The University of Manchester

MANCHESTER 1824

Economics Discussion Paper

EDP-0516

Random field models of microeconomic dynamics

by

I.V. Estigneev and M.I. Taksar

October 2005

igor.evstigneev@manchester.ac.uk
School of Social Sciences,
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom

Random field models of microeconomic dynamics*

I.V. Evstigneev**
Economics Department, School of Social Sciences, University of Manchester, Oxford Road, Manchester M13 9PL, UK
(e-mail: igor.evstigneev@manchester.ac.uk)

M.I. Taksar
Mathematics Department, University of Missouri, Columbia, MO 65211, USA
(e-mail: taksar@math.missouri.edu)

Abstract The paper analyzes stochastic models of dynamic economic equilibrium with locally interacting agents. The models are based on a control theory for random fields on directed graphs. The graphs involved serve to describe the spatio-temporal structure of commodity flows in the economy. The focus of the study is on questions of existence, uniqueness and stability of stochastic dynamic equilibria. In this paper results obtained previously for finite graphs are extended to infinite graphs.

Keywords: Economies with locally interacting agents, Stochastic equilibrium dynamics, Random fields on directed structures, Stationary (invariant) equilibria

JEL Classification Numbers: C62, D51, D90, O41

---

*Financial support from grants NSF DMS-0505435, NATO CRG 972115, from the Santa Fe Institute, the State of Missouri Research Board, the University of Missouri-Columbia Research Council, and the Manchester School Visiting Fellowship Fund is gratefully acknowledged.

**Corresponding author. Tel.: (+)44-161-2754275, fax: (+)44-161-2754812.

1 Introduction

The general objective of this study is to develop stochastic models of dynamic economic equilibrium taking into account local (micro-level) interactions between economic agents. The classical theory allows individuals to interact only on the macro level—through a price system prevailing in the market. However, much of the real economic process involves direct contacts between its participants. It is therefore of interest to provide equilibrium models taking into account possibilities of, as well as restrictions on, direct interactions between agents. The restrictions (specifying the notion of "locality") may be of various types: certain individuals may only be able to exchange commodities with certain others; some may only communicate with a group of the others; there may be spatial, temporal and informational constrains.

In the last 10-15 years, large work has been done on the integration of classical general equilibrium theory and local interaction models. Various views on the problem and various formal settings have been developed. Contributions to the field were made by Aoki [3], Blume [9], Blume and Durlauf [10], Brock and Durlauf [14, 15], Brock and Hommes [13], Durlauf [19, 20], Glaeser and Scheinkman [35], Horst and Scheinkman [37], Ioannides [38, 39], Kirman [41, 42], Kirman and Vriend [43], Lux [44], Lux and Marchesi [45], Verbrugge [61], Weisbuch, Kirman and Herreiner [62] and others. One can distinguish two large branches in this research area, corresponding to two different mathematical frameworks used. The first involves consideration of random graphs and is aimed, basically, at the modeling of the formation and evolution of socio-economic networks. The second applies in the economics context the methodology developed in statistical physics for the analysis of large interacting particle systems. We use another approach, distinct from the above-mentioned ones, describing the structure of local interactions between economic agents in terms of *random fields on directed structures* (directed graphs, partially ordered sets, etc.). Below we discuss this approach, in comparison with others, in detail.

In the present paper we examine a stochastic equilibrium model in which markets—where agents interact in the process of commodity exchange—are separated in space and time. The model is specified in terms of a fixed directed graph, T. The vertices of T correspond to agents acting at certain moments of time. The agents produce and consume commodities and deliver them to other agents. The directed arcs of the graph T describe the spatio-temporal structure of commodity flows in the economic system. Different agents in the economy are influenced by different random factors and possess different information. It is supposed that the stochastic structure of the model is in a sense compatible with the structure of the given graph T.

We analyze *equilibrium states* of the economy, i.e., those states in which all the agents implement their most preferred production and consumption decisions (given the local equilibrium prices) and balance constraints for the commodity flows are satisfied. The main results are existence and uniqueness theorems for such states. The results generalize those obtained in our previous work (Evstigneev [26], Evstigneev and Taksar [28, 29]), dealing with the case of a finite graph, to infinite graphs. This generalization requires some hypotheses regarding the infinite graph T. In particular, we need certain restrictions on the "branching rate" of T and the assumption that T is well-approximable in a proper sense by its finite subgraphs. These assumptions are used when passing to the limit as the number of nodes of T tends to infinity. A key role at this stage of the analysis is played by the stability results (turnpike theorems) for equilibria established in Dempster, Evstigneev and Pirogov [17].

The study of infinite graph models is motivated primarily by the fact that they reflect the idea of a "large" economy, developing over a long time interval. They provide a framework for analyzing such issues as stability, spatial and temporal homogeneity, and aggregation in the economic equilibrium context. The present paper is a step in our program of extending to the graph models the key results of the mathematical theory of economic dynamics and equilibrium over an infinite time horizon, as developed by Gale [32, 33, 34], Nikaido [51], McKenzie [47, 48], Brock [11], Brock and Mirman [16], Brock and Haurie [12], Radner [54, 55, 56], Polterovich [52], Bewley [7, 8], Majumdar and Zilcha [46] and others. This theory—rich in content and mathematically elegant—belongs to the classics of mathematical economics. The focus on the above specific goals is one of the main distinctions of the present line of studies from other research dealing with network models in economics, regional science, games, and operations management (e.g. Samuelson [57], Nagurney [49], Shapiro and Varian [58], Batten and Boyce [5], Nijkamp and Reggiani [50], and Bernstein *et al.* [6]).

The mathematical background of this work is the theory of random fields. By a "random field" one means a random function whose domain does not have a natural structure of linear ordering (Euclidean space, manifold, graph, etc.). The present study deals with functions of this kind defined on directed graphs or partially ordered sets. Such random fields arise in many applied problems, related and not related to economics, e.g. in the control of energy or fluid flows, the design of telecommunication networks, etc.

A powerful theory exists for random fields on *un*directed graphs. Originally, this theory has been built in connection with problems in statistical physics (Dobrushin [18], Preston [53] and others). Central concepts in that area are the notions of Gibbs and Markov fields. Methods related to such fields were first applied to problems in mathematical economics by Föllmer [31]. Results along similar lines have been obtained by Karmann [40] and Allen [1].

Our work is based on an entirely different approach dealing with random vector fields on directed, rather than undirected, graphs. In contrast with the above-mentioned line of studies, relying on the techniques of Gibbs–Markov fields, we do not have well-elaborated methods for working with random vector functions on directed graphs. The theory of such functions (especially, the analysis of their Markov properties) is far less developed than the theory of random fields on undirected graphs. Some general tools for the investigation of these classes of fields have been created (see the survey in Evstigneev and Greenwood [27]), but there are still many challenging open questions in this area. The present work not only exploits the methodology of this branch of probability theory, but also contributes to it in that it develops methods for the analysis of equilibrium stochastic control problems for random fields on directed graphs.

The paper is organized as follows. Section 2 describes the model. Section 3 states the main results. Section 4 presents formulations of some results of the previous work which are used in this paper. Section 5 contains the proof of the main theorem. Sections 6 and 7 discuss some specialized models.

2 The model

Let T be a finite or countable set. For each $t \in T$, let $K(t)$ be a subset of T containing t. The correspondence $t \mapsto K(t)$ determines the structure of a *directed graph* on T. The inclusion $s \in K(t)\backslash\{t\}$ means that there is a directed arc of the graph leading from t to s. We define

$$M(t) = \{s \in T : t \in K(s)\},\ K(t+) = K(t)\backslash\{t\},\ M(t-) = M(t)\backslash\{t\}. \qquad (1)$$

It is supposed that the sets $K(t)$ and $M(t)$ are finite for each $t \in T$, i.e., the graph under consideration is *locally finite.*

Elements of the set T represent economic agents (acting possibly at different moments of time). The agents produce and consume commodities. They supply their products to other agents. The inclusion $s \in K(t+)$, means that agent s depends directly on t via commodity supplies. For each $t \in T$, the set $M(t-) \cup K(t+)$ includes those agents which interact directly with t : they either supply commodities to t or receive them from t.

The model admits both a static and a dynamic interpretation. To introduce dynamics explicitly one can assume that the vertices of the graph T are pairs (n, b), $b \in B(n)$, $n = 1, 2, \ldots$, where elements of the set $B(n)$ represent economic agents acting at time n. It is natural to suppose that if $(n', b') \in K(n, b)$, then $n' > n$, i.e., all the economic transactions require positive time.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{F}_t \subseteq \mathcal{F}$, $t \in T$, be a family of σ-algebras and m_t, $t \in T$, a collection of positive integers. The integer m_t specifies the number of different types of commodities that can be received by agent t from other agents (or are contained in the initial endowment of t). For each $t \in T$, the σ-algebra $\mathcal{F}_t$ represents the class of events influencing agent t.

We suppose that each $\mathcal{F}_t$ is separable (i.e., generated by a countable number of events), and the following condition holds:

(**F**) If $s \in K(t)$, then $\mathcal{F}_t \subseteq \mathcal{F}_s$ $(s, t \in T)$.

Assumption (**F**) expresses the fact that random events influencing agent t may thereby influence agents $s \in K(t+)$, depending on t. (It is not assumed, however, that each agent $s \in K(t+)$ can directly observe all such events.) If a set $S \subseteq T$ is a cycle of the graph T, then, by virtue of (**F**), we have $\mathcal{F}_t = \mathcal{F}_s$ for any pair of vertices $s, t \in S$. Although we do not assume that the graph T does not have cycles, the last observation shows that the stochastic structure of the model over every cycle is in a sense trivial.

Consider the spaces of random vectors:

$$\mathcal{L}_t = L_1(\Omega, \mathcal{F}_t, P, R^{m_t}) \ , \ \ \mathcal{X}_t = L_\infty(\Omega, \mathcal{F}_t, P, R^{m_t}).$$

By definition, the space $\mathcal{L}_t$ consists of integrable $\mathcal{F}_t$-measurable m_t-dimensional random vectors. The space $\mathcal{X}_t$ includes those and only those elements of $\mathcal{L}_t$ which are essentially bounded. We set

$$\mathcal{P}_t = \{p \in \mathcal{L}_t : p \geq 0\}.$$

All equalities and inequalities for random vectors, such as $p \geq 0$ above, are understood coordinatewise and almost surely (a.s.). We often omit "a.s." if this does not lead to ambiguity.

Elements of $\mathcal{X}_t$ are interpreted as random *commodity vectors* and elements of $\mathcal{P}_t$ as random *price vectors*. For any $p = (p^1, ..., p^{m_t}) \in \mathcal{P}_t$ and $x = (x^1, ..., x^{m_t}) \in \mathcal{X}_t$, the scalar product $px = \sum_{j=1}^{m_t} p^j(\omega)x^j(\omega)$, computed for each $\omega \in \Omega$, characterizes the cost of the commodity bundle x in the price system p. It is supposed that commodities received by agent t from his suppliers $s \in M(t-)$, as well as the input of t, are evaluated in terms of the price vectors $p_t \in \mathcal{P}_t$. If, for example, the vertices of T represent regions, then p_t is the price system at the local regional market t.

We define

$$\mathcal{Z}_t = \prod_{s \in K(t)} \mathcal{X}_s \ , \ \ \mathcal{Q}_t = \prod_{s \in K(t)} \mathcal{P}_s.$$

Elements v in $\mathcal{Z}_t$, i.e., families of vectors $v = (v_s)_{s \in K(t)}$ $[v_s \in \mathcal{X}_s]$ are called *strategies* of agent t. Here, $-v_t$ is construed as the *input* vector and $(v_s)_{s \in K(t+)}$ as the set of *output* vectors. A strategy v reflects production and consumption decisions which lead to the inputs $-v_t(\omega)$ and the outputs $v_s(\omega)$, $s \in K(t+)$, depending on the random situation ω. Versions of the present model in which production and consumption decisions are considered explicitly are discussed in Section 6. It is natural in many economic interpretations of the model to assume the vectors $-v_t$ and $(v_s)_{s \in K(t+)}$ to be nonnegative. This assumption, however, is not needed for the validity of the assertions which we prove below.

Further, define

$$\mathcal{Q}_t = \prod_{s \in K(t)} \mathcal{P}_s.$$

Elements $q = (q_s)_{s \in K(t)}$ of the set $\mathcal{Q}_t$ are collections of those price vectors which correspond to the input and output vectors of agent t. Consider the sets

$$\mathcal{Z} = \prod_{t \in T} \mathcal{Z}_t, \; \mathcal{P} = \prod_{t \in T} \mathcal{P}_t.$$

Elements $z = (z_t)_{t \in T}$ $[z_t = (z_{ts})_{s \in K(t)} \in \mathcal{Z}_t]$ in $\mathcal{Z}$ are families of strategies of all agents and elements $p = (p_t)_{t \in T}$ $[p_t \in \mathcal{P}_t]$ in $\mathcal{P}$ are families of all price vectors. For any $z \in \mathcal{Z}$ and $p \in \mathcal{P}$, put

$$\mathbf{q}^t(p) = (p_s)_{s \in K(t)}, \;\; \mathbf{g}_t(z) = \sum_{s \in M(t)} z_{st}. \tag{2}$$

The mapping $p \mapsto \mathbf{q}^t(p)$ selects those components p_s of $p \in \mathcal{P}$ for which $s \in K(t)$. The vector $\mathbf{g}_t(z)$ can be written as

$$\mathbf{g}_t(z) = \sum_{s \in M(t-)} z_{st} - (-z_{tt}).$$

The sum $\sum_{s \in M(t-)} z_{st}$ specifies the amounts of commodities which agent t receives from his suppliers; the vector $-z_{tt}$ represents the input of agent t. If $M(t-) = \emptyset$, then we define $\sum_{s \in M(t-)} z_{st} = 0$ (all sums over empty sets of indices are supposed to equal zero).

Suppose we are given a family of random vectors

$$h = (h_t)_{t \in T}, \; h_t \in \mathcal{X}_t,$$

and a family of mappings

$$Z_t : \mathcal{Q}_t \to \mathcal{Z}_t, \; t \in T. \tag{3}$$

A pair $(z, p) \in \mathcal{Z} \times \mathcal{P}$ $[z = (z_t)_{t \in T}, \, p = (p_t)_{t \in T}]$ is said to form an *equilibrium* if, for all $t \in T$, the following relations hold:

$$z_t = Z_t(\mathbf{q}^t(p)), \tag{4}$$

$$\mathbf{g}_t(z) + h_t \geq 0 \text{ (a.s.)}, \tag{5}$$

and

$$p_t \mathbf{g}_t(z) + p_t h_t = 0 \text{ (a.s.)}. \tag{6}$$

The mappings (3) describe the economic behavior of agents depending on prices. For each $q = (q)_{s \in K(t)} \in \mathcal{Q}_t$, the collection of vectors $Z_t(q) = (Z_{ts}(q))_{s \in K(t)}$ is interpreted as *the most preferred* strategy of agent t given the price system q. We do not specify the preferences explicitly: what matters in this context is only the result of the agent's choice of inputs and outputs depending on his/her observation of the prices.

In an equilibrium, all the agents choose their most preferred strategies, and constraints (5), (6) are satisfied. Inequality (5), which can be written in the form

$$\sum_{s\in M(t-)} z_{st} + h_t \geq -z_{tt} \text{ (a.s.)},$$

expresses material balance: the input of every agent does not exceed the amount of commodities supplied plus the *initial endowment* h_t. (If some components of h_t are negative, this means that some amounts of commodities are withdrawn from the system.) Equality (6) represents a "complementary slackness" condition; it follows from (6) that inequalities (5) hold as equalities when all the coordinates of p_t are strictly positive with probability 1.

The collection of the data described above,

$$(T, K(\cdot)),\ \{\mathcal{F}_t\},\ \{m_t\},\ \{Z_t(\cdot)\},\ \{h_t\}, \tag{7}$$

will be called an *equilibrium model* and denoted by $\mathcal{M}$.

Note that $Z_t(q)$ depends only on those prices which are related to the input and output vectors of agent t. In this sense, we say that the model under consideration is *local*. Equilibrium states in this model and its non-local version were studied in [26] in the case of a finite graph T. That study extended to the equilibrium context the results obtained earlier for an optimal control scheme involving random fields on finite directed graphs [25]. Further investigation of models of the above type was undertaken in [17, 28, 29] and [30].

3 The assumptions and the main results

We fix some strictly positive constants A, A_0, δ, and a function

$$\zeta^H : T \to (0, \infty) \tag{8}$$

defined for each $H \in (0, \infty)$. We denote by $|a|$ the Euclidean norm of the finite-dimensional vector a and by $|a|_1$ the sum of the absolute values of the coordinates of a. For a random vector $x = x(\omega)$, the norms $||x||_1$ and $||x||_\infty$ are defined as $E|x(\omega)|_1$ and $\mathrm{ess\,sup}\ |x(\omega)|_1$, respectively.

We assume that the following conditions are satisfied.

(**A**) There exists a collection $\overset{o}{z}= (\overset{o}{z}_t)_{t\in T}$ of strategies $\overset{o}{z}_t= (\overset{o}{z}_{ts})_{s\in K(t)} \in \mathcal{Z}_t$ with properties (**A.1**) – (**A.4**) below.

(**A.1**) For any $t \in T$ and $q \in \mathcal{Q}_t$, the inequality $Z_{ts}(q) \geq \overset{o}{z}_{ts}$ holds for all $s \in K(t+)$.

(**A.2**) For each $s \in T$, we have

$$h_s + \sum_{t\in M(s)} [\overset{o}{z}_{ts} - \delta e_s] \geq 0,$$

where $e_s = (1, ..., 1) \in R^{m_s}$.

(**A.3**) The vectors $\overset{o}{z}_{ts}$ satisfy $-Ae_s \leq \overset{o}{z}_{ts} \leq Ae_s$ $(t \in T,\ s \in K(t+))$.

(**A.4**) For all $t \in T$ and $q \in \mathcal{Q}_t$, we have

$$E(qZ_t(q)|\mathcal{F}_t) \geq E(q\ \overset{o}{z}_t\ |\mathcal{F}_t) - A_0,$$

where $E(\cdot|\mathcal{F}_t)$ stands for the conditional expectation given the σ-algebra $\mathcal{F}_t$.

(**B**) The mappings $q \mapsto Z_t(q)$, $t \in T$, possess the following properties of boundedness, monotonicity and continuity.

(**B.1**) For all $q \in \mathcal{Q}_t$, $s \in K(t+)$ and $t \in T$, the vector $Z_{ts}(q)$ satisfies the inequalities $-Ae_s \leq Z_{ts}(q) \leq Ae_s$. For every $t \in T$, there exists a constant $A(t)$ such that $|Z_{tt}(q)| \leq A(t)$.

(**B.2**) For any $H \in (0, \infty)$ and $t \in S$, the inequality

$$E(q^1 - q^2)(Z_t(q^1) - Z_t(q^2)) \geq \zeta^H(t)\ E[\left|Z_t(q^1) - Z_t(q^2)\right|^2 + \left|q_t^1 - q_t^2\right|^2] \qquad (9)$$

holds for all $q^1 \in \mathcal{Q}_t$ and $q^2 \in \mathcal{Q}_t$ such that $q_t^1, q_t^2 \in \mathcal{P}_t(H)$, where

$$\mathcal{P}_t(H) = \{l \in \mathcal{P}_t :\ ||l||_\infty \leq H\}.$$

In formula (9), q_t^i is the component of the collection of vectors $q^i = (q_s^i)_{s \in K(t)}$ corresponding to the index t. The function $\zeta^H(t)$ is supposed to be given (see (8)); it takes on strictly positive values for all $H > 0$ and $t \in T$.

(**B.3**) If $q^k \in \mathcal{Q}_t$, $q \in \mathcal{Q}_t$ and $||q^k - q||_1 \to 0$, then $EwZ_t(q^k) \to EwZ_t(q)$ for all $w \in \mathcal{Z}_t$.

Most of the assumptions contained in (**A**) and (**B**) are similar to the assumptions used in [28, 29], where their meaning is discussed in detail. Here, we make only several brief remarks regarding the above hypotheses.

The family of vectors $\overset{o}{z}_{ts}$, $t \in T$, $s \in K(t+)$, described in (**A**) is interpreted as the set of "minimal outputs". The inequalities $Z_{ts}(q) \geq \overset{o}{z}_{ts}$ hold for all the output vectors $Z_{ts}(q)$, $s \in K(t+)$, of any strategy $Z_t(q)$ which might be chosen by agent t (see (**A.1**)). Condition (**A.2**) states that the family of strategies $\overset{o}{z}_t$, $t \in T$, allows one to produce the minimal outputs with excess. Assumption (**A.4**) is fulfilled when the production incentives of the agents do not differ "too much" from pure profit maximization; see Section 6. The strict monotonicity condition described in (**B.2**) is closely related to the Law of Demand (see Hildenbrand [36])—a strict monotonicity hypothesis for demand functions. Operators $Z_t(\cdot)$ having a property slightly weaker than (**B.2**) are examined in [28, 29]. In Section 6, we will consider a specialized model, where condition (**B.2**) will be deduced from some hypotheses regarding agents' production and consumption. Hypothesis (**B.3**) says that $Z_t(q^k)$ converges to $Z_t(q)$ weakly when q^k converges to q with respect to the norm $||\cdot||_1$.

We say that a sequence $t_0, \ldots, t_l \in T$ defines a *path* of length l from $s \in T$ to $t \in T$, if $t_0 = s$, $t_l = t$ and $t_{i+1} \in K(t_i)$, $i = 0, \ldots, l-1$. If there is at least one path from s to t, we say that t *majorizes* s and write $s \preceq t$. If $s \preceq t$, we denote by $l(s,t)$ the length of the shortest path $t_0, \ldots, t_l$ from s to t. For each $\kappa \in [0,1)$, we define

$$\Phi_T(\kappa) = \sup_{s \in T} \sum_{t \in T_s} \kappa^{l(s,t)}, \tag{10}$$

where $T_s = \{t \in T : s \preceq t\}$. The function $\Phi_T(\kappa)$, $\kappa \in [0,1)$, which may take on, in general, finite and infinite values, is called the *generating function* of the graph T.

(**T**) The graph T satisfies hypotheses (**T.1**) and (**T.2**) below.

(**T.1**) The generating function $\Phi_T(\kappa)$ is finite for any $\kappa \in [0,1)$.

This condition may be interpreted as a restriction on the "branching rate" of T. Denote by $\theta_m(s)$ ($s \in T$, $m = 1, 2, \ldots$) the number of elements in the set

$$T^m(s) \equiv \{t \in T : t \succeq s,\ l(s,t) = m\},$$

consisting of those elements t in T which can be reached from s along a path of length m. Hypothesis (**T.1**) holds if and only if, for each $s \in T$, the series

$$\Phi_T(s,\kappa) \equiv \sum_{m=0}^{\infty} \theta_m(s) \kappa^m \tag{11}$$

converges, and its sum, as a function of s, is bounded on T for each $\kappa \in [0,1)$. Clearly the supremum of $\Phi_T(s,\kappa)$ over $s \in T$ is equal to $\Phi_T(\kappa)$.

To formulate the next hypothesis we introduce some definitions and notation related to the graph T. In particular, we define the *closure*, $\mathrm{cl}B$, of a subset B of T as

$$\mathrm{cl}\, B = K(B) \bigcup M(B), \tag{12}$$

where $K(B)$ is the union of $K(t)$, $t \in B$, and $M(B)$ is the union of $M(t)$, $t \in B$. Further, we put

$$\partial_- B = \{(t,s) :\ t \in T \backslash B,\ s \in K(t),\ s \in B\}, \tag{13}$$

$$\partial_+ B = \{(t,s) :\ t \in B,\ s \in K(t),\ s \in T \backslash B\}, \tag{14}$$

$$\partial B = \partial_- B \bigcup \partial_+ B. \tag{15}$$

The set ∂B, consisting of the *boundary arcs* of B, includes those directed arcs of the graph T which either begin outside B and terminate inside B, or begin inside B and terminate outside B.

For a finite set $B \subseteq T$ and a number $H > 0$, we define

$$\mu(B) = \max_{s \in \mathrm{cl}B} (\#M(s)),\ \zeta^H(B) = \min_{t \in B} \zeta^H(t),\ \beta^H(B) = \frac{\mu(B)}{\zeta^H(B)}, \tag{16}$$

where the symbol $\#$ denotes the number of elements of a set.

The second hypothesis we impose on T is as follows.

(**T.2**) There exist a sequence $S_1 \subseteq S_2 \subseteq ... \subseteq T$ of finite subsets of T and a sequence of nonnegative integers λ_n, $n = 1, 2, ...$, such that

$$\mathrm{cl}\, S_n \subseteq S_{n+1},\ T = \bigcup_{n=1}^{\infty} S_n\,, \tag{17}$$

$$\lambda_n < n,\ n - \lambda_n \to \infty, \tag{18}$$

and, for any $H > 0$,

$$\Xi_n(H) \equiv \beta_n(H)^2 \nu_n (1 + \frac{1}{\beta_n(H)})^{1-\lambda_n} \to 0, \tag{19}$$

where

$$\beta_n(H) = \beta^H(S_n) \text{ and } \nu_n = \#\,\partial S_n.$$

According to the above assumption, the graph T can be approximated by a sequence $S_1, S_2, ...$ of its finite subsets such that condition (17) holds and certain characteristics of the sets S_n, defined in terms of the model under consideration, do not grow "too fast" as $n \to \infty$. These characteristics are as follows:

(a) the cardinality ν_n of the set ∂S_n;

(b) the maximum number $\mu_n \equiv \mu(S_n)$ of elements in $M(s)$, $s \in \mathrm{cl}\, S_n$;

(c) the maximum of the numbers $\zeta^H(t)^{-1}$, $t \in S_n$ (for any fixed $H > 0$).

In particular, hypothesis (**T.2**) holds if the following requirements (i) – (iii) are met:

(i) the number $\nu_n = \#\,\partial S_n$ is not greater than $c_0 n^c$ for some constants $c_0, c > 0$;

(ii) we have $\mu_n \leq b_0 n^b$ for some $b_0 > 0$ and $b \in [0, 1)$;

(iii) the numbers $\zeta^H(t)$, $t \in T$, are bounded away from 0 by some constant $\zeta^H > 0$ (for every fixed $H > 0$).

Under conditions (i) – (iii), we can define λ_n, for example, as the greatest integer that does not exceed $n/2$. Then, as is easily checked, we have $\lim_{n\to\infty} \Xi_n(H) = 0$ ($H > 0$), which yields (**T.2**).

The main result of the paper is Theorem 1 below. This theorem holds under the assumptions listed in (**A**), (**B**), (**F**), and (**T**).

Theorem 1. *Equilibrium states exist. There exists one and only one equilibrium state* $(z, p) = ((z_t, p_t))_{t\in T}$ *for which*

$$\sup_{t\in T} ||p_t||_\infty < \infty. \tag{20}$$

A comment is in order on the uniqueness part of the above result. Condition (20) says that the norms $||p_t||_\infty$ of the price vectors p_t are finite and, moreover, uniformly bounded on T. It can be proved (see Theorem 4 in the next section) that if the graph T is finite, then the equilibrium prices always have finite norms $||\cdot||_\infty$, satisfying $||p_t||_\infty \leq H^*$, where

$$H^* = \frac{A_0}{\delta}\Phi_T(\frac{2A}{\delta + 2A}). \tag{21}$$

Consequently, in the case of a finite T, the equilibrium state is unique in the class of all possible equilibria. If T is infinite, then, as can be shown, assumptions like (20) are needed for uniqueness even in the classical situation—when Ω consists of one point, $T = \{0, 1, 2, ...\}$ and $K(t) = \{t, t+1\}$. An example of a deterministic model in which there exist an equilibrium with bounded prices and another equilibrium with unbounded prices is presented in Polterovich [52, Section 5]. One can slightly modify Polterovich's example to satisfy all the assumptions we use in this paper.

4 Balanced states and boundedness of the equilibrium prices

The proof of Theorem 1 relies upon some results obtained in our previous work. These results are formulated in the present section. Here, we do not assume that all the hypotheses imposed in the previous sections hold. Rather, we indicate explicitly, in each particular case, what assumptions we use.

Let S be a subset of T. Let us say that $(z,p) \in \mathcal{Z} \times \mathcal{P}$ is an S-*balanced state* if constraints (5) and (6) are satisfied for all $t \in S$. According to this definition, an S-balanced state is a collection of strategies $z_t \in \mathcal{Z}_t$ and prices $p_t \in \mathcal{P}_t$ indexed by $t \in T$ such that the balance constrains hold on the given set S. Clearly, an equilibrium state is a T-balanced state satisfying conditions (4).

Suppose we are given a real-valued function $\zeta = \zeta(t) > 0$, defined on the set S. Let $(z^i, p^i) = ((z^i_t, p^i_t))_{t \in T} \in \mathcal{Z} \times \mathcal{P}$, $i = 1, 2$, be two S-balanced states. Let us say that these states are ζ-*comonotone* if, for each $t \in S$, we have

$$E(\mathbf{q}^t(p^1) - \mathbf{q}^t(p^2))(z^1_t - z^2_t) \geq \zeta(t) \cdot E[|z^1_t - z^2_t|^2 + |p^1_t - p^2_t|^2]. \tag{22}$$

For every finite set $B \subseteq S$, define the numbers $\mu(B)$, $\zeta(B)$ and $\beta(B)$ according to (16) with $\zeta(t)$ in place of $\zeta^H(t)$.

Theorem 2. *Let (z^1, p^1) and (z^2, p^2) be S-balanced states satisfying condition (22). Let $S_1 \subseteq S_2 \subseteq ... \subseteq S_N$ be a sequence of finite subsets of S such that*

$$\mathrm{cl}\, S_{n-1} \subseteq S_n,\ n = 2, ..., N. \tag{23}$$

Then for each $l \in \{0, 1, ..., N-1\}$, we have

$$\sum_{t \in S_{N-l}} E|(z^1_t, p^1_t) - (z^2_t, p^2_t)|^2 \leq$$

$$\frac{\beta(S_N)}{\zeta(S_{N-l})}\cdot(1+\frac{1}{\beta(S_N)})^{1-l}\sum_{(t,s)\in\partial S_N}E|(p_s^1-p_s^2)(z_{ts}^1-z_{ts}^2)|. \tag{24}$$

This result was obtained in [17, Theorem 1]. It may be regarded as a version of a stochastic turnpike theorem with an exponential estimate for the convergence rate (see [2]). Observe that the term $\zeta\,(S_{N-l})$ in the right hand side of (24) can be replaced by $\zeta\,(S_N)$, since the sequence $\zeta\,(S_n)$, $n = 1, 2, ..., N$, decreases—see (16). Furthermore, $\zeta(S_N)^{-1} = \beta(S_N)\mu(S_N)^{-1} \leq \beta(S_N)$. Therefore inequality (24) implies the following estimate for $r\,(t) \equiv E|(z_t^1, p_t^1) - (z_t^2, p_t^2)|_2^2$:

$$\sum_{t\in S_{N-l}} r\,(t) \leq C_N\,(1+\frac{1}{\beta\,(S_N)})^{1-l}, \tag{25}$$

where the number

$$C_N \equiv \beta(S_N)^2\sum_{(t,s)\in\partial S_N}E|(p_s^1-p_s^2)(z_{ts}^1-z_{ts}^2)| \tag{26}$$

does not depend on l. From this we can see that the sums $\sum_{t\in S_{N-l}} r\,(t)$, $l \in \{0, ..., N-1\}$, are bounded above by a sequence which *decreases at an exponential rate* as l varies from 0 to $N-1$. This rate,

$$1+\frac{1}{\beta(S_N)} = 1+\frac{\zeta(S_N)}{\mu(S_N)},$$

depends on the minimum value $\zeta\,(S_N)$ of $\zeta\,(t)$ on S_N and on the maximum value $\mu\,(S_N)$ of $\#M\,(t)$ on $\mathrm{cl}S_N$ (see (16)).

Note that the above result does not involve the mappings $Z_t(\cdot)$—the main ingredient of the equilibrium model described in Section 2—and so it does not rely upon any of the assumptions regarding $Z_t(\cdot)$.

Theorem 3. *Let the graph T be finite. Let the equilibrium model satisfy conditions* (**A**)*,* (**B**) *and* (**F**)*. Then an equilibrium state exists.*

This theorem is a direct consequence of Theorem 1 in [26].

Theorem 4. *Let the graph T be finite and let conditions* (**A**), (**B.1**)*, and* (**F**) *hold. Then for any equilibrium state (z, p) $[p = (p_t)_{t\in T}]$, we have*

$$\max ||p_t||_\infty \leq H^*, \tag{27}$$

where H^* is defined by (21).

For a proof of this assertion see [28, Theorem 5.2]. Note that the estimate for $||p_t||_\infty$ given by (21) does not involve the cardinality of T explicitly. Therefore one might conjecture that, by passing to the limit, one can derive an analogous estimate for an infinite graph T for which $\Phi_T(\kappa) < \infty$, $\kappa \in [0, \infty)$. However this conjecture fails to be true, since, as has already been noticed, if T is infinite, then the equilibrium prices may be unbounded.

Another comment concerns the definition of the norm $||\cdot||_\infty$. According to our definition, $||p_t||_\infty$ is the essential supremum of the finite-dimensional norm $|p_t|_1$ of the random vector p_t (and not of the Euclidean norm $|p_t|$, for example). Although all finite-dimensional norms are of course equivalent, the inequalities expressing their equivalence might contain constants depending on the dimension. In the present context, the dimension m_t may vary with t. The choice of the norm $|\cdot|_1$ allows us to obtain estimate (27) which is uniform in m_t.

5 Proof of the main theorem

The plan of proving the existence part of Theorem 1 is as follows. First we define a "restriction", $\bar{\mathcal{M}} = \mathcal{M}(\bar{T})$, of the given equilibrium model $\mathcal{M}$ to a subset $\bar{T}$ of T. We show that $\bar{\mathcal{M}}$ satisfies the same conditions as $\mathcal{M}$. If $\bar{T}$ is finite, Theorem 3 guarantees the existence of an equilibrium in $\bar{\mathcal{M}}$. We put $\bar{T} = S_N$, $N = 1, 2, ...$, where $\{S_N\}$ is the sequence of sets involved in (**T.2**), and obtain a sequence of equilibria in the models $\mathcal{M}(S_N)$. We extend them in a certain way to random vector fields on T so that these fields turn out to be S-balanced states for any finite $S \subseteq T$ and all N large enough. By using Theorems 2 and 4, we establish convergence of these fields in the norm $||\cdot||_2 \equiv (E|\cdot|^2)^{1/2}$ for each $t \in T$. Finally, we prove that the limit random field forms an equilibrium state.

In this section, we assume that all the assumptions listed in Sections 2 and 3 are fulfilled.

Let $\bar{T}$ be a subset of T. To define the restriction $\bar{\mathcal{M}} = \mathcal{M}(\bar{T})$ of the model $\mathcal{M}$ to $\bar{T}$, we fix the structure of a directed graph on $\bar{T}$ specified by the mapping $t \mapsto \bar{K}(t) \equiv K(t) \cap \bar{T}$, $t \in \bar{T}$. We set $\bar{M}(t) = M(t) \cap \bar{T}$, $t \in \bar{T}$. Then we have $\bar{M}(t) = \{s \in \bar{T} : t \in \bar{K}(s)\}$.

Further, define

$$\bar{\mathcal{Z}}_t = \prod_{s \in \bar{K}(t)} \mathcal{X}_s \,, \ \bar{\mathcal{Q}}_t = \prod_{s \in \bar{K}(t)} \mathcal{P}_s \,, \ t \in \bar{T}.$$

Consider the mapping $q \mapsto \tilde{q}$ of $\bar{\mathcal{Q}}_t$ into $\mathcal{Q}_t$ transforming any family of vectors $q = (q_s)_{s \in \bar{K}(t)} \in \bar{\mathcal{Q}}_t$ into the family of vectors $\tilde{q} = (\tilde{q}_s)_{s \in K(t)} \in \mathcal{Q}_t$ such that $\tilde{q}_s = q_s$, if $s \in \bar{K}(t)$, and $\tilde{q}_s = 0$, if $s \in K(t) \backslash \bar{T}$. (The mapping $q \mapsto \tilde{q}$ assigns the value 0 to those elements s in $K(t)$ for which q_s is not defined.) For any $q \in \bar{\mathcal{Q}}_t$, we put

$$\bar{Z}_{ts}(q) = Z_{ts}(\tilde{q}), \ s \in \bar{K}(t),$$

and

$$\bar{Z}_t(q) = (\bar{Z}_{ts}(q))_{s \in \bar{K}(t)}.$$

Observe that the mapping $q \mapsto \bar{Z}_t(q)$ acts from $\bar{\mathcal{Q}}_t$ into $\bar{\mathcal{Z}}_t$. When we replace $Z_t(\tilde{q})$ by $\bar{Z}_t(\tilde{q})$, we drop those components $Z_{ts}(\tilde{q})$ of the collection of vectors $Z_t(\tilde{q})$

for which $s \notin \bar{K}(t)$. Finally, we define

$$\bar{h}_t = h_t + \sum_{s \in M(t) \setminus \bar{T}} \overset{o}{z}_{st}, \ t \in \bar{T}. \tag{28}$$

(Sums taken over empty sets are supposed to be equal to zero.) The graph $\left(\bar{T}, \bar{K}(\cdot)\right)$, the σ-algebras $\mathcal{F}_t$, the natural numbers m_t, the vectors $\bar{h}_t$, and the mappings $\bar{Z}_t$ $(t \in \bar{T})$ specify the equilibrium model which we denote by $\bar{\mathcal{M}} = \mathcal{M}(\bar{T})$.

For every $t \in \bar{T}$, consider the vectors

$$z^0_{ts} = \overset{o}{z}_{ts}, \ s \in \bar{K}(t), \text{ and } z^0_t = (z^0_{ts})_{t \in \bar{K}(t)} \, .$$

Proposition 1. *The model $\bar{\mathcal{M}}$ satisfies conditions* **(A)** *and* **(B)** *with $(z^0_t)_{t \in \bar{T}}$ in place of $(\overset{o}{z}_t)_{t \in T}$. We have $\Phi_{\bar{T}}(\kappa) \le \Phi_T(\kappa)$, $\kappa \in [0,1)$.*

Proof. To verify **(A.2)**, we fix $t \in \bar{T}$ and write

$$\bar{h}_t + \sum_{s \in \bar{M}(t)} (z^0_{st} - \delta e_t) = h_t + \sum_{s \in M(t) \setminus \bar{T}} \overset{o}{z}_{st} + \sum_{s \in M(t) \cap \bar{T}} (\overset{o}{z}_{st} - \delta e_t) \ge$$

$$h_t + \sum_{s \in M(t) \setminus \bar{T}} (\overset{o}{z}_{st} - \delta e_t) + \sum_{s \in M(t) \cap \bar{T}} (\overset{o}{z}_{st} - \delta e_t) = h_t + \sum_{s \in M(t)} (\overset{o}{z}_{st} - \delta e_t) \ge 0.$$

The requirement

$$E(q \bar{Z}_t(q) | \mathcal{F}_t) \ge E(q z^0_t | \mathcal{F}_t) - A_0, \ q \in \bar{\mathcal{Q}}_t,$$

expressing hypothesis **(A.4)** for the model $\bar{\mathcal{M}}$, follows from hypothesis (A.4) for the model $\mathcal{M}$ and from the equalities

$$q \bar{Z}_t(q) = \sum_{s \in \bar{K}(t)} q_s \bar{Z}_{ts}(q) = \sum_{s \in \bar{K}(t)} q_s Z_{ts}(\tilde{q}) = \sum_{s \in K(t)} \tilde{q}_s Z_{ts}(\tilde{q}) = \tilde{q} Z_t(\tilde{q}),$$

$$q z^0_t = \sum_{s \in \bar{K}(t)} q_s z^0_{ts} = \sum_{s \in K(t)} \tilde{q}_s \, \overset{o}{z}_{ts} = \tilde{q} \, \overset{o}{z}_t,$$

holding for all $q \in \bar{\mathcal{Q}}_t$ because $\tilde{q}_s = 0$, $s \in K(t) \setminus \bar{T}$.

We claim that the mappings $\bar{Z}_t$ satisfy the condition of strong monotonicity analogous to **(B.2)** (with the same function $\zeta^H(t)$). Indeed, for any $t \in \bar{T}$ and $q^1, q^2 \in \bar{\mathcal{Q}}_t$ with $q^i_t \in \mathcal{P}_t(H)$, we have

$$E(\bar{Z}_t(q^1) - \bar{Z}_t(q^2))(q^1 - q^2) = \sum_{s \in \bar{K}(t)} E(\bar{Z}_{ts}(q^1) - \bar{Z}_{ts}(q^2))(q^1_s - q^2_s) =$$

$$\sum_{s \in K(t)} E(Z_{ts}(\tilde{q}^1) - Z_{ts}(\tilde{q}^2))(\tilde{q}^1_s - \tilde{q}^2_s) \ge \zeta^H(t) \cdot E[|Z_t(\tilde{q}^1) - Z_t(\tilde{q}^2)|^2 + |\tilde{q}^1_t - \tilde{q}^2_t|^2] \ge$$

$$\zeta^H(t)\cdot E[|\bar{Z}_t(q^1)-\bar{Z}_t(q^2)|^2+|q_t^1-q_t^2|^2],$$

since $q_t^i=\tilde{q}_t^i$ for all $q^i\in\bar{\mathcal{Q}}_t$ and since $|Z_t(\tilde{q}^1)-Z_t(\tilde{q}^2)|\geq|\bar{Z}_t(q^1)-\bar{Z}_t(q^2)|$.

Conditions (**A.1**), (**A.3**), (**B.1**) and (**B.3**) for the model $\bar{\mathcal{M}}$ are direct consequences of the analogous conditions for $\mathcal{M}$.

The truth of the inequality $\Phi_{\bar{T}}(\kappa)\leq\Phi_T(\kappa)$ ($\kappa\in[0,1)$) is clear from formula (10) and the following considerations: if there exists a path from $s\in\bar{T}$ to $t\in\bar{T}$ along arrows of the graph $\bar{T}$, then there exists a path from s to t along arrows of T; the length of the shortest path from s to t within $\bar{T}$ is not less than the length of the shortest path from s to t within T.

The proposition is proved.

Proposition 2. *For any finite $\bar{T}\subseteq T$, the model $\bar{\mathcal{M}}$ possesses an equilibrium state $(\bar{z},\bar{p})=((\bar{z}_t,\bar{p}_t))_{t\in\bar{T}}$ with $||\bar{p}_t||_\infty\leq H^*$, where H^* is defined by (21).*

Proof. The existence of $(\bar{z},\bar{p})$ follows from Theorem 3 and Proposition 1. By virtue of Theorem 4, we have that

$$||\bar{p}_t||_\infty\leq\frac{A_0}{\delta}\Phi_{\bar{T}}(\frac{2A}{\delta+2A}). \tag{29}$$

But $\Phi_{\bar{T}}(\kappa)\leq\Phi_T(\kappa)$ for all $\kappa\in[0,1)$, as has been shown in Proposition 1. Therefore (29) implies the inequality $||\bar{p}_t||_\infty\leq H^*$.

The proposition is proved.

We now describe a procedure of extending an equilibrium state in the model $\bar{\mathcal{M}}$ to a random vector field on T. Let $(\bar{z},\bar{p})=((\bar{z}_t,\bar{p}_t))_{t\in\bar{T}}$ be an equilibrium state in $\bar{\mathcal{M}}$. For each $t\in T$, set

$$p_t=\begin{cases}\bar{p}_t, & \text{if } t\in\bar{T},\\ 0, & \text{if } t\in T\backslash\bar{T},\end{cases}\quad p=(p_t)_{t\in T}\,, \tag{30}$$

$$z_t=Z_t(\mathbf{q}^t(p)),\ z=(z_t)_{t\in T}\,. \tag{31}$$

(Recall that $\mathbf{q}^t(p)=(p_s)_{s\in K(t)}$.)

Proposition 3. *Let S be a subset of $\bar{T}$ such that* $\mathrm{cl}\,S\subseteq\bar{T}$. *Then the random field $((z_t,p_t))_{t\in T}$ defined by (30) and (31) is an S-balanced state.*

Proof. Since $(\bar{z},\bar{p})$ is an equilibrium state in the model $\bar{\mathcal{M}}$, the vector

$$\bar{v}_t\equiv\bar{h}_t+\sum_{s\in\bar{M}(t)}\bar{z}_{st} \tag{32}$$

is nonnegative and $\bar{p}_t\bar{v}_t=0$ for all $t\in\bar{T}$. Since $\mathrm{cl}\,S\subseteq\bar{T}$, we have $\bar{K}(t)=K(t)$ and $\bar{M}(t)=M(t)$, $t\in S$. Therefore, if $t\in S$, then $\bar{h}_t=h_t$ and

$$\bar{v}_t=h_t+\sum_{s\in M(t)}\bar{z}_{st}. \tag{33}$$

Observe that

$$\bar{z}_{ts}=z_{ts},\ t,s\in\bar{T}. \tag{34}$$

Indeed, fix some $t \in \bar{T}$ and define $\bar{q} = (\bar{p}_s)_{s \in \bar{K}(t)}$. Then, for any $s \in \bar{T}$, we have $\bar{z}_{ts} = \bar{Z}_{ts}(\bar{q}) = Z_{ts}(\tilde{q})$, where $\tilde{q}$ is the family of vectors $\tilde{q}_r$, $r \in K(t)$, such that $\tilde{q}_r = \bar{p}_r$ for $r \in \bar{T}$ and $\tilde{q}_r = 0$ for $r \in T \backslash \bar{T}$. In view of the definition of p, we have $\tilde{q} = \mathbf{q}^t(p)$. Consequently,

$$\bar{z}_{ts} = Z_{ts}(\tilde{q}) = Z_{ts}(\mathbf{q}^t(p)) = z_{ts}$$

(see (31)), which yields (34).

By applying (34), with s and t interchanged, to equality (33), we find

$$h_t + \sum_{s \in M(t)} z_{st} = v_t \geq 0, \text{ and } p_t(h_t + \sum_{s \in M(t)} z_{st}) = \bar{p}_t v_t = 0,$$

for all $t \in S$.

The proof is complete.

Let $S_1 \subseteq S_2 \subseteq ... \subseteq S_N \subseteq ... \subseteq T$ be a sequence of finite subsets of T involved in hypothesis (**T.2**).

Proposition 4. *Let N be a natural number. Let (z^1, p^1) and (z^2, p^2) be S_N-balanced states satisfying*

$$z_t^i = Z_t(\mathbf{q}^t(p^i)) \; for \; all \; t \in S^N \tag{35}$$

and

$$p_t^i \in \mathcal{P}_t(H), \; t \in T, \tag{36}$$

for some constant H and $i = 1, 2$. Then we have

$$\sum_{t \in S_{N-\lambda_N}} E|(z_t^1, p_t^1) - (z_t^2, p_t^2)|^2 \leq 4AH\Xi_N(H). \tag{37}$$

Here, A is the constant specified in hypotheses (**A**) and (**B**). The numbers λ_N and $\Xi_N(H)$ are described in (**T.2**).

Proof of Proposition 4. In view of (35), (36) and (**B.2**), relation (22) holds with $\zeta(t) = \zeta^H(t)$ for all $t \in S^N$. By using Theorem 2 (with $S = S_N$) and formulas (25), (26), we obtain that the left-hand side of (37) is not greater than

$$\beta_N(H)^2 \cdot (1 + \frac{1}{\beta_N(H)})^{1-\lambda_N} \sum_{(t,s) \in \partial S_N} E|(p_s^1 - p_s^2)(z_{ts}^1 - z_{ts}^2)| \leq$$

$$\beta_N(H)^2 \cdot (1 + \frac{1}{\beta_N(H)})^{1-\lambda_N} \cdot (\#\partial S_N) \cdot 4AH = 4AH\Xi_N(H),$$

which proves the proposition.

Now everything is prepared to prove Theorem 1.

Proof of Theorem 1. For each $N \geq 1$, consider the restriction $\mathcal{M}(S_{N+1})$ of the model $\mathcal{M}$ to the set S_{N+1}. By virtue of Proposition 2 and Theorem 3, this model possesses an equilibrium $(\bar{z}^N, \bar{p}^N) = ((\bar{z}^N_t, \bar{p}^N_t))_{t \in S_{N+1}}$. In view of Theorem 4, we have $||\bar{p}^N_t||_\infty \leq H^*$.

Define

$$p^N_t = \begin{cases} \bar{p}^N_t, & \text{if } t \in S_{N+1}, \\ 0, & \text{if } t \in T \backslash S_{N+1}, \end{cases} \quad (t \in T),\ p^N = (p^N_t)_{t \in T},$$

$$z^N_t = Z_t(\mathbf{q}^t(p^N))\ (t \in T),\ z^N = (z^N_t)_{t \in T}. \tag{38}$$

According to Proposition 3, (z^N, p^N) is an S_N-balanced state (since $\operatorname{cl} S_N \subseteq S_{N+1}$), and we have $||p^N_t||_\infty \leq H^*$ for all $t \in T$.

Let us show that, for every $t \in T$, (z^N_t, p^N_t) is a Cauchy sequence with respect to the L_2-norm $||\cdot||_2 = (E|\cdot|)^{1/2}$. To prove this, fix some $t = t_0 \in T$ and consider the sequence $\{\lambda_n\}$ described in (**T.2**). Let N_0 be such that $t_0 \in S_{N-\lambda_N}$ for all $N > N_0$. The number N_0 exists since $N - \lambda_N \to \infty$ (see (18)) and T is the union of the sets $S_1 \subseteq S_2 \subseteq \ldots$. Fix an arbitrary $\epsilon > 0$ and consider a number $N = N(\epsilon) > N_0$ such that $4AH^*\Xi_N(H^*) < \epsilon$, where $\Xi_N(\cdot)$ is defined by (19). The existence of $N = N(\epsilon)$ follows from the fact that $\Xi_N(H) \to 0$ for each $H > 0$. Let $n, m > N = N(\epsilon)$. Then (z^n, p^n) and (z^m, p^m) are S_N-balanced states. By virtue of (38), we have (35) with $i = m, n$. Furthermore, (36) holds with $i = m, n$ and $H = H^*$. Thus, Proposition 4 can be applied, which yields

$$r(n, m) \equiv E|(z^n_{t_0}, p^n_{t_0}) - (z^m_{t_0}, p^m_{t_0})|^2 \leq$$

$$\sum_{t \in S_{N-\lambda_N}} E|(z^n_t, p^n_t) - (z^m_t, p^m_t)|^2 \leq 4AH^*\Xi_N(H^*) < \epsilon.$$

Thus, $r(n, m) < \epsilon$ for all n and m large enough, which means that $\{(z^N_{t_0}, p^N_{t_0})\}$ is a Cauchy sequence.

Since the sequence (z^N_t, p^N_t) is Cauchy with respect to the L_2-norm $||\cdot||_2$, we have $z^N_t \to z_t$ and $p^N_t \to p_t$ for some $z_t \in \mathcal{Z}_t$ and $p_t \in \mathcal{P}_t(H^*)$, where the sequences z^N_t and p^N_t converge in L_2 and hence in L_1. This implies that $z_t = Z_t(\mathbf{q}^t(p))$ in view of (**B.3**) and of the equality $z^N_t = Z_t(\mathbf{q}^t(p^N))$, which holds for all N and t. For every $t \in T$, we have $t \in S_N$ for all N large enough. Consequently,

$$\mathbf{g}_t(z^N) + h_t \geq 0 \text{ (a.s.)},\ p^N_t(\mathbf{g}_t(z^N) + h_t) = 0 \text{ (a.s.)}.$$

for all sufficiently large N, because (z^N, p^N) is an S_N-balanced state. By passing to the limit as $N \to \infty$ in the above inequalities, we obtain the relations (5), (6) and $||p_t||_\infty \leq H^*$. Consequently, $(z, p) \equiv ((z_t, p_t))_{t \in T}$ is an equilibrium in the model $\mathcal{M}$ satisfying (20).

To establish the uniqueness of an equilibrium state with bounded prices, suppose that there are two such states $(z^i, p^i) = ((z^i_t, p^i_t))_{t \in T}$, $i = 1, 2$, satisfying $||p^i_t||_\infty \leq H$. Then conditions (35) and (36) hold for all N and $i = 1, 2$. Furthermore, (z^1, p^1) and (z^2, p^2) are S_N-balanced states for each N. Consequently, inequality (37) is true for all N. Fix some $t \in T$. Since $N - \lambda_N \to \infty$, (37) implies

$$E|(z_t^1, p_t^1) - (z_t^2, p_t^2)|^2 \le 4AH\Xi_N(H)$$

for all N large enough. This is possible only if $(z_t^1, p_t^1) = (z_t^2, p_t^2)$ (a.s.) because $\Xi_N(H) \to 0$.

The proof is complete.

6 Some specialized models

Suppose that the mappings $Z_{ts}(q)$ involved in the description of the equilibrium model in Section 2 are of the following form:

$$Z_{ts}(q) = \begin{cases} W_{ts}(q), & s \in K(t+), \\ W_{tt}(q) - C_t(q_t), & s = t, \end{cases} \tag{39}$$

where $W_t(q) = (W_{ts}(q))_{s \in K(t)}$ is defined on $\mathcal{Q}_t$ and takes values in $\mathcal{Z}_t$, while $C_t(\cdot)$ acts from $\mathcal{P}_t$ into $\mathcal{X}_t$. The mapping $q \mapsto W_t(q)$ characterizes the production activity of the economic agent t depending on the prices $q = (q_s)_{s \in K(t)}$. The vector $-W_{tt}(q)$ specifies the production input, and the vectors W_{ts}, $s \in K(t+)$, represent the production outputs (delivered to the agents s in $K(t+)$). The collection of vectors $W_t(q)$ is interpreted as the *most preferred production strategy* of agent t. For each $t \in T$, the mapping $l \mapsto C_t(l)$ specifies the *demand function.* The vector $C_t(l)$ describes the commodity bundle which agent t chooses for consumption, provided $l \in \mathcal{P}_t$ is the price system prevailing at the local market t. According to formula (39), the *total input* $-Z_{tt}(q)$ of agent t is the sum of the *production input* $-W_{tt}(q)$ and the *consumption vector* $C_t(q_t)$. In the present context, elements of T may represent economic units of various kinds (e.g. regions), so that the term "economic agent" should be understood in a sufficiently broad sense.

Further, suppose that the structure of the mapping $W_t(q)$ is as follows. Let a convex set $W_t \subseteq \mathcal{Z}_t$ and a strictly concave functional $F_t(w)$, $w \in W_t$, be given. Assume that, for each $q \in \mathcal{Q}_t$, the functional

$$F_t(w) + Eqw, \; w \in W_t, \tag{40}$$

attains its maximum on W_t. Let $W_t(q)$ be defined as (the unique) point of maximum of (40). The model described above—with the mapping $W_t(q)$ defined in terms of the functional (40) and a general mapping $C_t(\cdot)$—was investigated in much detail in [29]. This class of models includes as special cases those considered by Gale [33, 34], Dynkin [21, 22], Polterovich [52], Radner [55], Arkin and Evstigneev [4], Taksar [59, 60], Evstigneev and Katyshev [24]. In the paper [29], conditions on W_t, $F_t(\cdot)$, $C_t(\cdot)$ were presented guaranteeing the fulfillment of hypotheses (**A**), (**B.1**), (**B.3**) and a somewhat weaker version of hypothesis (**B.2**).

Below, we provide additional assumptions which imply (**B.2**). Specifically, we fix some $t \in T$, and examine conditions on the above data under which the inequality

$$E(Z_t(q^1) - Z_t(q^2))(q^1 - q^2) \geq$$

$$\zeta(H) \cdot E[|Z_t(q^1) - Z_t(q^2)|^2 + |q_t^1 - q_t^2|^2] \ \ (H > 0) \tag{41}$$

holds for some function $\zeta : (0, \infty) \to (0, \infty)$ and for all $q^i \in \mathcal{Q}_t(H) \equiv \{q^i \in \mathcal{Q}_t : q_t^i \in \mathcal{P}_t(H)\}$, where q_t^i stands for the tth component of the collection of vectors $q^i = (q_s)_{s \in K(t)}$. The conditions guaranteeing (41) are discussed in Remarks 1-3 below.

Remark 1. If

$$E(C_t(l^1) - C_t(l^2))(l^1 - l^2) \leq$$

$$-\gamma(H) \cdot E[|C_t(l^1) - C_t(l^2)|^2 + |l^1 - l^2|^2], \tag{42}$$

for $l^1, l^2 \in \mathcal{P}_t(H)$, and

$$E(W_t(q^1) - W_t(q^2))(q^1 - q^2) \geq \zeta \cdot E[|W_t(q^1) - W_t(q^2)|^2, \tag{43}$$

where $\gamma(H) > 0$ and ζ is a fixed strictly positive number, then the mapping (39) satisfies (41). Indeed, we may assume without loss of generality that $\gamma(H) \leq \zeta$ (replace $\gamma(H)$ by $\min(\gamma(H), \zeta)$). From (43) and (42) with $l^i = q_t^i$, we find

$$E(Z_t(q^1) - Z_t(q^2))(q^1 - q^2) \geq \gamma(H) \cdot E[|W_t(q^1) - W_t(q^2)|^2 +$$

$$|C_t(q_t^1) - C_t(q_t^2)|^2 + |q_t^1 - q_t^2|^2] \geq \frac{1}{2}\gamma(H) \cdot E[|Z_t(q^1) - Z_t(q^2)|^2 + |q_t^1 - q_t^2|^2],$$

which yields (41) with $\zeta(H) = \gamma(H)/2$.

Remark 2. Hypothesis (42) is a consequence of the following one

$$E(C_t(l^1) - C_t(l^2))(l^1 - l^2) \leq -\hat{\gamma}(H) \cdot E|l^1 - l^2|^2, \ l^i \in \mathcal{P}_t(H), \tag{44}$$

and the Lipschitz property of $C_t(\cdot)$ on $\mathcal{P}_t(H)$:

$$||C_t(l^1) - C_t(l^2)||_2 \leq B(H) \cdot ||l^1 - l^2||_2, \ l^i \in \mathcal{P}_t(H), \tag{45}$$

where $B(H) > 0$ is some number independent of l^i. To show this, let us write (45) in the form

$$E|l^1 - l^2|^2 \geq B(H)^{-2} \cdot E|C_t(l^1) - C_t(l^2)|^2.$$

This, combined with (44), yields

$$E(C_t(l^1) - C_t(l^2))(l^1 - l^2) \leq$$

$$-\frac{\hat{\gamma}(H)}{2} \cdot E|l^1 - l^2|^2 - \frac{\hat{\gamma}(H)}{2} B(H)^{-2} \cdot E|C_t(l^1) - C_t(l^2)|^2.$$

The last inequality implies (42) with $\gamma(H) = \hat{\gamma}(H) \cdot \min\{1, B(H)^{-2}\}/2$.

Remark 3. Hypothesis (43) holds when $F_t(w)$ is *strongly concave* in the following sense:

$$F_t((w^1 + w^2)/2) \geq (F_t(w^1) + F_t(w^2))/2 + \rho \cdot E|w^1 - w^2|^2, \; w^i \in W_t, \tag{46}$$

Indeed, suppose (46) is true. Consider any $q^1, q^2 \in \mathcal{Q}_t$ and set $w^i = W_t(q^i)$, $i = 1, 2$. Then, in view of (46), we have

$$F_t(w^i) + Eq^i w^i \geq (F_t(w^1) + F_t(w^2))/2 + Eq^i((w^1 + w^2)/2) + \rho \cdot E|w^1 - w^2|^2.$$

By adding up the above inequality with $i = 1$ and the analogous inequality with $i = 2$, we obtain, after easy computations, formula (43) with $\zeta = 4\rho$.

7 Invariant equilibria

In the remainder of the paper, we will consider an *invariant* version of the general model described in Section 2. Let G be a group of transformations of the graph T, that is: every $g \in G$ is a one-to-one mapping $T \to T$; the identity mapping belongs to G; if $g_1 \in G$ and $g_2 \in G$, then $g_1 g_2 \in G$; if $g \in G$, then $g^{-1} \in G$. The result of applying $g \in G$ to $t \in T$ will be denoted by gt. Suppose that the graph structure on T is invariant with respect to the action of the group G:

(**G.1**) We have $gK(t) = K(gt)$, $t \in T$, $g \in G$.

Clearly (**G.1**) holds if and only if $gM(t) = M(gt)$ for all $t \in T$ and $g \in G$. Further, assume that, to each $g \in G$, there corresponds a one-to-one transformation $\Theta_g : \Omega \to \Omega$ of the space Ω such that Θ_g and Θ_g^{-1} are $\mathcal{F}$-measurable, preserve the measure P, and satisfy $\Theta_g \Theta_{g'} = \Theta_{gg'}$ for all $g, g' \in G$. The correspondence $g \mapsto \Theta_g$ specifies a *representation of the group G by automorphisms Θ_g* of the probability space $(\Omega, \mathcal{F}, P)$. By a standard abuse of notation, we will denote by the same symbol Θ_g the operator acting on functions of ω according to the formula $\Theta_g f(\omega) = f(\Theta_g \omega)$.

The model $\mathcal{M}$ is said to be *invariant* if the following condition is fulfilled.

(**G.2**) For all $g \in G$ and $t \in T$, we have

$$\mathcal{F}_{gt} = \Theta_g^{-1} \mathcal{F}_t, \; m_t = m_{gt}, \tag{47}$$

and

$$h_{gt} = \Theta_g h_t, \; Z_{gt}(\Theta_g q) = \Theta_g Z_t(q)), \; q \in \mathcal{Q}_t. \tag{48}$$

Observe that (**G.1**), (47) and the fact that Θ_g is measure preserving for each g imply

$$\mathcal{X}_{gt} = \Theta_g \mathcal{X}_t, \; \mathcal{P}_{gt} = \Theta_g \mathcal{P}_t, \; \mathcal{Z}_{gt} = \Theta_g \mathcal{Z}_t, \text{ and } \mathcal{Q}_{gt} = \Theta_g \mathcal{Q}_t.$$

The notion of an invariant model can be used to express the idea of temporal and spatial homogeneity of the system under consideration. For example, let the graph T be of the form $\mathbf{Z} \times B$, where $\mathbf{Z} = \{0, \pm 1, \pm 2, ...\}$ is the set of moments of time and B is the set of agents. Let the group G include time shifts $(n, b) \mapsto (n+1, b)$ and mappings $(n, b) \mapsto (n, \gamma b)$, $\gamma \in G_B$, where G_B is some group of transformations of B. Invariance of the model $\mathcal{M}$ under the time shifts corresponds to the notion of *stationarity* of a stochastic model defined in ergodic theory terms, i.e. in terms of measure-preserving transformations of the underlying probability space (see Evstigneev [23], Radner [55], Bewley [7], and Arkin and Evstigneev [4, Chapter IV]). Invariance under the transformations $(n, b) \mapsto (n, \gamma b)$ expresses spatial homogeneity of the system. A typical example of B is a d-dimensional integer lattice $\mathbf{Z}^d$ (such spaces of agents are often considered in the studies of local economic interactions: see, e.g., Föllmer [31] and Kirman [41]). The group G_B may consist of translations or of Euclidean transformations of the lattice.

In an invariant model, we can define the notion of an *invariant equilibrium state*. Such states $(z, p) = ((z_t), (p_t))$ are defined as those satisfying

$$z_{gt} = \Theta_g z_t, \; p_{gt} = \Theta_g p_t, \; t \in T, \; g \in G. \tag{49}$$

This definition implies that the random vector function (z_t, p_t) is stationary on every *orbit* $T(t_0, G) \equiv \{gt_0 : \; g \in G\}$ of the group G (here t_0 is any fixed vertex of the graph). In particular, (49) implies that the distributions of the random vectors (z_t, p_t) are the same for all $t \in T(t_0, G)$. If $T(t_0, G)$ coincides with T for some (and hence for all) $t_0 \in T$, which means that T is a *homogenous space of the group* G, then the probabilistic structure of the strategies and the prices involved in an invariant equilibrium state is the same at all the vertices of the graph T.

The uniqueness result contained in Theorem 1 enables one to obtain the following theorem (in which hypotheses (**A**), (**B**), (**T**) and (**F**) are supposed to hold).

Theorem 5. *If the equilibrium model $\mathcal{M}$ is invariant, then any equilibrium state with prices p_t satisfying $\sup_{t \in T} ||p_t||_\infty < \infty$ is invariant.*

In combination with the existence part of Theorem 1, the last result establishes the existence of an invariant equilibrium state in an invariant model.

Proof of Theorem 5. Let $((z_t, p_t))_{t \in T}$ be an equilibrium state with $||p_t||_\infty \leq H < \infty$. To verify its invariance, it is sufficient to establish the relations $\Theta_g^{-1} z_{gt} = z_t$ and $\Theta_g^{-1} p_{gt} = p_t$ $(t \in T, \; g \in G)$. To prove these relations, in view of the uniqueness of an equilibrium with bounded prices, it suffices to show that the random field $(z'_t, p'_t) \equiv (\Theta_g^{-1} z_{gt}, \;\Theta_g^{-1} p_{gt})$, $t \in T$, forms an equilibrium state. (Indeed, we have $||p'_t||_\infty = ||p_t||_\infty$, since Θ_g^{-1} is measure preserving, and so the prices p'_t are bounded.)

Let us show that $((z'_t, p'_t))_{t \in T}$ is an equilibrium state. We have

$$\mathbf{g}_t(z') + h_t = \sum_{s \in M(t)} z'_{st} + h_t = \sum_{s \in M(t)} \Theta_g^{-1} z_{gs,gt} + h_t =$$

$$\Theta_g^{-1}[\sum_{s\in M(t)} z_{gs,gt} + \Theta_g h_t] = \Theta_g^{-1}[\sum_{s\in M(t)} z_{gs,gt} + h_{gt}] =$$

$$\Theta_g^{-1}[\sum_{r\in M(gt)} z_{r,gt} + h_{gt}] = \Theta_g^{-1}(\mathbf{g}_{gt}(z) + h_{gt}) \geq 0 \text{ (a.s.)}, \tag{50}$$

since $\Theta_g h_t = h_{gt}$ and $r \in M(gt) \Leftrightarrow r = gs,\ s \in M(t)$, by virtue of (**G.1**). The last inequality in (50) holds a.s. because $\mathbf{g}_{gt}(z) + h_{gt} \geq 0$ a.s. and Θ_g preserves the measure P. Furthermore,

$$p'_t(\mathbf{g}_t(z') + h_t) = \Theta_g^{-1}[p_{gt}(\mathbf{g}_{gt}(z) + h_{gt}] = 0 \text{ (a.s.)}$$

by virtue of (50) and the definition of p'_t. Finally,

$$z'_t = \Theta_g^{-1} z_{gt} = \Theta_g^{-1} Z_{gt}(\mathbf{q}^{gt}(p)) = Z_t(\Theta_g^{-1}\mathbf{q}^{gt}(p)) = Z_t(\mathbf{q}^t(p')),$$

in view of the relations

$$\Theta_g^{-1}\mathbf{q}^{gt}(p) = \Theta_g^{-1}(p_s)_{s\in K(gt)} = (\Theta_g^{-1} p_{gs})_{s\in K(t)} = (p'_s)_{s\in K(t)} = \mathbf{q}^t(p').$$

The theorem is proved.

References

[1] Allen, B.: Some stochastic processes of interdependent demand and technological diffusion of an innovation exhibiting externalities among adopters. International Economic Review **23**, 595–608 (1982)

[2] Amir, R., Evstigneev, I. V.: Stochastic version of Polterovich's model: Exponential turnpike theorems for equilibrium paths. Macroeconomic Dynamics **3**, 149–166 (1999)

[3] Aoki, M.: A new approach to macroeconomic modelling. New York: Cambridge University Press 1996

[4] Arkin, V. I., Evstigneev, I. V.: Stochastic models of control and economic dynamics. London: Academic Press 1987

[5] Batten, D. F., Boyce, D. E.: Spatial interaction, transportation, and interregional commodity flow models. In: Nijkamp, P. (ed.) Handbook of regional and urban economics, vol. 1, pp. 357–406. Amsterdam: North-Holland 1986

[6] Bernstein, D., Friesz, T. L., Smith, T. E., Tobin, R. L., Wie, B-W.: A variational inequality formulation of the dynamic network user equilibrium problem. Operations Research **41**, 179–191 (1993)

[7] Bewley, T.: Stationary equilibrium. Journal of Economic Theory **24**, 265–295 (1981)

[8] Bewley, T.: An integration of equilibrium theory and turnpike theory. Journal of Mathematical Economics **10**, 233–267 (1982)

[9] Blume, L. E.: The statistical mechanics of strategic interaction. Games and Economic Behavior **5**, 387–424 (1993)

[10] Blume, L. E., Durlauf, S. N.: Equilibrium concepts for social interaction models. International Game Theory Review **5**, 193–209 (2003)

[11] Brock, W. A.: On existence of weakly maximal programs in a multisector economy. Review of Economic Studies **37**, 275–280 (1970)

[12] Brock, W. A., Haurie, A.: On existence of overtaking optimal trajectories over an infinite time horizon. Mathematics of Operations Research **1**, 337–346 (1976)

[13] Brock, W. A., Hommes C.: A rational route to randomness. Econometrica **65**, 1059–1095 (1997)

[14] Brock, W. A., Durlauf, S. N.: Interactions based models. In: Heckman, J. J., Leamer E. E. (eds.) Handbook of econometrics, vol. 5, pp. 3297–3380. Amsterdam: North-Holland 2001

[15] Brock W. A., Durlauf S. N.: Discrete choice with social interactions. Review of Economic Studies **68**, 235–260 (2001)

[16] Brock, W. A., Mirman, L. J.: Optimal economic growth and uncertainty: The no discounting case. International Economic Review **14**, 560–573 (1973)

[17] Dempster, M. A. H., Evstigneev I. V., Pirogov, S. A.: Balanced states in stochastic economies with locally interacting agents. Stochastics **64**, 235–253 (1998)

[18] Dobrushin, R. L.: The existence of a phase transition in two- and three-dimensional Ising models. Theory of Probability and its Applications **10**, 209–230 (1968)

[19] Durlauf, S. N.: Non-ergodic economic growth, Review of Economic Studies **60**, 349–366 (1993)

[20] Durlauf, S. N.: Statistical mechanics approaches to socioeconomic behavior. In: Arthur, W. B., Durlauf, S. N., Lane, D. (eds.) The economy as an evolving complex system II, pp. 81–104. Menlo Park, CA: Addison-Wesley 1997

[21] Dynkin, E. B.: Some probability models for a developing economy. Soviet Mathematics (Doklady) **12**, 1422–1425 (1971)

[22] Dynkin, E. B.: Economic equilibrium under uncertainty. In: Los, J., Los, M. W. (eds.) Computing equilibria: How and why, pp. 41–60. Amsterdam: North-Holland 1976

[23] Evstigneev, I. V.: Optimal economic planning taking account of stationary random factors. Soviet Mathematics (Doklady) **13**, 1357–1359 (1972)

[24] Evstigneev, I. V., Katyshev, P. K.: Equilibrium paths in stochastic models of economic dynamics. Theory of Probability and its Applications **27**, 127–135 (1982)

[25] Evstigneev, I. V.: Controlled random fields on an oriented graph. Theory of Probability and its Applications **33**, 433–445 (1988)

[26] Evstigneev, I. V.: Controlled random fields on graphs and stochastic models of economic equilibrium. In: Sazonov V. V. and Shervashidze, T. (eds.) New trends in probability and statistics, pp. 391–412. Utrecht: VSP 1991

[27] Evstigneev, I. V., Greenwood, P. E.: Markov fields over countable partially ordered sets: Extrema and splitting. Memoirs of the American Mathematical Society **112**, No. 537, 1–101 (1994)

[28] Evstigneev, I. V., Taksar, M. I.: Stochastic equilibria on graphs I. Journal of Mathematical Economics **23**, 401–433 (1994)

[29] Evstigneev, I. V., Taksar, M. I.: Stochastic equilibria on graphs II. Journal of Mathematical Economics **24**, 383–406 (1995)

[30] Evstigneev, I. V., Taksar, M. I.: Equilibrium states of random economies with locally interacting agents and solutions to stochastic variational inequalities in $< L_1, L_\infty >$. Annals of Operations Research (Special issue "Stochastic equilibrium problems in economics and game theory") **114**, 145–165 (2002)

[31] Föllmer, H.: Random economies with many interacting agents. Journal of Mathematical Economics **1**, 51–62 (1974)

[32] Gale, D.: A closed linear model of production. In: Kuhn, H. W., *et al.* (eds.) Linear inequalities and related systems. Annals of Mathematical Studies, vol. 38, pp. 285–303. Princeton: Princeton University Press 1956

[33] Gale, D.: On optimal development of a multisector economy. Review of Economic Studies **34**, 1–18 (1967)

[34] Gale, D.: A mathematical theory of optimal economic development. Bulletin of the American Mathematical Society **74**, 207–223 (1968)

[35] Glaeser, E., Scheinkman, J., Non-market interactions. In: Dewatripont, M., Hansen, L. P., Turnovsky, S. (eds.) Advances in economics and econometrics: Theory and applications. Cambridge: Cambridge University Press 2000

[36] Hildenbrand, W.: Market demand: Theory and empirical evidence. Princeton: Princeton University Press 1994

[37] Horst, U., Scheinkman, J.: Equilibria in systems of social interactions. Preprint, Department of Economics, Princeton University, May 2005.

[38] Ioannides, Y. M.: The evolution of trading structures. In: Arthur, W. B., Durlauf, S. N., Lane, D. (eds.) The economy as an evolving complex system II, pp. 129–167. Menlo Park, CA: Addison-Wesley 1997

[39] Ioannides, Y. M.: Topologies of social interactions. To appear in Economic Theory **28**, 559–584 (2006)

[40] Karmann, A.,: Stochastische Interaktionsökonomien – Tauschwirtschaften mit grossen Agentenzahl. Operations Research Verfahren **XXII**, 51–69 (1976)

[41] Kirman, A. P.: The economy as an interactive system. In: Arthur, W. B., Durlauf, S. N., Lane, D. (eds.) The economy as an evolving complex system II, pp. 491–531. Menlo Park, CA: Addison-Wesley 1997

[42] Kirman A. P.: The economy as an evolving network. Journal of Evolutionary Economics **7**, 339–353 (1997)

[43] Kirman, A. P., Vriend N. J.: Evolving market structure: An ACE model of price dispersion and loyalty. Journal of Economic Dynamics and Control **25**, 459–502 (2001)

[44] Lux, T.: The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distributions. Journal of Economic Behavior and Organization **33**, 143–165 (1998)

[45] Lux, T., Marchesi M.: Scaling and criticality in a stochastic multi-agent model of a financial market. Nature **397**, 498–500 (1999)

[46] Majumdar, M., Zilcha, I.: Optimal growth in a stochastic environment: Some sensitivity and turnpike results. Journal of Economic Theory **27**, 101–136 (1987)

[47] McKenzie, L. W.: Optimal economic growth, turnpike theorems and comparative dynamics. In: Arrow, K. J., Intrilligator, M. D. (eds.) Handbook of mathematical economics, vol. III, pp. 1281–1355. Amsterdam: North Holland 1986

[48] McKenzie, L.W.: Turnpikes. American Economic Review Papers and Proceedings **88**, 1–14 (1998)

[49] Nagurney, A.: Network economics: A variational inequality approach. Dordrecht: Kluwer Academic Publishers 1993

[50] Nijkamp, P., Reggiani, A.: The economics of complex spatial systems. Amsterdam: North Holland 1998

[51] Nikaido, H.: Convex structures and economic theory. New York: Academic Press 1968

[52] Polterovich, V. M.: Equilibrium trajectories of economic growth. In: Efimov, B. A. (ed.) Methods of functional analysis in mathematical economics, pp. 56–97. Moscow: Nauka 1978 (in Russian). English translation: Econometrica **51**, 693–729 (1983)

[53] Preston, C. J.: Gibbs states on countable sets. Cambridge: Cambridge University Press 1974

[54] Radner, R.: Balanced stochastic growth at the maximum rate. In: Contributions to the von Neumann growth model (Conference proceedings, Institute for Advanced Studies, Vienna, 1970), Zeitschrift für Nationalökonomie, Suppl. No. **1**, 39–53 (1971)

[55] Radner, R.: Optimal stationary consumption with stochastic production and resources. Journal of Economic Theory **6**, 68–90 (1973)

[56] Radner, R.: Equilibrium under uncertainty. In: Arrow, K. J., Intrilligator, M. D. (eds.) Handbook of mathematical economics, pp. 923–1006. Amsterdam: North-Holland 1982

[57] Samuelson, P. A.: Spatial price equilibrium and linear programming. American Economic Review **42**, 283–303 (1952)

[58] Shapiro, C., Varian, H. R.: Information rules: A strategic guide to the network economy. Harvard: Harvard Business School Press 1998

[59] Taksar, M. I.: Optimal planning over infinite time interval under random factors. In: Los, J., Los, M. W. (eds.) Mathematical models in economics, pp. 289–298. Amsterdam: North Holland 1974

[60] Taksar, M. I.: Price and income regulation under uncertainty in the model with one producer. Journal of Information and Optimization Sciences **7**, 235–244 (1986)

[61] Verbrugge, R.: Interactive-agent economies: An elucidative framework and survey of results. Macroeconomic Dynamics **7**, 424–472 (2003)

[62] Weisbuch, G., Kirman, A., Herreiner, D.: Market organisation and trading relationships. Economic Journal **110** 411–436 (2000)