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by

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ABSTRACT

When seasonal time series are periodically integrated, we show that the any cointegration is either full periodic cointegration or full nonperiodic cointegration, with no possibility of cointegration applying for only some seasons. In contrast, seasonally integrated series can be seasonally, periodically or nonperiodically cointegrated, with the possibility of cointegration applying for a subset of seasons. Cointegration tests are analysed for periodically integrated series. A residual-based test is examined and its asymptotic distribution is derived under the null hypothesis of no cointegration. A Monte Carlo analysis confirms the size properties of the test and shows it to have good power. The role of deterministic terms in the cointegrating test regression are also investigated. Further, we show that the asymptotic distribution of the error-correction test for periodic cointegration derived by Boswijk and Franses (1995) does not apply for periodically integrated processes, and we derive this distribution for the case of uncorrelated periodically integrated processes.

1. Introduction

To date, cointegration analyses of long run relationships in seasonal time series have been conducted primarily in terms of the separate (zero and seasonal frequency) unit roots implied by the seasonal differencing filter, which leads to the concept known as seasonal cointegration; see Hylleberg, Engle, Granger and Yoo (1990) (HEGY), Engle, Granger, Hylleberg, Lee (1993), Lee (1992), Johansen and Schuamberg (1999), Cubadda (2001) among others. However, cointegration may also be considered season by season, and this route leads to so-called periodic cointegration, which is examined by Birchenhall, Bladen-Hovell, Chui, Osborn and Smith (1989), Franses and Kloek (1995), Franses and Boswijk (1995), and others.

Seasonal cointegration can apply only for seasonally integrated (*SI*) processes, which are nonstationary processes which are made stationary by the application of annual differencing. In an analogous way, periodic cointegration can apply for periodically integrated (*PI*) processes, which are nonstationary but rendered stationary by application of a seasonally varying quasi-difference filter. In an *SI* process, nonstationary unit root behaviour exists not only at the longrun (or zero) frequency, but also at all the seasonal frequencies. Although not always discussed, the implication of these seasonal unit roots is that the seasons of the year are not cointegrated with each other, and hence “summer may become winter”; see, for example, Osborn (1991) or Ghysels and Osborn (2001). From an economic perspective, this implication may be unattractive. On the other hand periodically integrated processes may be more plausible than seasonally integrated ones, because they allow for nonstationarity in conjunction with cointegration applying between the separate seasons of the year (Osborn, 1991, Franses, 1996).

Although there has been little analysis of seasonal versus periodic cointegration, Franses (1993, 1995) shows that these imply different parameter restrictions on the cointegrating relationships when *SI* processes are considered. In other words, periodic cointegration can apply between seasonally integrated, as well as between periodically integrated, processes. Boswijk and Franses (1995) propose a Wald test for periodic cointegration in *SI* processes and derive its asymptotic distribution, which they assert also applies when the individual series are *PI*. However, the present paper shows that this test has a different asymptotic distribution under the null hypothesis when applied to *PI*, rather than *SI*, processes. Indeed, since quarterly *PI* and *SI* processes differ in that the former implies one underlying unit root process across the four seasons whereas the latter implies four distinct unit root processes, we might anticipate that these cases will give rise to different asymptotic distributions.

From a theoretical perspective, the distribution of tests for periodic integration in *PI* processes is unresolved. A system approach, in which an equation is estimated for observations relating to each season for each variable, can theoretically be applied (see, for example, Ghysels and Osborn, 2001, pp.171-176). However, this is likely to be feasible in practice only where data are a relatively high frequency are available, as in the application of Haldrup *et* al. (2005). Although a two-step approach of the Engle-Granger (1987) type can be adopted (as in Birchenhall *et al*., 1989, or Franses and Kloek, 1995), unless testing is undertaken separately for each season, which is likely to be inefficient, the asymptotic distribution of the test statistic is again unknown. Franses (1996, p.182) proposes testing for periodic cointegration through the application of the Boswijk-Franses (1996) *PI* test to the first-stage residuals over all seasons and speculates as to its asymptotic distribution. The present paper contributes to this strand of literature by establishing that this test statistic follows the Phillips and Ouliaris (1990) distribution, which enables asymptotically valid inference to be undertaken.

Prior to deriving the distributions of the test statistics for *PI* processes, Section 2 discusses the cointegration possibilities for these processes, which formalises the discussion in Ghysels and Osborn (2001, pp.168-171). When the series are *PI*, we show that the only cointegration possibilities are periodic cointegration or nonperiodic cointegration, with cointegration for any one season implying cointegration for all seasons. The section also compares this to the wider set of possibilities for *SI* processes. Section 3 then derives the asymptotic distribution of the residual-based cointegration test for *PI* processes, which is followed (Section 4) by an analysis of the asymptotic distribution of the Boswijk-Franses (1995) cointegration test when applied to uncorrelated *PI* processes. A Monte Carlo analysis in Section 5 examines the finite sample distribution of the residual-based test, including an analysis of the role of deterministic terms in the regression, with a concluding section completing the paper.

2. Periodic Integration and Cointegration

2.1 Properties of Periodically Integrated Processes

Consider a univariate time series $x_{s\tau}$, where the first subscript refers to the season (s) and the second subscript to the year (τ). For simplicity of exposition, we assume that the data are quarterly and that observations are available for precisely N years, so that the total sample size is $T = 4N$, with initial values $x_{40} = 0$. The annual difference operator is $\Delta_4 = 1 - L^4$, where L is the usual lag operator that works on the seasons ($L^k x_{s\tau} = x_{s-k,\tau}$). Note that, through out the paper, it is understood that $x_{s-k,\tau} = x_{4-k+s,\tau-1}$ for $s - k \leq 0$.

We assume, for simplicity of exposition, a first-order periodic process

$$x_{s\tau} = \phi_s\, x_{s-1,\tau} + e_{s\tau} \tag{1}$$

where $e_{s\tau}$ is white noise, then $x_{s\tau}$ is a periodically integrated process, if $\phi_1\phi_2\phi_3\phi_4 = 1$. In such a *PI*(1) process, nonstationarity arises from a single common trend shared by the four quarterly observations of the time series; equivalently; there are three cointegration relationships between the quarters. It is convenient to explore this through the representation referred to as the vector of quarters (VQ) representation by Franses (1994), which is based on the vector $X_\tau = (x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau})'$ and disturbance process $E_\tau = (e_{1\tau}, e_{2\tau}, e_{3\tau}, e_{4\tau})'$. The VQ representation is

$$\Phi_0 X_\tau = \Phi_1 X_{\tau-1} + E_\tau \tag{2}$$

where

$$\Phi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_2 & 1 & 0 & 0 \\ 0 & -\phi_3 & 1 & 0 \\ 0 & 0 & -\phi_4 & 1 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} 0 & 0 & 0 & \phi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

To reduce the VQ system for the *PI*(1) process in (1) to stationarity, consider the vector MA [VMA] in the annual difference $\Delta_4 X_\tau = X_\tau - X_{\tau\text{-}1}$, given by

$$\Delta_4 X_\tau = (\Theta_0 - \Theta_1 L^4)E_\tau = C(L^4)E_\tau. \tag{4}$$

This VMA process is noninvertible, because the matrix $C(L^4)$ has three unit roots. Therefore, $C(1)$ is of rank one and it is possible to write

$$C(1) = \Theta_0 - \Theta_1 = ab' \tag{5}$$

where $a = (1,\ \phi_2,\ \phi_2\phi_3,\ \phi_2\phi_3\phi_4)'$, $b = (1,\ \phi_1\phi_3\phi_4,\ \phi_1\phi_4,\ \phi_1)'$. Further details of the above can be found in Boswijk and Franses (1996), Franses (1994), Ghysels and Osborn (2001), among others.

An implication of (4) and (5) is that the four elements of X_τ share a single common stochastic trend, given by $b'\sum_{t=1}^{\tau} E_t$, to which they adjust with periodic adjustment coefficients given by the elements of a; see Boswijk and Franses (1996) for further discussion.

2.2 Cointegration for *PI* Processes

Now consider the $m\times 1$ vector process $x_{s\tau} = [x_{s\tau}^{(1)}, ..., x_{s\tau}^{(m)}]'$, for which we assume that each $x_{s\tau}^{(j)}$ is a *PI*(1) process, satisfying

$$x_{s\tau}^{(j)} = \phi_s^{(j)} x_{s-1,\tau}^{(j)} + e_{s\tau}^{(j)} \quad with \quad \prod_{s=1}^{4} \phi_s^{(j)} = 1, j = 1, \ldots, m \tag{6}$$

where $E_{s\tau} = (e_{s\tau}^{(1)}, ..., e_{s\tau}^{(m)})'$ is vector white noise with $\text{E}[E_{s\tau}E_{s\tau}'] = \Sigma$ positive definite[1]. Therefore, we can define the VQ representation as in (2)/(3) for each $X_\tau^{(j)} = (x_{1\tau}^{(j)}, x_{2\tau}^{(j)}, x_{3\tau}^{(j)}, x_{4\tau}^{(j)})', j = 1, \ldots, m$.

Cointegration can then be defined as follows:

DEFINITION 1. *The* $m\times 1$ *vector* $x_{s\tau}$ *of PI processes satisfying* (6) *is periodically cointegrated if there exist* $m\times r$ *matrices* α_s *such that the linear combinations* $\alpha_s' x_{s\tau}$ *are stationary for each* $s = 1, \ldots, 4$.

Although not formally defined in this way, the idea of periodic cointegration appears to have been applied first by Birchenhall *et al*. (1989).

In their discussion of periodic cointegration, Boswijk and Franses (1995) distinguish full and partial periodic cointegration, where the former corresponds to Definition 1 and the latter to the situation where stationary linear combinations $\alpha_s' x_{s\tau}$ exist for only some $s = 1, \ldots, 4$. However, Ghysels and Osborn (2001) show that partial periodic cointegration cannot apply between two *PI*(1) processes; such processes are either (fully) periodically cointegrated or no cointegrating relationship exists for any $s = 1, \ldots, 4$. This result is generalized in Lemma 1 for the case of *m PI*(1) processes.

LEMMA 1. *Consider the* $m\times 1$ *vector* $x_{s\tau}$ *of PI processes of* (6), *such that the* $m\times r$ *matrix* α_s *of rank r defines all linearly independent stationary linear combinations* $\alpha_s' x_{s\tau}$ *for some* $s = 1, \ldots, 4$. *Then:*

(i) α_s, *together with the coefficients* $\phi_s^{(j)}$ ($s = 1, 2, 3, 4; j = 1, \ldots, m$) *of* (6), *determine the* $m\times r$ *matrix* α_q *of rank r, which must exist for each* $q = 1, 2, 3, 4, q \neq s$ *such that* $\alpha_q' x_{q\tau}$ *is stationary;*

[1] Although periodic variation in Σ can be permitted, the purpose of our analysis is to analyse the implications of periodically varying coefficients.

(ii) *Nonperiodic cointegration with* $\alpha_s = \alpha$ (s = 1, 2, 3, 4) *applies if and only if* $\phi_s^{(j)} = \phi_s$, j = 1, …, m *in* (6).

The first part of Lemma 1 implies that there must be the same number of cointegrating relationships between $PI(1)$ processes for all seasons s = 1, 2, 3, 4. Hence, as in the bivariate case considered by Ghysels and Osborn (2001), partial periodic contegration cannot apply between $PI(1)$ processes. Further, given the cointegrating vectors that apply for one season and the univariate PI coefficients of (1), then all four sets of cointegrating relations can be determined. Part (ii) further establishes that the same (nonperiodic) cointegrating relations can apply over seasons if and only if all processes have identical univariate PI coefficients. The proof of this Lemma[2], rests on the fact that the VQ process corresponding to a $PI(1)$ variable is driven by a single unit root process. The stationary relationships between observations for the seasons that exist for each $X_\tau^{(j)} = [x_{1\tau}^{(j)}, x_{2\tau}^{(j)}, x_{3\tau}^{(j)}, x_{4\tau}^{(j)}]'$ then imply that cointegrating relations for the vector $x_{s\tau}$ can be mapped from season to season.

Conventional cointegration between $I(1)$ processes provides a special case of Lemma 1, where the same cointegrating relations apply for all seasons (quarters) of the year and all $\phi_s^{(j)}$ of (6) are unity.

2.3 Cointegration for *SI* Processes

Unlike the quasi-differencing $x_{s\tau} - \phi_s x_{s-1,\tau}$ of (1) with $\phi_1\phi_2\phi_3\phi_4 = 1$ that renders the $PI(1)$ process stationary, first order seasonally integrated, or $SI(1)$, processes are made stationary and invertible by annual differencing. As is now well know, such processes contain four unit roots, implying that the quarters of the year are not cointegrated with each other; Osborn (1993) and Franses (1994) provide discussions of some of the implications. In the context of cointegration between elements of an $m\times 1$ vector $x_{s\tau}$, if each element is $SI(1)$ then the lack of cointegration

[2] All proofs can be found in the Appendix.

across the seasons implies that distinct contegrating relations can apply for each of the vectors $x_{s\tau}$ for s = 1, 2, 3, 4. This is the essence of Lemma 2.

LEMMA 2. *Consider the* $m\times 1$ *vector* $x_{s\tau} = [x_{s\tau}^{(1)}, ..., x_{s\tau}^{(m)}]'$ *of SI processes. Then the existence of an* $m\times r$ *matrix* α_s *of rank* r *such that* $\alpha_s' x_{s\tau}$ *is stationary for some* s = 1, …, 4 *has no implications for the existence or nature of cointegration across the elements of* $x_{q\tau}$ *for* $q \neq s$.

An immediate consequence of Lemma 2 is that full and partial periodic cointegration are possibilities for *SI*(1) processes.

So-called seasonal cointegration, which corresponds to cointegration at the distinct seasonal spectral frequencies, is another possibility for *SI* processes and is analysed by Engle, Granger, Hylleberg and Lee (1993), Cubadda (2001), Johansen and Schaumberg (1999) and Lee (1992). However, our analysis focuses on testing for periodic cointegration. More specifically, we are particularly interested in testing for cointegration for *PI* processes. However, the case of *SI* processes is relevant, since Boswijk and Franses (1995) claim that the same asymptotic distribution applies for their test for *SI* and *PI* processes.

3. Residual-Based Test for Periodic Cointegration

This section analyses the periodic analogue of the Engle-Granger (1987) test, which applies a test for *PI* to the residuals from a first-stage regression. We first set out the test regression and then, before obtaining the distribution of the test statistic in subsection 3.3, subsection 3.2 examines the properties of the processes in (6) in the absence of cointegration.

3.1 The Test Regression

As usual, a residual-based test requires that the potential cointegrating relationship being examined is unique. That is, either there exists at most one cointegrating vector or, if there potentially exist $1< k < m$ cointegrating vectors between the separate series, then (exclusion) restrictions are imposed to ensure uniqueness. From the analysis of the previous section, we know that cointegration applying for one season between *PI* processes implies cointegration for all seasons. Therefore, it is anticipated that efficiency gains will result by considering all seasons jointly.

To keep notation simple, our analysis in Section 3.3 below assumes only one cointegrating relationship may exist across the *m* variables. Arbitrarily normalising on the first element of *x*, we propose fitting the regression

$$x_{s\tau}^{(1)} = \sum_{i=2}^{m} \sum_{s=1}^{4} \beta_{is} D_{s\tau} x_{s\tau}^{(i)} + v_{s\tau} \tag{7}$$

and then applying the periodic integration test of Boswijk and Franses (1996) to the residuals $\hat{v}_{s\tau} = x_{s\tau}^{(1)} - \sum_{i=2}^{m} \hat{\beta}_{is} x_{s\tau}^{(i)}$. The intuition is that, in the absence of cointegration, the residuals $\hat{v}_{s\tau}$ follow a nonstationary *PI* process (see Franses, 1996, pp.181-182).

Now, partition $x_{s\tau}$ as

$$x_{s\tau} = \left(x_{s\tau}^{(1)}, z_{s\tau}'\right)', \qquad z_{s\tau}' = \left(x_{s\tau}^{(2)}, \ldots, x_{s\tau}^{(m)}\right)' \tag{8}$$

so that $z_{s\tau}$ comprises the vector of right-hand side variables in (7). We assume that all $j = 1, \ldots, m$ variables are $PI(1)$ processes, as in (6), with variance-covariance disturbance matrix corresponding to the system being

$$E\left[E_{s\tau} E_{s\tau}'\right] = \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{1z}' \\ \sigma_{1z} & \Sigma_{zz} \end{bmatrix} \tag{9}$$

where Σ in (9) partitioned conformably with $x_{s\tau}$ in (8).

3.2 Properties of the *PI* System

As discussed in Section 2.1 above, the matrix $C^{(j)}(L^4)$ in the VMA representation for each process $j = 1, \ldots, m$ has three unit roots, and hence

$$C^{(j)}(1) = \left(\Theta_0^{(j)} - \Theta_1^{(j)}\right) = a^{(j)} b^{(j)}{}', \quad j = 1, ..., m \tag{10}$$

where $a^{(j)} = [1, \phi_2^{(j)}, \phi_2^{(j)}\phi_3^{(j)}, \phi_2^{(j)}\phi_3^{(j)}\phi_4^{(j)}]'$, $b^{(j)} = [1, \phi_1^{(j)}\phi_3^{(j)}\phi_4^{(j)}, \phi_1^{(j)}\phi_4^{(j)}, \phi_1^{(j)}]'$.

Stacking the processes and using the annual difference representation of (4), we can write

$$\Delta_4 X_\tau = \Theta_0^x E_\tau^x - \Theta_1^x E_{\tau-1}^x \tag{11}$$

where $\Delta_4 X_\tau = \left(\Delta_4 X_\tau^{(1)'}, \Delta_4 X_\tau^{(2)'}, \cdots, \Delta_4 X_\tau^{(m)'}\right)'$, $\Delta_4 X_\tau^{(j)} = \left(\Delta_4 x_{1\tau}^{(j)}, \Delta_4 x_{2\tau}^{(j)}, \Delta_4 x_{3\tau}^{(j)}, \Delta_4 x_{4\tau}^{(j)}\right)'$ and corresponding definitions apply for E_τ^x. The MA coefficient matrices in (11) are block diagonal, of the form

$$\Theta_i^x = \begin{bmatrix} \Theta_i^{(1)} & 0 & \cdots & 0 \\ 0 & \Theta_i^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Theta_i^{(m)} \end{bmatrix}, \qquad i = 1, 2$$

and

$$E[E_\tau^x E_\tau^x{}'] = \Sigma \otimes I_4 .$$

The long-run covariance matrix between the $m \times 4$ random walk processes in X_τ is then given by (see also Boswijk & Franses, 1995, p.440)

$$\begin{aligned} \Omega &= \sum_{i=-\infty}^{\infty} E[\Delta_4 X_\tau \Delta_4 X_{\tau-i}^{'}] \\ &= \sum_{i=-\infty}^{\infty} E[\Theta_0^x E_\tau^x - \Theta_1^x E_{\tau-1}^x][\Theta_0^x E_{\tau-i}^x - \Theta_1^x E_{\tau-i-1}^x]' \\ &= [\Theta_0^x - \Theta_1^x](\Sigma \otimes I_4)[\Theta_0^x - \Theta_1^x]' \end{aligned} \tag{12}$$

In the absence of cointegration between the $x_{s\tau}^{(j)}$ $j = 1, 2, \cdots, m$, cointegration applies only across the seasons separately within each $x_{s\tau}^{(j)}$ $j = 1, 2, \cdots, m$ and we have

$$C^x(1) = \Theta_0^x - \Theta_1^x = a^x b^x{}' \tag{13}$$

where the $(4m)\times m$ matrices a^x, b^x are defined by (in an obvious notation)

$$a^x = \begin{bmatrix} a^{(1)} & 0 & \cdots & 0 \\ 0 & a^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{(m)} \end{bmatrix}, \qquad b^x = \begin{bmatrix} b^{(1)} & 0 & \cdots & 0 \\ 0 & b^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^{(m)} \end{bmatrix}. \tag{14}$$

and all sub-matrices in (14) are 4×1. In this case, $C^x(1)$ is of rank m.

However, if cointegration exists across processes, then $C^x(1)$ is of rank $k < m$ and hence a^x and b^x do not have the block diagonal form of (14). Specifically, a^x and b^x are then matrices of rank k, with dimension $4m \times k$.

Returning to the case of no cointegration, Lemma 3 establishes the asymptotic distribution of the scaled vector $X_\tau = [X_\tau^{(1)}{}', Z_\tau{}']'$ relevant for the regression (7). The result is obtained by accounting for the contemporaneous correlation between the disturbances through the decomposition $\Sigma = PP'$ where P is upper triangular.

LEMMA 3. *Consider the vector of m PI*(1) *processes defined in* (6), (8) *and* (9), *with no cointegration applying across the m processes. Also define the* $4m \times 1$ *vector Brownian motion* $W^x(r)$ *with covariance matrix* I_{4m}, *where* $W^x(r) = [W^{(1)}(r)', W^z(r)']'$ *in which* $W^{(1)}(r)$ is 4×1, $W^z(r)$ *is* $4n \times 1$ *and* $n = m - 1$. *Then, as* $N = T/4 \to \infty$:

$$\begin{aligned} \frac{1}{\sqrt{N}} X_{[rN]} &= \frac{1}{\sqrt{N}} \begin{bmatrix} X_{[rN]}^{(1)} \\ Z_{[rN]} \end{bmatrix} \Rightarrow B(r) = \begin{bmatrix} B^{(1)}(r) \\ B^z(r) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11}^{1/2} a^{(1)} b^{(1)}{}' \left(\sqrt{1 - \rho_{1z}{}' \rho_{1z}}\, W^{(1)}(r) + (\rho_{1z}{}' \otimes I_4) W^z(r) \right) \\ a^z b^z{}' (P_{zz} \otimes I_4) W^z(r) \end{bmatrix} \end{aligned} \tag{15}$$

where $[rN]$ *is the integer part of rN,* $\rho_{1z} = \sigma_{11}^{-1/2} P_{zz}^{-1} \sigma_{1z}$, *the upper triangular matrix* P_{zz} *satisfies* $P_{zz}P_{zz}' = \Sigma_{zz}$, *and* a^z, b^z *are the lower right-hand* $4n \times n$ *blocks of* a^x, b^x, *respectively, in* (14).

Here and throughout the paper $\Rightarrow$ indicates convergence in distribution.

Note that we can define standard Brownian motions underlying (15) as

$$\widetilde{w}^{(1)}(z) = \left(b^{(1)}{}'b^{(1)}\right)^{-1/2} b^{(1)}{}'\left(\sqrt{1-\rho_{1z}{}'\rho_{1z}}\, W^{(1)}(r) + \left(\rho_{1z}{}'\otimes I_4\right)W^{z}(r)\right)$$
$$\widetilde{w}^{(j)}(r) = \left(b^{(j)}{}'b^{(j)}p^{(j)}p^{(j)}{}'\right)^{-1/2} b^{(j)}{}'(p^{(j)}\otimes I_4)W^{z}(r) \qquad j = 2, 3, \ldots, m \tag{16}$$

where $p^{(j)}$ is the $(j\text{-}1)^{th}$ row of P_{zz}. Using (16),

$$B^{(1)}(r) = \sigma_{11}^{1/2}\left(b^{(1)}{}'b^{(1)}\right)^{1/2} a^{(1)}\widetilde{w}^{(1)}(z)$$
$$B^{(j)}(r) = \left(p^{(j)}p^{(j)}{}'b^{(j)}{}'b^{(j)}\right)^{1/2} a^{(j)}\widetilde{w}^{(j)}(z), \quad j = 2, 3, \ldots, m \tag{17}$$

provides an alternate representation to (15). Therefore, the scalar standard Brownian motions $\widetilde{w}^{(j)}(r)$ in (16) can be thought of as the stochastic trends underlying the *m* individual *PI* processes, which in turn derive from the vector Brownian motion processes $W^{(1)}(r)$ and $W^{z}(r)$.

It is clear from (15) or (16) that, in general, Brownian motions processes relating to $x^{(1)}$ and *z* are correlated. That is, when the contemporaneous covariance σ_{1z} in (9) is nonzero, $W^{z}(r)$ influences $\widetilde{w}^{(1)}(r)$. This effect disappears in the special case of $\sigma_{1z} = 0$, since $\rho_{1z} = 0$ when $x^{(1)}$ is uncorrelated with $x^{(2)}, \ldots, x^{(m)}$.

3.3 Asymptotic Distribution of the Test Statistic

We now turn to the properties of the residuals resulting from OLS estimation of (7), which are summarized in Lemma 4.

LEMMA 4. *Consider the vector of PI*(1) *processes defined in* (6), (8) *and* (9), *with no cointegration applying across the m processes. The* 4×1 *vector* $\hat{V}_\tau = [\hat{v}_{1\tau}, \hat{v}_{2\tau}, \hat{v}_{3\tau}, \hat{v}_{4\tau}]'$ *of residuals from* (7) *for year τ then satisfy, as N = T/*4 → ∞:

$$\frac{1}{\sqrt{N}}\hat{V}_{[rN]} \Rightarrow l_{11}\sigma_{11}^{1/2}(b^{(1)}{}'b^{(1)})^{0.5} a^{(1)}\overline{w_m}(r) \tag{18}$$

where [*rN*] *is the integer part of rN and the univarite Brownian motion* $\overline{w_m}(r)$ *is defined by*

$$\overline{w_m}(r) = \overline{w^{(1)}}(r) - \int \overline{w^{(1)}}(r)\overline{W^{z}}(r)'dr\left[\int \overline{W^{z}}(r)\overline{W^{z}}(r)'dr\right]^{-1}\overline{W^{z}}(r) \tag{19}$$

in which $\overline{W^x}(r) = [\overline{w^{(1)}}(r), \overline{W^z}(r)']'$ *is* $m \times 1$ *standard Brownian motion with covariance matrix* I_m, *the* 4×1 *vectors* $a^{(1)}$, $b^{(1)}$ *are defined in* (14) *and* l_{11} *is a scalar.*

Lemma 4 implies that the residuals from (7) asymptotically retain the same nonstationary periodic coefficients as the univariante process for $x_{s\tau}^{(1)}$ in (6). This is easily seen by comparing (18) with the first equation of (17).

Building on the implication of Lemma 4 that the residuals of (7) retain the *PI* properties of $x_{s\tau}^{(1)}$ in (6), the strategy of testing for periodic integration in the residuals of (7) is clear. More specifically, following Franses (1996, pp. 181-182), we propose testing the null hypothesis $\phi_1 \phi_2 \phi_3 \phi_4 = 1$ against the alternative $\phi_1 \phi_2 \phi_3 \phi_4 < 1$ in the unrestricted dummy variable regression[3]

$$\hat{v}_{s\tau} = \sum_{s=1}^{4} \phi_s D_{s\tau} \hat{v}_{s-1,\tau} + \varepsilon_{s\tau} \tag{20}$$

where $D_{s\tau}$ is the usual binary seasonal dummy variable for season s. Under the null hypothesis the residuals $\hat{v}_{s\tau} \sim PI(1)$, so that there is no cointegration between the two *PI* processes. Under the alternative hypothesis, the residuals are stationary, implying that either full periodic cointegration or full nonperiodic cointegration exists between the processes.

We employ the test of periodic integration proposed by Boswijk and Franses (1996), which uses the Likelihood Ratio statistic

$$LR = N \ln\left(\tilde{\sigma}_0^2 / \tilde{\sigma}^2\right) \tag{21}$$

where $\tilde{\sigma}^2$ is the unrestricted maximum likelihood estimator of the disturbance variance in (7) and $\tilde{\sigma}_0^2$ is the corresponding estimator when the restriction $\phi_1\phi_2\phi_3\phi_4 = 1$ is imposed. The usual

[3] Note that, although (18) implies the *PI* coefficients for the residuals are identical to those of the univariate process for $x_{s\tau}^{(1)}$ in (6), we do not propose that equality between these should be imposed.

practice is to impose the restriction by setting $\phi_4 = 1/(\phi_1\phi_2\phi_3)$, with (20) then estimated by nonlinear least squares.

Theorem 1 establishes the asymptotic distribution of this periodic cointegration test statistic.

THEOREM 1. *Under the null hypothesis of no cointegration between the PI processes of* (6)*,* (8) *and* (9)*, the likelihood ratio test statistic of* (21) *applied to the residuals from* (7) *has asymptotic distribution*:

$$LR \Rightarrow \left\{\int \overline{w_m}(r)^2 dr\right\}^{-1} \left\{\int \overline{w_m}(r)\, d\overline{w_m}(r)\right\}^2 \qquad (22)$$

where $\overline{w_m}(r)$ *is defined in* (19).

The distribution of the test statistic in (22) is the square of the Dickey-Fuller test for cointegration using the residuals of a (nonperiodic) regression, as derived by Phillips and Ouliaris (1990). It is clear from (19) that this asymptotic distribution depends on the number of regressors in (7), namely $n = m - 1$. Consequently, the distribution of the *LR* test statistic in (22) also depends on m. However, the distribution is invariant to the values of the *PI* coefficients for the processes in (6) and nonzero disturbance covariances in (9).

Also in common with Phillips and Ouliaris (1990), this asymptotic distribution will continue to hold in the presence of stationary autocorrelation in the processes of (6), provided that the test regression for the residuals in (20) is sufficiently augmented to take account of this autocorrelation.

4. The Boswijk and Franses Test

Boswijk and Franses (1995) propose Wald tests for periodic cointegration relating to a specific season s and over all seasons with an error-correction mechanism (ECM) framework. This test is built on the cointegration test of Boswijk (1994), which was developed in a nonperiodic context. To avoid issues Boswijk and Franses encounter concerned with the possible dependence of some asymptotic distributions on correlation between the disturbances of the processes considered, we confine our attention to the "spurious regression" case where the variables are mutually uncorrelated. Also for simplicity, we continue to assume that all variables have zero means, with no deterministic terms included in the estimated ECM model. Although a special case, this is sufficient to establish that, contrary to the statement of Boswijk and Franses (1995), the asymptotic distribution of this test when applied to *PI* processes differs from the result they obtain for *SI* processes.

Therefore, using the notation of the previous section, and arbitrarily assuming that the first variable of $x_{s\tau}$ is the dependent variable, our periodic ECM model is

$$\Delta_4 x_{s\tau}^{(1)} = \gamma_s \left(x_{s,\tau-1}^{(1)} - \kappa_s ' z_{s,\tau-1} \right) + u_{s\tau}, \qquad s = 1, 2, 3, 4$$

where κ_s is an $n \times 1$ vector. Employing seasonal dummy variables, this can be written

$$\Delta_4 x_{s\tau}^{(1)} = \sum_{s=1}^{4} \left(\delta_{0s} D_{s\tau} x_{s,\tau-1}^{(1)} + \delta_{1s} ' D_{s\tau} z_{s,\tau-1} \right) + u_{s\tau} \tag{23}$$

where $\delta_{0s} = \gamma_s$ and $\delta_{1s} = -\gamma_s \kappa_s$. We also continue to assume that the variables are generated by the $PI(1)$ processes of (6), (8) and (9), with the additional assumption $\sigma_{1z} = 0$ and Σ_{zz} being a diagonal matrix. In comparison with Boswijk and Franses (1995), no conditioning on the $n \times 1$ vector $\Delta z_{s\tau}$ is included in (23), due to our simplifying assumption of zero covariance between $x^{(1)}$ and z.

Using a similar notation to Boswijk and Franses (1995), the Wald statistic to test the null of no cointegration in season s, or equivalently to test $\delta_{0s} = 0$, $\delta_{1s} = 0$ in (23), is

$$Wald_s = \hat{\delta}_s ' \left(V\hat{a}r\left[\hat{\delta}_s \right] \right)^{-1} \hat{\delta}_s \tag{24}$$

where $\hat{\delta}_s = (\hat{\delta}_{0s}, \hat{\delta}_{1s}')'$ is the ordinary least squares estimator of the relevant coefficients and $V\hat{a}r[\hat{\delta}_s]$ is the corresponding estimated OLS covariance matrix. When all seasons are considered, the joint cointegration test statistic for the null hypothesis $\delta = 0$, where $\delta = [\delta_1', \delta_2', \delta_3', \delta_4']'$, is given by (in an obvious notation)

$$Wald = \hat{\delta}' \left(V\hat{a}r\left[\hat{\delta}\right]\right)^{-1} \hat{\delta} = \sum_{s=1}^{4} \hat{\delta}_s' \left(V\hat{a}r[\hat{\delta}_s]\right)^{-1} \hat{\delta}_s \ . \tag{25}$$

Note that *Wald* statistic in (25) is the sum of the individual $Wald_s$ of (24) due to the block orthogonality of the regressors in (23) that is a consequence of the seasonal dummy variables.

As noted by Ghysels and Osborn (2001, pp.176-179), the null distribution obtained by Boswijk and Franses (1995) assumes $x_{s\tau}$ is a vector of *SI* processes. More specifically, Assumption 1 of Boswijk and Franses (1995, p.440) does not require the longrun variance-covariance matrix Ω of the vector Brownian motion process corresponding to $(x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau})'$ to be positive definite, which allows the possibility of one or more components being *PI* processes. However, the proof of their Theorem 2 assumes that $\tilde{C}_s'\Omega\tilde{C}_s$ is strictly positive[4]. Consequently, the asymptotic distributions derived by Boswijk and Franses require Ω to have full rank, ruling out the possibility that any element of $x_{s\tau}$ is periodically integrated.

Under the null hypothesis of no periodic cointegration, and assuming *SI* processes, Boswijk and Franses (1995) establish that the distribution of the $Wald_s$ statistic used to test for cointegration relating to an individual season s is identical to that obtained by Boswijk (1994) for the nonperiodic case. Theorem 2 below shows that this result does not carry over to the case of *PI*(1) processes. Indeed, for such processes, the theorem shows that the distribution of Boswijk (1994) emerges in relation to the test statistic for full periodic cointegration.

[4] See the paragraph between expressions (A.11) and (A.12) of Boswijk and Franses (1995, p.451).

THEOREM 2. *Under the null hypothesis of no cointegration between the PI processes of* (6)*,* (8) *and* (9) *with* $\sigma_{1z} = 0$ *and* Σ_{zz} *being a diagonal matrix, the asymptotic distributions of the Wald test statistics proposed by Boswijk and Franses are:*

(i) for the $Wald_s$ *test of* $\delta_{0s} = \delta_{1s} = 0$ *for an individual s*

$$Wald_s \Rightarrow 4\frac{(a_s^{(1)})^2}{a^{(1)}{}'a^{(1)}}\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'dr\right)^{-1}\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right), \qquad (26)$$

(ii) for the joint Wald test of $\delta_{0s} = \delta_{1s} = 0$*,* $s = 1, 2, 3, 4$,

$$Wald \Rightarrow 4\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'dr\right)^{-1}\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right) \qquad (27)$$

where $\overline{W^x}(r)' = [\overline{w^{(1)}}(r), \overline{W^z}(r)']'$ *is m-vector standard Brownian motion and* $a^{(1)} = [1, \phi_2^{(1)}, \phi_2^{(1)}\phi_3^{(1)}, \phi_2^{(1)}\phi_3^{(1)}\phi_4^{(1)}]'$ *which has* s^{th} *element* $a_s^{(1)}$.

There are two important differences between the distributional results given (26) and (27) and those of Boswijk and Franses (1995) for *SI* processes. Firstly, the statistic in (26) does not follow the distribution of Boswijk (1994), due to the multiplicative factor $\lambda_s = 4\left(a_s^{(1)}\right)^2 / a^{(1)}{}'a^{(1)}$. Since, these λ_s average unity over s = 1, 2, 3, 4, the scaling will inflate or deflate values relative to the Boswijk (1994) distribution, depending on the specific *PI* coefficients and the season *s*.

Secondly, the distribution defined by (27) is four times the distribution obtained by Boswijk (1994). Intuitively, this arises because there is only one underlying stochastic trend for each vector process $X_\tau^{(j)}$ and hence, as discussed in Section 2, there can be only one linearly independent cointegrating relationship over the four quarters of the year. Consequently, when the *Wald* test is applied to the *PI*(1) variables, effectively a single cointegration relationship is being tested four times (once for each quarter).

The asymptotic distribution of (27) is not that derived by Boswijk and Franses (1995) for *SI* processes. To be specific, because an *SI* process for a quarterly series involves four distinct unit root processes, these are reflected in the asymptotic distribution. For uncorrelated *SI* processes, the asymptotic Boswijk-Franses distribution is (Ghysels and Osborn, 2001, p.178)

$$Wald \Rightarrow \sum_{s=1}^{4}\left(\int W_s^x(r)dw_s^{(1)}(r)\right)'\left(\int W_s^x(r)W_s^x(r)'\right)^{-1}\left(\int W_s^x(r)dw_s^{(1)}(r)\right) \qquad (28)$$

where $W_s^x(r) = [w_s^{(1)}(r), W_s^z(r)']'$ is formed by selecting elements of the $4m \times 1$ vector standard Brownian motion $W^x(r)$ corresponding to season s. It is obvious that (27) and (28) differ, with the former being four times the Boswijk (1994) distribution whereas the latter is the sum of four independent distributions of this type. Indeed, this comparison also clarifies the role played by the four distinct unit roots underlying an *SI* process and which therefore appear in (28) as against the single unit root underlying a *PI* process.

5. Monte Carlo Analysis

In this section we present Monte Carlo results relating to the empirical size and power of the residual-based test for periodic cointegration analysed in Section 3. Subsection 5.1 considers zero-mean processes, with the analysis of Subsection 5.2 allowing the possibility of nonzero trends.

5.1 Zero Mean Processes

We first investigate empirical size[5] for zero-mean processes generated through the bivariate model, where $x_{s\tau} = (y_{s\tau}, z_{s\tau})'$, such that

[5] All results presented are based on the 5 percent critical value of 7.3 obtained through a Monte Carlo simulation for T = 200 observations. However, use of the asymptotic critical value of $(-2.76)^2$ from Phillips and Ouliaris (1990, Table IIa) does not alter the substantive conclusions.

$$
\begin{aligned}
&y_{s\tau} = \phi_s^y y_{s-1,\tau} + u_{s\tau}^y, \quad \prod_{s=1}^{4} \phi_s^y = 1, \quad u_{s\tau}^y = \gamma u_{s-1\tau}^y + e_{s\tau}^y \\
&z_{s\tau} = \phi_s^z z_{s-1,\tau} + u_{s\tau}^z, \quad \prod_{s=1}^{4} \phi_s^z = 1, \quad u_{s\tau}^z = \gamma u_{s-1\tau}^z + e_{s\tau}^z
\end{aligned} \tag{29}
$$

for $\gamma = \{0.0, 0.5\}$ and

$$
E\left[\begin{bmatrix} e_{s\tau}^y \\ e_{s\tau}^z \end{bmatrix}\begin{bmatrix} e_{s\tau}^y & e_{s\tau}^z \end{bmatrix}\right] = \begin{bmatrix} 1 & \sigma_{\mathrm{yz}} \\ \sigma_{\mathrm{yz}} & 1 \end{bmatrix}, \quad \sigma_{\mathrm{yz}} = \{0.0, 0.4, 0.8\} \tag{30}
$$

As show in Lemma 4, the DGP of (29) implies that the residuals follow a *PI*(1) process when the regression of (7) is estimated.

Note that (30) permits three levels of contemporaneous correlation between the innovations $e_{s\tau}^y$ and $e_{s\tau}^z$. Further, the processes of (29) are PAR(1) processes when $\gamma = 0$, and in this case the residuals also follow PAR(1) processes. However, a second-order model is required when $\gamma = 0.5$, so that we also test the *PI* restriction using PAR(2) models for the residuals.

The empirical power is obtained from the DGP

$$
\begin{aligned}
&y_{s\tau} = k_s z_{s,\tau} + u_{s\tau}, \quad u_{s\tau} = \frac{e_{s\tau}^y}{(1-\gamma_1 L)(1-\gamma_2 L)} \\
&z_{s\tau} = \phi_s^z z_{s-1,\tau} + e_{s\tau}^z, \quad \prod_{s=1}^{4} \phi_s^z = 1
\end{aligned} \tag{31}
$$

$\gamma_1 = \{0.0, 0.5\}$, $\gamma_2 = \{0.0, 0.8\}$ and the periodic cointegrating relationship has coefficients $k_4 = 0.4$, $k_{s-1} = 0.4\,\phi_s^z / \phi_s^y$ for s = 3, 2, 1, with ϕ_s^y (s = 1, 2, 3, 4) being the *PI* coefficients for $y_{s\tau}$. The innovation covariance matrix is again given by (30). When γ_1, γ_2 are both zero in (31), the residuals from (7) follow a white noise process. On the other hand, the residuals are an AR(1) when one of these coefficients is zero, so that a PAR(1) model is sufficient to account for this autocorrelation. However, $\gamma_1 = 0.5$, $\gamma_2 = 0.8$ leads to an AR(2) process for the residuals, which is accommodated by estimating a PAR(2) model. Thus, (31) allows the same levels of serial dependence as considered in (29).

Table 1 shows the combinations of coefficients used in (29) and (31) to compute the empirical size and power respectively. The size and power results are collected in Table 2, for a sample size of 50 years (200 observations), and based on 5,000 replications.

The results of Table 2 verify that, even in finite samples, the residual-based test for periodic cointegration has good size properties, provided that the appropriate order of periodic autoregressive model is selected. This is true across all sets of *PI* coefficients considered and irrespective of the extent of correlation between the disturbances. However, the test is badly undersized when the PAR order is underspecified.

Further, again provided that an appropriate order of periodic process is fitted, the test has power approaching unity, especially when $\gamma_1 = 0.5$, $\gamma_2 = 0.8$. The relatively low power obtained for a PAR(1) specification in this case is a reflection of the undersizing that results when a model of too low order is employed.

5.2 Deterministic Terms

In order to facilitate the theoretical analysis above, we omit deterministic terms and assume the initial value $x_{40} = 0$, thereby implying $E[x_{s\tau}] = 0$. Here we relax these restrictions by considering the addition of deterministic terms to the cointegrating test regression.

In the case of standard (nonperiodic) cointegration, the appropriate form of the cointegration test regression depends on the properties of the time series under study; see Phillips and Ouliaris (1990) and Hansen (1992). The inclusion of an intercept allows for possibly nonzero starting values, with means constant over time, by demeaning the variables used in the longrun regression. The null distribution of the LR test for (nonperiodic) cointegration then satisfies (22), with $\overline{w_m}(r)$ as defined in (19), where it is understood that $\overline{W^x}(r) = [\overline{w^{(1)}}(r), \overline{W^z}(r)']'$ is a vector of demeaned standard Brownian motions. The addition of a trend allows for a nonzero drift, and the vector of Brownian motions is then demeaned and detrended.

Turning to the case of *PI*(1) processes, a nonzero starting value in (1), with no deterministic terms, implies a seasonally varying mean $E[x_{s\tau}]$ that is, however, constant over years $\tau = 1, 2, \ldots$ However, as shown by Paap and Franses (1999), the addition of an intercept to (1) leads to a seasonally-varying trend in $E[x_{s\tau}]$, and hence an annual growth rate $\Delta_4 x_{s\tau}$ that varies over $s = 1, 2, 3, 4$. Further, excluding the special case of an *I*(1) process, they show that a *PI*(1) process with an intercept cannot have a trend that is common over $s = 1, 2, 3, 4$, irrespective of whether the intercept is constant over seasons or is seasonally-varying. On the other hand, the univariate first-order process

$$x_{s\tau} = \mu_s + \vartheta_s \tau + \phi_s x_{s-1,\tau} + e_{s\tau}, \quad \prod_{i=1}^{s} \phi_s = 1 \tag{32}$$

with $e_{s\tau}$ white noise and trend coefficients that satisfy

$$\vartheta_s = (1 - \phi_s)[\mu_4 + \phi_4\mu_3 + \phi_3\phi_4\mu_2 + \phi_2\phi_3\phi_4\mu_1] \quad s = 1, 2, 3, 4 \tag{33}$$

has a common linear trend shared by all quarters (Paap and Franses, 1999). However, with unrestricted trend coefficients, (32) implies seasonally-varying quadratic trends in $E[x_{s\tau}]$.

In the context of testing for periodic cointegration, the above discussion implies that the relevant cointegrating regressions that may be considered in place of (7) are

$$x_{s\tau}^{(1)} = \sum_{s=1}^{4} \beta_{0s} D_{s\tau} + \sum_{i=2}^{m} \sum_{s=1}^{4} \beta_{is} D_{s\tau} x_{s\tau}^{(i)} + v_{s\tau} \tag{34}$$

and

$$x_{s\tau}^{(1)} = \sum_{s=1}^{4} \beta_{0s} D_{s\tau} + \sum_{s=1}^{4} \beta_{1s} D_{s\tau} \tau + \sum_{i=2}^{m} \sum_{s=1}^{4} \beta_{is} D_{s\tau} x_{s\tau}^{(i)} + v_{s\tau} \tag{35}$$

More specifically, (34) is appropriate when the variables in the regression are known to have constant (possibly periodically-varying) mean over time, while the use of (35) permits the possibility that the variables may trend linearly over time[6].

In addition to the case with unrestricted trends in (35), we also investigate cointegrating regressions using restricted trend coefficients such that $\beta_{11} = \beta_{12} = \beta_{13} = \beta_{14}$. This last case is considered when ϑ_s satisfies the restrictions of (33), and hence the linear trend in each $PI(1)$ process is constant over seasons.

The results of Panel a of Table 3 verify that, for the three bivariate $PI(1)$ DGPs considered there, the inclusion of deterministic terms has the anticipated effect on the residual-based test for periodic cointegration. That is, for zero-mean processes, the inclusion of periodically-varying intercepts or periodically-varying intercepts and trends, as in (34) or (35) respectively, causes the distribution of the LR test for periodic cointegration to shift, with the percentiles of the test statistic under the null hypothesis being effectively the same as the corresponding values obtained by Phillips and Ouliaris (1990) for the nonperiodic case (with the latter values squared).

Since the inclusion of unrestricted intercepts leads to seasonally-varying trends in a $PI(1)$ process, a cointegrating test regression of the form of (35), with unrestricted intercepts and trends, takes account of these deterministic effects. Panel b of Table 3 verifies that, in this case, the (squared) Phillips-Ouliaris (1990) critical values for nonperiodic random walks with drifts continue to apply in this periodic case. As seen in Panel c, these critical values also apply if the individual processes within the DGP have trends restricted to be identical across seasons, provided that no restrictions are imposed when (35) is estimated. However, imposition of the restriction of nonperiodic trends in the cointegrating test regression of (35) causes the Phillips-Ouliaris critical values to be inappropriate for these DGPs.

[6] In common with much of the unit root literature, the possibility of quadratic trends over time is excluded on *a priori* grounds.

In contrast to the effects of restricted trends in Panel c, Panel d shows that, whether the trend coefficients of the cointegrating test regression are restricted to be identical over seasons or not, the Phillips-Ouliaris (1990) critical values can be used when testing cointegration between two $PI(1)$ processes which have identical periodic coefficients, $\phi_s^{(j)} = \phi_s, j = y, z$. However, the case of identical coefficients across PI separate processes is a special one, for which Lemma 1 shows that any cointegration must be nonperiodic.

To investigate this further, consider the $PI(1)$ vector $x_{s\tau}$, where all elements have constant trends over seasons. Separating the deterministic and stochastic components of each element, we can write

$$x_{s\tau}^{(i)} = c_{0s}^{(i)} + c_1^{(i)}\tau + \xi_{s\tau}^{(i)}, \quad s = 1, 2, 3, 4; i = 1, 2, ..., m \tag{36}$$

where $\xi_{s\tau}^{(i)}$ is a zero-mean $PI(1)$ process and $E[x_{s\tau}^{(i)}] = c_{0s}^{(i)} + c_1^{(i)}\tau$, which has a periodically-varying intercept but nonperiodic trend. The regression relevant for testing periodic cointegration between the zero-mean stochastic unit root processes $\xi_{s\tau}^{(i)}$ is

$$(x_{s\tau}^{(1)} - c_{0s}^{(1)} - c_1^{(1)}\tau) = \sum_{i=2}^{m} \beta_{is}(x_{s\tau}^{(i)} - c_{0s}^{(i)} - c_1^{(i)}\tau) + u_{s\tau},$$

that is,

$$\begin{aligned} x_{s\tau}^{(i)} &= (c_{0s}^{(1)} - \sum_{i=2}^{m} \beta_{is} c_{0s}^{(i)}) + (c_1^{(1)} - \sum_{i=2}^{m} \beta_{is} c_1^{(i)})\tau + \sum_{i=2}^{m} \beta_{is} x_{s\tau}^{(i)} + u_{s\tau} \\ &= \beta_{0s} + \beta_{1s}\tau + \sum_{i=2}^{m} \beta_{is} x_{s\tau}^{(i)} + u_{s\tau} \end{aligned} \tag{37}$$

which is identical in form to (35). Notice, however, that although (37) has periodically varying intercepts and periodic trend coefficients, the trend coefficients in the latter satisfy

$$\beta_{1s} = c_1^{(1)} - \sum_{i=2}^{m} \beta_{is} c_1^{(i)}, \quad s = 1, 2, 3, 4. \tag{38}$$

If the $PI(1)$ coefficients are identical across processes, and hence any cointegrating relationship is nonperiodic, then $\beta_{i1} = \beta_{i2} = \beta_{i3} = \beta_{i4}$ (i = 2, …, m) and (38) implies nonperiodic trends in the cointegrating regression of (35) or (37).

The Monte Carlo results of Panels c and d support this analysis. In particular, the $PI(1)$ processes in Panel d with identical coefficients and individual nonperiodic trends imply that any trend in (35) is also nonperiodic. Therefore, the restriction of identical trends derives from the nonperiodic nature of any cointegration in this case, with the imposition of this restriction effectively having no impact on the distribution of the residual-based test statistic.

On the other hand, when the $PI(1)$ coefficients differ over processes, (38) implies that the imposition of the nonperiodic trend restriction is inappropriate when the β_{is} are not correspondingly restricted. However, from Lemma 1, nonperiodic cointegration can apply only when the separate processes have identical *PI* coefficients. Therefore, the trend coefficients in (35) should not be restricted to be nonperiodic when testing for cointegration between periodic processes, except for the special case analysed in Panel d.

6. Concluding Remarks

This paper has provided an analysis of cointegration for periodically integrated processes. We first establish that the only cointegration possibilities are so-called full periodic or full nonperiodic cointegration. Due to the cointegration between seasons that exists for a univariate periodically integrated variable, if no cointegration between variables applies for a specific individual season, then no cointegration applies at all. Further, if the periodically integrated processes have identical coefficients over processes, then any cointegration that exists is nonperiodic, with identical cointegrating relationships over seasons.

Two tests of cointegration have been proposed as appropriate in previous literature for periodically integrated processes. However, this paper is the first to obtain analytical results for the asymptotic distributions of these tests.

The analytical results previously available for cointegration related to seasonal processes have focussed on the case of seasonally integrated processes, including Boswijk and Franses (1995), Hylleberg *et al*. (1990), Johansen and Schaumburg (1999). However, the greater economic plausibility of periodic processes in some contexts suggests that attention should also be devoted to this case. The present paper provides results that contribute to our understanding of cointegration for seasonal processes, while also emphasising that periodic and seasonal integration have distinct longrun implications. In particular, although the Boswijk-Franses (1995) periodic cointegration test can be applied for both types of seasonal nonstationarity, the test statistic follows different distributions in the two cases. Therefore, a careful prior univariate analysis should be undertaken before considering cointegration for seasonal processes.

Our analysis also formally establishes the asymptotic distribution of a residual-based test of cointegration for periodically integrated processes, showing this distribution to be the same as for the nonperiodic case. Moreover, our Monte Carlo analysis verifies that the critical values of Phillips and Ouliaris (1990) can be used in the context of periodic processes, provided that potentially relevant trend terms included in the cointegration test regression are not restricted to be constant over the quarters of the year when the potential cointegration is periodic. Therefore, the test can be employed by applied workers in realistic contexts where the periodic series under analysis exhibit nonzero means and possible trends.

As in the case of univariate periodically integrated processes analysed by Paap and Franses (1999), the use of trend terms in testing for periodic cointegration tests requires some care. Specifically, when testing for cointegration in periodic processes which contain nonperiodic trends, we show that the trend coefficients in the cointegration test regression should be restricted to be identical over seasons only when the individual processes have identical periodic coefficients. Since the situation where identical coefficients apply over the different univariate processes may not occur widely in practice, we recommend that the trend (as well as intercept) coefficients should be unrestricted over seasons when using the residual-based test for cointegration between periodically integrated processes.

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Table 1. DGPs used for size and power calculations of Table 2

.

DGP	ϕ_1^y	ϕ_2^y	ϕ_3^y	ϕ_4^y	ϕ_1^z	ϕ_2^z	ϕ_3^z	ϕ_1^z
1	1.200	0.700	1.000	1.190	0.800	0.900	1.200	1.157
2	1.200	1.000	0.800	1.042	0.800	1.000	1.200	1.042
3	0.800	0.800	1.200	1.302	1.200	0.700	1.000	1.190
4	1.200	0.700	1.000	1.190	1.200	0.700	1.000	1.190
5	0.800	1.000	1.200	1.042	0.800	1.000	1.200	1.042
6	0.800	0.900	1.200	1.157	0.800	0.900	1.200	1.157

Table 2. Size and power of residual-based test for periodic cointegration

		Size $\gamma = 0$	Power $\gamma_1 = 0$ $\gamma_2 = 0$	Power $\gamma_1 = 0$ $\gamma_2 = 0.8$	Size $\gamma = 0.5$		Power $\gamma_1 = 0.5$ $\gamma_2 = 0.8$	
σ_{yz}^2	DGP	PAR(1)	PAR(1)	PAR(1)	PAR(1)	PAR(2)	PAR(1)	PAR(2)
0	1	0.050	1.000	0.989	0.006	0.056	0.365	0.998
0	2	0.047	1.000	0.985	0.007	0.057	0.364	0.997
0	3	0.050	1.000	0.991	0.006	0.061	0.352	0.999
0	4	0.055	1.000	0.991	0.005	0.055	0.376	0.998
0	5	0.053	1.000	0.988	0.007	0.062	0.347	0.998
0	6	0.052	1.000	0.989	0.004	0.057	0.336	0.998
0.4	1	0.050	1.000	0.982	0.007	0.053	0.352	0.998
0.4	2	0.049	1.000	0.979	0.005	0.054	0.368	0.998
0.4	3	0.053	1.000	0.986	0.006	0.052	0.345	0.998
0.4	4	0.048	1.000	0.984	0.005	0.054	0.350	0.998
0.4	5	0.053	1.000	0.984	0.006	0.050	0.346	0.998
0.4	6	0.054	1.000	0.985	0.007	0.055	0.342	0.998
0.8	1	0.046	1.000	0.972	0.004	0.051	0.321	0.998
0.8	2	0.052	1.000	0.965	0.006	0.049	0.338	0.998
0.8	3	0.051	1.000	0.983	0.005	0.050	0.320	0.999
0.8	4	0.059	1.000	0.978	0.006	0.061	0.321	0.998
0.8	5	0.050	1.000	0.979	0.007	0.054	0.308	0.999
0.8	6	0.048	1.000	0.979	0.007	0.065	0.298	0.998

Note: The residual-based test is applied to (7). Results are based 5,000 replications, for a sample of 200 observations (N=50). The DGPs used for size and power are given in (29) and (31) respectively, using the coefficients of Table 1. PAR(1) and PAR(2) indicate that periodic autoregressive models of order 1 or 2, respectively, are fitted to the residuals in order to obtain the LR statistic used to test periodic cointegration at a nominal significance level of 5 percent. The critical value used is 7.3, which has been obtained from a Monte Carlo based on 15,000 replications of two uncorrelated *PI*(1) processes with a sample size of 200 observations (N = 50).

Table 3. Effect of deterministic terms on the empirical distribution of the residual-based cointegration test

DGP	Deterministic terms in regression	Percentile						
		0.85	0.875	0.9	0.925	0.95	0.975	0.99
Panel a. Zero-mean DGPs								
1	None	5.024	5.447	5.958	6.626	7.503	8.994	10.922
	Intercepts	8.062	8.565	9.154	9.943	10.981	12.709	14.970
	Intercepts & trends	10.824	11.386	12.113	12.992	14.149	16.047	18.484
2	None	5.001	5.402	5.904	6.572	7.438	8.908	10.748
	Intercepts	7.939	8.443	9.066	9.840	10.825	12.635	14.934
	Intercepts & trends	10.768	11.361	12.036	12.882	14.080	15.990	18.524
3	None	5.036	5.456	5.980	6.658	7.517	9.037	11.042
	Intercepts	8.034	8.553	9.158	9.884	10.949	12.767	15.014
	Intercepts & trends	10.840	11.382	12.067	12.910	14.110	16.026	18.492
Panel b. Periodic-trend DGPs								
1	Intercepts & trends	10.844	11.421	12.108	12.948	14.158	16.062	18.505
2	Intercepts & trends	10.828	11.417	12.097	12.968	14.139	16.091	18.428
3	Intercepts & trends	10.789	11.389	12.073	12.930	14.089	16.023	18.488
Panel c. Nonperiodic trend DGPs								
1	Intercepts & trends	10.724	11.292	11.996	12.838	13.962	16.015	18.635
	Inter./restr. trend	26.281	28.984	32.485	37.367	44.460	56.823	75.590
2	Intercepts & trends	10.826	11.404	12.087	12.938	14.133	16.123	18.638
	Inter./restr. trend	16.689	18.021	19.562	21.827	25.219	31.502	40.791
3	Intercepts & trends	10.796	11.400	12.106	12.955	14.095	16.012	18.508
	Inter./restr. trend	17.053	18.498	20.272	22.663	26.285	32.921	43.194
Panel d. Identical PI processes with nonperiodic trends								
1*	Intercepts & trends	10.854	11.419	12.187	13.014	14.132	16.061	18.761
	Inter./restr. trend	10.887	11.470	12.230	13.058	14.205	16.095	18.833
2*	Intercepts & trends	10.900	11.466	12.167	13.052	14.262	16.261	18.477
	Inter./restr. trend	10.937	11.505	12.223	13.110	14.285	16.351	18.613
3*	Intercepts & trends	10.787	11.440	12.163	13.051	14.230	16.020	18.523
	Inter./restr. trend	10.882	11.515	12.238	13.128	14.342	16.077	18.668
Phillips-Ouliaris critical values								
	None	5.100	5.538	6.005	6.668	7.628	9.331	11.468
	Intercept	8.202	8.744	9.399	10.228	11.326	13.264	15.696
	Intercept & trend	11.078	11.701	12.379	13.298	14.440	16.583	19.034

Notes: The residual-based test is applied to (7) when no deterministic terms are included, and to (34) or (35) as appropriate when intercepts or intercepts and trends are included in the regression. Intercepts and relevant trend coefficients in (34) or (35) are unrestricted, unless otherwise stated; restricted trends impose $\beta_{11} = \beta_{12} = \beta_{13} = \beta_{14}$. All DGPs are uncorrelated (both serially and contemporaneously) bivariate *PI*(1) processes. The coefficients for the processes of Panels a, b, and c are:

1: $\phi_1^y = 0.8, \phi_2^y = 0.9, \phi_3^y = 1.2, \phi_4^y = 1.157$; $\phi_1^z = 1.2, \phi_2^z = 0.7, \phi_3^z = 1, \phi_4^z = 1.190$;

2: $\phi_1^y = 1.25, \phi_2^y = 0.8, \phi_3^y = 0.9, \phi_4^y = 1.111$; $\phi_1^z = 1, \phi_2^z = 0.8, \phi_3^z = 1.2, \phi_4^z = 1.042$;

3: $\phi_1^y = 1.2, \phi_2^y = 0.7, \phi_3^y = 1, \phi_4^y = 1.190; \phi_1^z = 0.8, \phi_2^z = 0.8, \phi_3^z = 1.2, \phi_4^z = 1.302;$

The identical *PI* DGPs 1*, 2* and 3* of Panel d have periodic integration coefficients for both processes that are identical to the coefficients for $y_{s\tau}$ for the DGPs 1, 2 and 3 respectively. The DGPs of Panels b, c and d use

$\mu_1^y = 1, \mu_2^y = 1.2, \mu_3^y = 0.5, \mu_4^y = 0.2; \mu_1^z = 0.2, \mu_2^z = 0.5, \mu_3^z = 1.2, \mu_4^z = 1$

in the notation of (32). These intercept values are also used in the nonperiodic-trend DGPs of Panels c and d, with the trend coefficients restricted through (33). Results are based on 25,000 replications for a sample of size 2,000 observations (N = 500 years). The Phillips and Ouliaris (1990) percentiles are the squares of critical values given in their Tables IIa, IIb and IIc corresponding to no deterministic terms, intercept and intercept and trend, respectively, for n = 1 explanatory variable.

Appendix: Proofs

Lemma 1

To prove *(i)*, and without loss of generality, assume that the linear combination $\alpha_1' x_{1\tau}$ is stationary, with α_1 of rank r. Also, for ease of exposition, assume two seasons per year τ, s = 1, 2.

The *PI* process of (6) then implies

$$x_{1\tau} = \Phi_1^+ x_{2,\tau\text{-}1} + E_{1\tau} \tag{A.1}$$

where Φ_1^+ is a diagonal $m \times m$ matrix with jth diagonal element $\phi_1^{(j)}$ and the $m \times 1$ vector $E_{s\tau}$ has jth element $e_{s\tau}^{(j)}$. Premultiplying (A.1) by α_1' yields

$$\begin{aligned} \alpha_1' x_{1\tau} &= \alpha_1' \Phi_1^+ x_{2,\tau-1} + \alpha_1' E_{1\tau} \\ &= \alpha_2' x_{2,\tau-1} + \alpha_1' E_{1\tau} \end{aligned} \tag{A.2}$$

where the $m \times r$ matrix $\alpha_2 = \Phi_1^+ \alpha_1$ defined by (A.2) has rank r, since Φ_1^+ is nonsingular and α_1 is of rank r. Further, the columns of α_2 must contain r cointegrating vectors for $x_{2,\tau\text{-}1}$, as otherwise the right-hand side of (A.2) would be nonstationary.

However, we need to establish that there are no additional linearly independent cointegrating vectors for $x_{2\tau}$, beyond those in the columns of α_2. Say one such cointegrating vector exists, and append this as an additional column of α_2 to form the $m \times (r+1)$ matrix α_2^* of rank r +1. Then, analogously to (A.2), and where Φ_2^+ is a diagonal $m \times m$ matrix with jth diagonal element $\phi_2^{(j)}$, we have

$$\begin{aligned} \alpha_2^{*\prime} x_{2\tau} &= \alpha_2^{*\prime} \Phi_2^+ x_{1\tau} + \alpha_2^{*\prime} E_{2\tau} \\ &= \alpha_1^{*\prime} x_{1\tau} + \alpha_2^{*\prime} E_{2\tau}. \end{aligned}$$

By the same argument as above, $\alpha_1^* = \Phi_2^+ \alpha_2^*$ must be a matrix of r + 1 cointegrating vectors for $x_{2\tau}$. This, however, contradicts the assumption that there are exactly r cointegrating vectors for $x_{1\tau}$. Consequently, there can be only r cointegrating linearly independent cointegrating vectors for $x_{2\tau}$.

Recognizing that α_2 on the right-hand side of (A.2) relates to season s-1 for s = 1, the generalization to four seasons, s = 1, 2, 3, 4 is straightforward, with the r cointegrating vectors for each quarter satisfying

$$\alpha_{s\text{-}1} = \Phi_s^+ \alpha_s \qquad s = 1, 2, 3, 4. \tag{A.3}$$

Note that for s = 4, it is understood that s+1 = 1. By repeated substitution in (A.3), it is clear that given any α_s and the periodic coefficients, the cointegrating vectors for all other quarters can be determined. Also note that the *PI* property of (6) implies that

$$\Phi_1^+ \Phi_2^+ \Phi_3^+ \Phi_4^+ = I_4.$$

In order to establish *(ii)*, first note that, for this first-order case, each of the m processes having identical *PI* coefficients implies $\Phi_s^+ = \phi_s I_m$, for s = 1, 2, 3, 4. Therefore, from (A.3), $\alpha_{s\text{-}1} = \phi_s \alpha_s$, and since the scaling is irrelevant, the cointegrating relationships are identical over s = 1, 2, 3, 4. Conversely, since Φ_s^+ is nonsingular, $\alpha_s = c_s \alpha_{s\text{-}1}$ for some scalar constant c_s only if $\Phi_s^+ = c_s I_m$, s = 1, 2, 3, 4. In turn, $\Phi_s^+ = c_s I_m$ implies that the m PI(1) processes have identical periodic coefficients.

♦

Lemma 2

Define the vector of observations for process j of $x_{s\tau}$ in year τ as $X_\tau^{(j)} = [x_{1\tau}^{(j)}, x_{2\tau}^{(j)}, x_{3\tau}^{(j)}, x_{4\tau}^{(j)}]'$. As all elements of $x_{s\tau}$ are SI, then the series for the quarters of the year are not cointegrated, so that no 4×r matrix of cointegrating vectors β_j exists such that $\beta_j' X_\tau^{(j)}$ is stationary for any j = 1, …, m. Since no cointegration connects the I(1) processes $x_{s\tau}$ and $x_{q\tau}$ ($q \neq s$), the existence of cointegration between the elements of $x_{s\tau}$ has no implications for cointegration between the elements of $x_{q\tau}$.

♦

Lemma 3

For the process of (6), (8) and (9), and as in Boswijk (1994) or Oulariaris and Phillips (1990), we use the decomposition $\Sigma = PP'$ where the upper triangular matrix P is

$$P = \begin{bmatrix} \sigma_{11}^{1/2}\sqrt{1-\rho_{1z}'\rho_{1z}} & \sigma_{11}^{1/2}\rho_{1z}' \\ 0 & P_{zz} \end{bmatrix} \tag{A.4}$$

in which the n×1 (with $n = m$ -1) vector ρ_{1z} is defined from the elements of Σ in (9) as

$$\rho_{1z} = \sigma_{11}^{-1/2} P_{zz}^{-1} \sigma_{1z}\,. \tag{A.5}$$

For a 4m × 1 vector white noise sequence $\{U_\tau\}$ with mean zero and variance matrix I_{4m}, the multivariate invariance principle (see Phillips and Durlauf, 1986) implies that

$$\frac{1}{\sqrt{N}}\sum_{j=1}^{[rN]} U_j \Rightarrow W(r) \tag{A.6}$$

where $W(r)$ is a 4m × 1 vector standard Brownian motion process. For later use, define

$$W(r) = [W^1(r)', W^2(r)', \cdots, W^m(r)']' = [W^1(r)', W^z(r)']',$$

where $W^j(r)$, j = 1, …, m, are 4×1 vectors whose elements we associate with the seasons, while $W^z(r)$ is 4n×1. From these, define the 4m × 1 vector Brownian motion with covariance matrix $\Sigma \otimes I_4$ as

$$E^x(r) = (P \otimes I_4)W(r) = \begin{bmatrix} \sigma_{11}^{1/2}\{\sqrt{1-\rho_{1z}'\rho_{1z}}\, W^{(1)}(r) + (\rho_{1z}'\otimes I_4)W^z(r)\} \\ (P_{zz}\otimes I_4)W^z(r) \end{bmatrix}. \tag{A.7}$$

As in Lemma 1 of Boswijk and Franses (1996),

$$\begin{aligned} \frac{1}{\sqrt{N}} X_{[rN]} &= \left(\Theta_0^x - \Theta_1^x\right)\frac{1}{\sqrt{N}}\sum_{j=1}^{[rN]} E_j^x + o_p(1) \\ &\Rightarrow B^x(r) = \left(\Theta_0^x - \Theta_1^x\right)E^x(r) = a^x b^{x\,\prime} E^x(r). \end{aligned} \tag{A.8}$$

so that, from (A.7) and (A.8), we have

$$\begin{aligned} B^{(1)}(r) &= \sigma_{11}^{1/2} a^{(1)} b^{(1)'} \left(\sqrt{1-\rho_{1z}'\rho_{1z}}\, W^{(1)}(r) + (\rho_{1z}' \otimes I_4) W^{z}(r) \right) \\ B^{z}(r) &= a^{z} b^{z}{}' (P_{zz} \otimes I_4) W^{z}(r) \end{aligned} \qquad \text{(A.9)}$$

as in (15) of the text, where a^z and b^z are defined as the lower right-hand $4n \times n$ blocks of a^x and b^x in (14).

♦

Lemma 4

Consider, first the OLS estimates of the coefficients of (7), for each season denoted $\hat{\mathrm{B}}_s = [\hat{\beta}_{2s}, \hat{\beta}_{3s}, \cdots, \hat{\beta}_{ms}]'$, where

$$\hat{\mathrm{B}}_s = \left[N^{-2} \sum_{\tau=1}^{N} z_{s\tau} z_{s\tau}' \right]^{-1} \left[N^{-2} \sum_{\tau=1}^{N} x_{s\tau}^{(1)} z_{s\tau} \right].$$

Then

$$\hat{\mathrm{B}}_s \Rightarrow \left[\int B_s^z(r) B_s^z(r)' dr \right]^{-1} \int B_s^z(r) B_s^{(1)}(r) dr \qquad \text{(A.10)}$$

where $B_s(r) = [B_s^{(1)}(r), B_s^z(r)']'$ is $m \times 1$ vector Brownian motion, with $n \times 1$ $B_s^z(r) = [B_s^{(2)}(r), \ldots, B_s^{(m)}(r)]'$.

From (17), we can write

$$\begin{aligned} B_s^{(1)}(r) &= \omega_1 a_s^{(1)} \tilde{w}^{(1)}(r) \\ B_s^{(j)}(r) &= \omega_j a_s^{(j)} \tilde{w}^{(j)}(r) \quad j = 2, 3, \cdots, m \end{aligned}$$

in which

$$\omega_1 = \sigma_{11}^{0.5} \left(b^{(1)}{}' b^{(1)} \right)^{0.5}, \quad \omega_j = \left(p^{(j)} p^{(j)}{}' b^{(j)}{}' b^{(j)} \right)^{0.5}, \quad j = 2, 3, \cdots, m$$

and each $\tilde{w}^{(j)}(r)$, $j = 1, 2, \ldots, m$ is univariate standard Brownian motion.

Therefore, defining the $n \times 1$ vector $\tilde{W}^z(r) = [\tilde{w}^{(2)}(r), \tilde{w}^{(3)}(r), \ldots, \tilde{w}^{(m)}(r)]'$, we have

$$\begin{aligned} \int B_s^z(r) B_s^z(r)'\, dr &= A_s [\int \tilde{W}^z(r) \tilde{W}^z(r)'\, dr] A_s \\ \int B_s^z(r) B_s^{(1)}(r)\, dr &= A_s [\int \tilde{W}^z(r) \tilde{w}^{(1)}(r) dr] \omega_1 a_s^{(1)} \end{aligned} \qquad \text{(A.11)}$$

where A_s is as $n \times n$ diagonal matrix such that $A_s = diag\{\omega_2 a_s^{(2)}, \omega_3 a_s^{(3)}, \ldots, \omega_m a_s^{(m)}\}$.

Then, from (A.10) and (A.11) it is easy to see that:

$$\hat{B}_s \Rightarrow \omega_1 a_s^{(1)} A_s^{-1} \left[\tilde{W}^z(r) \tilde{W}^z(r)' \right]^{-1} \int \tilde{W}^z(r) \tilde{w}^{(1)}(r) dr \qquad \text{(A.12)}$$

The appropriately scaled residuals from (7) can be expressed as:

$$\frac{1}{\sqrt{N}} \hat{v}_{s[rN]} = \frac{1}{\sqrt{N}} x_{s[rN]}^{(1)} - \hat{\mathrm{B}}_s' \frac{1}{\sqrt{N}} z_{s[rN]} \qquad \text{(A.13)}$$

where $z_{s[rN]} = [x_{s[rN]}^{(2)}, x_{s[rN]}^{(3)}, \ldots, x_{s[rN]}^{(m)}]'$ and, from (17),

$$
\begin{aligned}
&\frac{1}{\sqrt{N}} z_{s[rN]} \Rightarrow B_s^z(r) = A_s \widetilde{W}^z(r) \\
&\frac{1}{\sqrt{N}} x_{s[rN]}^{(1)} \Rightarrow B_s^{(1)}(r) = \omega_1 a_s^{(1)} \widetilde{w}^{(1)}(r)
\end{aligned}
\qquad \text{(A.14)}
$$

Hence, from (A.12), (A.13) and (A.14),

$$
\begin{aligned}
\frac{1}{\sqrt{N}} \hat{v}_{s[rN]} &\Rightarrow \omega_1 a_s^{(1)} \left\{ \widetilde{w}^{(1)}(r) - \int \widetilde{w}^{(1)}(r) \widetilde{W}^z(r)' dr \left[\int \widetilde{W}^z(r) \widetilde{W}^z(r)' dr \right]^{-1} \widetilde{W}^z(r) \right\} \\
&= \omega_1 a_s^{(1)} \eta' \widetilde{W}^x(r)
\end{aligned}
\qquad \text{(A.15)}
$$

where $\eta' = \left[1, -\int \widetilde{w}^{(1)}(r) \widetilde{W}^z(r)' dr \left[\int \widetilde{W}^z(r) \widetilde{W}^z(r)' dr \right]^{-1} \right]$. Each element of the $m \times 1$ vector of Brownian motions $\widetilde{W}^x(r) = [\widetilde{w}^{(1)}(r), \widetilde{W}^z(r)']'$ has unit variance, while the $m \times m$ long-run covariance matrix Π of $\widetilde{W}^x(r)$ can be expressed as

$$
\Pi = \begin{array}{c} \begin{bmatrix} 1 & \varpi_{1z}' \\ \varpi_{1z} & \Pi_{zz} \end{bmatrix} \end{array} \begin{array}{l} 1 \\ n \end{array}
$$

with elements on the principal diagonal of Π_{zz} equal to one.

Defining the $m \times m$ matrix L such that $\Pi = LL'$, and where the first column of L is given by (l_{11}, 0) then, using part (a) of Lemma 2.2 by Phillips and Ouliaris (1990), we have that

$$
\widetilde{W}^x(r) = L\overline{W^x}(r)
$$

where $\overline{W^x}(r) = [\overline{w^{(1)}}(r), \overline{w^{(2)}}(r), \ldots, \overline{w^{(m)}}(r)]' = [\overline{w^{(1)}}(r), \overline{W^z}(r)']'$ is an $m \times 1$ vector of standard Brownian motions with covariance matrix I_m. Finally from part (b) of Lemma 2.2 of Phillips and Ouliaris (1990), it is possible to write:

$$
\begin{aligned}
&\eta' \widetilde{W}^x(r) = l_{11} \kappa' \overline{W^x}(r) \\
&\kappa' = \left[1, -\int \overline{w^{(1)}}(r) \overline{W^z}(r)' dr \left[\int \overline{W^z}(r) \overline{W^z}(r)' dr \right]^{-1} \right].
\end{aligned}
$$

Recalling that $\omega_1 = \sigma_{11}^{0.5} \left(b^{(1)}{}' b^{(1)} \right)^{0.5}$, the result in (18)/(19) is obtained by substituting these last two expressions into (A.15) and stacking the residuals for s = 1, 2, 3, 4 to define the vector $\hat{V}_{[rN]} = [\hat{v}_{1[rN]}, \hat{v}_{2[rN]}, \hat{v}_{3[rN]}, \hat{v}_{4[rN]}]'$.

♦

Theorem 1

It follows from Lemma 4 that, in the absence of cointegration, we can write

$$
\hat{v}_{s\tau} = \phi_s^{(1)} \hat{v}_{s-1,\tau} + \varepsilon_{s\tau}, \qquad \prod_{s=1}^{4} \phi_s^{(1)} = 1. \qquad \text{(A.16)}
$$

Hence, as in Lemma 1 of Boswijk and Franses (1996) or Lemma 3 of the text, we have that:

$$\frac{1}{\sqrt{N}}\hat{V}_{[rN]}=\left(\Theta_0^{(1)}-\Theta_1^{(1)}\right)\frac{1}{\sqrt{N}}\sum_{j=1}^{[rN]}E_j^m+o_p(1)$$
$$\Rightarrow\sigma\left(\Theta_0^{(1)}-\Theta_1^{(1)}\right)E^m(r)=\sigma\, a^{(1)}b^{(1)}{'}E^m(r). \tag{A.17}$$

where $E_\tau^m=[\varepsilon_{1\tau},\varepsilon_{2\tau},\varepsilon_{3\tau},\varepsilon_{4\tau}]'$ with $\frac{1}{\sqrt{N}}\sum_{j=1}^{[rN]}E_j^m\Rightarrow\sigma E^m(r)$ and $E^m(r)$ is 4×1 vector Brownian motion that is a function of the elements of $W(r)$. Comparing (18) and (A.17) it follows that

$$\sigma\, a^{(1)}b^{(1)}{'}E^v(r)=\sigma_{11}^{1/2}\left(b^{(1)}{'}b^{(1)}\right)^{0.5}a^{(1)}\overline{w_m}(r)$$

and hence

$$\overline{w_m}(r)=\frac{\sigma}{l_{11}\sigma_{11}^{1/2}}\left(b^{(1)}{'}b^{(1)}\right)^{-0.5}b^{(1)}{'}E^m(r) \tag{A.18}$$

provides an alternative definition of $\overline{w_m}(r)$. For notational convenience in what follows, we omit the superscripts referring to the parameters relating to process $x_{s\tau}^{(1)}$.

Now, turning to the test statistic of (21) for the null hypothesis $\phi_1\phi_2\phi_3\phi_4=1$ in (20), note that we can write

$$LR=\left\{q_{n0}Q_{n0}^{-1}G_0\right\}'\left\{G_0{'}Q_{n0}^{-1}G_0\right\}^{-1}\left\{G_0{'}Q_{n0}^{-1}q_{n0}\right\}+o_p(1) \tag{A.19}$$

where q_0 and Q_0 are the gradient and Hessian matrices, respectively, for the log-likelihood function under the null hypothesis and G_0 is a 4×1 vector with elements $\partial(\phi_1\phi_2\phi_3\phi_4=1)/\partial\phi_s$ for s = 1, 2, 3, 4. It is straightforward to see that:

$$q_0'=\sigma^{-2}\left[\Sigma\hat{v}_{4,\tau-1}\varepsilon_{1\tau},\Sigma\hat{v}_{1\tau}\varepsilon_{2\tau},\Sigma\hat{v}_{2\tau}\varepsilon_{3\tau,},\Sigma\hat{v}_{3\tau}\varepsilon_{4\tau}\right]$$

$$Q_0=\sigma^{-2}\begin{bmatrix}\Sigma\hat{v}_{4,\tau-1}^2 & 0 & 0 & 0\\ 0 & \Sigma\hat{v}_{1\tau}^2 & 0 & 0\\ 0 & 0 & \Sigma\hat{v}_{2\tau}^2 & 0\\ 0 & 0 & 0 & \Sigma\hat{v}_{3\tau}^2\end{bmatrix} \tag{A.20}$$

$$G_0'=\left[\phi_2\phi_3\phi_4,\phi_1\phi_3\phi_4,\phi_1\phi_2\phi_4,\phi_1\phi_2\phi_3\right]$$

From (18) of Lemma 4, together with (A.18) and the continuous mapping theorem, it is possible to write:

$$N^{-1}\sum\hat{v}_{s-1,\tau}\varepsilon_{s\tau}\Rightarrow l_{11}\sigma_{11}^{1/2}\sigma(b'\,b)^{0.5}a_{s-1}\int\overline{w_m}(r)dE_s^m(r)$$
$$N^{-1}\sum\hat{v}_{s-1\tau}^2\Rightarrow l_{11}^2\sigma_{11}(b'b)(a_{s-1})^2\int\overline{w_m}(r)^2dr$$

where the elements of a are given by $a=[1,\phi_2,\phi_2\phi_3,\phi_2\phi_3\phi_4]'$. Substituting these expressions in (A.20), we obtain

$$N^{-1}q_0^{'} \Rightarrow \frac{l_{11}\sigma_{11}^{1/2}(b'b)^{0.5}}{\sigma}\left[a_4\int \overline{w_m}(r)dE_1^m(r),\ a_1\int \overline{w_m}(r)dE_2^m(r),\ a_2\int \overline{w_m}(r)dE_3^m(r),\ a_3\int \overline{w_m}(r)dE_4^m(r)\right]$$

$$N^{-2}Q_0 \Rightarrow \frac{l_{11}^2\sigma_{11}(b'b)}{\sigma^2}\begin{bmatrix} (a_4)^2\int \overline{w_m}(r)^2 dr & 0 & 0 & 0 \\ 0 & (a_1)^2\int \overline{w_m}(r)^2 dr & 0 & 0 \\ 0 & 0 & (a_2)^2\int \overline{w_m}(r)^2 dr & 0 \\ 0 & 0 & 0 & (a_3)^2\int \overline{w_m}(r)^2 dr \end{bmatrix}$$

and, therefore,

$$NG_0'Q_0^{-1}q_0 \Rightarrow \frac{\sigma}{l_{11}\sigma_{11}^{1/2}(b'b)^{0.5}\int \overline{w_m}(r)^2 dr}\times$$
$$\int \overline{w_m}(r)\left(\frac{\phi_2\phi_3\phi_4\, dE_1^m(r)}{\phi_2\phi_3\phi_4} + \frac{\phi_1\phi_3\phi_4\, dE_2^m(r)}{1} + \frac{\phi_1\phi_2\phi_4\, dE_3^m(r)}{\phi_2} + \frac{\phi_1\phi_2\phi_3\, dE_4^m(r)}{\phi_2\phi_3}\right)$$

Using (A.18) and the definition $b = [1, \phi_1\phi_3\phi_4, \phi_1\phi_4, \phi_1]'$ it follows that

$$NG_0'Q_0^{-1}q_0 \Rightarrow \frac{\int \overline{w_m}(r)d\left[\sum_{s=1}^{4} b_s E_s^m(r)\right]}{(b'b)^{0.5}\int \overline{w_m}(r)^2 dr} = \frac{\int \overline{w_m}(r)d\overline{w_m}(r)}{\int \overline{w_m}(r)^2 dr} \qquad \text{(A.21)}$$

since $\sigma \to \sigma_{11}^{1/2} l_{11}$.

Using similar arguments,

$$N^2\left\{G_0'Q_0^{-1}G_0\right\}$$
$$\Rightarrow \frac{1}{(b'b)\int \overline{w_m}(r)^2 dr}\left\{\frac{(\phi_2\phi_3\phi_4)^2}{(\phi_2\phi_3\phi_4)^2} + \frac{(\phi_1\phi_3\phi_4)^2}{1} + \frac{(\phi_1\phi_2\phi_4)^2}{(\phi_2)^2} + \frac{(\phi_1\phi_2\phi_3)^2}{(\phi_2\phi_3)^2}\right\}$$
$$= \frac{1}{(b'b)\int \overline{w_m}(r)^2 dr}\left\{1 + (\phi_1\phi_3\phi_4)^2 + (\phi_1\phi_4)^2 + (\phi_1)^2\right\} \qquad \text{(A.22)}$$
$$= \frac{b'b}{(b'b)\int \overline{w_m}(r)^2 dr} = \frac{1}{\int \overline{w_m}(r)^2 dr}.$$

The required result is easily obtained by substituting (A.21) and (A.22) into (A.19).
♦

Theorem 2

For *(i)*, note, first, that the Wald statistic (24) to test the null for no cointegration in season *s* is

$$\begin{aligned} Wald_s &= \hat{\delta}'\left(V\hat{a}r[\delta_s]^{-1}\right)\hat{\delta}_s \\ &= \hat{\sigma}_u^{-2}\left(\sum_{\tau=1}^{N}\Delta_4 x_{s\tau}^{(1)} x_{s,\tau-1}'\right)\left(\sum_{\tau=1}^{N} x_{s,\tau-1}x_{s,\tau-1}'\right)^{-1}\left(\sum_{\tau=1}^{N}\Delta_4 x_{s\tau}^{(1)} x_{s,\tau-1}\right). \end{aligned} \quad \text{(A.23)}$$

Then, from Lemma 3,

$$N^{-1}\sum_{\tau=1}^{N}\Delta_4 x_{s\tau}^{(1)} x_{s,\tau-1} = \int B_s^x(r)dB_s^{(1)}(r)$$

$$N^{-2}\sum_{\tau=1}^{N} x_{s,\tau-1}'x_{s,\tau-1} \Rightarrow \int B_s^x(r)B_s^x(r)'dr$$

where $B_s^x(r) = [B_s^{(1)}, B_s^{(2)}, ..., B_s^{(m)}]'$. Using (17) and the fact that in the spurious regression case $\Sigma = diag\{\sigma_{11}, \sigma_{22}, \ldots, \sigma_{mm}\}$, it is possible to see that,

$$B_s^{(j)}(r) = \sigma_{jj}^{1/2} a_s^{(j)}\left(b^{(j)}{}'b^{(j)}\right)^{1/2}\overline{w^{(j)}}(r) \quad j = 1, 2, \ldots, m$$

Substituting into (A.23) it then follows that

$$\begin{aligned} Wald_s &= \hat{\delta}_s'\left(V\hat{a}r[\delta_s]^{-1}\right)\hat{\delta}_s \\ &\Rightarrow \sigma_u^{-2}\left(\int B_s^x(r)dB_s^{(1)}(r)\right)'\left(\int B_s^x(r)B_s^x(r)'dr\right)^{-1}\left(\int B_s^x(r)dB_s^{(1)}(r)\right) \\ &= \sigma_u^{-2}\sigma_{11}(a_s^{(1)})^2 b^{(1)}{}'b^{(1)}\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'dr\right)^{-1}\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right) \end{aligned}$$

(A.24)

where, as in Boswijk and Franses (1996),

$$\hat{\sigma}_u^2 \to \sigma_u^2 = \frac{1}{4}\sum_{s=1}^{4} Var\left(x_{s\tau}^{(1)}\right) = \frac{1}{4}\sigma_{11}b^{(1)}{}'b^{(1)}a^{(1)}{}'a^{(1)}. \quad \text{(A.25)}$$

Substituting (A.25) in (A.24) yields the result in (26).

For the joint test statistic, due to the seasonal dummy variables, then

$$\begin{aligned} Wald &= \sum_{s=1}^{4} Wald_s \\ &= 4\,\frac{\sum_{s=1}^{4}(a_s^{(1)})^2}{a^{(1)}{}'a^{(1)}}\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'dr\right)^{-1}\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right) \\ &= 4\left(\int \overline{W^x}(r)d\overline{w^x}(r)\right)'\left(\int \overline{W^x}(r)\overline{W^x}(r)'dr\right)^{-1}\left(\int \overline{W^x}(r)d\overline{w^{(1)}}(r)\right) \end{aligned}$$

as given in (27).

♦