingness is another meaningful domain over the real line that guarantees the existence of strategy-proof social choice functions. More precisely, it proves that each *positional dictator* is group strategy-proof on single-crossing domains. Conversely, every social choice function that satisfies anonymity, unanimity and strategy-proofness is shown to be a member of this family, with the natural consequence that A, U and SP imply Pareto efficiency and tops-onliness.

As we argue in the text, strategy-proofness on single-crossing preferences requires that the social choice be always an individual's most preferred alternative. This is necessary to rule out orderings that might produce incentives for manipulation, because the argument exploits (i) that the outcome is an individual's ideal point, (ii) the ordering of that agent, and (iii) the *correlation* among individual preferences in single-crossing domains. Remarkably, *no additional information about the shape of each preference relation is necessary to guarantee strategy-proofness.*

To put it in other terms, the results of this paper show that, in the case of public goods, convexity of individual preferences is not necessary to prevent manipulation, provided that a "certain amount of correlation" among preferences is simultaneously imposed. Unfortunately, this is no longer true when the collective choice problem refers to the allocation of a private good among a finite number of agents. In that case, Saporiti [31] have shown that *intermediateness*, a preference restriction closely related to single-crossingness, is not sufficient to ensure the existence of Pareto efficient, anonymous and strategy-proof allocation rules.

Furthermore, even in the case of public goods relaxing convexity is costly, because any extended median rule is A, U and SP on single-peaked preferences, without any restriction on the distribution of fixed ballots. However, in our framework, the family characterized by A, U and SP coincides with the class of positional dictators, which is a subset of extended median rules.

Finally, the paper also shows that the Representative Voter Theorem, i.e. "the single-crossing version" of the Median Voter Theorem, has a well defined strategic foundation, in the sense that its prediction can be implemented in dominant strategies. However, this result only holds on a subdomain of single-crossing preferences, the rectangular one. So, relaxing sincere voting is not free either. Moreover, the implementation itself may demand a substantial amount of information from the planner.[20] Thus, it also follows that the RVT would not probably have the same appeal as its counterpart on single-peakedness.

6 Appendix: Missing proofs and auxiliary results

In order to prove the main result of this paper, namely that a social choice function is anonymous, unanimous and strategy-proof on single-crossing domains if and only if it is an extended median rule with $n-1$ parameters distributed on the end points of the feasible set of alternatives, we first show that unanimity and strategy-proofness imply tops-onliness. To do this, the following preliminary result will be useful.

Proposition 4 *If a social choice function f is strategy-proof on $\mathcal{SC}$, then $\forall\, S \subset I$, and $\forall\, (\theta_S, \theta_{\bar{S}}) \in \mathcal{SC}$ such that $\tau|_{O^f_S(\theta_{\bar{S}})}(\theta_i) = x\ \forall\, i \in S$, $f(\theta) = x$.*

Proof: The proof is domain independent, and it is based on Le Breton and Weymark [22]. Assume f is SP on $\mathcal{SC}$, and consider any coalition $S \subset I$, and any profile $(\theta_S, \theta_{\bar{S}}) \in \mathcal{SC}$ such that $\tau|_{O^f_S(\theta_{\bar{S}})}(\theta_i) = x$ for all $i \in S$. Suppose, by contradiction, $f(\theta_S, \theta_{\bar{S}}) = y \neq x$. Define the social choice function $g : p(\Theta)^{|S|} \rightarrow X$, where for all $\theta'_S \in p(\Theta)^{|S|}$, $g(\theta'_S) = f(\theta'_S, \theta_{\bar{S}})$. It is easy to show that g is SP on $p(\Theta)^{|S|}$, with $r_g = O^f_S(\theta_{\bar{S}})$.

Since $x \in O^f_S(\theta_{\bar{S}})$, there exists a subprofile $\widetilde{\theta}_S \in p(\Theta)^{|S|}$ such that $g(\widetilde{\theta}_S) = f(\widetilde{\theta}_S, \theta_{\bar{S}}) = x$. Consider the sequence

$$\theta^0_S \;=\; (\theta_1, \ldots, \theta_{|S|}) = \theta_S,$$

[20]Recall that the function p that produces single-crossing profiles is assumed to be commonly known.

$$
\begin{aligned}
\theta_S^1 &= (\widetilde{\theta}_1, \theta_2, \ldots, \theta_{|S|}),\\
\vdots \quad & \qquad \vdots \\
\theta_S^{|S|-1} &= (\widetilde{\theta}_1, \ldots, \widetilde{\theta}_{|S|-1}, \theta_{|S|}),\\
\theta_S^{|S|} &= (\widetilde{\theta}_1, \ldots, \widetilde{\theta}_{|S|}) = \widetilde{\theta}_S.
\end{aligned}
$$

For all $k = 0, 1, \ldots, |S|$, let $z^k = g(\theta_S^k) = f(\theta_S^k, \theta_{\bar{S}})$. Suppose $j = \inf\{1, \ldots, |S|\}$ such that $g(\theta_S^j) = f(\theta_S^j, \theta_{\bar{S}}) = x$. Such a j exists because $g(\theta_S^{|S|}) = f(\theta_S^{|S|}, \theta_{\bar{S}}) = f(\widetilde{\theta}_S, \theta_{\bar{S}}) = x$. Moreover, $j \neq 0$, because by hypothesis $g(\theta_S^0) = f(\theta_S^0, \theta_{\bar{S}}) = f(\theta_S, \theta_{\bar{S}}) = y \neq x$. Hence, person j can manipulate g at θ_S^{j-1} via $\widetilde{\theta}_j$, which contradicts SP. □

Corollary 2 *If a social choice function f is strategy-proof on $\mathcal{SC}$, then for all $i \in I$, and all $(\theta_i, \theta_{-i}) \in \mathcal{SC}$ such that $\tau|_{O_i^f(\theta_{-i})}(\theta_i) = x$, $f(\theta_i, \theta_{-i}) = x$.*

Proof: Immediate from Proposition 4, by setting $S = \{i\}$. □

To prove Proposition 5 below, we also use the following remark:

Remark 1 *If f is tops-only on $\mathcal{SC}$, then the next statements are equivalent:*

(A) For all $\theta, \bar{\theta} \in \mathcal{SC}$ such that $\tau|_{r_f}(\theta_i) = \tau|_{r_f}(\bar{\theta}_i)$ for all $i \in I$, $f(\theta) = f(\bar{\theta})$;

(B) For all $i \in I$, all $(\theta_i, \theta_{-i}) \in \mathcal{SC}$, and all $\bar{\theta}_i \in p(\Theta)$ such that $\tau|_{r_f}(\bar{\theta}_i) = \tau|_{r_f}(\theta_i)$, $f(\theta_i, \theta_{-i}) = f(\bar{\theta}_i, \theta_{-i})$.

Proof:

(A) $\Rightarrow$ (B): On the contrary, assume $\exists\ i \in I$, $(\theta_i, \theta_{-i}) \in \mathcal{SC}$, and $\bar{\theta}_i \in p(\Theta)$ such that $\tau|_{r_f}(\bar{\theta}_i) = \tau|_{r_f}(\theta_i)$ and $f(\theta_i, \theta_{-i}) \neq f(\bar{\theta}_i, \theta_{-i})$. It is immediate to see that this contradicts (A).

(B) $\Rightarrow$ (A): Assume $\exists\, \theta, \theta' \in \mathcal{SC}$ such that $\tau|_{r_f}(\theta_i) = \tau|_{r_f}(\theta_i')$ for all $i \in I$, and $f(\theta) \neq f(\theta')$. Then, $\exists\, j \in I$ such that, $f(\theta_1', \ldots, \theta_{j-1}', \theta_j, \theta_{j+1}, \ldots, \theta_n) = f(\theta)$, while $f(\theta_1', \ldots, \theta_{j-1}', \theta_j', \theta_{j+1}, \ldots, \theta_n) \neq f(\theta)$. Denote $\theta_{-j}'' = (\theta_1', \ldots, \theta_{j-1}', \theta_{j+1}, \ldots, \theta_n)$. Note that $f(\theta_j, \theta_{-j}'') = f(\theta)$ and $f(\theta_j', \theta_{-j}'') \neq f(\theta)$. However, this contradicts (B), since $\tau|_{r_f}(\theta_j) = \tau|_{r_f}(\theta_j')$. □

Proposition 5 *Suppose $|I| = 2$. A social choice function f is unanimous and strategy-proof on $\mathcal{SC}$ only if it satisfies tops-only on $\mathcal{SC}$.*

Proof: Assume, by contradiction, there exists a social choice function f that is U and SP on $\mathcal{SC}$, but not TO. Then, using Remark 1, there must exist $(\theta_1, \theta_2) \in \mathcal{SC}$ and $\bar{\theta}_1 \in p(\Theta)$ such that $\tau(\bar{\theta}_1) = \tau(\theta_1)$ and $f(\bar{\theta}_1, \theta_2) = y \neq x = f(\theta_1, \theta_2)$.[21] Without loss of generality, assume $x < y$. By SP, $x\, p(\theta_1)\, y$ and $y\, p(\bar{\theta}_1)\, x$. Hence, $\tau(\theta_1) \neq x$ and $\tau(\bar{\theta}_1) \neq y$. Moreover, note that $x\, p(\theta_2)\, \tau(\theta_1)$. Otherwise, 2 might manipulate f at (θ_1, θ_2) via $\hat{\theta}_2 = \theta_1$, generating by U $\tau(\theta_1)$. Using a similar argument, $y\, p(\theta_2)\, \tau(\theta_1)$.

Two cases are possible, depending on the location of x, y and $\tau(\theta_1)$:[22]

Case 1: $x < \tau(\theta_1) < y$. Then, if $\theta_2 > \theta_1$, we have that $x < \tau(\theta_1)$ and $x\, p(\theta_2)\, \tau(\theta_1)$ imply $x\, p(\theta_1)\, \tau(\theta_1)$: contradiction. Thus, $\theta_2 < \theta_1$. But then $\tau(\theta_1) < y$ and $y\, p(\theta_2)\, \tau(\theta_1)$ imply, by SC, that $y\, p(\theta_1)\, \tau(\theta_1)$: contradiction.

Case 2: $\tau(\theta_1) < x < y$. Suppose $\tau(\theta_2) = x$. Then, $\theta_2 > \theta_1$. Otherwise, $\theta_2 < \theta_1$, $\tau(\theta_1) < x$ and $x\, p(\theta_2)\, \tau(\theta_1)$ would imply, by SC, $x\, p(\theta_1)\, \tau(\theta_1)$. Similarly, $\bar{\theta}_1 > \theta_2$, since $x\, p(\theta_2)\, y$ implies $x\, p(\theta)\, y$ for all $\theta \leq \theta_2$, and $y\, p(\bar{\theta}_1)\, x$ by hypothesis. But, $\tau(\theta_1)\, p(\bar{\theta}_1)\, x$ implies $\tau(\theta_1)\, p(\theta)\, x$ for all $\theta \leq \bar{\theta}_1$, contradicting the fact that $\tau(\theta_2) = x$. Therefore, $x \neq \tau(\theta_j)$ for all $j = 1, 2$.

Next, suppose that $\theta_2 < \theta_1$. Then, $\tau(\theta_1)\, p(\theta_1)\, x$ implies $\tau(\theta_1)\, p(\theta_2)\, x$: contradiction. Thus, $\theta_2 > \theta_1$, and $\tau(\theta_2) > \tau(\theta_1)$. Furthermore, $\theta_2 > \bar{\theta}_1$: (i) If $\theta_2 = \bar{\theta}_1$, by U, $f(\bar{\theta}_1, \theta_2) = \tau(\theta_1)$: contradiction; (ii) If $\theta_2 < \bar{\theta}_1$, $\tau(\theta_1)\, p(\bar{\theta}_1)\, \tau(\theta_2)$ implies $\tau(\theta_1)\, p(\theta_2)\, \tau(\theta_2)$: contradiction. But then $y\, p(\theta_2)\, x$. Otherwise, $x\, p(\theta_2)\, y$ implies $x\, p(\bar{\theta}_1)\, y$. And, $\tau(\theta_2) \geq y$. Otherwise, $\tau(\theta_2) < y$ and $\tau(\theta_2)\, p(\theta_2)\, y$ would imply $\tau(\theta_2)\, p(\bar{\theta}_1)\, y$, contradicting again SP, because 1 would manipulate f at $(\bar{\theta}_1, \theta_2)$ via $\hat{\theta}_1 = \theta_2$. So, we have a situation like in Figure 6.

Note that $y \notin O_2^f(\theta_1)$. Otherwise, $\exists\, \theta_2'$ such that $f(\theta_1, \theta_2') = y$. And because $y\, p(\theta_2)\, x$, it follows that 2 would manipulate f at (θ_1, θ_2) via θ_2'. Now define a preference ordering $\bar{\theta}_2$ "between" $\bar{\theta}_1$ and θ_2, such that (i) $\tau|_{O_2^f(\bar{\theta}_1)}(\bar{\theta}_2) = y$, and (ii) $\tau|_{O_2^f(\theta_1)}(\bar{\theta}_2) = \tau(\theta_1)$ (see Figure 7). By Corollary 2, $f(\bar{\theta}_1, \bar{\theta}_2) = y$ and $f(\theta_1, \bar{\theta}_2) = \tau(\theta_1)$. But then 1 would manipulate f at $(\bar{\theta}_1, \bar{\theta}_2)$ via θ_1, contradicting that f is SP.

Therefore, using Cases 1 and 2, it follows that, if $|I| = 2$ and f is U and SP, then f is also TO on $\mathcal{SC}$. $\square$

Now, before generalizing Proposition 5 to $|I| > 2$, we first prove Proposition 3 of the main text, which extends the tops-only property to the option sets generated by a unanimous and strategy-proof social choice function.

[21] Note that, under unanimity, $\tau|_{r_f}(\bar{\theta}_1) = \tau(\bar{\theta}_1)$ and $\tau|_{r_f}(\theta_1) = \tau(\theta_1)$. In the rest of the paper, we use a similar argument on several occasions.

[22] The remaining situation, where $x < y < \tau(\theta_1)$, is similar to Case 2.

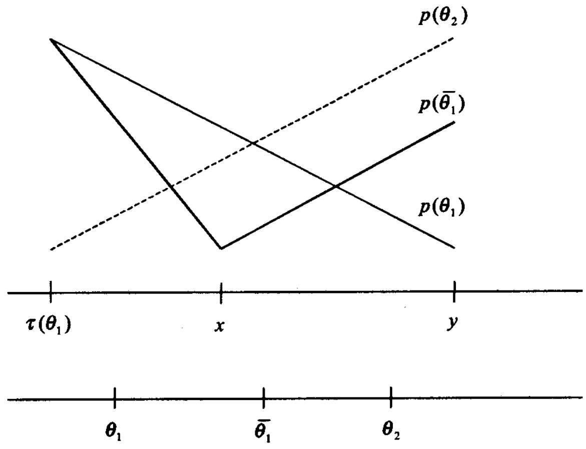

Figure 6:

Proof of Proposition 3: We make the proof in five steps:

Step 1: Fix any $i \in I$ and two subprofiles $\overline{\theta}_{-i}, \underline{\theta}_{-i} \in p(\Theta)^{n-1}$,

$$\begin{aligned} \overline{\theta}_{-i} &= (\overline{\theta}_1, \ldots, \overline{\theta}_{i-1}, \overline{\theta}_{i+1}, \ldots, \overline{\theta}_n), \text{ and} \\ \underline{\theta}_{-i} &= (\underline{\theta}_1, \ldots, \underline{\theta}_{i-1}, \underline{\theta}_{i+1}, \ldots, \underline{\theta}_n), \end{aligned}$$

such that for all $j, k \in I \backslash \{i\}$, $j \neq k$, $\overline{\theta}_j = \overline{\theta}_k$ and $\underline{\theta}_j = \underline{\theta}_k$, and for all $j \in I \backslash \{i\}$, $\tau(\overline{\theta}_j) = \tau(\underline{\theta}_j) \equiv z$. We want to show that $O_i^f(\overline{\theta}_{-i}) = O_i^f(\underline{\theta}_{-i})$. To simplify the notation, let us write

$$\begin{aligned} \overline{\theta}_{-i} &= (\underbrace{\overline{\theta}, \ldots, \overline{\theta}}_{n-1 \; times}), \text{ and} \\ \underline{\theta}_{-i} &= (\underbrace{\underline{\theta}, \ldots, \underline{\theta}}_{n-1 \; times}). \end{aligned}$$

Define the sequence

$$\begin{aligned} \theta_{-i}^0 &= (\overline{\theta}, \ldots, \overline{\theta}) = \overline{\theta}_{-i}, \\ \theta_{-i}^1 &= (\underline{\theta}, \overline{\theta}, \ldots, \overline{\theta}), \\ \theta_{-i}^2 &= (\underline{\theta}, \underline{\theta}, \overline{\theta}, \ldots, \overline{\theta}), \\ &\vdots \\ \theta_{-i}^{n-1} &= (\underline{\theta}, \ldots, \underline{\theta}) = \underline{\theta}_{-i}. \end{aligned}$$

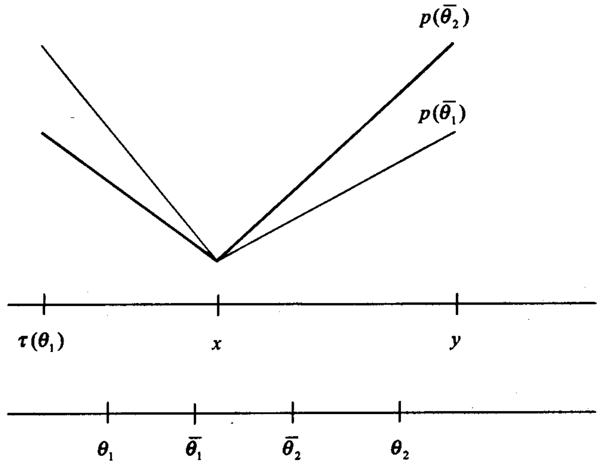

Figure 7:

To establish the result, it is enough to prove that, for all $j = 1, \dots, n-1$, $O_i^f(\theta_{-i}^j) = O_i^f(\theta_{-i}^{j-1})$. Assume, by contradiction, there exists $x \in X$ such that for some $1 \leq j^* \leq n-1$, $x \in O_i^f(\theta_{-i}^{j^*-1})$ and $x \notin O_i^f(\theta_{-i}^{j^*})$. By U, $z \in O_i^f(\theta_{-i}^{j^*-1}) \cap O_i^f(\theta_{-i}^{j^*})$, since θ_i can always be chosen as being either $\overline{\theta}$ or $\underline{\theta}$. Therefore $z \neq x$.

Let θ_i^* be such that $\tau|_{O_i^f(\theta_{-i}^{j^*-1})}(\theta_i^*) = x$ and $\tau|_{O_i^f(\theta_{-i}^{j^*})}(\theta_i^*) = z$. This type belongs to $p(\Theta)$. Suppose not. That is, assume there exists a type $\hat{\theta}_i \in p(\Theta)$ with a *dip* at x on $O_i^f(\theta_{-i}^{j^*-1})$, so that for some $v, w \in O_i^f(\theta_{-i}^{j^*-1})$, $v < x < w$, $v\, p(\hat{\theta}_i)\, x$ and $w\, p(\hat{\theta}_i)\, x$. Since $x \in O_i^f(\theta_{-i}^{j^*-1})$, there exists $\widetilde{\theta}_i \in p(\Theta)$ such that $f(\widetilde{\theta}_i, \theta_{-i}^{j^*-1}) = x$. Thus, $\tau|_{O_i^f(\theta_{-i}^{j^*-1})}(\widetilde{\theta}_i) = x$ and, therefore, $\widetilde{\theta}_i \neq \hat{\theta}_i$. If $\widetilde{\theta}_i < \hat{\theta}_i$, SC implies $v\, p(\widetilde{\theta}_i)\, x$. Instead, if $\hat{\theta}_i < \widetilde{\theta}_i$, again SC $\Rightarrow$ $w\, p(\widetilde{\theta}_i)\, x$. However, since $v, w \in O_i^f(\theta_{-i}^{j^*-1})$, this means that i can manipulate f at $(\widetilde{\theta}_i, \theta_{-i}^{j^*-1})$: contradiction. Hence, $\theta_i^* \in p(\Theta)$.

By Corollary 2, $f(\theta_i^*, \theta_{-i}^{j^*-1}) = x$ and $f(\theta_i^*, \theta_{-i}^{j^*}) = z$. However, since $z\, p(\overline{\theta})\, x$, the agent of type $\overline{\theta}$ who deviates at round j^* can manipulate f at $(\theta_i^*, \theta_{-i}^{j^*-1})$ by reporting a type $\underline{\theta}$, which contradicts that f is SP. Therefore, $O_i^f(\overline{\theta}_{-i}) = O_i^f(\underline{\theta}_{-i})$.

Step 2: Fix $j \neq i$ and $\theta'_j, \theta''_j \in p(\Theta)$, such that $\tau(\theta'_j) = \tau(\theta''_j)$. We want to show that $O_i^f(\theta'_j, \overline{\theta}_{-\{i,j\}}) = O_i^f(\theta''_j, \overline{\theta}_{-\{i,j\}})$. Define the two-person social choice function $g : p(\Theta)^2 \to X$, such that for all $(\theta_i, \theta_j) \in p(\Theta)^2$, $g(\theta_i, \theta_j) = f(\theta_i, \theta_j, \overline{\theta}_{-\{i,j\}})$. Since f is SP on $\mathcal{SC}$, g is SP on $p(\Theta)^2$ and, by Proposition 4, unanimous over $r_g = O^f_{\{i,j\}}(\overline{\theta}_{-\{i,j\}})$. Two cases are possible.

Case 1: If $\tau|_{r_g}(\theta'_j) = \tau|_{r_g}(\theta''_j)$, Step 1 implies $O_i^g(\theta'_j) = O_i^g(\theta''_j)$. By construction, $O_i^g(\theta'_j) = O_i^f(\theta'_j, \overline{\theta}_{-\{i,j\}})$ and $O_i^g(\theta''_j) = O_i^f(\theta''_j, \overline{\theta}_{-\{i,j\}})$. Therefore, $O_i^f(\theta'_j, \overline{\theta}_{-\{i,j\}}) = O_i^f(\theta''_j, \overline{\theta}_{-\{i,j\}})$.

Case 2: If $\tau|_{r_g}(\theta'_j) = a \neq b = \tau|_{r_g}(\theta''_j)$, Step 1 cannot be applied, because this step rests on the existence of a common peak on the range of the social choice function for the whole subprofile θ^{k-1}_{-i}, $k = 1, \ldots, n$. Assume, by contradiction, $O_i^g(\theta'_j) \neq O_i^g(\theta''_j)$. Without loss of generality, suppose there exists $c \in r_g$ such that $c \in O_i^g(\theta'_j)$ and $c \notin O_i^g(\theta''_j)$. Then, $\exists\, \widetilde{\theta}_i \in p(\Theta)$ such that $g(\widetilde{\theta}_i, \theta'_j) = c$. Let $g(\widetilde{\theta}_i, \theta''_j) = d \neq c$ (because $c \notin O_i^g(\theta''_j)$). Consider $\widehat{\theta}_j \in p(\Theta)$ such that $\tau(\widehat{\theta}_j) = z$ and $\tau|_{r_g}(\widehat{\theta}_j) = a$.[23] This type exists because $\tau(\overline{\theta}) = z$ and $\tau|_{r_g}(\theta'_j) = a$. By Proposition 5, g is TO $\Rightarrow g(\widetilde{\theta}_i, \widehat{\theta}_j) = c$. By U of f, $z \in O_i^g(\widehat{\theta}_j)$. From Step 1, $\tau|_{r_g}(\widehat{\theta}_j) = \tau|_{r_g}(\theta'_j) \Rightarrow O_i^g(\widehat{\theta}_j) = O_i^g(\theta'_j) \Rightarrow z \in O_i^g(\theta'_j)$. Repeating the argument, let $\theta^+_j \in p(\Theta)$ be such that $\tau(\theta^+_j) = z$ and $\tau|_{r_g}(\theta^+_j) = b$.[24] This type exists in $p(\Theta)$ because $\tau(\overline{\theta}) = z$ and $\tau|_{r_g}(\theta''_j) = b$. By TO, $g(\widetilde{\theta}_i, \theta^+_j) = d$. By U, $z \in O_i^g(\theta^+_j)$. From Step 1, $O_i^g(\theta^+_j) = O_i^g(\theta''_j)$, meaning that $z \in O_i^g(\theta''_j)$. Therefore, $z \in O_i^g(\theta'_j) \cap O_i^g(\theta''_j)$ $\Rightarrow z \neq c$. Moreover, $z \neq d$. Otherwise, j would manipulate g at $(\widetilde{\theta}_i, \widehat{\theta}_j)$ via θ''_j. Note that $\tau|_{r_g}(\widetilde{\theta}_i) \neq z$. Otherwise, $g(\widetilde{\theta}_i, \widehat{\theta}_j) = z \neq c$. Furthermore, since $c \neq d$, either $\tau|_{r_g}(\widetilde{\theta}_i) \neq c$ or $\tau|_{r_g}(\widetilde{\theta}_i) \neq d$. Suppose the latter, i.e. $\tau|_{r_g}(\widetilde{\theta}_i) \neq d$. Consider $\theta^+_i \in p(\Theta)$ such that $\tau|_{r_g}(\theta^+_i) = \tau|_{r_g}(\widetilde{\theta}_i)$, and $z\, p(\theta^+_i)\, d$. By TO, $g(\theta^+_i, \theta^+_j) = d$. Thus, i can manipulate g at (θ^+_i, θ^+_j) via $\overline{\theta}$: contradiction.

Hence, using Cases 1 and 2 above, we conclude that $O_i^f(\theta'_j, \overline{\theta}_{-\{i,j\}}) = O_i^f(\theta''_j, \overline{\theta}_{-\{i,j\}})$. And, following a similar reasoning, we also have that $O_i^f(\theta'_j, \underline{\theta}_{-\{i,j\}}) = O_i^f(\theta''_j, \underline{\theta}_{-\{i,j\}})$.

Step 3: Next we prove that $O_i^f(\theta'_j, \overline{\theta}_{-\{i,j\}}) = O_i^f(\theta''_j, \underline{\theta}_{-\{i,j\}})$. From Step 2, we know that $O_i^f(\theta''_j, \underline{\theta}_{-\{i,j\}}) = O_i^f(\theta'_j, \underline{\theta}_{-\{i,j\}})$. Hence, it is enough to show that $O_i^f(\theta'_j, \overline{\theta}_{-\{i,j\}}) = O_i^f(\theta'_j, \underline{\theta}_{-\{i,j\}})$. Proceeding as in Step 1, define

[23] Note that we are not assuming that $z \neq a$.

[24] Again we are not assuming $z \neq b$.

the sequence

$$\begin{aligned}
\theta^0_{-\{i,j\}} &= (\overline{\theta},\dots,\overline{\theta}) = \overline{\theta}_{-\{i,j\}},\\
\theta^1_{-\{i,j\}} &= (\underline{\theta},\overline{\theta},\dots,\overline{\theta}),\\
&\vdots \qquad\qquad \vdots\\
\theta^{n-2}_{-\{i,j\}} &= (\underline{\theta},\dots,\underline{\theta}) = \underline{\theta}_{-\{i,j\}}.
\end{aligned}$$

To show that $O_i^f(\theta'_j,\overline{\theta}_{-\{i,j\}}) = O_i^f(\theta'_j,\underline{\theta}_{-\{i,j\}})$, it is enough to prove that for all $k = 1,\dots,n-2$, $O_i^f(\theta'_j,\theta^{k-1}_{-\{i,j\}}) = O_i^f(\theta'_j,\theta^{k}_{-\{i,j\}})$. Suppose, by contradiction, there exists $1 \leq k^* \leq n-2$ such that $O_i^f(\theta'_j,\theta^{k^*-1}_{-\{i,j\}}) \neq O_i^f(\theta'_j,\theta^{k^*}_{-\{i,j\}})$. Without loss of generality, let $x \in X$ be such that,

$$x \in O_i^f(\theta'_j,\theta^{k^*-1}_{-\{i,j\}}) \text{ and } x \notin O_i^f(\theta'_j,\theta^{k^*}_{-\{i,j\}}) \tag{1}$$

Recall that

$$\begin{aligned}
\theta^{k^*-1}_{-\{i,j\}} &= (\underbrace{\underline{\theta},\dots,\underline{\theta}}_{k^*-1},\underbrace{\overline{\theta},\dots,\overline{\theta}}_{n-k^*-1}), \text{ and}\\
\theta^{k^*}_{-\{i,j\}} &= (\underbrace{\underline{\theta},\dots,\underline{\theta},\underline{\theta}}_{k^*},\underbrace{\overline{\theta},\dots,\overline{\theta}}_{n-k^*-2}).
\end{aligned}$$

That is, profiles $\theta^{k^*-1}_{-\{i,j\}}$ and $\theta^{k^*}_{-\{i,j\}}$ differ only in one preference relation (but both rankings have the same peak, because $\tau(\underline{\theta}) = \tau(\overline{\theta}) = z$). Abusing a bit the notation, we assume this ordering corresponds to agent k^*. Now fix the preferences of everybody, except i and k^*, at $(\theta'_j,\theta^{k^*-1}_{-\{i,j,k^*\}})$, and define the two-person social choice function $g : p(\Theta)^2 \to X$, such that for all $(\theta_i,\theta_{k^*}) \in p(\Theta)^2$, $g(\theta_i,\theta_{k^*}) = f(\theta_i,\theta_{k^*},\theta'_j,\theta^{k^*-1}_{-\{i,j,k^*\}})$. Note that g is SP on $p(\Theta)^2$ and unanimous over $r_g = O^f_{\{i,k^*\}}(\theta'_j,\theta^{k^*-1}_{-\{i,j,k^*\}})$. By Proposition 5, g is TO on $p(\Theta)^2$. By definition, $\tau(\underline{\theta}_{k^*}) = \tau(\overline{\theta}_{k^*}) = z$. From (1), there exists $\widetilde{\theta}_i \in p(\Theta)$ such that $g(\widetilde{\theta}_i,\overline{\theta}_{k^*}) = x$ and $g(\widetilde{\theta}_i,\underline{\theta}_{k^*}) \neq x$. Thus, repeating the argument of Step 2, we get the desired contradiction with SP. Hence, $O_i^f(\theta'_j,\overline{\theta}_{-\{i,j\}}) = O_i^f(\theta''_j,\underline{\theta}_{-\{i,j\}})$.

Step 4: Suppose $O_i^f(\theta'_K,\overline{\theta}_{\overline{K}\backslash\{i\}}) = O_i^f(\theta''_K,\underline{\theta}_{\overline{K}\backslash\{i\}})$ for some $K \subset I\backslash\{i\}$ and $\theta'_K, \theta''_K \in p(\Theta)^K$ such that $\tau(\theta'_j) = \tau(\theta''_j)$ for all $j \in K$.[25] Fix

[25] Note that Step 3 deals with the particular case where $K = \{j\}$.

any $k \in \overline{K}$, and $\theta'_k, \theta''_k \in p(\Theta)$ such that $\tau(\theta'_k) = \tau(\theta''_k)$. We want to show that $O_i^f(\theta'_{K\cup\{k\}}, \overline{\theta}_{\overline{K}\setminus\{i,k\}}) = O_i^f(\theta''_{K\cup\{k\}}, \underline{\theta}_{\overline{K}\setminus\{i,k\}})$, which is equivalent to prove that $O_i^f(\theta'_{K\cup\{k\}}, \overline{\theta}_{\overline{K}\setminus\{i,k\}}) = O_i^f(\theta''_{K\cup\{k\}}, \overline{\theta}_{\overline{K}\setminus\{i,k\}})$. Define the $(|K|+2)$-person social choice function $g : p(\Theta)^{|K|+2} \to X$, such that for all $(\theta_i, \theta_{K\cup\{k\}}) \in p(\Theta)^{|K|+2}$, $g(\theta_i, \theta_{K\cup\{k\}}) = f(\theta_i, \theta_{K\cup\{k\}}, \overline{\theta}_{\overline{K}\setminus\{i,k\}})$. By Step 3, $O_i^g(\theta'_K, \theta'_k) = O_i^g(\theta''_K, \theta''_k)$. Hence, $O_i^f(\theta'_{K\cup\{k\}}, \overline{\theta}_{\overline{K}\setminus\{i,k\}}) = O_i^f(\theta''_{K\cup\{k\}}, \overline{\theta}_{\overline{K}\setminus\{i,k\}})$. In particular, since this is true for any $K \subset I\setminus\{i\}$, we have that $O_i^f(\theta'_1, \ldots, \theta'_{i-1}, \theta'_{i+1}, \ldots, \theta'_n) = O_i^f(\theta''_1, \ldots, \theta''_{i-1}, \theta''_{i+1}, \ldots, \theta''_n)$ or, more compactly, $O_i^f(\theta'_{-i}) = O_i^f(\theta''_{-i})$.

Step 5: Finally, assume $O_S^f(\theta'_{\bar{S}}) = O_S^f(\theta''_{\bar{S}})$ for some $S \subset I$, where $\tau(\theta'_j) = \tau(\theta''_j)$ for all $j \in \bar{S}$. Notice that, if $S = \{i\}$, then we have the previous result, i.e. $O_i^f(\theta'_{-i}) = O_i^f(\theta''_{-i})$. Fix $h \in \bar{S}$. We want to show that $O^f_{S\cup\{h\}}(\theta'_{\bar{S}\setminus\{h\}}) = O^f_{S\cup\{h\}}(\theta''_{\bar{S}\setminus\{h\}})$. Suppose not. Without loss of generality, assume there exists $x \in X$ such that $x \in O^f_{S\cup\{h\}}(\theta'_{\bar{S}\setminus\{h\}})$ and $x \notin O^f_{S\cup\{h\}}(\theta''_{\bar{S}\setminus\{h\}})$. Then, $\exists\ \widetilde{\theta}_{S\cup\{h\}} \in p(\Theta)^{|S|+1}$ such that $f(\widetilde{\theta}_{S\cup\{h\}}, \theta'_{\bar{S}\setminus\{h\}}) = x$. Fix $\widetilde{\theta}_S \in p(\Theta)^{|S|}$ and define the $|\bar{S}|$-person social choice function $g : p(\Theta)^{|\bar{S}|} \to X$, such that for all $\theta_{\bar{S}} \in p(\Theta)^{|\bar{S}|}$, $g(\theta_{\bar{S}}) = f(\widetilde{\theta}_S, \theta_{\bar{S}})$. Since g is SP and U over r_g, from Step 4 it follows that $O_h^g(\theta'_{\bar{S}\setminus\{h\}}) = O_h^g(\theta''_{\bar{S}\setminus\{h\}})$. Hence, by definition, $O_h^f(\widetilde{\theta}_S, \theta'_{\bar{S}\setminus\{h\}}) = O_h^f(\widetilde{\theta}_S, \theta''_{\bar{S}\setminus\{h\}}) \Rightarrow x \in O_h^f(\widetilde{\theta}_S, \theta''_{\bar{S}\setminus\{h\}})$. That is, $\exists \widehat{\theta}_h \in p(\Theta)$ such that $f(\widehat{\theta}_h, \widetilde{\theta}_S, \theta''_{\bar{S}\setminus\{h\}}) = x \Rightarrow x \in O^f_{S\cup\{h\}}(\theta''_{\bar{S}\setminus\{h\}})$: contradiction. Therefore, for all $h \in \bar{S}$, $O^f_{S\cup\{h\}}(\theta'_{\bar{S}\setminus\{h\}}) = O^f_{S\cup\{h\}}(\theta''_{\bar{S}\setminus\{h\}})$. And, since $S \subset I$ and $\theta', \theta'' \in \mathcal{SC}$ were arbitrarily chosen, this completes the proof. $\square$

Finally, before proving Theorem 2, we show that a U and SP social choice function must also satisfy *top-monotonicity* on $\mathcal{SC}$. Roughly speaking, this property ensures that collective choices do not respond perversely to changes in individuals' ideal points.

Definition 8 (TM) *A social choice function f is top-monotonic on $\mathcal{SC}$ if for all $i \in I$, all $(\theta_i, \theta_{-i}) \in \mathcal{SC}$, and all $\theta'_i \in p(\Theta)$ such that $\tau|_{r_f}(\theta'_i) \geq \tau|_{r_f}(\theta_i)$, $f(\theta'_i, \theta_{-i}) \geq f(\theta_i, \theta_{-i})$.*

Like in the text, now let us assume until the end of the Appendix that $p(\underline{\theta}_i)$ (respectively, $p(\overline{\theta}_i)$) denote agent i's most leftist (respectively, rightist) preference relation on X.

Lemma 2 *If a social choice function f is unanimous and strategy-proof on $\mathcal{SC}$, then f is top-monotonic.*

Proof: Let f be U and SP on $\mathcal{SC}$. Consider any individual $i \in I$, any profile $(\theta_i, \theta_{-i}) \in \mathcal{SC}$ and any admissible deviation $\theta_i' \in p(\Theta)$, such that $\tau(\theta_i') \geq \tau(\theta_i)$. We want to show that $f(\theta_i', \theta_{-i}) \geq f(\theta_i, \theta_{-i})$. Three cases are possible:

1. If $\tau(\theta_i) \geq f(\overline{\theta}_i, \theta_{-i}) \Rightarrow m^3(\tau(\theta_i), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i})) = m^3(\tau(\theta_i'), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i}))$, because SP implies that $f(\underline{\theta}_i, \theta_{-i}) \leq f(\overline{\theta}_i, \theta_{-i})$, and $\tau(\theta_i) \leq \tau(\theta_i')$ by hypothesis. Therefore, by Lemma 1, $f(\theta_i', \theta_{-i}) = f(\theta_i, \theta_{-i})$;

2. If $f(\underline{\theta}_i, \theta_{-i}) < \tau(\theta_i) < f(\overline{\theta}_i, \theta_{-i})$, then $m^3(\tau(\theta_i), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i})) = \tau(\theta_i)$ and, given that $\tau(\theta_i') \geq \tau(\theta_i)$, $m^3(\tau(\theta_i'), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i})) \geq \tau(\theta_i)$. Therefore, by Lemma 1, $f(\theta_i', \theta_{-i}) \geq f(\theta_i, \theta_{-i})$;

3. Finally, if $\tau(\theta_i) \leq f(\underline{\theta}_i, \theta_{-i})$, then $m^3(\tau(\theta_i), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i})) = f(\underline{\theta}_i, \theta_{-i}) \leq m^3(\tau(\theta_i'), f(\underline{\theta}_i, \theta_{-i}), f(\overline{\theta}_i, \theta_{-i}))$. Hence, by Lemma 1, $f(\theta_i', \theta_{-i}) \geq f(\theta_i, \theta_{-i})$. $\square$

We are now ready to prove Theorem 2.

Proof of Theorem 2: **(Sufficiency)** Immediate from Proposition 2 and the definition of positional dictators.

(Necessity) Suppose f is U, A and SP on $\mathcal{SC}$. We want to show that $f \in PD$. By Theorem 3, f is TO on $\mathcal{SC}$. Consider first the case where $|I| = 2$. Fix a profile $\theta \in \mathcal{SC}$. Without loss of generality, assume $\tau(\theta_1) \leq \tau(\theta_2)$. By Lemma 1, $f(\theta_1, \theta_2) = m^3(\tau(\theta_1), f(\underline{\theta}_1, \theta_2), f(\overline{\theta}_1, \theta_2))$. Applying Lemma 1 once again, $f(\underline{\theta}_1, \theta_2) = m^3(\tau(\theta_2), f(\underline{\theta}_1, \underline{\theta}_2), f(\underline{\theta}_1, \overline{\theta}_2))$, and $f(\overline{\theta}_1, \theta_2) = m^3(\tau(\theta_2), f(\overline{\theta}_1, \underline{\theta}_2), f(\overline{\theta}_1, \overline{\theta}_2))$. By unanimity, $f(\underline{\theta}_1, \underline{\theta}_2) = \underline{X}$ and $f(\overline{\theta}_1, \overline{\theta}_2) = \overline{X}$. By anonymity, $f(\underline{\theta}_1, \overline{\theta}_2) = f(\overline{\theta}_1, \underline{\theta}_2)$. Furthermore, by SP, $f(\underline{\theta}_1, \overline{\theta}_2), f(\overline{\theta}_1, \underline{\theta}_2) \in \{\underline{X}, \overline{X}\}$. Suppose not. That is, assume for instance that $f(\underline{\theta}_1, \overline{\theta}_2) = z \in X \backslash \{\underline{X}, \overline{X}\}$.

Then, as we show in Figure 8 below, there must exist a type $\theta_1' \in p(\Theta)$ such that $\tau(\theta_1') = \tau(\underline{\theta}_1)$, and $\overline{X}\, p(\theta_1')\, z$. By TO, $f(\theta_1', \overline{\theta}_2) = f(\underline{\theta}_1, \overline{\theta}_2) = z$ $\Rightarrow$ agent 1 would manipulate f at $(\theta_1', \overline{\theta}_2)$ via $\overline{\theta}_1$: contradiction. Thus, $f(\underline{\theta}_1, \overline{\theta}_2), f(\overline{\theta}_1, \underline{\theta}_2) \in \{\underline{X}, \overline{X}\}$. Furthermore, if $f(\underline{\theta}_1, \overline{\theta}_2) = f(\overline{\theta}_1, \underline{\theta}_2) = \underline{X}$, $f(\underline{\theta}_1, \theta_2) = m^3(\tau(\theta_2), \underline{X}, \underline{X}) = \underline{X}$, and $f(\overline{\theta}_1, \theta_2) = m^3(\tau(\theta_2), \underline{X}, \overline{X}) = \tau(\theta_2)$. Thus, $f(\theta_1, \theta_2) = m^3(\tau(\theta_1), \underline{X}, \tau(\theta_2)) = \tau(\theta_1)$. Instead, if

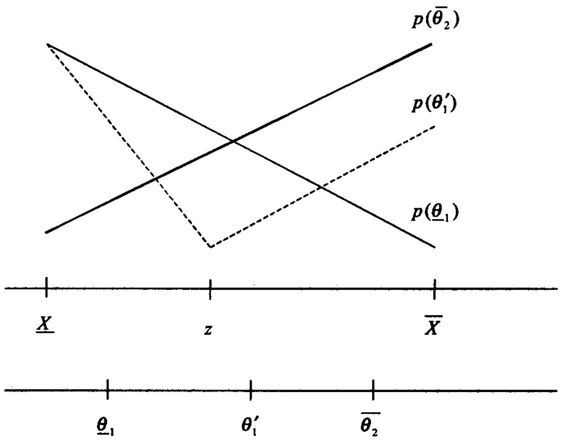

Figure 8:

$f(\underline{\theta}_1, \overline{\theta}_2) = f(\overline{\theta}_1, \underline{\theta}_2) = \overline{X}$, then a similar reasoning shows that $f(\theta_1, \theta_2) = m^3(\tau(\theta_1), \tau(\theta_2), \overline{X}) = \tau(\theta_2)$.

Thus, if $|I| = 2$ and f satisfies the hypotheses of Theorem 2, (i.e. f is U, A and SP), the previous paragraphs show that there exists a parameter (or fixed ballot) $\alpha \in \{\underline{X}, \overline{X}\}$ such that, for all $\theta \in \mathcal{SC}$, $f(\theta) = m^3(\tau(\theta_1), \tau(\theta_2), \alpha)$. Hence, $f \in PD$.

Now, suppose $|I| = 3$. Take any profile $\theta \in \mathcal{SC}$. Without loss of generality, relabel I if necessary so that $\tau(\theta_1) \leq \tau(\theta_2) \leq \tau(\theta_3)$. Using Lemma 1, it is easy to see that,

$$f(\theta) = m^3 \Big[\tau(\theta_1),\, m^3 \Big(\tau(\theta_2),\, m^3(\tau(\theta_3), a_3, a_2),\, m^3(\tau(\theta_3), a_2, a_1)\Big),$$

$$m^3 \Big(\tau(\theta_2),\, m^3(\tau(\theta_3), a_2, a_1),\, m^3(\tau(\theta_3), a_1, a_0)\Big)\Big], \tag{2}$$

where $a_3 = f(\underline{\theta}_1, \underline{\theta}_2, \underline{\theta}_3)$, $a_0 = f(\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3)$, and

$$a_2 = f(\underline{\theta}_1, \underline{\theta}_2, \overline{\theta}_3) = f(\underline{\theta}_1, \overline{\theta}_2, \underline{\theta}_3) = f(\overline{\theta}_1, \underline{\theta}_2, \underline{\theta}_3), \tag{3}$$

and

$$a_1 = f(\underline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3) = f(\overline{\theta}_1, \overline{\theta}_2, \underline{\theta}_3) = f(\overline{\theta}_1, \underline{\theta}_2, \overline{\theta}_3), \tag{4}$$

where the equalities in (3) and in (4), respectively, follow from the fact that f is A on $\mathcal{SC}$. By U and TM, we have that $\overline{X} = a_0 \geq a_1 \geq a_2 \geq a_3 = \underline{X}$.

By SP, a_1, $a_2 \in \{\underline{X}, \overline{X}\}$. Otherwise, if for example $f(\underline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3) = z \in X \backslash \{\underline{X}, \overline{X}\}$, we can define an ordering $\theta_1' \in p(\Theta)$ such that $\tau(\theta_1') = \tau(\underline{\theta}) = \underline{X}$ and $\overline{X}\, p(\theta_1')\, z$. By TO, $f(\theta_1', \overline{\theta}_2, \overline{\theta}_3) = f(\underline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3) \Rightarrow$ agent 1 would like to manipulate f at $(\theta_1', \overline{\theta}_2, \overline{\theta}_3)$ via $\overline{\theta}_1$. Then,

i If $\tau(\theta_1) \geq a_0$, then $\forall\, i = 1, 2, 3,\ \tau(\theta_i) = \overline{X}$. Thus, independently of the distribution of a_1 and a_2, it follows from (2) that $f(\theta) = \overline{X}$;

ii Similarly, if $\tau(\theta_3) \leq a_3$, then $\forall\, i = 1, 2, 3,\ \tau(\theta_i) = \underline{X}$, and $f(\theta) = \underline{X}$;

iii If $a_1 = \underline{X}$, then $a_2 = \underline{X}$, because, by TM, $a_1 \geq a_2$. Therefore, (2) can be rewritten as $f(\theta) = m^3(\tau(\theta_1), \underline{X}, \tau(\theta_2)) = \tau(\theta_1)$;

iv Similarly, if $a_2 = \overline{X}$, then $a_1 = \overline{X}$, and $f(\theta) = m^3(\tau(\theta_1), \tau(\theta_3), \overline{X}) = \tau(\theta_3)$;

v Finally, if $a_1 = \overline{X}$ and $a_2 = \underline{X}$, then (2) can be rewritten as $f(\theta) = m^3(\tau(\theta_1), \tau(\theta_2), \tau(\theta_3)) = \tau(\theta_2)$.

Thus, since θ was arbitrarily chosen, (i)-(v) imply that, if $|I| = 3$ and f is A, U and SP, then there exists $\alpha_1, \alpha_2 \in \{\underline{X}, \overline{X}\}$ such that, for all $\theta \in \mathcal{SC}$, $f(\theta) = m^5(\tau(\theta_1), \tau(\theta_2), \tau(\theta_3), \alpha_1, \alpha_2)$. Hence, $f \in PD$.

Now let us extend the proof to $|I| = n > 3$. For all $K \subseteq I$, let $a_{|K|} = f(\underline{\theta}_K, \overline{\theta}_{\bar{K}})$, where $\bar{K} = I \backslash K$. By unanimity, $K = \emptyset$ implies $a_0 = f(\overline{\theta}_1, \ldots, \overline{\theta}_n) = \overline{X}$. Similarly, if $K = I$, then $a_n = f(\underline{\theta}_1, \ldots, \underline{\theta}_n) = \underline{X}$. By anonymity,

$$
\begin{array}{rcl}
a_1 & = & f(\underline{\theta}_i, \overline{\theta}_{-i}),\ \forall \{i\} \subset I, \\
a_2 & = & f(\underline{\theta}_{\{i,j\}}, \overline{\theta}_{-\{i,j\}}),\ \forall \{i,j\} \subseteq I, \\
\vdots & & \quad\vdots \\
a_{n-1} & = & f(\underline{\theta}_{-j}, \overline{\theta}_j),\ \forall \{j\} \subset I.
\end{array}
$$

Thus, by top-monotonicity, $a_0 \geq a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq a_n$. Moreover, for all $k = 0, 1, \ldots, n$, $a_k \in \{\underline{X}, \overline{X}\}$. In effect, if either $k = 0$ or $k = n$, then the result follows immediately from U. So, assume that $a_k \in \{\underline{X}, \overline{X}\}$ for some $k = 0, 1, \ldots, n-2$, and let us prove the claim for a_{k+1}. On the contrary, suppose $a_{k+1} \notin \{\underline{X}, \overline{X}\}$. Specifically, assume $a_{k+1} = f(\underline{\theta}_1, \ldots, \underline{\theta}_{k+1}, \overline{\theta}_{k+2}, \ldots, \overline{\theta}_n) = z \in X \backslash \{\underline{X}, \overline{X}\}$. Without loss of generality, let $a_k = f(\underline{\theta}_1, \ldots, \underline{\theta}_k, \overline{\theta}_{k+1}, \ldots, \overline{\theta}_n) = \overline{X}$. Consider $\theta_{k+1}' \in p(\Theta)$ such that $\tau(\theta_{k+1}') = \tau(\underline{\theta}_{k+1})$ and $\overline{X}\, p(\theta_{k+1}')\, z$ (recall Figure 8 above). By TO,

$f(\underline{\theta}_1,\ldots,\underline{\theta}_k,\theta'_{k+1},\overline{\theta}_{k+2},\ldots,\overline{\theta}_n)=z\Rightarrow$ agent $k+1$ would like to manipulate f at $(\underline{\theta}_1,\ldots,\underline{\theta}_k,\theta'_{k+1},\overline{\theta}_{k+2},\ldots,\overline{\theta}_n)$ via $\overline{\theta}_{k+1}$: contradiction.

Now, fix any profile $\theta\in\mathcal{SC}$, and relabel I if necessary, so that $\tau(\theta_1)\leq\tau(\theta_2)\leq\ldots\leq\tau(\theta_n)$. By repeated application of Lemma 1, for all $n>3$,

$$f(\theta)=m^3\Big[\tau(\theta_1),m^3\Big(\tau(\theta_2),\ldots,m^3\Big(\tau(\theta_{n-1}),m^3\left(\tau(\theta_n),a_n,a_{n-1}\right),m^3\left(\tau(\theta_n),a_{n-1},a_{n-2}\right)\Big),$$
$$,\ldots,m^3\Big(\tau(\theta_{n-1}),m^3\left(\tau(\theta_n),a_3,a_2\right),m^3\left(\tau(\theta_n),a_2,a_1\right)\Big),\ldots,$$
$$m^3\Big(\tau(\theta_2),\ldots,m^3\Big(\tau(\theta_{n-1}),m^3\left(\tau(\theta_n),a_2,a_1\right),m^3\left(\tau(\theta_n),a_1,a_0\right)\Big)\Big)\Big]. \quad (5)$$

The following cases are possible:

i If $\tau(\theta_1)\geq a_0$, then $\forall\, i=1,\ldots,n,\ \tau(\theta_i)=\overline{X}$, and it follows from (5) that $f(\theta)=m^3(\tau(\theta_1),a_1,a_0)=\overline{X}$;

ii If $\tau(\theta_n)\leq a_n$, then $\forall\, i=1,\ldots,n,\ \tau(\theta_i)=\underline{X}$, and we have from (5) that $f(\theta)=m^3(\tau(\theta_1),a_n,a_{n-1})=\underline{X}$;

iii If $\forall\, k=1,\ldots,n-1,\ a_k=\overline{X}$, then $f(\theta)=m^3(\tau(\theta_1),\tau(\theta_n),\overline{X})=\tau(\theta_n)$;

iv If $\forall\, k=1,\ldots,n-1,\ a_k=\underline{X}$, then $f(\theta)=m^3(\tau(\theta_1),\underline{X},\tau(\theta_2))=\tau(\theta_1)$;

v Finally, if for some $k=1,2,\ldots,n-2$, $a_1=\ldots=a_k=\overline{X}$ and $a_{k+1}=\ldots=a_{n-1}=\underline{X}$, then (5) implies that $f(\theta)=m^3(\tau(\theta_1),\tau(\theta_{k+1}),\tau(\theta_{k+2}))=\tau(\theta_{k+1})$.

Therefore, since $\theta\in\mathcal{SC}$ was arbitrarily chosen and, for every $k=0,1\ldots,n$, a_k is independent of θ, if f is A, U and SP, then items (i)-(v) imply that there exist $n-1$ parameters $\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$ on $\{\underline{X},\overline{X}\}$ such that, for all $\theta\in\mathcal{SC}$, $f(\theta)=m^{2n-1}(\tau(\theta_1),\tau(\theta_2),\ldots,\tau(\theta_n),\alpha_1,\ldots,\alpha_{n-1})$. Hence, $f\in PD$. □

References

[1] Austen-Smith, D., and J. Banks, (1999), *Positive Political Theory I: Collective Preference*, Ann Arbor: The University of Michigan Press.

[2] Barberà, S., Gul, F., and E. Stacchetti, (1993), Generalized median voter schemes and committees, *Journal of Economic Theory* 61, 262-89.

[3] Barberà, S., and M. Jackson, (1994), A characterization of social choice functions for economies with pure public goods, *Social Choice and Welfare* 11, 241-52.

[4] Berga, D., (1998), Strategy-proofness and single-plateaued preferences, *Mathematical Social Sciences* 35, 105-20.

[5] Black, D., (1948), On the rationale of group decision making, *Journal of Political Economy* 56, 23-34.

[6] Border, K., and J. Jordan, (1983), Straightforward elections, unanimity and phantom voters, *Review of Economic Studies* 50, 153-70.

[7] Calabrese, S., Epple, D., Romer, T., and H. Sieg, (2006), Local public good provision: voting, peer effects, and mobility, *Journal of Public Economics* 90, 959-81.

[8] Campbell, D., and J. Kelly, (2003b), A strategy-proofness characterization of majority rule, *Economic Theory* 22, 557-68.

[9] Ching, S., (1997), Strategy-proofness and "median voters", *International Journal of Game Theory* 26, 473-90.

[10] Demange, G., (1994), Intermediate preferences and stable coalition structures, *Journal of Mathematical Economics* 23, 45-58.

[11] Eguia, J., (2007), Citizen candidates under uncertainty, *Social Choice and Welfare* 29, 317-31.

[12] Ehlers, L., Peters, H., and T. Storcken, (2002), Strategy-proof probabilistic decision schemes for one-dimensional single-peaked preferences, *Journal of Economic Theory* 105, 408-34.

[13] Epple, D., Filimon, R., and T. Romer, (1993), Existence of voting and housing equilibrium in a system of communities with property taxes, *Regional Science and Urban Economics* 23, 585-610.

[14] Epple, D., and G. Platt, (1998), Equilibrium among jurisdictions when households differ by preferences and income, *Journal of Urban Economics* 43, 23-51.

[15] Epple, D., Romer, T., and H. Sieg, (2001), Interjurisdictional sorting and majority rule: an empirical analysis, *Econometrica* 69, 1437-65.

[16] Epple, D., Romano, R., and H. Sieg, (2006), Admission, tuition and financial aid policies in the market for higher education, *Econometrica* 74, 885-928.

[17] Gans, J., and M. Smart, (1996), Majority voting with single-crossing preferences, *Journal of Public Economics* 59, 219-37.

[18] Gibbard, A., (1973), Manipulation of voting schemes: a general result, *Econometrica* 41, 587-601.

[19] Grandmont, J., (1978), Intermediate preferences and the majority rule, *Econometrica* 46, 317-30.

[20] Kung, F., (2006), An algorithm for stable and equitable coalition structures with public goods, *Journal of Public Economic Theory* 8, 345-55.

[21] List, C., (2001), A possibility theorem on aggregation over multiple interconnected propositions, *Mathematical Social Sciences* 45, 1-13.

[22] Le Breton, M., and J. Weymark, (1999), Strategy-proof social choice with continuous separable preferences, *Journal of Mathematical Economics* 32, 47-85.

[23] Meltzer, A., and S. Richard, (1981), A rational theory of the size of government, *Journal of Political Economy* 89, 914-27.

[24] Moulin, H., (1980), On strategy-proofness and single-peakedness, *Public Choice* 35, 437-55.

[25] Moulin, H., (1988), *Axioms of Cooperative Decision Making*, Cambridge: Cambridge University Press.

[26] Myerson, R., (1996), Fundamentals of social choice theory, Discussion Paper No 1162, Math Center, Northwestern University.

[27] Persson, T., and G. Tabellini, (2000), *Political Economics: Explaining Economic Policy*, Cambridge, MA: MIT Press.

[28] Roberts, K., (1977), Voting over income tax schedules, *Journal of Public Economics* 8, 329-40.

[29] Rothstein, P., (1990), Order-restricted preferences and majority rule, *Social Choice and Welfare* 7, 331-42.

[30] Rothstein, P., (1991), Representative voter theorems, *Public Choice* 72, 193-212.

[31] Saporiti, A., (2005), Allocation rules on intermediate preference domains, manuscript.

[32] Saporiti, A., (2007), On single-crossing and order-restriction, manuscript.

[33] Saporiti, A., and F. Tohmé, (2006), Single-crossing, strategic voting and the median choice rule, *Social Choice and Welfare* 26, 363-83.

[34] Satterthwaite, M., (1975), Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, *Journal of Economic Theory* 10, 187-217.

[35] Schummer, J., and R. Vohra, (2002), Strategy-proof location on a network, *Journal of Economic Theory* 104, 405-28.

[36] Westhoff, F., (1977), Existence of equilibria in economies with a local public good, *Journal of Economic Theory* 14, 84-112.

[37] Weymark, J., (2006), Strategy-proofness and the tops-only property, Discussion Paper No 04-W09R, Vanderbilt University.

[38] Zhou, L., (1991), Impossibility of strategy-proof mechanisms in economies with pure public goods, *Review of Economic Studies* 58, 107-19.