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Private Information, Transferable Utility, and the Core

by

S. D. Flåm and L. Koutsougeras

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leonidas@manchester.ac.uk
School of Social Sciences,
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom

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Abstract. Considered here are transferable-utility, coalitional production (or market) games, featuring differently informed players. It is assumed that personalized contracts must comply with idiosyncratic information. The setting may create two sorts of shadow prices: one for material endowments, the other for knowledge. Focus is on specific, computable core solutions, generated by such prices. Solutions of that sort obtain under standard regularity assumptions.

Keywords: cooperative games, transferable utility, differential information, private core, Lagrangian duality, value of information.
JEL classification: C62, C71, D51, D82.

1. Introduction

Economics deals with various ways to handle scarcity. Prominent problems, and corresponding institutions, regard production, valuation or allocation of limited *material* items. Equally important issues revolve though, around acquisition, distribution and sharing of *information*. The latter object is, however, just like other more tangible commodities, often unevenly distributed, scarce, or quite simply lacking.

Efficient instruments that handle lacking but *symmetric* information come as contracts offered say, by insurers or financial bodies. In contrast, presence of *asymmetric* information frequently impedes efficiency, eliminating maybe good opportunities for concerted actions, exchange or insurance.

That observation has inspired many studies on contracts under differential knowledge about the state of the world. Main concerns were always with efficiency, incentive compatibility, and existence of appropriate solutions. In particular, the appropriateness and properties of various core versions have been scrutinized in this context.[1] This paper pursues that vein, placing the *private core* at center stage and specializing to transferable utility.

Our approach goes as follows. Manifold instances involve agents who worship maximization of quasi-linear utility or monetary payoff. So, for the argument, let us talk hereafter about profit-maximizing producers, each willing to accept side-payments. These agents hold different technologies, endowments, and informations.

*School of Economics, University of Manchester, Oxford Road M13 9PL UK. The first author is visiting, on leave from University of Bergen; sjur.flaam@econ.uib.no, and he thanks Finansmarkedsfondet for financial support.

[1]Studies include [1], [2], [9], [15], [18], [19], [30], [31] and references therein.

In short, everybody possibly acts in three intertwined roles: as producer, owner, and "informer." It appears natural therefore, that contracts pay each party in three corresponding capacities.

When inquiring whether acceptable and feasible payment schemes of such sort exist, a leading maxim says that scarcity commands a price. Another guideline tells that prices of private, perfectly divisible, material commodities typically emerge as shadow items, brought to the fore by differential calculus. There is however, no direct counterpart concerning marginal amounts of information. A rich theory notwithstanding [7], [17], to measure information content still seems difficult - and to divide it even harder.

These pessimistic observations seemingly preclude differentiation, classical or not, as our main or only vehicle. Closer scrutiny shows however, that *Lagrangian duality*, already known to furnish standard shadow prices, may help to evaluate information as well.[2] Instrumental to this end are multipliers that relax informational constraints.

The prospect of such relaxation motivated our inquiry on several grounds. First, since dual problems often come more tractable than their primal counterparts, one may more easily use them to compute or display *explicit* core outcomes. Another bonus of duality is that inquiries about existence of equilibrium *prices* can be divorced from those concerning *allocations*. Further, it seems worthwhile to have handy some simple or practical instances to test intuition - and to isolate crucial solution properties, in particular the presence or size of informational rent. It is noteworthy that such rent may accrue to totally unproductive, quite poor but complementary informed parties. Also noteworthy is that a player may be at disadvantage by becoming better informed. Finally, but admittedly on a more technical note, we find it interesting to see precisely where and how - and to what degree - the availability of price-generated imputations depends on convex preferences.

The paper addresses several groups of readers. One comprises economists and game theorists who wish to analyze, compute or display some "quantifiable" effects of differential information. Another group include actuaries and finance theorists dealing with differentially informed agents. Also addressed are mathematicians and operations researchers interested in how convex analysis applies to parts of game theory.

Section 2 formalizes the setting. Section 3 relates core solutions to shadow prices. A few properties of price-generated imputations are investigated in Section 4. Section 5 deals with non-transferable utility. Some comments and examples are found in Section 6, and Section 7 concludes.

2. Formulation

The subsequent model requires several sorts of data, presented next. Some readers might contend with perusing this section, returning to details when needed.

[2]This observation has long been central in stochastic programming. See in particular the nice papers [11], [12].

Players form a finite, fixed set I of economic agents, each construed as a producer who aims at maximal expected profit.

Uncertainty prevails as to which scenario will materialize next. These constitute a comprehensive set S of mutually exclusive states. For example, S might be seen as the range of a stochastic vector. All parties understand that *one* $s \in S$ will come about in a while. Since our chief concerns are with analysis, computation and modelling, we hesitate not in assuming S finite. Doing so saves discussion of purely mathematical or technical issues - albeit, of course, at the loss of some generality.[3]

The occurrence of the state separates time in two periods. During the first, decisions are committed in face of non-negligible uncertainty. Ex post, when the state already has occurred, players honor contracts and collect proceeds.

Information ex post about the realized state may differ in degree or nature among players. For example, when s is a vector, various agents may get to see different components. Formally, at the second stage, individual i can only ascertain to which part $P_i(s)$ in a prescribed *partition* $\mathbb{P}_i$ of S the true state belongs. Evidently, the finer his partition, the better informed he is then. At one extreme, if some player's partition consists exclusively of singletons, he fully knows the state after it has happened. Reflecting this, we refer to each instance $\mathbb{P}_i = \{\{s\}\}$ as one of *perfect information*. In contrast, if the entire space S constitutes a player's partition $\mathbb{P}_i = \{S\}$, then even ex post he knows virtually nothing about s.

For the subsequent analysis let $\mathcal{F}_i$ denote the field formed by taking unions of parts $P_i \in \mathbb{P}_i$. Such parts are also called *atoms*. More generally, a nonempty family $\mathcal{F}$ of subsets in S is declared a *field* if stable under complements and unions. Minimal members of $\mathcal{F}$ are then referred to as atoms. A field $\mathcal{F}$ embodies coarser information than the (finer) field $\hat{\mathcal{F}}$ iff $\mathcal{F} \subsetneq \hat{\mathcal{F}}$.

The polar instance of symmetric information has all fields $\mathcal{F}_i$ equal. Plainly, partitions then coincide across players, and everybody knows that ex post merely one and the same part of the state space will be worth caring about. This case is covered below but not especially considered - except as a good case for mutual insurance.

Commodity bundles are codified as vectors in a standard Euclidean space X with coordinates indexed by the goods in question. A *contingent commodity bundle* $x(\cdot)$ is a mapping $s \in S \mapsto x(s) \in X$. Oftentimes, when confusion cannot result, we write simply x instead of $x(\cdot)$. Let $\mathbb{X} := X^S$ denote the space of all contingent commodity bundles.[4] $x \in \mathbb{X}$ is declared *adapted* - or *measurable* with respect - to a field $\mathcal{F}$ iff x is constant on each atom of $\mathcal{F}$.

A priori agent i owns a $\mathcal{F}_i$-adapted *endowment* $e_i \in \mathbb{X}$. Clearly, $e_i(s) \in X$ is

[3] We take care though, to state things in ways that invite generalizations.

[4] As a notational matter, whenever S, T are nonempty sets, T^S denotes the set of all mappings from S into T.

construed as the resource bundle privately owned by i in state s. If e_i, as conceived ex ante, were not adapted to the information structure of agent i, then $\mathcal{F}_i$ should be replaced by a finer field.

Given any function f defined on S, its "level sets" constitute a partition that generates a minimal field $\mathcal{F}(f)$ with respect to which f is adapted. Thus we require that $\mathcal{F}(e_i) \subseteq \mathcal{F}_i$. A strict inclusion is acceptable. It would mean that i has at hand more private information than imbedded in e_i.

The essential objective of player i is to maximize a proper monetary payoff $\pi_i(x_i)$ with respect to his $\mathcal{F}_i$-adapted bundle x_i.[5] To account conveniently for other constraints that might apply only to him - be it nonnegativity, capacity limits etc. - we allow π_i to take the value $-\infty$.[6] This simple device accounts for constraint violation by means of an infinite penalty. It serves as Occam's razor, allowing us to focus on essential objectives - and to shy away from particular features. We refrain therefore, from spelling out what feasibility might mean in full and quite varied detail.

The nature and properties of $\pi_i(\cdot)$ are, for now, left fairly unspecified. Emphasized though, is that $\pi_i(\cdot)$ incorporates all but *one* constraint of player i. The single exception is that his contract x_i be adapted to own information. There are two reasons for stating this explicit constraint in that form: First, the only treaties he can credibly commit to, are constant across contingencies he cannot discriminate. Second, only such treaties are enforceable. In short, imperfect information makes for incomplete contracts or partial commitments.[7]

Accommodated are, of course, additively separable payoffs

$$\pi_i(x_i) = \sum_{s \in S} \Pi_i(s, x_i(s))\mu(s), \tag{1}$$

featuring a state-dependent function $\Pi_i(s, \cdot)$, and a positive probability measure μ over S. The "integrand" $s \mapsto \Pi_i(s, \chi)$ in (1) need not be $\mathcal{F}_i$-measurable. But, because only $\mathcal{F}_i$-adapted x_i are feasible, we may just as well replace $\Pi_i(\cdot, \chi)$ with its adapted version $E\left[\Pi_i(\cdot, \chi)\,|\mathcal{F}_i\right]$. Format (1) is more general than it might first appear. To wit, evident modifications of Π_i allow one to replace μ by a measure μ_i that better mirror agent i's subjective beliefs.

Exchange and sharing of commodities is presumed free of restrictions. That is, all goods are construed as perfectly divisible and transferable. So, ex ante a *coalition* $C \subseteq I$ might allocate any $\mathcal{F}_i$-adapted x_i to its member $i \in C$ subject to $\pi_i(x_i) > -\infty$

[5] Any function taking values in $\mathbb{R} \cup \{-\infty\}$ is called *proper* iff not identical to $-\infty$.

[6] Although somewhat unlikely, it is conceivable, and not precluded, that some agent cannot survive in autarchy, this meaning $\pi_i(e_i) = -\infty$.

[7] Two competing models deviate at this point. In one, all contracts are written on *common information* $\wedge_{i \in I}\mathcal{F}_i = \cap_{i \in I}\mathcal{F}_i$, this leaving fairly few or slim possibilities for mutual agreements. In the other, all information is pooled into $\vee_{i \in I}\mathcal{F}_i$. But then, quite likely, some parties must commit ex ante to terms they cannot verify ex post.

and

$$\sum_{i \in C} x_i = e_C := \sum_{i \in C} e_i. \tag{2}$$

We envisage that this sort of agreement comes as an ensemble of *contracts*, one for each member $i \in C$, specifying, *in terms verifiable by him*, precisely what bundle $x_i(s)$ he is entitled to in state s.

Denote by $\vee_{i \in C} \mathcal{F}_i$ the smallest field that contains all $\mathcal{F}_i, i \in C$. Evidently, both sides of (2) are adapted to $\vee_{i \in C} \mathcal{F}_i$. It may well happen though, that $\mathcal{F}(e_C)$ is strictly coarser than $\vee_{i \in C} \mathcal{F}_i$. Indeed, it is interesting, and not precluded, that $\mathcal{F}(e_C)$ be totally uninformative, meaning that e_C is a constant.

Pooling mechanism (2) has two economic advantages. First, it allows resource transfers across C. Second, if some member $i \in C$ is strictly better informed than others in that $\mathcal{F}(e_{C \setminus i}) \subsetneq \mathcal{F}(e_C)$, then greater flexibility becomes possible for C in adapting pro-actively to various contingencies.

Prices on contingent commodity bundles are linear functionals, mapping $\mathbb{X}$ into $\mathbb{R}$. These functionals constitute a vector space $\mathbb{X}^*$ called *dual* to $\mathbb{X}$. Presence of a *star* henceforth signals that the object in point is a price - or an operator on such items.

It's convenient to have an explicit representation of members $x^* \in \mathbb{X}^*$. For that purpose fix hereafter a probability measure μ on S with full support. That is, posit $\mu(s) > 0$ for all $s \in S$. Naturally, if some positive μ reflects prior and common probabilistic beliefs, then that μ becomes most appropriate to use. In any case, each strictly positive measure μ on S generates a positive definite, bilinear form

$$\langle x', x \rangle := \sum_{s \in S} x'(s) \cdot x(s) \mu(s) \tag{3}$$

on $\mathbb{X}$, the dot indicating the standard (or another) inner product on the underlying commodity space X. By the Riez representation theorem a dual vector corresponds to a unique linear form $\langle x^*, \cdot \rangle$. With this sort of identification the space at hand becomes self-dual; that is: $\mathbb{X} = \mathbb{X}^*$.

Expectations - and conditional versions of these - are essential below. The positive probability measure μ, just mentioned, gives rise to an *unconditional expectation* $E : \mathbb{X} \to X$ by $Ex := \sum_{s \in S} x(s)\mu(s)$. Further, for each field $\mathcal{F}$ in S, generated by a partition $\mathbb{P}$, there is a *conditional expectation* operator $E[\cdot \,|\mathcal{F}] : \mathbb{X} \to \mathbb{X}$, defined by

$$\mu(P) E[x \,|\mathcal{F}](s) := E[\mathbf{1}_P x] \quad \text{for each state } s \in P \text{ and every part } P \in \mathbb{P}.$$

Here the indicator $\mathbf{1}_P$ equals 1 on P and 0 elsewhere. Since by assumption $\mu(P) > 0$, the customary formula applies:

$$E[x \,|\mathcal{F}](s) = \frac{E[\mathbf{1}_P x]}{\mu(P)} = \sum_{s' \in P} x(s') \frac{\mu(s')}{\mu(P)} \quad \text{when } s \in P \in \mathbb{P}.$$

Most important, writing $E_i := E[\cdot|\mathcal{F}_i]$, we see that $\mathcal{F}_i$-measurability of x_i amounts to

$$x_i = E_i x_i. \tag{4}$$

Note that $E_{\mathcal{F}} := E[\cdot|\mathcal{F}]$, when seen as a linear operator from $\mathbb{X}$ to $\mathbb{X}$, has a standard $S \times S$ real matrix representation with $\frac{\mu(s')}{\mu(P)}$ in entry $(s, s') \in S \times S$ when $s, s' \in P$, and 0 otherwise.[8] To operator $E_{\mathcal{F}} : \mathbb{X} \to \mathbb{X}$ is associated a *transpose* $E^*_{\mathcal{F}} : \mathbb{X}^* \to \mathbb{X}^*$, represented by the transposed matrix, featuring $\frac{\mu(s)}{\mu(P)}$ in entry $(s, s') \in S \times S$ when $s, s' \in P$, and 0 otherwise. It's impact is given by

$$(E^*_{\mathcal{F}} x^*)(s) = \frac{\mu(s)}{\mu(P(s))} x^*(P(s))$$

where $P(s)$ is the part of $\mathbb{P}$ to which s belong, and where $x^*(P) := \sum_{s \in P} x^*(s)$.

For any transformation $T \in \{E_{\mathcal{F}}, E^*_{\mathcal{F}}\}$ it holds that $T^2 = T$. Consequently, either transformation projects its domain onto a subspace, alias its *range*. Further, using inner product (3), either transformation decomposes the domain orthogonally into the direct sum of its range and the kernel of its transpose:

$$domT = RangeT \quad \oplus \quad KerT^* \ \text{ with } \ RangeT \quad \perp \quad KerT^*. \tag{5}$$

For example, any $x \in \mathbb{X}$ admits a unique representation

$$x = x_{\mathcal{F}} + x^{\perp}_{\mathcal{F}} \quad \text{with} \quad \left\langle x_{\mathcal{F}}, x^{\perp}_{\mathcal{F}} \right\rangle = 0, \quad x_{\mathcal{F}} = E_{\mathcal{F}} x, \quad x^{\perp}_{\mathcal{F}} = x - x_{\mathcal{F}} \in \ker E^*_{\mathcal{F}}. \tag{6}$$

3. THE GAME AND CORE SOLUTIONS

Recall that payoffs and resources are regarded as transferable. Also recall that ex ante the triples $(\pi_i, \mathbb{P}_i, e_i), i \in I$, are commonly known.[9]

The above data generates a *transferable-utility*, *cooperative game* with player set I and a *characteristic function* that associates to coalition $C \subseteq I$ aggregate value

$$v_C := \sup \left\{ \sum_{i \in C} \pi_i(x_i) : \sum_{i \in C} x_i = e_C \ \text{ and } \ x_i = E_i x_i \ \text{ for all } i \in C \right\}. \tag{7}$$

As before, $e_C := \sum_{i \in C} e_i$ is shorthand for the aggregate endowment held by coalition C. Note that "excess demand" $x_i - e_i$ of any agent i is adapted to his information.

The economic attractions of pooling objectives and endowments, as done in (7), are evident and twofold. First, the most efficient producers can utilize resources furnished by others. Second, complementary production factors can be brought together. Formally, the advantages of coordination reflect in superadditive values:

$$v_{C_1 \cup C_2} \geq v_{C_1} + v_{C_2} \quad \textit{whenever nonempty } C_1, C_2 \subset I \ \textit{ are disjoint.}$$

[8] The finer partition that underlies $\mathcal{F}$, the more sparse the matrix representation of $E_{\mathcal{F}}$. In particular, when each s is an atom, the said matrix equals the identity.

[9] Ex post, private information, in the hand of agent i, amounts to his certainty as to which part $P_i(s) \in \mathbb{P}_i$ does indeed prevail.

A function $C \subseteq I \mapsto v_C$, satisfying $v_\varnothing = 0$, is called *convex* or *supermodular* [21] if

$$v_{C_1 \cup C_2} + v_{C_1 \cap C_2} \geq v_{C_1} + v_{C_2}.$$

The marginal value $v_{C \cup i} - v_C$ of player i joining coalition C then increases with C. Instance (7) is, however, not generally convex. To see this, let $\mathcal{F}_i = \{\varnothing, S\}$ and posit

$$\pi_i(x_i) := \sup\{\langle \bar{y}, y_i \rangle : Ay_i = x_i, y_i \geq 0\} \tag{8}$$

where A maps an ordered Hilbert space $\mathbb{Y}$ linearly into $\mathbb{X}$, and $\bar{y} \in \mathbb{Y}$. Then

$$v_C = v(e_C) := \sup\{\langle \bar{y}, y \rangle : Ay = e_C, \ y \geq 0\} \tag{9}$$

with $v_C = -\infty$ whenever the linear program is infeasible. Since the reduced function $e \mapsto v(e)$ so defined is concave, its generalized differential $\partial v(\cdot)$ is monotone decreasing [6]. This points to possible disadvantages of joining a coalition as its last member.

Anyway, whenever he enters, a new member may bring *three* benefits to a coalition. First, if endowed, he adds to the aggregate holding. Second, if productive, he expands the joint capacity. Third, if additionally informed, he makes for more flexible exchanges.[10]

Note that problem (7) is linearly constrained. This feature is most convenient for theoretical analysis and practical computation. To wit, the Kuhn-Tucker optimality conditions then come without any constraint qualification.

Given the characteristic function $C \mapsto v_C$, defined in (7), we want to "solve" the game, using the core as solution concept. Specifically, a payment pattern $(u_i) \in \mathbb{R}^I$ is said to reside in the *private-information core* iff

Pareto efficient:	$\sum_{i \in I} u_i = v_I$,	and
stable against blocking:	$\sum_{i \in C} u_i \geq v_C$	for all $C \subset I$.

A chief concern is, of course, that the core could be empty. Put differently: the question is whether the game is *balanced* or not? In that regard the following result can be established along well known lines; see [22]:

Proposition 3.1. (**Balanced games**) *Suppose all payoff functions $\pi_i(\cdot)$ are concave. Then the core is nonempty in every subgame which involves a player community $C \subseteq I$ that has finite value v_C. In particular, when v_C is finite for all $C \subseteq I$, the entire game becomes totally balanced* [27].

Proposition 3.1 just deals with existence; it's non-constructive. Moreover, it demands that every party has convex preferences. In contrast, we seek *computable* core

[10] Broadly speaking, the more varied private information is across coalition members, the less cumbersome the restriction that components $x_i(s)$ must stay constant over parts $P_i \in \mathbb{P}_i$.

solutions, brought to the fore in constructive and explicit manner. Besides, in this endeavor, it would be expedient that convexity, if not already present, emerges exactly where and when needed.[11] $\square$

So, to identify explicit solutions, if any, consider problem (7) from a dual and price-oriented vantage-ground. In doing so, associate a multiplier (price) vector $x^* \in \mathbb{X}^*$ to constraint (2) and a similar vector $x_i^* \in \mathbb{X}^*$ to constraint (4). Thus, related to problem (7) is a standard *Lagrangian*

$$L_C(\vec{x}, \vec{x}^*) := \sum_{i \in C} \{\pi_i(x_i) + \langle x^*, e_i - x_i \rangle + \langle x_i^*, E_i x_i - x_i \rangle\}, \tag{10}$$

or equivalently,

$$L_C(\vec{x}, \vec{x}^*) = \sum_{i \in C} \{\langle x^*, e_i \rangle + \pi_i(x_i) - \langle x^* + x_i^* - E_i^* x_i^*, x_i \rangle\}$$

where $\vec{x} := (x_i)$, and $\vec{x}^* := (x^*, x_i^*, i \in I)$. The interpretation of L_C is commonplace but worth recalling all the same. Suppose individual $i \in C$ could add a perturbation $\Delta e_i \in \mathbb{X}$ to his endowment at cost $\langle x^*, \Delta e_i \rangle$. Upon doing so constraint (2) would take the relaxed form

$$\sum_{i \in C} x_i = \sum_{i \in C} (e_i + \Delta e_i). \tag{11}$$

Further imagine that instead of (4) member $i \in C$ could face the looser constraint

$$x_i = E_i x_i + \Delta x_i, \tag{12}$$

with $\Delta x_i \in \mathbb{X}$ chosen freely but at extra cost $\langle x_i^*, \Delta x_i \rangle$. In that relaxed setting coalition C could achieve overall payoff

$$\sup_{(\Delta e_i, \Delta x_i), i \in C} \left\{ \sum_{i \in C} [\pi_i(x_i) - \langle x^*, \Delta e_i \rangle - \langle x_i^*, \Delta x_i \rangle] : (11) \text{ \& } (12) \text{ hold} \right\} = L_C(\vec{x}, \vec{x}^*).$$

Given x^*, observe that no $\mathcal{F}_i$-adapted perturbation Δx_i commands any extra value. Indeed, during optimization all worthwhile $\mathcal{F}_i$-adapted substitutions have already been considered. So, by (6) only choices $\Delta x_i \in \ker E_i^*$ merit attention. That is, $\Delta x_i(P) := \sum_{s \in P} \Delta x_i(s) = 0$ for all $P \in \mathbb{P}_i$.

Anyway, the more freedom in choosing perturbations, the richer in detail the corresponding price regimes. For such reasons we face a crucial modelling choice at this juncture, namely: *Should perturbed versions* (11) *of equations that, in essence, account for material balances, also embody extra information*? We choose to block this avenue, our motivation being to separate payments for tangible endowments from those concerning information. Reflecting this choice, and since formation of the *grand*

[11] For more on this issue, see [13], [14]. It will be seen though, that we can hardly accommodate any agent whose objective $\pi_i(\cdot)$ is globally convex.

coalition $C = I$ is of chief interest, we insist from here on that *any endowment price* x^* *be* $\mathcal{F}(e_I)$*-measurable.*

After these considerations declare now $\vec{x}^* = (x^*, x_i^*, i \in I)$ a *shadow price* or *Lagrange multiplier vector* iff, under that price regime, access to a competitive market for

$$\begin{cases} \text{material perturbations:} & \Delta e = E\left[\Delta e \,|\mathcal{F}(e_I)\right] \quad \text{and} \\ \text{informational perturbations:} & \Delta x_i, i \in I, \end{cases}$$

offers the grand coalition no advantage. Formally, and much simpler, $\vec{x}^*$ qualifies as Lagrange multiplier iff

$$\sup_{\vec{x}} L_I(\vec{x}, \vec{x}^*) \leq v_I. \tag{13}$$

In mathematical terms $\vec{x}^*$ is a shadow price iff it realizes the *saddle value* of L_I:

$$\sup_{\vec{x}} L_I(\vec{x}, \vec{x}^*) = \inf_{\vec{x}^*} \sup_{\vec{x}} L_I(\vec{x}, \vec{x}^*) = \sup_{\vec{x}} \inf_{\vec{x}^*} L_I(\vec{x}, \vec{x}^*). \tag{14}$$

To bring out economic and game-theoretic implications of this concept let

$$f^{(*)}(x^*) := \sup \left\{ f(x) - \langle x^*, x \rangle : x \in \mathbb{X} \right\}$$

denote the *conjugate* of a proper function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$.[12] Plainly, $f^{(*)}(x^*)$ records the profit that accrue to a producer who enjoys revenue function $f(\cdot)$ and faces linear price regime x^* for inputs.[13]

Applying these terms, the additive separability of L_C together with the defining relations $\langle x_i^*, E_i x_i \rangle = \langle E_i^* x_i^*, x_i \rangle$ imply that

$$\sup_{\vec{x}} L_C(\vec{x}, \vec{x}^*) = \sum_{i \in C} \left\{ \langle x^*, e_i \rangle + \pi_i^{(*)}(x^* + x_i^* - E_i^* x_i^*) \right\}. \tag{15}$$

We can now state a chief result forthwith:

Theorem 3.1. (**Price-supported core solutions**) *Each shadow price* $\vec{x}^* = (x^*, x_i^*, i \in I)$ *generates a solution* $(u_i) \in \mathbb{R}^I$ *in the private-information core by the formula*

$$u_i = u_i(\vec{x}^*) := \langle x^*, e_i \rangle + \pi_i^{(*)}(x^* + x_i^* - E_i^* x_i^*). \tag{16}$$

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[12] $f^{(*)}$ is convex and lower semicontinuous. In terms of the standard *Fenchel conjugate* f^* of convex analysis $f^{(*)}(x^*) = (-f)^*(-x^*)$; see [25]. Using instead the *concave conjugate*

$$f_*(x_*) := \inf_x \left\{ \langle x_*, x \rangle - f(x) \right\},$$

which reflects minimization of expenses $\langle x_*, x \rangle$ net of revenues $f(x)$, we get $f^{(*)} = -f_*$, and f_{**} is the smallest concave upper semicontinuous function $\geq f$.

[13] Note that separable instances $f(x) = \sum_{s \in S} f_s(x(s))\mu(s)$ gives

$$f^{(*)}(x^*) = \sum_{s \in S} f_s^{(*)}(x^*(s))\mu(s).$$

Proof. For any coalition $C \subseteq I$ and any multiplier vector $\vec{x}^*$ it holds via (15) that

$$\sum_{i \in C} u_i := \sum_{i \in C} \left\{ \langle x^*, e_i \rangle + \pi_i^{(*)}(x^* + x_i^* - E_i^* x_i^*) \right\} = \sup_{\vec{x}} L_C(\vec{x}, \vec{x}^*)$$
$$\geq \inf_{\vec{x}^*} \sup_{\vec{x}} L_C(\vec{x}, \vec{x}^*) \geq \sup_{\vec{x}} \inf_{\vec{x}^*} L_C(\vec{x}, \vec{x}^*) = v_C.$$

Thus $\sum_{i \in C} u_i \geq v_C$. Since $C \subseteq I$ was arbitrary, this takes care of stability against blocking. Further, for Pareto optimality we need now only verify that $\sum_{i \in I} u_i \leq v_I$. But the last inequality follows from (13). $\square$

A few words on the economic significance of formula (16) are in order. Plainly, that formula reimburses agent i the value $\langle x^*, e_i \rangle$ of his endowment. We shall argue later that the resource price x^* emerges as a (generalized) derivative of a concave function with respect to the aggregate endowment. Since concavity makes for a decreasing derivative, the larger aggregate, the smaller the price. Formally, $e \mapsto x^*(e)$ is monotone decreasing in the sense that

$$\langle e - e', x^*(e) - x^*(e') \rangle \leq 0 \tag{17}$$

for all e, e' at which the derivatives exist. When $X = \mathbb{R}^G$ for a finite set G of goods,

$$\langle x^*, e_i \rangle = \sum_{g \in G} E(x_g^* \cdot e_{ig}) = \sum_{g \in G} \left\{ Ex_g^* \cdot Ee_{ig} + cov(x_g^*, e_{ig}) \right\}. \tag{18}$$

Consequently, like in finance, i receives a covariance premium for his endowment e_{ig} of good g provided that holding be anti-correlated with $e_{I \setminus i, g}$.

As said, the second component in (16) reflects production profit, calculated at a common resource price x^* distorted by an additive, idiosyncratic component $x_i^* - E_i^* x_i^*$, stemming from private information. One might call

$$p_i := x^* + x_i^* - E_i^* x_i^*$$

the *information-corrected shadow price* facing agent i. If player i holds perfect information, $p_i := x^*$. Plainly, an agent i benefits from collaboration iff $u_i > \pi_i(e_i)$. A sufficient condition for this to happen is that

$$\pi_i^{(*)}(p_i) > \pi_i^{(*)}(x^*)$$

because then $u_i > \langle x^*, e_i \rangle + \pi_i^{(*)}(x^*) \geq \pi_i(e_i)$.

While equal treatment is standard in the customary core (and in Walras equilibrium as well), differential information may overthrow that property; see [1]. Here though, transferable utility restores it: *Agents who have equal endowments, information, and preferences, receive the same price-generated imputation* (16).

Theorem 3.1 begs the question whether Lagrange multipliers exist? To ensure existence, as expected and seen below, concavity of each $\pi_i(\cdot)$ would be most convenient. That property would reflect wide-spread risk aversion but it will really not be required here. Instead comes a somewhat weaker assumption about *convoluted preferences* - often assigned to who is called a *representative agent*.

Before regarding his preferences recall that sup-convolution (7) contributes towards concavity of the resulting, reduced function. Broadly, by admitting many and small agents the optimal value $v_I = v(e_I)$ becomes "more concave" in e_I. The curvature or global support of $e \mapsto v(e)$ is what decides existence of shadow prices. To bring this out consider the aggregate but perturbed payoff function

$$\pi(\Delta e, \Delta x) := \sup \left\{ \sum_{i \in I} \pi_i(x_i) : \sum_{i \in I} x_i = e_I + \Delta e \ \& \ x_i = E_i x_i + \Delta x_i \ \forall i \in I \right\}. \quad (19)$$

Here Δe is $\mathcal{F}(e_I)$-measurable. Observe that $\pi(0,0) = v_I$. Since shadow prices bear on differential properties of π, recall that a proper function f, mapping a Hilbert space $\mathbb{Y}$ into $\mathbb{R} \cup \{-\infty\}$, has a *supergradient* y^* at a point y iff

$$f(\cdot) \leq f(y) + \langle y^*, \cdot - y \rangle .$$

We then write $y^* \in \partial f(y)$ and declare f *superdifferentiable* at y.

Theorem 3.2. (**Characterization and existence of solutions**)
• $\vec{x}^*$ *is a shadow price iff* $\vec{x}^* \in \partial \pi(0,0)$. *Thus existence of a shadow price is ensured iff the perturbation function* π *is superdifferentiable at* $(0,0)$.
• *Denote by* $\hat{\pi}$ *the smallest concave function* $\geq \pi$, *the latter defined in* (19). *It suffices for existence of a shadow price, whence of a core solution* (16), *that* $\hat{\pi}(\cdot,\cdot)$ *be finite-valued near* $(0,0)$ *with* $\hat{\pi}(0,0) = v_I$. *In particular, if all* π_i *are concave, with* $\pi(\cdot,\cdot)$ *finite near* $(0,0)$, *then at least one shadow price regime exists.*
• *No core solution of the sort* (16) *exists if there is a strictly positive duality gap:*

$$\delta := \inf_{\vec{x}^*} \sup_{\vec{x}} L_I(\vec{x}, \vec{x}^*) - v_I.$$

In this case, any scheme (16) *entails aggregate overpayment* $\geq \delta$.[14] □

Proof. Plainly, $\vec{x}^* = (x^*, x_i^*, i \in I) \in \partial \pi(0,0)$ iff

$$\pi(\Delta e, \Delta x) - \langle x^*, \Delta e \rangle - \sum_{i \in I} \langle x_i^*, \Delta x_i \rangle \leq \pi(0,0)$$

for all $\Delta x = (\Delta x_i)$ and all $\mathcal{F}(e_I)$-measurable Δe.In turn, via substitutions $\Delta e = \sum_{i \in I}(x_i - e_i)$, $\Delta x_i = x_i - E_i x_i$, and $\pi(0,0) = v_I$, this is equivalent to

$$L_I(\vec{x}, \vec{x}^*) = \sum_{i \in I} \{\pi_i(x_i) + \langle x^*, e_i - x_i \rangle + \langle x_i^*, E_i x_i - x_i \rangle\} \leq v_I \quad \text{for all } \vec{x},$$

[14]Note that an empty core is not precluded. In particular, if $v_I = -\infty$, imputations (16) would yield $\sum_{i \in I} u_i = +\infty$ for any feasible multiplier $\vec{x}^*$. Presumably such explosion of aggregate imputations (dual problem values) may assist in detecting emptiness of the core.

whence to (13). This takes care of the first bullet. For the second simply note that the "concavification" $\hat{\pi}$ of π has a supergradient at each point near which it is finite-valued, and evidently, $\partial\hat{\pi}(0,0) \subseteq \partial\pi(0,0)$ because $\hat{\pi}(0,0) = \pi(0,0)$. Finally, the assertion after the third bullet is justified by the fact that each instance of (16) yields $\sum_{i\in I} u_i > v_I$. $\square$

Example 3.1. A single producer facing resource owners has a state-dependent revenue function $f_s : X \to \mathbb{R}\cup\{-\infty\}$ which is concave. Each agent $i \in I_{-0}$ is non-productive when quite alone, but he owns an endowment profile $e_i \in \mathbb{X}$. Let $I := \{0\} \cup I_{-0}$ and put $e_0 := 0$. Then

$$v_C := \begin{cases} 0 & \text{if } 0 \notin C \\ Ef_\bullet(e_C(\bullet)) & \text{otherwise.} \end{cases}$$

For any $i \neq 0$ we have $\pi_i(0) = 0$, and $\pi_i(x_i) = -\infty$ otherwise. Consequently, $\pi_i^{(*)} = 0$, and resource owner i receives $u_i = \langle x^*, e_i\rangle$. Any resource price x^* reflects marginal revenues, that is, $x^*(s) \in \partial f_s(e_I(s))$ for each s. Therefore, $i \neq 0 \Rightarrow u_i = E\left[x^* e_i\right] = \sum_s x^*(s) \cdot e_i(s)\mu(s)$. The producer receives

$$u_0 = Ef_\bullet^{(*)}(x^* + x_0^* - E_0^* x_0^*)(\bullet) = \sum_{s\in S} \left\{f_s^{(*)}(x^*(s) + x_0^*(s) - E_0^* x_0^*(s))\right\}\mu(s).$$

Quite generally, an agent with $\pi_i^{(*)} \equiv 0$ qualifies as a *pure resource owner*. When $X = \mathbb{R}^G$ for a finite set G of goods, any shadow price gives a pure resource owner core payment given by formula (18).

A this point a consequence of refined information is noteworthy. Suppose *one* agent i gets a nondegenerate part P_i of his partition split into smaller sets - and that his endowment e_i be redefined with some variation across P_i. Also suppose that each other player already has P_i fully contained in an atom of his. As a result $\mathcal{F}(e_I)$ becomes more refined, whence x^* is likely to vary inside P_i. Pursuing this line, one may straightforwardly design instances where $(Ex^*) \cdot (Ee_i)$ remain constant, but $\sum_{g\in G} cov(x_g^*, e_{ig})$ does not. Broadly, if $s \mapsto \partial f_s$ and $s \longmapsto e_i(s)$ are positively (negatively) correlated, then refined information is likely to advantage (disadvantage) i in his capacity as resource owner. So, while information refinement increases v_I, the distributional impact is not clear cut. $\square$

Some discussion of the Theorem 3.2 is in order. The superdifferentiability - that is, the global support - of the perturbed function π at $(0,0)$ does not demand that all underlying π_i be concave. Also, on a technical note, such support unhinges arguments for existence of equilibrium from appeals to fixed point theorems. If some π_i isn't concave, one may "board up its holes" by employing instead the smallest concave function $\hat{\pi}_i \geq \pi_i$. This done, each price regime $\vec{x}^*$ generates imputations $\hat{u}_i(\vec{x}^*) \geq u_i(\vec{x}^*)$. In terms of the duality gap δ any shadow price $\vec{x}^*$ for the concavified game gives

$$\begin{array}{ll} \sum_{i\in I} \hat{u}_i(\vec{x}^*) \leq v_I + \delta, & \text{and} \\ \sum_{i\in C} \hat{u}_i(\vec{x}^*) \geq v_C & \text{for all } C \subseteq I. \end{array}$$

The Shapley-Folkman theorem [10] asserts that concavification of payoffs will be effective on at most $\dim X + 1$ agents. For more on this issue, and for estimates of the duality gap (or *core deficit*) δ, see [3], [13], [14], [28].

The upshot is that there is room for agents whose payoffs are non-concave in regions of no economic interest. It is hard however, to accommodate players having globally *convex* payoff functions. Indeed, presence of merely *one* individual of that sort suffices to render the perturbed function π convex. When moreover, that π has a supergradient somewhere, it must be affine. Definitely, such an instance has little of interest or realism.

Uniqueness of a Lagrange multiplier amounts, of course, to have $\pi(\cdot,\cdot)$ differentiable at $(0,0)$. We shall not explore this issue.

Typically, the commodity space X is ordered, and most often at least some agent has monotone payoff. Then, provided $\sum_{i\in I} x_i$ always be $\mathcal{F}(e_I)$-measurable, *free disposal* becomes possible because it suffices for material balance to insist that $\sum_{i\in I} x_i \leq e_I$. So, under such qualifications, the resource price x^* must be nonnegative.

Nothing was said so far about solvability of (primal) problems (7). For completeness, recorded next are some propositions on existence of an optimal allocation across the grand coalition:

Proposition 3.2. (**Existence of optimal allocations**) *Optimal allocations exist and the value v_I is attained in each the following three cases:*
1) *The upper level set*

$$\left\{ (x_i) : \sum_{i\in I} \pi_i(x_i) \geq r,\ x_i = E_i x_i,\ \sum_{i\in I} x_i = e_I \right\} \tag{20}$$

is closed for all $r \in \mathbb{R}$ with respect to convergence in each state-dependent bundle $x_i(s)$.[15] *Also suppose that the said set is nonempty compact for at least one $r \in \mathbb{R}$.*
2) *Each π_i is upper semicontinuous and concave on $Range E_i$, and the recession functions*

$$0^-\pi_i(d_i) := \inf_{r>0} \frac{\pi_i(x_i + r d_i) - \pi_i(x_i)}{r}, \quad \pi_i(x_i) \textit{ finite},$$

satisfy

$$\sum_{i\in I} 0^-\pi_i(d_i) \geq 0 \ \ \& \ \ \sum_{i\in I} 0^-\pi_i(-d_i) < 0 \ \text{ implies } \sum_{i\in I} d_i \neq 0.$$

3) *Each π_i is upper semicontinuous with a conjugate $\pi_i^{(*)}$ that is finite-valued continuous at 0.*

[15]Since S is finite, this amounts of course to have $\pi_i(\cdot)$ upper semicontinuous with respect to the product topology on X^S.

Proof. Statement 1) is standard. For 2) see Rockafellar (1970) Corollary 9.2.1. For 3) Let

$$f_*(y_*) := \inf_y \{\langle y_*, y - f(y)\rangle\}$$

denote the *concave conjugate* of a proper function f that maps a Hilbert space into $[-\infty, +\infty)$. Then, on the same space, $\hat{f} := (f_*)_*$ equals the smallest concave upper semicontinuous function $\geq f$. The fact that π_{i*} is finite-valued and continuous at 0 implies, by Moreau-Rockafellar theorem [6], that each upper level set $\{\hat{\pi}_i \geq r_i\}$ is compact.

Now consider any maximizing, feasible sequence $x^k = (x_i^k)$. Since v_I is finite there exist real numbers r_i such that $x_i^k \in \{\hat{\pi}_i \geq r_i\}$ for all k and i. Extract a convergent subsequence to get the targeted conclusion. $\square$

Having cared so far about existence of shadow prices and attainment of values, it seems fitting to mention a case that causes little of such concerns:

Example 3.2. Linear Production Games. The computational and expressive power of linear programming, with modern extensions [5], motivates a brief look at cooperative producers who have linear technologies [23]. A special instance was already considered in (8), (9). Here, more generally posit

$$v_i := \sup\{\langle c_i, y_i\rangle : A_i y_i \leq e_i,\ y_i \geq 0\}. \tag{P_i}$$

The objective

$$\langle c_i, y_i\rangle := E[c_i \cdot y_i] = \sum_{s\in S} c_i(s)\cdot y_i(s)\mu(s),$$

features $\mathcal{F}_i$-adapted vectors $c_i(s)$ and $y_i(s)$ that reside in an Euclidean space Y_i. The constraints in (P_i) mean that $A_i(s)y_i(s) \leq e_i(s)$, $y_i(s) \geq 0$ for all s. The $\mathcal{F}_i$-adapted operator (or matrix) $A_i(s)$ maps Y_i into X, and both these spaces are ordered.

Problem (7) now amounts to the following aggregate linear program:

$$v_C := \sup\left\{\sum_{i\in C}\langle c_i, y_i\rangle : \sum_{i\in C} A_i y_i \leq e_C \text{ with } y_i \geq 0 \text{ and } \mathcal{F}_i\text{-adapted}\right\}. \tag{P_C}$$

Proposition 3.3. (**Linear imputations**) *Suppose the aggregate linear problem (P_I) has finite optimal value v_I. Let x^* and $y_i^*, i \in I$, be Lagrange multipliers - alias optimal dual variables - associated to the material balance $\sum_{i\in I} A_i y_i \leq e_I$ and the information restrictions $y_i = E_i y_i$, $i \in I$, respectively. Then the payment pattern*

$$i \in I \to \langle x^*, e_i\rangle$$

belongs to the private-information core. This happens if x^* and $y_i^*, i \in I$, optimally solve the dual problem

$$\min \langle x^*, e_I\rangle \ \ s.\ t.\ x^* \geq 0 \text{ and } c_i \leq A_i^* x^* + y_i^* - E_i^* y_i^* \ \ \textit{for all} \ \ i. \ \square$$

For a simple numerical illustration, consider two agents, two states, with asymmetric informations and endowments described as follows:

Agent i	*Partition* $\mathbb{P}_i$	state s:	s_1	s_2
1	$\{s_1\}, \{s_2\}$	endowment $e_1(s)$:	2	0
2	$\{s_1, s_2\}$	endowment $e_2(s)$:	1	1

When $\pi_i(x_i) = \sum_s x_i(s)$ for all i, s, we get $v_i = 2$ for each i, and $v_I = 4$. This specification makes the cooperative game perfectly additive: Pooling among the risk neutral agents offers nobody any benefit over autarky.

As one would expect, *no direct information rent accrue when all players are risk-neutral.* Also note that individual payoff was defined here as a reduced function: $\pi_i(x_i) := \sup_{y_i} \mathbf{\Pi}_i(x_i, y_i)$. This feature, and the importance of such instances, speaks against presuming π_i smooth.

Linear objectives belong to the wide and important class of *polyhedral functions*, defined as those whose hypograph equals the intersection of finitely many closed half-spaces [25]. Since the conjugate of such functions are polyhedral as well formula (16) become tractable.

As is well known, presence of players with linear objectives facilitate risk sharing. Likewise, when information is symmetric, the prospects of mutual insurance appear good:

Proposition 3.4. (**Symmetric information and mutual insurance**) *Suppose all $\mathcal{F}_i = \mathcal{F}$ are equal and generated by a partition $\mathbb{P}$. Also suppose π_i is of separable form* (1) *with $\Pi_i(s, \cdot)$ adapted to the common $\mathcal{F}$. Then coalition C has value $v_C = \sum_{P \in \mathbb{P}} v_C(P)\mu(P)$ where*

$$v_C(P) := \sup \left\{ \sum_{i \in C} \pi_i(s, \chi_i) : \sum_{i \in C} \chi_i = e_C(s) \right\} \text{ for each } s \in P.$$

Moreover, $u_i = \sum_{P \in \mathbb{P}} u_i(P)\mu(P)$ *with*

$$u_i(P) = x^*(s) \cdot e_i(s) + \pi_i^{(*)}(s, x^*(s)) \text{ for each } s \in P.$$

Thus, cooperative solutions are sustained merely via state-contingent transfers.

Proof. With no loss of generality replace S with $\mathbb{P}$. After such replacement everybody has a perfect information structure whence the information constraints can all be ignored. □

4. Some Properties of Price-generated Imputations

We have stressed the advantages of cooperation. It may happen though, that some player prefers to take no part:

Proposition 4.1. (**On dummies or outsiders**) *Imputation* (16) *offers agent i autarky payment $u_i = \pi_i(e_i)$ iff the information-corrected shadow price "coincides" with his marginal payoff; that is, iff*

$$p_i := x^* + x_i^* - E_i^* x_i^* \in \partial \pi_i(e_i). \tag{21}$$

Proof. Since $x_i^* - E_i^* x_i^* \in KerE_i^*$ and e_i is $\mathcal{F}_i$-measurable, we have $\langle x_i^* - E_i^* x_i^*, e_i \rangle = 0$. Therefore autarky payment happens iff

$$\langle p_i, e_i \rangle + \pi_i^{(*)}(p) = \pi_i(e_i),$$

or equivalently, precisely when

$$\pi_i^{(*)}(p_i) := \sup \{\pi_i(x_i) - \langle p_i, x_i \rangle\} = \pi_i(e_i) - \langle p_i, e_i \rangle .$$

Plainly, the function $x_i \mapsto \pi_i(x_i) - \langle p_i, x_i \rangle$ is maximal at $x_i = e_i$ iff (21) holds. $\square$

As customary Lagrange multipliers relate to geometry, information, and willingness to pay. These features are recorded next. For the statement denote by $f'(y; \Delta y)$ the *directional derivative* of $f : \mathbb{Y} \to \mathbb{R} \cup \{-\infty\}$ at $y \in Y$ in the direction Δy.

Proposition 4.2. (**Properties of shadow prices**)
1) *x^* must be orthogonal on the affine subspace spanned by equation $\sum_{i \in I} x_i = e_I$. More precisely, the replicated vector $(x^*, ..., x^*) \in \mathbb{X}^{*I}$ is normal to the manifold $\mathcal{M} := \left\{ \vec{x} = (x_i) \in \mathbb{X}^I : \sum_{i \in I} x_i = e_I \right\}$ in that*

$$\left\langle x^*, \sum_{i \in I} x_i - e_I \right\rangle = 0 \;\; \textit{for all } \mathcal{F}(e_I)\textit{-adapted aggregates } \sum_{i \in I} x_i \tag{22}$$

Further, x_i^* must be orthogonal to $\ker E_i^*$:

$$\langle x_i^*, x_i - E_i x_i \rangle = 0 \;\; \textit{for all } x_i. \tag{23}$$

2) *The material component x^* is, or can be made, $\mathcal{F}(e_I)$-measurable. Also, x_i^* must be $\mathcal{F}_i$-adapted.*
3) *If the perturbed function π (19) is differentiable at $\mathbf{0}$ in the direction $(\Delta e, \Delta x)$, then*

$$\pi'(\mathbf{0}; \Delta e, \Delta x) \leq \inf \left\{ \langle x^*, \Delta e \rangle + \sum_{i \in I} \langle x_i^*, \Delta x_i \rangle : \vec{x}^* \text{ shadow price} \right\}. \tag{24}$$

In case π is concave and finite near $(0,0)$, equality holds in (24).

Proof. 1) The saddle property (14) of shadow prices opens, as usual, a perspective on a two-person, zero-sum, non-cooperative game. To wit, there is a fictitious, price-setting agent who wants to minimize $L_I(\vec{x}, \vec{x}^*)$ with respect to $\vec{x}^*$. In doing so, he obtains no advantage in letting x^* have a non-zero component normal to manifold $\mathcal{M}$. And quite similarly, he cannot benefit from letting x_i^* have a non-trivial component orthogonal on $RangeE_i$.

(22) tells that, given price regime x^*, it is not worth coalition I while to contemplate additions Δx_i to e_i that violate $\sum_{i\in I} \Delta x_i = 0$. (23) says that information prices for agent i blocks him from straying outside the adapted subspace $E_i\mathbb{X} =: RangeE_i$. (22) and (23) are commonly called *complementarity conditions*.

2) The first assertion derives from the hypothesis that only $\mathcal{F}(e_I)$-measurable perturbations of the aggregate endowment were accommodated. Plainly, the dual space to $E_{\mathcal{F}(e_I)}\mathbb{X}$ comprises only functionals of corresponding measurability. Concerning x_i^* we already observed in Proposition 8 that this price must stand orthogonally on $\ker E_i^*$. But, by decomposition (5), the orthogonal complement to $\ker E_i^*$ is $RangeE_i$, that is, the space of $\mathcal{F}_i$-adapted contingent vectors.

3) By Theorem 3.2 each shadow price $\vec{x}^*$ is a supergradient of π at $(0,0)$. This implies that

$$\pi(t\Delta e, t\Delta x) - \pi(0,0) \leq t\left\{ \langle x^*, \Delta e\rangle + \sum_{i\in I} \langle x_i^*, \Delta x_i\rangle \right\}$$

for any $t > 0$ and shadow price $\vec{x}^*$. Consequently,

$$\frac{\pi(t\Delta e, t\Delta x) - \pi(0,0)}{t} \leq \inf\left\{ \langle x^*, \Delta e\rangle + \sum_{i\in I} \langle x_i^*, \Delta x_i\rangle : \vec{x}^* \text{ shadow price} \right\}$$

and the first assertion follows. The second one is a standard result of convex analysis. □

For interpretation of assertion 3) assume the shadow price be unique. Then $\langle x^*, \Delta e\rangle$ is an upper bound on the willingness of grand coalition I to pay for resource additions in direction Δe. Further, if agent i could commit a more flexible contract x_i, beyond his information structure, then $\langle x_i^*, \Delta x_i\rangle$ estimates what his adaptability is worth in direction Δx_i.

We conclude this section by inquiring about the robustness or stability of core imputations (16). In particular, how do they fare under perturbations of endowments, payoffs and information structures?

The issue can be formalized as follows: Let $\vec{x}^{*n}$ be a shadow price of a game $\Gamma^n := (e_i^n, \pi_i^n, E_i^n)_{i\in I}$. Suppose the latter converges to $\Gamma := (e_i, \pi_i, E_i)_{i\in I}$ in a sense to be made precise. Then, will each cluster point $\vec{x}^*$ of the sequence $(\vec{x}^{*n})$ be a shadow price for Γ? Further, will $u_i^n = u_i^n(\vec{x}^{*n}) \to u_i(\vec{x}^*)$?

Plainly, in asking these questions, there is no ambiguity or choice as to what $(e_i^n, \vec{x}^{*n}, u_i^n) \to (e_i, \vec{x}^*, u_i)$ should mean. Also, $E_i^n \to E_i$ amounts to have the matrix representation of E_i^n converge in each entry to that of E_i. But some care is needed

in defining the appropriate notion of convergence $\pi_i^n \to \pi_i$. We say that a sequence of functions $f^n : X \to \mathbb{R}\cup\{\pm\infty\}$ *epi-converges* to $f : X \to \mathbb{R}\cup\{\pm\infty\}$, and write $f^n \to^{epi} f$, iff
• $\forall x \in X\ \forall x^n \to x$ it holds that $\liminf f^n(x^n) \geq f(x)$ and
• $\forall x \in X\ \exists x^n \to x$ such that $\limsup f^n(x^n) \leq f(x)$.

Proposition 4.3. (**Variational convergence of shadow prices and imputations**) *Suppose* $(e_i^n, E_i^n) \to (e_i, E_i)$, *and*
$\forall i \in I, \forall x_i^* \in \mathbb{X}^*, \forall x_i^{n*} \to x_i^*$ *it holds that* $\liminf \pi_i^{n(*)}(x_i^{n*}) \geq \pi_i^{(*)}(x_i^*)$, *and* $\pi_i^{n(*)}(x_i^*) \to \pi_i^{(*)}(x_i^*)$. *Also suppose each lower level set* $\left\{\pi_i^{(*)} \leq r\right\}$ *is bounded for every* $r \in \mathbb{R}$ *and every* i.

Let $\vec{x}^{*n}$ *be a shadow price of game* $\Gamma^n = (e_i^n, \pi_i^n, E_i^n)_{i\in I}$. *Then each cluster point* $\vec{x}^*$ *of the sequence* $(\vec{x}^{*n})$ *is a shadow price of the unperturbed game* $\Gamma = (e_i, \pi_i, E_i)_{i\in I}$. *Moreover,* $u_i^n = u_i^n(\vec{x}^{*n}) \to u_i = u_i(\vec{x}^*)$ *for each* i.

Proof. Denote by $L_i^n : \mathbb{X}^* \times \mathbb{X}^* \to \mathbb{X}^*$ the linear mapping defined by $L_i^n(x^*, x_i^*) := x^* + x_i^* - E_i^* x_i^*$. Plainly, $E_i^n \to E_i$ implies $E_i^{n*} \to E_i^*$. Thus $L_i^n \to L_i$ for each i. Now define

$$F^n(\vec{x}^*) := \sum_{i\in I} \langle x^*, e_i^n\rangle + \pi_i^{n(*)} \circ L_i^n(x^*, x_i^*) \text{ and } F(\vec{x}^*) := \sum_{i\in I} \langle x^*, e_i\rangle + \pi_i^{(*)} \circ L_i(x^*, x_i^*).$$

Observe that $F^n \to^{epi} F$. Since $\vec{x}^{*n} \in \arg\min F^n$, the conclusion follows from Theorem 7.33 in [26]. $\square$

5. NON-TRANSFERABLE UTILITY

So far arguments hinged upon utility being transferable. This section drops that assumption at the cost of a less constructive approach to core solutions.

As hitherto, by a *price* system is understood a profile $\vec{x}^* := (x^*, x_i^*, i \in I)$ such that $x^* \in \mathbb{X}$ is $\mathcal{F}(e_I)$-measurable, and each $x_i^* \in \mathbb{X}$ is $\mathcal{F}_i$-measurable. For any price system let

$$c_i(\vec{x}^*, x_i) := \langle x^*, x_i\rangle + \langle x_i^*, x_i - E_i x_i\rangle$$

denote the cost incurred by player i when he purchases $x_i \in \mathbb{X}$. Note that $c_i(\vec{x}^*, e_i) = \langle x^*, e_i\rangle$. A feasible price-allocation pair $(\vec{x}^*, \vec{x})$ is called a *Walras equilibrium* if for each i
• $c_i(\vec{x}^*, x_i) \leq \langle x^*, e_i\rangle$, and $\pi_i(x_i') > \pi_i(x_i) \Rightarrow c_i(\vec{x}^*, x_i') > \langle x^*, e_i\rangle$.
It is declared a *quasi-equilibrium* if for each i
• $c_i(\vec{x}^*, x_i) = \langle x^*, e_i\rangle$, and $\pi_i(x_i') \geq \pi_i(x_i) \Rightarrow c_i(\vec{x}^*, x_i') \geq \langle x^*, e_i\rangle$.
A feasible allocation $\vec{x}$ is in the *Core* if no proper coalition $C \subseteq I$ can find another feasible allocation $(x_i')_{i\in C}$ such that $\pi_i(x_i') \geq \pi_i(x_i)$ for each $i \in C$, with at least one inequality strict.

Proposition 5.1. (**Existence of quasi-equilibrium**) *Assume each* π_i *is Lipschitz* *continuous, concave on its effective domain* $dom\pi_i := \{x_i : \pi_i(x_i) > -\infty\}$*, and that the latter set is nonempty compact. Then there exists a quasi-equilibrium.*

Proof. Let $(\vec{x}^{*\delta}, \vec{x}^{\delta})$ be a saddle-point for the Lagrangian

$$L^{\delta}(\vec{x}^*, \vec{x}) := \sum_{i\in I} \{\delta_i \pi_i(x_i) - c_i(\vec{x}^*, x_i) + \langle x^*, e_i\rangle\}.$$

Then

$$\delta_i \left\{\pi_i(x_i^{\delta}) - \pi_i(x_i)\right\} \geq c_i(\vec{x}^{*\delta}, x_i^{\delta}) - c_i(\vec{x}^{*\delta}, x_i). \tag{25}$$

Let $\mathbb{S}(\delta)$ equal the set of all saddle point of L^{δ}, and posit

$$D(\vec{x}^*, \vec{x}) := \{\delta \in \Delta : \delta_i = 0 \text{ if } c_i(\vec{x}^*, x_i) > c_i(\vec{x}^*, e_i)\},$$

Since each π_i is Lipschitz continuous on its domain, so are all functions $(x_i) \mapsto \sum_{i\in I} \delta_i \pi_i(x_i)$ on $K := \Pi_{i\in I} dom\pi_i$ with a modulus that doesn't depend on δ. Consequently, the components of the multiplier vectors $\vec{x}^{*\delta}$, having the nature of a supergradients

$$x^{*\delta} + x_i^{*\delta} - E_i^* x_i^{*\delta} \in \partial \left[\delta_i \pi_i(x_i^{\delta})\right],$$

must be uniformly bounded. This entails that, modulo the transformation $\vec{x}^* \to (x^{*\delta} + x_i^{*\delta} - E_i^* x_i^{*\delta})$, we can restrict $\vec{x}^*$ to belong to a compact convex set K^*. Then $\mathbb{S} \times D$ has a fixed point $(\vec{x}^*, \vec{x}, \delta)$ on the set $K^* \times K \times \Delta$.

We claim that $c_i(\vec{x}^*, x_i) = c_i(\vec{x}^*, e_i)$ for all i. Indeed, if some $c_i(\vec{x}^*, x_i) > c_i(\vec{x}^*, e_i)$, then by construction $\delta_i = 0$, and (25) would yield the contradiction $c_i(\vec{x}^*, x_i) \leq c_i(\vec{x}^*, e_i)$. Consequently, $c_i(\vec{x}^*, x_i) \leq c_i(\vec{x}^*, e_i)$ for all i. But, if some such inequality were strict, there is the contradiction $\sum_{i\in I} c_i(\vec{x}^*, x_i) < \sum_{i\in I} c_i(\vec{x}^*, e_i)$. This proves the claim. Similarly, if $\pi_i(x_i') \geq \pi_i(x_i)$ and $c_i(\vec{x}^*, x_i') < c_i(\vec{x}^*, x_i)$ for some $\mathcal{F}_i$-measurable x_i', then $L^{\delta}(\vec{x}^*, \cdot)$ cannot be maximal at $\vec{x}$. □

Proposition 5.2. (**Walras equilibrium**) *Suppose each* π_i *is lower semicontinuous on its effective domain* $dom\pi_i$ *and that this set is starshaped with respect to* 0. *Then each quasi-equilibrium for which all* $\langle\vec{x}^*, e_i\rangle > 0$, *is a Walras equilibrium.*

Proof. If a quasi-equilibrium $(\vec{x}^*, \vec{x})$ is not a Walras equilibrium, then some agent i has a $\mathcal{F}_i$-measurable x_i' such that $\pi_i(x_i') > \pi_i(x_i)$ and $c_i(\vec{x}^*, x_i') = c_i(\vec{x}^*, e_i)$. Since $dom\pi_i$ is starshaped with respect to 0, we have $rx_i' \in dom\pi_i$ for all $r \in [0, 1]$. By the lower semicontinuity of π_i on its effective domain, for $r < 1$ sufficiently close to 1 we still get $\pi_i(rx_i') > \pi_i(x_i)$ but $c_i(\vec{x}^*, rx_i') < c_i(\vec{x}^*, e_i)$ which contradicts the quasi-equilibrium. □

Proposition 5.3. (**Nonempty core**) *Under the hypotheses of Propositions 5.1-2 there exists a core solution.*

Proof. Pick any quasi-equilibrium $(\vec{x}^*, \vec{x})$. If $\vec{x}$ is not in the core, some proper coalition C has an alternative feasible allocation $(x_i')_{i\in C}$ satisfying $\pi_i(x_i') \geq \pi_i(x_i)$ for all $i \in C$, with at least one inequality is strict. By quasi-equilibrium $c_i(\vec{x}^*, x_i') \geq c_i(\vec{x}^*, e_i)$ for all i. By Walras equilibrium, $c_i(\vec{x}^*, x_i') > c_i(\vec{x}^*, e_i)$ for each strictly improving agent. The upshot is the contradiction $\sum_{i\in C} c_i(\vec{x}^*, x_i') > \sum_{i\in C} c_i(\vec{x}^*, e_i)$. $\square$

6. Some Comments and Examples

Since payment $u_i = \langle x^*, e_i\rangle + \pi_i^{(*)}(x^* + x_i^* - E_i^* x_i^*)$ is convex in $\vec{x}^*$, impacts of changes in measurability become interesting. For the argument suppose first that e_i remains unaltered but let the triple $[x^*, x_i^* - E_i^* x_i^*, \mathcal{F}(e_I)]$ be replaced by a "finer" version $\left[\hat{x}^*, \hat{x}_i^* - \hat{E}_i^* \hat{x}_i^*, \mathcal{F}(\hat{e}_I)\right]$, satisfying $\hat{x}^* + \hat{x}_i^* - \hat{E}_i^* \hat{x}_i^* \neq x^* + x_i^* - E_i^* x_i^*$ and

$$E\left[\hat{x}^* + \hat{x}_i^* - \hat{E}_i^* \hat{x}_i^* \middle| \mathcal{F}(\hat{e}_I)\right] = x^* + x_i^* - E_i^* \hat{x}_i^*.$$

Then, if $\pi_i^{(*)}$ is strictly convex, $\hat{u}_i := \langle \hat{x}^*, e_i\rangle + \pi_i^{(*)}(\hat{x}^* + \hat{x}_i^* - \hat{E}_i^* \hat{x}_i^*) > u_i$. In particular, if player i is propertyless, perfectly informed, and has $\pi_i^{(*)}$ is strictly convex, he is likely to benefit form a refinement of the field $\mathcal{F}(e_I)$. In short, anybody who causes an expansion of $\mathcal{F}(e_I)$ to $\hat{\mathcal{F}}$, seems to create positive externalities each other player who has and maintains $\mathcal{F}_i \subseteq \mathcal{F}(e_I)$.

If e_i changes, there is, of course, a material effect, but possibly also repercussions via the information structure. To better isolate the latter, let i be a pure resource owner. He has $\pi_i^{(*)} \equiv 0$ and gets $u_i = \langle x^*, e_i\rangle$. A pair y_1, y_2 of real-valued random variables, defined on the same probability space, is said to exhibit *negative* (*positive*) *dependence* if

$$\Pr\{y_1 \leq r_1 | y_2 \leq r_2\} \quad \leq (\geq) \quad \Pr\{y_1 \leq r_1\} \cdot \Pr\{y_2 \leq r_2\} \text{ for all real } r_1, r_2,$$

with strict inequality for at least one choice r_1, r_2. When $X = \mathbb{R}^G$ for a finite set G of goods, formula (18) gives u_i. Thanks to (17), upon ignoring possible nonsmoothness, we can, as an instance of the *law of demand*, quite reasonably posit that resource price x_g^* be a decreasing function of total abundance e_{Ig}. Then, if e_{Ig} and e_{ig} are negatively dependent, $cov(x_g^*, e_{ig}) > 0$; see [20] Proposition 16.9. Consequently, if each pair $(e_{Ig}, e_{ig}), g \in G$, shows negative (positive) dependence, agent i experiences some bonus (loss) over the average payment $E(x^*) \cdot E(e_i)$.

It is noteworthy that *the first fundamental theorem of welfare economics* is no longer valid. The reason is that (rational expectation) Walras equilibria, by ascribing value merely to commodities, need not belong to the private core. For example, a propertyless agent i with perfect information structure $\mathcal{F}_i = \{\{s\}\}$ gets production profit $\pi_i^{(*)}(x^*)$. So, provided $\pi_i^{(*)}(x^*) > 0$, he is left with some purchasing power.

Plainly, Walras equilibrium, in giving any propertyless agent zero wealth, nullifies his consumption - irrespective of what information he brings. In contrast, the private core is apt to reward him for information that allow risk averters to write more detailed or diversified contracts. It also deserves mention that Walras equilibrium may fail to exist in cases where the core is nonempty:

Example 6.1. An instance with no Walras equilibrium but nonempty core: Let there be two goods, two players, and two equally likely states. Posit $e_1(s) = (1, 0)$, $e_2(s) = (1, 1)$ in each state s, and

$$\pi_i(x_i) := \begin{cases} Ex_{i,g=i}(s) & \text{if } x_i(s) \in \mathbb{R}^2_+ \text{ for all } s \\ -\infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}_i$ be generated by a perfect partition. Then any allocation that always gives player 1 an amount $\alpha \in [1, 2]$ of the first good - and invariably player 2 all the rest, is in the core. Plainly, $x^*(s) \equiv (1, 1)$ is the unique, and state-independent shadow price, and

$$\pi_i^*(x^*) := \begin{cases} 0 & \text{if } x^*(1) \geq 1 \text{ and } x^*(2) \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Consequently, $u_1 = 1, u_2 = 2$ is a price-generated core imputation. There is however, no competitive equilibrium. Indeed, an equilibrium price vector $p = [p(s)] = [p(1), p(2)]$ cannot have $p(1) = 0$, leaving agent 1 destitute. Further, if $p(1) > 0$, then agent 2 will demand more of good 2 than available. Changing $\mathcal{F}_1$ to $\{\varnothing, S\}$ would not upset this conclusion. □

It must be emphasized that differential information easily comes in the way of good contracts - as illustrated next.

Example 6.2. A case for autarky: Accommodated are two agents, one good, and three states as follows:

Agent i	$\mathbb{P}_i$	state s :	s_1	s_2	s_3
1	$\{s_1\}, \{s_2, s_3\}$	endowment $e_1(s)$:	ξ_1	0	0
2	$\{s_2\}, \{s_1, s_3\}$	endowment $e_2(s)$:	0	ξ_2	0

Posit format (1) with $\Pi_i(s, 0) = 0$ and $x_i(s) \geq 0$, to get $v_i = \Pi_i(s_i, \xi_i)\mu(s_i)$ for each i. Both players get 0 is state s_3. Therefore, by measurability $x_1(s_2) = 0$ and $x_2(s_1) = 0$, - to the effect that no contract becomes possible apart from the autarkic one. Information structures are unequal here but symmetric across players. While both parties might want to write contracts in terms of s_1, s_2, either is unable to disentangle s_3 as a special contingency. Note that $\mathcal{F}_1 \vee \mathcal{F}_2$ amounts to perfect ex post information. □

This example illustrates that the private core is a second-best equilibrium concept. Indeed, Pareto efficiency often does not obtain. The simple reason is that agents might be unable to unveil the pooled information $\vee_{i \in I} \mathcal{F}_i$.

Example 6.3. On the advantage of being informed. Accommodated are two agents, two goods, and two states as follows:

$\underline{Agent\ i}$	$\underline{\mathbb{P}_i}$	state s :	s_1	s_2
1	$\{s_1\},\{s_2\}$	endowment $e_1(s)$:	$(1,0)$	$(0,0)$
2	$\{s_1,s_2\}$	endowment $e_2(s)$:	$(0,2)$	$(0,2)$

The two goods, referred to by second-place subscripts $g=1,2$, are *perfect complements*, i.e. completely useful only when available in equal quantities. Thus

$$\pi_i(x_i)=\sum_s \min\{x_{i1}(s),x_{i2}(s)\}\,\mu(s)$$

with $x_i(s)\geq 0$, to get $v_i=0$ for each i. It is impossible to offer player 2 a positive constant amount of good 1. Thus

$$v_I=\max\left\{\sum_s \min\{x_{11}(s),x_{12}(s)\}\,\mu(s):0\leq x_1(s)\leq e_I(s)\right\}=\mu(s_1).$$

The shadow price x^* on resources is the constant vector $(1,0)$. Thus the price-generated core payments are

$$u_1=\langle x^*,e_1\rangle=\mu(s_1)\ \text{ and }\ u_2=\langle x^*,e_2\rangle=0.$$

When probability measure $\mu(s_1)\leq 0.5$, player 1 is not superior in terms of endowment and technology. But his information advantage allows him to produce the cake - and have it all. □

Example 6.4. On syndication. It is known that players who hold relatively scarce resources may loose by forming a syndicate. It appears though that price-generated core solutions may mitigate this. To wit, following [24], let there be 5 players, one good, and two states:

$\underline{Agent\ i}$	$\underline{\mathbb{P}_i}$	state s :	s_1	s_2
$i=1=2$	$\{s_1,s_1\}$	endowment $e_i(s)$:	1	0
$i=3=4=5$	$\{s_1\},\{s_2\}$	endowment $e_i(s)$:	0	1/2

Posit $\pi_i(x_i):=\min\ \{x_i(s_1),x_i(s_2)\}$ to have $v_S=\min\{|S\cap\{1,2\}|,|S\cap\{3,4,5\}|\}$, and the private core reduces to the single profile $(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Here the resource price $x^*=[x^*(s_1),x^*(s_2)]=[0,1/\mu(s_2)]$, and $u_i=\langle x^*,e_i\rangle$.

If owners of the scarce resource form a syndicate $\{1,2,3\}$, the core becomes larger, and it contains imputations $u_{\{3,4,5\}}<3/2$. However, since syndication does not affect x^*, for the price-generated selection we still get $u_{\{3,4,5\}}(x^*)=\left\langle x^*,e_{\{3,4,5\}}\right\rangle=3/2$. This attests to the competitive nature of formula (16). □

Example 6.5. Superior information. Accommodate here a player who owns nothing but some valuable information:

Agent i	$\mathbb{P}_i$	state s :	s_1	s_2	s_3
1	$\{s_1\},\{s_2,s_3\}$	endowment $e_1(s)$:	2	1	1
2	$\{s_1,s_3\},\{s_2\}$	endowment $e_2(s)$:	1	2	1
3	$\{s_1,s_2\},\{s_3\}$	endowment $e_3(s)$:	0	0	0

Use payoff

$$\pi_i(x_i)=\sum_{s\in S}\left\{x_i(s)-x_i^2(s)/2\right\}$$

with $x_i(s)\geq 0$ for all i,s, to get $v_i=1$ for $i=1,2$, and $v_3=0$. Further, value $v_{\{1,2\}}=\sum_{i=1,2}v_i=2$, is supported by status quo (no trade) because otherwise $x_1(s_1)$ must maximize $\sum_{i=1,2}\{x_i(s_1)-x_i^2(s_1)/2\}$ s.t. $\sum_{i=1,2}x_i(s_1)=e_I(s_1)$ which entails $x_1(s_1)=x_2(s_1)=3/2$. By measurability this implies $x_2(s_3)=3/2$ whence via material balance $x_1(s_3)=1/2$. Finally, again by measurability and material balance, in that order, $x_1(s_2)=1/2$ and $x_2(s_2)=5/2$. But symmetry considerations say $x_1(s_1)=x_2(s_2)$ - so we have a contradiction at hand. Note that either player $i=1,2$ finds it interesting to collude with the utterly poor but somewhat informative agent 3 : For $i=1,2$

$$v_{\{i,3\}}=2>v_i+v_3=1.$$

To explain this, let $C=\{1,3\}$ (the instance $\{2,3\}$ is quite similar) and suppose $x_1(s_1)>0$. Since $x_1(s_1)$ must maximize $\sum_{i\in C}\{x_i(s_1)-x_i^2(s_1)/2\}$ s.t. $\sum_{i\in C}x_i(s_1)=e_I(s_1)$, we get $x_1(s_1)=x_3(s_1)=1$. Considerations of measurability and material balance thereafter yield:

state s :	s_1	s_2	s_3
allocation $x_1(s)$:	1	0	0
allocation $x_3(s)$:	1	1	1

and thereby $v_{\{1,3\}}=2$. Finally, for the grand coalition assume each $x_i(s_i)>0$ to get optimal allocation

state s :	s_1	s_2	s_3
allocation $x_1(s)$:	3/2	1/2	1/2
allocation $x_2(s)$:	1/2	3/2	1/2
allocation $x_3(s)$:	1	1	1

and consequently, $v_I=3.75$. The upshot is that players $1,2$, although they have no genuine reason to collude with each other, find it in their interest to join the grand coalition, this making the propertyless player a right honorable member.

7. CONCLUDING REMARKS

The core, a most popular solution concept of cooperative game theory, occupied center stage here. Moreover, a price-generated selection was made within the core. Such selection points to the first welfare theorem and to various ways of shrinking the core. In fact, to ensure a nonempty or small core, possible avenues include replication of players [8], accommodation of a nonatomic player set [4], convexification of preferences [14], or tolerance for fuzzy coalitions [16].

None of these approaches were pursued here. Instead we simply presumed that aggregate payoff was superdifferentiable at the point of reference. Economic agents - all facing uncertainty but differing in capacities, efficiencies, endowments or informations - can then benefit from cooperation. To bring this out the paper reconsidered coalitional production (or market) games introduced by Shapley and Shubik (1969). When contracts comply with private information, and utility is transferable, explicit core solutions obtain. These are defined in terms of Lagrange multipliers that relate to material resources, information, and production.

As said, existence of appropriate multipliers requires that the perturbed function π (19) be concave at the point of reference. Such concavity could come about via aggregation of a representative agent economy as follows: Let $I := \{1, ..., |I|\}$ and introduce for each $t \in (i-1, i]$, $i \in I$, a player with endowment $e_t = e_i$, upper semicontinuous payoff $\pi_t = \pi_i$, and partition $\mathbb{P}_t = \mathbb{P}_i$. Thus player i becomes a representative for a continuum of identical agents. Introduce next the functions

$$\hat{\pi}_i(x_i) := \sup \left\{ \int_{i-1}^{i} \pi_t(x_t) dt : x_t = E_i x_i \text{ and } \int_{i-1}^{i} x_t dt = x_i \right\}.$$

The functions $\hat{\pi}_i$ so constructed are all concave [29], and

$$\sup \left\{ \int_0^{|I|} \pi_t(x_t) dt : x_t = E_i x_i \text{ and } \int_0^{|I|} x_t dt = e_I \right\}$$
$$= \sup \left\{ \sum_{i \in I} \hat{\pi}_i(x_i) : x_i = E_i x_i \text{ and } \sum_{i \in I} x_i = e_I \right\}.$$

The resulting, "representative" game $\hat{\Gamma} = (e_i, \hat{\pi}_i, \mathbb{P}_i)_{i \in I}$ has a concave perturbed function $\hat{\pi}$ (19), and the preceding analysis applies.

References

[1] B. Allen and N. C. Yannelis, Differential information economies: Introduction, *Economic Theory* 18, 263-273 (2001).

[2] B. Allen, Incentives in market games with asymmetric information: the core, in C. D. Aliprantis et al. (eds.) *Assets, Beliefs, and Equilibria in Economic Dynamics*, Studies in Economic Theory 18, Springer (2004).

[3] J.P. Aubin and I. Ekeland, Estimates of the duality gap in nonconvex optimization, *Mathematics of Operations Research* 1, 225-245 (1976).

[4] R. J. Aumann, Markets with a continuum of traders, *Econometrica* 32, 29-50 (1964).

[5] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization*, SIAM, Philadelphia (2001).

[6] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization*, Springer, Berlin (2000).

[7] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, New York (1991).

[8] G. Debreu and H. Scarf, A limit theorem on the core of an economy, *Int. Economic Review* 4, 235-246 (1963).

[9] E. Einy, D. Moreno and B. Shitovitz, Competitive and core allocation in large economies with differential information, *Economic Theory* 18, 321-332 (2001).

[10] I. Ekeland and R. Teman, *Convex Analysis and Variational Problems*, North-Holland, Elsevier (1976).

[11] I. V. Evstigneev, Lagrange multipliers for the problems of stochastic programming, in M. W. Los et al. (eds.) *Lecture Notes in Econ. and Math. Syst. Springer*, Berlin 133, 34-48 (1976).

[12] I.V. Evstigneev, W.K. Klein Haneveld and L. J. Mirman, Robust insurance mechanisms and the shadow price of information constraints, *J. Applied Math. & Decision Sciences* 3, 1 85-128 (1999).

[13] I.V. Evstigneev and S. D. Flåm, Sharing nonconvex costs, *J. of Global Optimization* 20, 257-271(2001).

[14] S. D. Flåm, G. Owen and M. Saboyá, Large production games and approximate core solutions, Typescript (2005).

[15] D. Glycopantis, A. Muir and N. C. Yannelis, An extensive form interpretation of the private core, *Economic Theory* 18, 293-319 (2001).

[16] F. Hüsseinov, Interpretation of Aubin's fuzzy coalitions and their extensions, *J. Math. Econ.* 23, 499-516 (1994).

[17] A. I. Khinchin, *Mathematical foundations of information theory*, Dover, New York (1957).

[18] T. Kobayashi, Equilibrium contracts for syndicates with differential information, *Econometrica* 48, 7, 1635-1665 (1980).

[19] L. Koutsougeras and N. C. Yannelis, Incentive compatibility and information superiority of the core of an economy with differential information, *Economic Theory* 3, 195-216 (1995).

[20] M. Magill and M. Quinzii, *Theory of Incomplete Markets*, MIT Press (1998).

[21] K. Murota, *Discrete Convex Analysis*, SIAM, Philadelphia (2003).

[22] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*, MIT Press, Cambridge (1994).

[23] G. Owen, On the core of linear production games, *Mathematical Programming* 9, 358-370 (1975).

[24] A. Postlewaite and R. W. Rosenthal, Disadvantageous Syndicates, *J. Economic Theory* 9, 324-326 (1974).

[25] R. T. Rockafellar, *Convex Analysis*, Princeton University Press (1970).

[26] R. T. Rockafellar and J-B. Wets, *Variational Analysis*, Springer, Berlin (1998).

[27] L. S. Shapley and M. Shubik, On market games, *J. Economic Theory* 1, 9-25 (1969).

[28] R. M. Starr, Quasi-equilibria in markets with non-convex preferences, *Econometrica* 37, 15-38 (1969).

[29] M. Valadier, Integration de convexes fermes notamment d'epigraphes inf-convolution continue, *R.I.R.0* 57-73 (1970).

[30] R. Wilson, Information, efficiency, and the core of an economy, *Econometrica* 47, 807-816 (1978).

[31] N. C. Yannelis, The core of an economy with differential information, *Economic Theory* 1, 183-198 (1991).