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Affine Price Expectations and Equilibrium in Strategic Markets

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**Abstract.** This paper considers equilibrium in imperfect markets, featuring agents who exchange property rights. Important cases include trade in emission permits of greenhouse gases, or exchange of catch quotas of fish. Some players act strategically while others are price-takers. The "demand curve" is endogenous, and it affects all parties. The resulting, reduced objectives need not be concave. Therefore, existence of equilibrium is a delicate matter. To simplify things, and to ensure availability of "equilibria up to first order", we presume that all strategic agents form affine price expectations.

**Keywords**: Noncooperative games, Cournot oligopoly, emissions trading, second order optimality conditions.
*JEL Classification*: C72, L13, Q50.

## 1. Introduction

Economic theory has given some pre-eminence to large, anonymous markets for price-takers, each participant immediately knowing at precisely which places he will be a net supplier. Modern economies feature however, many markets that share neither of these properties. Indeed, some important settings comprise fairly few, non-anonymous, and price-affecting parties - many of whom must reason a bit before knowing on what side of the counter they will stand.

Included are exchange markets for various fish quotas. There, before making their bids, owners of significant fish resources might deliberate on how own supply/demand will affect clearing prices. Also important are the emerging markets for emission rights of greenhouse gases.

Two issues complicate modelling and analysis of such situations. First, unlike the Cournot oligopoly, there is no "demand curve." That object must rather be derived endogenously. Second, the said curve may affect players in ways that upset desirable curvature properties. To wit, some reduced objectives, after incorporation of the demand curve, could become non-concave in proper decisions. And then, as is well known, existence of pure Nash equilibrium is hard to ensure.

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Reflecting on this problem, the present paper has two parts, following after the model set-up in Sections 2 and 3. The leading and larger part, Section 4, while somewhat pessimistic in tenor, stresses the difficulties in getting hold of a full-fledged Nash Equilibrium. The second and smaller and decidedly more optimistic part, Section 5, brings out that if players contend with affine price expectations, then indeed, there are situations that comply with equilibrium up to first-order optimality.

For ease of interpretation - and comparison - the main story is phrased as dealing only with trade of emissions permits.[3] Montgomery [13] first modelled such markets and proved existence of equilibrium, assuming that all agents were price-takers. That assumption was later relaxed by Hahn [9] who accommodated *one* dominant firm. His study spurred a large literature on permit markets, with various extensions, including Westskog's [18] oligopoly case.[4]

That literature inspires at least *two* queries. *First*, on a technical note, revenue or cost functions are commonly assumed smooth - say, at least twice continuously differentiable. As a matter of fact, reduced functions, generated by optimization - and notably by linear programs - are often merely piecewise differentiable.[5] Admittedly, differentiability might be seen mainly as a technical issue, carrying less of economic substance. But, in any case, it had better be dealt with. For example, the slope of the demand curve, so crucial in optimality conditions, how can it be identified? Indeed, even if all objective functions were indeed differentiable, one cannot claim that derived demand be likewise.[6] *Second*, one can hardly contend with merely characterizing equilibrium, leaving questions about existence and uniqueness somewhat in the lurk.

This paper looks into both these issues. While subscribing to convex analysis, it accommodates non-smooth functions and explores existence and uniqueness of equilibrium. If agents must form perfect opinions (rational expectations) about entire price curves, results on existence of equilibrium are non-conclusive and not very encouraging. As said, chief concerns are with the resulting curvature of objective functions. Besides, special provisions are needed for either side of the permit market.

On a decidedly more positive note we offer a convenient and constructive - and, in our opinion, quite natural - escape from those complexities. It comes by assuming that players behave *as though* the price curve were globally affine.[7] This assumption amounts, on the psychological side, to relieve players from much cognitive effort - and, on the mathematical side, to simplify analysis considerably. The price to be

[3] Adaptations to quota markets are straightforward.

[4] In the emerging market for greenhouse gas emissions permits under the Kyoto Protocol, a broad literature (see e.g. Springer [17]) suggests that trade will occur between relatively few parties. This provided part of the motivation for the present study.

[5] This feature is dealt with in the literature that concerns *estimation* of emissions (abatement) cost functions. For the case of greenhouse gases, see e.g. [4].

[6] So, we hesitate in presuming as much smoothness as do, for example, Kolstad and Mathiesen [11]. A constructive and usefull exception is the study of Murphy et al. [14].

[7] If it really is - as in linearly constrained, quadratic programming - then so much the better. Otherwise, how strategic traders preceive price curves is an issue for experimental economics and cognitive physcology.

paid is that the resulting solution concept might produce spurious "equilibria."[8]

## 2. The Setting

Accommodated here is a fixed, finite set $I$ of economic agents, each construed as a cost minimizing producer. These agents are collectively constrained (say, by various treaties) to keep the aggregate emissions of a finite group $G$ of pollutants below specified levels.

Individual $i \in I$ is endowed with a permit vector $e_i = (e_{ig}) \in \mathbb{R}^G$. We use the convention that $e_{ig} > 0$ implies the right, on the part of $i$, to emit this amount of "gas" $g \in G$ into global commons. Similarly $e_{ig} \leq 0$ accounts for his obligation to absorb/capture $|e_{ig}|$ from the environment.[9]

Clearly, agent $i$ may want a permit pattern $x_i \in \mathbb{R}^G$ that differs from $e_i$. In the aggregate however, such demand cannot exceed supply, i.e., $\sum_{i \in I} x_i \leq \sum_{i \in I} e_i$, this inequality being understood to hold component-wise. Permits are homogeneous, non-storable, perfectly divisible and exchanged in a common market where a price vector $p \in \mathbb{R}^G$ prevails.

Some agents are particularly well endowed; others are plagued by significant shortages. Such "large" agents presumably act strategically à la Cournot, accounting for how their choice of quantity will influence prices. Others simply take prices for granted; they are minor, and, most likely, they figure as strategic dummies. Thus, each agent $i \in I$ belongs either to a nonempty *competitive fringe* $F \subset I$ of *price-takers* or to a complementary society $S \subset I$ of *strategists*. Since presumably no one can figure both "small" and "large" - or both as dummy and strategist - at the same time, $I$ is the disjoint union $F \cup S$, $F \neq \emptyset$.

Agent $i$ incurs convex cost $c_i(x_i)$ by keeping $x_i$ for personal use.[10] Technological restrictions, capacity limits and other constraints will of course affect costs. It is convenient therefore, both in notation and analysis, to account for these by means of infinite penalties. Thus $c_i(x_i) = +\infty$ iff $x_i$ is infeasible. This device save us repeated mention of evident or implicit constraints (for example non-negativity).

Producers interact over two stages in Stackelberg manner as follows: "First", each strategist $i \in S$ chooses to his heart the permit bundle $x_i$ he wants to retain for himself. "Second", the fringe members regard the residual quantity

$$Q := \sum_{i \in I} e_i - \sum_{i \in S} x_i$$

as total supply, to be allocated via perfect competition. Thereby they minimize their

[8] If any, most of those might not survive posterior analysis - or pertubations.

[9] For any vector $v = (v_g) \in \mathbb{R}^G$, when writing $v \geq 0$, we mean that $v_g \geq 0$ for all $g \in G$. The inner product on $\mathbb{R}^G$ is defined by $v \cdot w := \sum_{g \in G} v_g w_g$.

[10] When instead payoff $\pi_i(x_i)$ obtains from input $x_i$, posit $c_i(x_i) := -\pi_i(x_i)$.

aggregate cost:

$$c_F(Q) := \inf\left\{\sum_{i\in F} c_i(x_i) : \sum_{i\in F} x_i \le Q\right\}. \tag{1}$$

Associated to *primal problem* (1) is a *dual*, aimed at appropriate pricing, namely: Maximize the reduced function

$$\inf_{x_F}\left\{\sum_{i\in F} c_i(x_i) + p\cdot(\sum_{i\in F} x_i - Q)\right\} \tag{2}$$

with respect to $p \in \mathbb{R}^G$. Here $x_F := (x_i)_{i\in F}$ is the allocation of permits across the fringe, and the dot signals the standard inner product. To relate problem (1) to (2) recall that the *Fenchel conjugate*

$$c^*(p) := \sup_x \{p\cdot x - c(x)\} \tag{3}$$

of a cost function $c : \mathbb{R}^G \to \mathbb{R}\cup\{+\infty\}$ reports the competitive profit under price $p$. Now, any vector $p \in \mathbb{R}^G$ that makes the dual value (2) coincide with the primal value (1) is declared a *shadow price* at $Q$, and we write $p = P(Q)$. Thus emerges an implicitly defined, vector-valued, inverse "market curve" $Q \mapsto P(Q)$, bound to become a main object here below. Now is the best time to list some properties of the primal-dual problem pair (1)-(2). The following, which derive from standard convex analysis, are stated without proof.[11]

**Proposition 1** (On fringe costs, duality and shadow prices)
• (Attainment of fringe cost) *If the essential objective in* (1) *is inf-compact, meaning that the lower level set*

$$\left\{(x_i)_{i\in F} : \sum_{i\in F} c_i(x_i) \le r \text{ and } \sum_{i\in F} x_i \le Q\right\}$$

*is compact for every real number $r$, then the infimal value $c_F(Q)$ in* (1) *is attained. In that case, $c_F$ is a proper function (that is, finite in at least one point and never $-\infty$).*
• (Preservation of convexity and lower semicontinuity) *If all terms $c_i$ in* (1) *are convex, lower semicontinuous (lsc for short), then so is $c_F$.*
• (Bi-conjugacy and attainment of fringe cost) *If $c_F$ is convex lsc, and all $c_i^*, i \in F$, are finite and continuous at a common $p$, then $c_F^{**} = c_F$, and the minimum in* (1) *is attained.*
• (Shadow prices are dual optimal) *Any shadow price solves problem* (2) *optimally.*

[11] Particularly useful and informative is the material on infimal convolution in Laurent [12].

*Moreover, given existence of at least one shadow price $p = P(Q)$, the optimal value of problem* (2) *equals $c_F(Q)$.*
• (Shadow prices are negative subgradients) $p = P(Q)$ *is a shadow price iff*

$$c_F(\hat{Q}) \geq c_F(Q) - p \cdot (\hat{Q} - Q) \text{ for all } \hat{Q}. \tag{4}$$

*This happens iff $-p$ is a* ***subgradient*** *of $c_F$ at $Q$, that is, $-p \in \partial c_F(Q)$. Moreover, if $p = P(Q)$ is a shadow price and $c_F(Q)$ is attained at $x_i, i \in F$, then $-p \in \partial c_i(x_i)$ for all $i \in F$, meaning that "marginal" costs are equal across the fringe.*
• (Finite fringe cost and existence of shadow prices) *Existence of a shadow price $p = P(Q)$ is ensured whenever $c_F$ in finite-valued near $Q$ and convex.*
• (Prices slope downwards) (4) *implies the* (*implicit*) ***law of demand***:

$$-p \in \partial c_F(Q), -\hat{p} \in \partial c_F(\hat{Q}) \Rightarrow (p - \hat{p}) \cdot (Q - \hat{Q}) \leq 0. \ \square$$

Focus is throughout on strategic interaction. At this point however, since fringe members are price-takers, a semi-collusive feature might be noted as follows. Given total supply $\mathbb{S} = \sum_{i \in S}(e_i - x_i)$ from the strategists, suppose fringe member $i$ demands $d_i$ with $\sum_{i \in F} d_i = \mathbb{S}$. Then, for any nonempty subset $\mathcal{F} \subseteq F$ of fringe members define its (coalitional) cost

$$c_{\mathcal{F}}(Q_{\mathcal{F}}) := \inf \left\{ \sum_{i \in \mathcal{F}} c_i(x_i) : \sum_{i \in \mathcal{F}} x_i \leq Q_{\mathcal{F}} \right\}$$

where $Q_{\mathcal{F}} := \sum_{i \in \mathcal{F}}(e_i + d_i)$. Observe that $Q_F = Q$. Thus emerges a cooperative transferable-utility (market or) *production game* inside the fringe. From [5] one immediately derives

**Proposition 2** (Shadow prices generate core solutions) *For any shadow price $p = P(Q)$ and demand profile $d_i, i \in F$, the imputations*

$$i \in F \mapsto \gamma_i := -c_i^*(-p) - p \cdot (e_i + d_i)$$

*belongs to the core of the game having characteristic function $\mathcal{F} \mapsto c_{\mathcal{F}}(Q_{\mathcal{F}})$. That is,*

$$\begin{array}{ll} \sum_{i \in F} \gamma_i = c_F(Q_F) & \text{and} \\ \sum_{i \in \mathcal{F}} \gamma_i \leq c_{\mathcal{F}}(Q_{\mathcal{F}}) & \text{for each } \mathcal{F} \subset F. \ \square \end{array}$$

Take hereafter existence of inverse demand *correspondence* $Q \to P(Q)$ for granted. Note that $P(Q)$ is a closed convex *set*, depending upper semicontinuously on $Q$. Plainly, a multi-valued price curve becomes non-tractable in analysis. We need that $Q \to P(Q)$ be a single-valued *function* (whence continuous). This motivates the following

**Definition 1** (Essential smoothness and strict convexity) *A proper convex function*

*$c$ is **essentially smooth** if it is differentiable on the interior of its proper domain $domc := \{x : c(x) < +\infty\}$ and $\|\nabla c(x)\| \to +\infty$ whenever $x$ approaches the boundary of that domain. Furthermore, $c$ is **essentially strictly convex** if strictly convex on every convex subset of $\{x : \partial c(x) \neq \emptyset\}$.* $\square$

Suppose henceforth that some $c_i, i \in F$, is monotone decreasing (i.e., non-increasing) with respect to the customary order on $\mathbb{R}^G$, that is, $x_{ig} \leq x'_{ig}$ for all $g \Rightarrow c_i(x_i) \geq c_i(x'_i)$.

**Proposition 3** (Concerning mainly single-valuedness and continuity of inverse demand)

- (On fringe profit) *When some function $c_i$, $i \in F$, is monotone decreasing, the Fenchel conjugate $c_F^*$* (3) *of the function $c_F$ defined in* (1) *is given by $c_F^* = \sum_{i \in F} c_i^*$, that is, the aggregate profit of the fringe under any price regime equals the sum of the members' optimal, price-taking profits.*
- *If at least one $c_i^*, i \in F$, is essentially strictly convex, then so is $c_F^*$.*
- *If $c_F^*$ is strictly convex with all $c_i$, $i \in F$, lsc convex proper, and $c_F$ is lsc proper, then $c_F$ becomes continuously differentiable on the interior of its domain, and its derivative equals $-p$ there, i.e. $c'_F(Q) = -P(Q)$.*
- *If $c_F^*$ is essentially strictly convex, then $c_F$ is essentially smooth.*

**Proof.** The monotonicity assumption entails that the constraint in (1) can be replaced by $\sum_{i \in F} x_i = Q$. Now the first bullet follows from Rockafellar [16, Theorem 16.4]. The second bullet is trivial, and the third follows from Hirriart-Urruty and Lemaréchal [10, §X, Theorem 4.1.1, p. 79]. For the fourth and last, see Borwein and Lewis [2, Theorem 4.2.5, p. 78]. $\square$

To make good use of Propositions 1 and 3 we shall invoke some

**Standing assumptions:** (On the properties of the cost and profit functions)

- *Each agent $i$ incurs lsc convex cost $c_i(x_i)$. Furthermore, each effective domain*

$$X_i := domc_i := \{x_i : c_i(x_i) < +\infty\}$$

*is nonempty compact and $c_i$ is monotone decreasing on $X_i$.*

- *The profit function $c_i^*(\cdot)$ is essentially strictly convex for at least one $i \in F$.*
- *There exists at least one price $p$ at which all profit functions $c_i^*$, $i \in F$, are finite-valued and continuous.*
- *At any such $p = P(Q)$, the cost function $c_F$ is finite near $Q$.* $\square$

## 3. The game with perfect price expectations

Under the above standing assumptions a unique market clearing permit price $p = P(Q) \in \mathbb{R}^G$ does indeed exist for each $Q$. We posit that each agent is so clever and well informed as to foresee the upcoming $p$. To begin with we make an even stronger

**Assumption on rational price expectations:** *Until further notice posit that each strategist knows the entire function $Q \mapsto P(Q)$. Also presume that this fact is common knowledge among the strategists.*

Admittedly, with $I$ small, the fringe agents behave naively indeed, ignoring their market power and acting as non-strategic dummies. The following notion of equilibrium reflects this feature:

**Definition 2** (Cournot-Nash equilibrium) *A feasible emission profile $(\bar{x}_i)_{i\in I}$ constitutes a* ***Cournot-Nash equilibrium*** *iff $\bar{x}_i$ then minimizes agent $i$'s cost plus his proceeds from sale. That is, $\bar{x}_i$ minimizes*

$$\left.\begin{array}{ll} c_i(x_i) + P(\sum_{i\in I} e_i - \sum_{j\in S\diagdown i} \bar{x}_j - x_i)\cdot(x_i - e_i) & \textit{when } i \in S \\ c_i(x_i) + P(\sum_{i\in I} e_i - \sum_{j\in S} \bar{x}_j)\cdot(x_i - e_i) & \textit{otherwise.} \end{array}\right\} \tag{5}$$

*Here $P(Q) = -\partial c_F(Q)$.* $\square$

## 4. Existence, Characterization, and Uniqueness of equilibrium

As said, existence of equilibrium is a delicate matter, to be discussed first:

### 4.1. Existence of equilibrium.

Concerning this crucial issue the following result serves as a benchmark.

**Proposition 4** (Existence in case of convex preferences) *Suppose each strategist $i \in S$ faces a quasi-convex or uni-modal*[12] *objective* (5). *Then there exists a Cournot-Nash Equilibrium.*

**Proof.** The best response $B_i(x_{-i})$ of any agent $i \in I$ to the choice profile $x_{-i} := (x_j)_{j\neq i}$ committed by his "rivals" is a nonempty closed convex subset of $X_i = domc_i$. Moreover, the correspondence $x_{-i} \rightsquigarrow B_i(x_{-i})$ has closed graph. The existence of a *fixed point* $(x_i) = x \in B(x) := \Pi_{i\in I} B_i(x_{-i})$ now follows from Kakutani's theorem. Any such point is an equilibrium. $\square$

The key hypothesis in Proposition 4 is hard to defend.[13] Instead, one may be tempted to avoid it by means of randomized strategies. Glicksberg's theorem [8] then gives:

**Proposition 5** (Existence of mixed equilibrium) *Suppose players randomize over pure strategies while seeking to minimize expected costs. Then there exists a Cournot-Nash Equilibrium in mixed strategies.* $\square$

[12] *Uni-modal* means here that the set of minimizing $x_i$ is convex.

[13] If, as in Montgomery [13], *all* players were price-takers, existence would obtain easily.

As said, it's hard to argue for convex preferences. But plainly, it seems equally, if not more, difficult to justify mixed equilibria. So, the rest of this section falls back to explore whether strategists' preferences might reasonably be convex. A main result in this subsection is that affine price curves pose no problems with existence of Nash equilibrium. If the reader prefers to accept that assertion, he may move on to the next subsection 4.2 without loosing main arguments.

To identify sufficient conditions for

$$x_i \mapsto P(Q) \cdot (x_i - e_i) \tag{6}$$

to be convex in $x_i$, it is convenient first to discuss some curvature properties at the aggregate level. For this purpose denote by

$$Z := \sum_{i \in S} (x_i - e_i) = \sum_{i \in F} e_i - Q$$

the total net amount of permits bought by the strategists and name

$$\mathbb{P}(Z) := P(\sum_{i \in F} e_i - Z)$$

the *residual price function.* Assume henceforth that $\mathbb{P}(Z)$ be increasing componentwise in $Z$. We shall need that the total market expenditure $\mathbb{P}(Z) \cdot Z$ incurred by the strategists (henceforth named the *industry*), be convex:

**Lemma 1** (The properties of industry expenditure) *The expenditure function $\mathbb{P}(Z) \cdot Z$, is convex in $Z$, if one of the following holds.*
*(i) $\mathbb{P}(Z)$ is affine in $Z$,*
*(ii) $\mathbb{P}(Z)$ is convex (concave) in $Z$, and $Z \geq 0$ ($\leq 0$), that is, the strategists are net buyers (sellers) of permits,*

**Proof.** Let $\alpha, \hat{\alpha} \in (0, 1)$ be arbitrary but satisfy $\alpha + \hat{\alpha} = 1$. Pick any feasible $Z$, $\hat{Z}$ and define

$$\begin{aligned} \sigma &:= \mathbb{P}(\alpha Z + \hat{\alpha}\hat{Z}) \cdot (\alpha Z + \hat{\alpha}\hat{Z}) \text{ and} \\ \mu &:= \alpha \mathbb{P}(Z) \cdot Z + \hat{\alpha}\mathbb{P}(\hat{Z}) \cdot \hat{Z}. \end{aligned}$$

Industry expenditure $\mathbb{P}(Z) \cdot Z$ will be convex in $Z$ iff $\mu \geq \sigma$. Otherwise,

$$\begin{aligned} 0 &< \sigma - \mu \\ &= \mathbb{P}(\alpha Z + \hat{\alpha}\hat{Z}) \cdot (\alpha Z + \hat{\alpha}\hat{Z}) - \alpha \mathbb{P}(Z) \cdot Z - \hat{\alpha}\mathbb{P}(\hat{Z}) \cdot \hat{Z}. \\ &= \mathbb{P}(\alpha Z + \hat{\alpha}\hat{Z}) \cdot (\alpha Z + \hat{\alpha}\hat{Z}) \\ &- \alpha \mathbb{P}(Z) \cdot (\alpha Z + \hat{\alpha} Z + \hat{\alpha}\hat{Z} - \hat{\alpha}\hat{Z}) \\ &- \hat{\alpha}\mathbb{P}(\hat{Z}) \cdot (\alpha Z - \alpha Z + \alpha \hat{Z} + \hat{\alpha}\hat{Z}), \end{aligned}$$

which after a bit of algebra

$$\left.\begin{array}{rl} = & \left[\mathbb{P}(\alpha Z+\hat{\alpha}\hat{Z})-\alpha\mathbb{P}(Z)-\hat{\alpha}\mathbb{P}(\hat{Z})\right]\cdot(\alpha Z+\hat{\alpha}\hat{Z}) \\ - & \alpha\hat{\alpha}\left[\mathbb{P}(Z)-\mathbb{P}(\hat{Z})\right]\cdot\left[Z-\hat{Z}\right] =: \gamma \end{array}\right\} \tag{7}$$

Thus $\sigma-\mu>0$ implies $\gamma>0$. Since $\mathbb{P}(Z)$ is increasing in $Z$, the second line in (7) is negative. Thus,
$(i)$ if $\mathbb{P}(Z)$ is affine in $Z$, the first term in (7) is zero, hence $\gamma>0$ is a contradiction.
$(ii)$ If $\mathbb{P}(Z)$ is convex (concave) in $Z$, the first part of the first line of (7) is negative (positive) respectively. Hence if $Z,\hat{Z}$ are positive (negative) in each coordinate, this furnishes a contradiction once again. □

Note that whenever $\mathbb{P}(Z)$ is *strictly* increasing in $Z$, the second line in (7) is *strictly* negative. It is therefore obvious that when $(i)$ or $(ii)$ are satisfied, $\mathbb{P}(Z)\cdot Z$ is *strictly* convex when $\mathbb{P}(Z)$ is *strictly* increasing in $Z$.

Lemma 1 does not preclude strategic behavior on either side of each market. That diversity motivates a closer scrutiny of the curvature properties of the individual market expenditure $\mathbb{P}(Z)\cdot z_i$ with $z_i := x_i - e_i$ for strategists $i\in S$. Let $Z_{-i} := Z - z_i$ denote what the other strategic agents demand in aggregate.

**Lemma 2** (On the properties of individual expenditure of strategists) *For each $i\in S$, his expenditure $\mathbb{P}(z_i+Z_{-i})\cdot z_i$ is convex in $x_i$ (and $z_i$) in each of the following two cases:*
$(i)$ *The assumptions of Lemma 1 are satisfied such that $\mathbb{P}(Z)\cdot Z$ is convex and any one of the following conditions hold:*
- *a) $\mathbb{P}$ is affine,*
- *b) $\mathbb{P}$ is concave, and $Z_{-i}\geq 0$,*
- *c) $\mathbb{P}$ is convex, and $Z_{-i}\leq 0$.*

$(ii)$ *$\mathbb{P}$ is increasing and at least one of the following holds*
- *a) $\mathbb{P}$ is affine,*
- *b) $\mathbb{P}$ is concave, and $(x_i-e_i)\geq 0$,*
- *c) $\mathbb{P}$ is convex, and $(x_i-e_i)\leq 0$.*

**Proof.** Fix any strategist $i\in S$. Pick arbitrary $\alpha,\hat{\alpha}\in(0,1)$ such that $\alpha+\hat{\alpha}=1$. Select any two feasible $x_i$, $\hat{x}_i$. Let $z_i := x_i-e_i$, $\hat{z}_i := \hat{x}_i - e_i$. Define

$$\begin{array}{rl} \beta & := \mathbb{P}(\alpha z_i+\hat{\alpha}\hat{z}_i+Z_{-i})\cdot(\alpha z_i+\hat{\alpha}\hat{z}_i), \text{ and} \\ \theta & := \alpha\mathbb{P}(z_i+Z_{-i})\cdot z_i+\hat{\alpha}\mathbb{P}(\hat{z}_i+Z_{-i})\cdot\hat{z}_i. \end{array}$$

The assertion follows if $\beta\leq\theta$. Assume on the contrary that

$$\left.\begin{array}{rcl} 0 & < & \beta-\theta \\ & = & \mathbb{P}(\alpha z_i+\hat{\alpha}\hat{z}_i+Z_{-i})\cdot(\alpha z_i+\hat{\alpha}\hat{z}_i)-\theta \\ & = & \mathbb{P}(\alpha z_i+\hat{\alpha}\hat{z}_i+Z_{-i})\cdot(\alpha z_i+\hat{\alpha}\hat{z}_i+Z_{-i}) \\ & - & \mathbb{P}(\alpha z_i+\hat{\alpha}\hat{z}_i+Z_{-i})\cdot Z_{-i}-\theta. \end{array}\right\} \tag{8}$$

$(i)$ By convexity of industry expenditure granted by Lemma 1, the last expression is

$$\begin{array}{ll} \leq & \alpha\mathbb{P}(z_i + Z_{-i}) \cdot (z_i + Z_{-i}) + \hat{\alpha}\mathbb{P}(\hat{z}_i + Z_{-i}) \cdot (\hat{z}_i + Z_{-i}) \\ - & \mathbb{P}(\alpha z_i + \hat{\alpha}\hat{z}_i + Z_{-i}) \cdot Z_{-i} - \theta \\ =: & Z_{-i} \cdot \phi, \end{array}$$

where $\phi = \alpha\mathbb{P}(z_i + Z_{-i}) + \hat{\alpha}\mathbb{P}(\hat{z}_i + Z_{-i}) - \mathbb{P}(\alpha z_i + \hat{\alpha}\hat{z}_i + Z_{-i})$. Hence, if $\beta > \theta$, then $Z_{-i} \cdot \phi > 0$. This furnishes a contradiction when any one of $(i)$ $a) - c)$ are satisfied. It also follows immediately that $\mathbb{P} \cdot (x_i - e_i)$ is strictly convex if under any one $(i)$ $a) - c)$ industry revenue is strictly convex
$(ii)$ By expanding (8) as in (7) we get that

$$\begin{array}{rcl} 0 & < & \beta - \theta \\ & = & -\phi \cdot (\alpha z_i + \hat{\alpha}\hat{z}_i) \\ & - & \alpha\hat{\alpha}[\mathbb{P}(z_i + Z_{-i}) - \mathbb{P}(\hat{z}_i + Z_{-i})] \cdot [z_i - \hat{z}_i] \\ & =: & \xi. \end{array}$$

Thus, $\beta > \theta$ implies $\xi > 0$. Since $\mathbb{P}$ is increasing in $x_i$, we get $\alpha\hat{\alpha}[\mathbb{P}(z_i + Z_{-i}) - \mathbb{P}(\hat{z}_i + Z_{-i})] \cdot [z_i - \hat{z}_i] \geq 0$. Hence $\xi > 0$ is a contradiction whenever $\phi \cdot (\alpha z_i + \hat{\alpha}\hat{z}_i) \geq 0$. The latter is clearly satisfied under setting $(ii)$ when at least one among conditions $a) - c)$ hold. In that same setting, if $\mathbb{P}$ is monotone increasing in $x_i$, then $\mathbb{P} \cdot (x_i - e_i)$ is strictly convex under any one condition $a) - c)$. $\square$

Note that if all other strategic agents than $i$ are net buyers of permits in aggregate, then $Z_{-i} \geq 0$ for agent $i$ if he buys less than the other strategic agents or is a permit seller. On the contrary if the other strategic agents are net sellers of permits in aggregate, then $Z_{-i} \geq 0$ is true if agent $i$ sells more permits than the other strategists in aggregate. The converse is of course true for $Z_{-i} \leq 0$.

After all these preparations comes a first main result:

**Theorem 1** (Existence and of Cournot-Nash equilibrium) *Assume that the conditions of Lemma 2 are satisfied and that all strategy sets are nonempty compact convex. Then there exists a Cournot-Nash equilibrium.*

**Proof.** Each agent $i$, whether strategist or price-taker, has an objective which is convex in own variable and jointly continuous. The fringe members can be "eliminated" from the game. They only serve to generate a well-behaved market curve $Q \mapsto P(Q)$. In the reduced game, featuring only strategists, since strategy sets are of the desirable sort and objectives are bona fide, equilibrium existence follows from Borwein and Lewis [2, p. 206]. $\square$

**4.2. Characterization of equilibrium.** Granted existence, equilibrium calls for closer scrutiny. Of particular importance is characterization and uniqueness of equilibrium. For those purposes, some additional properties of inverse demand merit to be brought out:

**Lemma 3** (The slope of the market curve) *Given $Q$ suppose $(p, x_F)$ solves problem (2) such that locally around $(p, x_F)$, all functions $\sum_{i\in F} c_i^*$and $c_i$, $i \in F$, are twice continuously Fréchet-differentiable. Suppose also that $\left(\sum_i c_i^*(p)\right)''$ ,$[c_i''(x_i)]$, $i \in F$, are all non-singular there. Then, $P(\cdot)$ is differentiable at the given $Q$ and*

$$P'(Q) = \left[\sum_{i\in F} [c_i''(x_i)]^{-1}\right]^{-1}$$

*where $P' := \frac{\partial P}{\partial x_i} = -\frac{\partial P}{\partial Q}$, $i \in S$. If in addition, $\left(\sum_i c_i^*(p)\right)''$, $[c_i''(x_i)]$ are continuous in a neighborhood of $(p, x_F)$ then $P'$ is continuous in a neighborhood of $Q$.*

**Proof.** From the Theorem of Crouzeix [3], and by applying the envelope theorem on (1) respectively, we have $(c_F^*)''(p) = [c_F''(Q)]^{-1} = [P'(Q)]^{-1}$. Thus,

$$\left.\begin{array}{rcl} P'(Q) & = & \left[(c_F^*)''(p)\right]^{-1} = \left[\left(\sum_{i\in F} c_i^*(p)\right)''\right]^{-1} \\ & = & \left[\sum_{i\in F} c_i^{*\prime\prime}(p)\right]^{-1} = \left[\sum_{i\in F} [c_i''(x_i)]^{-1}\right]^{-1}. \end{array}\right\} \tag{9}$$

The second equality in the first line of (9) comes from the first bullet of Proposition 2, while the second equality in the second line results from applying Crouzeix [3] once again. □

The differentiability assumption in Lemma 3 is not particularly attractive - and worrisome if cost functions were "piecewise linear". One way around this obstacle could be opened by regularizing the cost functions by integral of infimal convolution using smooth enough kernels. With Lemma 3 in vigor, the following obtains forthwith:

**Proposition 6** (Characterization of Cournot-Nash equilibrium) *Under the assumptions of Theorem 1, together with those of Lemma 3, a Cournot-Nash equilibrium is characterized by,*

$$\left.\begin{array}{l} p = P(Q) \in -\partial c_i(x_i) \quad \text{for all} \quad i \in F, \; Q = \sum_{i\in I} e_i - \sum_{i\in S} x_i, \text{ and} \\ p \in -\partial c_i(x_i) - P'(Q)\cdot(x_i - e_i) \quad \text{for all} \quad i \in S, \end{array}\right\} \tag{10}$$

*where $P'(Q) = \left[\sum_{i\in F} [c_i''(x_i)]^{-1}\right]^{-1}$.*

**Proof.** This follows immediately form the first order optimality conditions of (5) and from Lemma 3. □

**4.3. Uniqueness of equilibrium.** For the purpose of uniqueness, we shall follow the lines of Murphy et al. [14], by defining an auxiliary **equilibrating problem**

related to (5). It is parametric in nature and assumes the following form:

$$\left.\begin{array}{l}\min_x \sum_{i\in I}\{c_i(x_i)+p\cdot(x_i-e_i)\}+\frac{1}{2}\sum_{i\in S}(x_i-e_i)\cdot p'(x_i-e_i)\\ \text{subject to } Q=\sum_{i\in I}e_i-\sum_{i\in S}x_i.\end{array}\right\}\quad\text{(EP)}$$

In (EP) $Q\in\mathbb{R}^G$ is taken as a datum and $p=P(Q)$, $p'=P'(Q)$.

Suppose that all $c_i(\cdot)$ besides already being convex also are continuously differentiable in the relevant domain. Then, $x:=(x_i)_{i\in I}$ solves (EP) if and only if there exist a multiplier vector $\lambda\in\mathbb{R}^G$ associated to the material balance in (EP) such that

$$\left.\begin{array}{l}0=c_i'(x_i)+p \text{ for all } i\in F, \text{ and}\\ \lambda=c_i'(x_i)+p+p'(x_i-e_i) \text{ for all } i\in S.\end{array}\right\}\quad(11)$$

**Lemma 4** (Equilibrium amounts to vanishing multipliers)
• *Let $Q$ be such that the optimal solution $(x_i)_{i\in I}$ to* (EP) *yields $\lambda=0$, in* (11). *Then, $(x_i)_{i\in I}$ is a Cournot-Nash equilibrium. Conversely, if $(x_i)_{i\in I}$ is an equilibrium solution, then $(x_i)_{i\in I}$ solves* (EP) *for $Q=\sum_{i\in I}e_i-\sum_{i\in S}x_i=\sum_{i\in F}\bar{x}_i$, $p=P(Q), p'=P'(Q)$, and $\lambda=0$.*
• *Suppose that the additional condition of Lemma 3 holds, then $\lambda(Q)$ and $x_i(Q), i\in S$, are upper semi-continuous correspondences in $Q$ with compact values. In particular, under uniqueness they depend continuously on $Q$.*

**Proof.** For the first bullet simply note that when $\lambda=0$, (11) comprises all first-order necessary optimality conditions that go along with interior solutions to problems (5). When appropriate convexity prevails as well, the same conditions are also sufficient. The last bullet follows directly for $x_i(Q)$ when applying Berge's Maximum Theorem (see e.g. Aliprantis and Border [1, Theorem 14.30]) to problem (EP). By that same theorem, it follows that the dual function associated to problem (EP) is continuous, whence so is its maximizer $\lambda(Q)$. □

Clearly, for the purpose of uniqueness, one would want the relation $Q\mapsto\lambda(Q)$ to be monotonous, which would also fit the price interpretation of $\lambda$:

**Lemma 5** (On the monotonicity of $\lambda(Q)$) *Suppose $\mathbb{P}(Z)\cdot Z$ is convex in $Z$; that $c_i(\cdot)$ be continuously differentiable in the relevant domain for all $i\in S$, and that at least one agent is a strategist. Then, $\lambda(Q)$ is a monotone decreasing function of $Q$. Moreover, $\lambda(Q)$ is strictly monotone in $Q$ when at least one of the additional holds.*
*(i) $\mathbb{P}(Z)\cdot Z$ is strictly convex in $Z$,*
*(ii) $c_i(x_i)$ are strictly convex for all $i\in S$,*
*(iii) $P(Q)$ is strictly decreasing in $Q$.*

**Proof.** Summing in (11) over all $i\in S$ and write $s:=|S|$ we get

$$s\lambda=\sum_{i\in S}c_i'(x_i)+sp+p'Z.$$

Pick any distinct and feasible $Q$ and $\hat{Q}$, and solve (EP) for each of these. If $\lambda(\cdot)$ is monotone decreasing, it implies that

$$\omega := s[\lambda(Q) - \lambda(\hat{Q})] \cdot [Q - \hat{Q}] \leq 0.$$

Assume on the contrary that $\omega > 0$, that is

$$\begin{aligned} 0 < \omega &= [\textstyle\sum_{i \in S} c_i'(x_i(Q)) - \sum_{i \in S} c_i'(x_i(\hat{Q}))] \cdot [Q - \hat{Q}] \\ &+ s[P(Q) - P(\hat{Q})] \cdot [Q - \hat{Q}] \\ &+ [P'(Q)Z - P'(\hat{Q})\hat{Z}] \cdot [Q - \hat{Q}] \end{aligned}$$

where $\hat{Z} := \sum_{i \in F} e_i - \hat{Q}$. Now, let $\mathbb{P}'(Z)$ be the derivative of $\mathbb{P}(Z)$ taken with respect to $Z$ and recall that for $P'(Q)$ the derivative was taken with respect to $x_i$, $i \in S$. It then follows that $\mathbb{P}'(Z) = P'(Q)$. As $x_i$, $i \in S$ is non-increasing in $Q$ and $c_i(x_i)$ is convex in $x_i$, that is $c_i'(x_i)$ is increasing in $x_i$, it follows that

$$\begin{aligned} \omega &\leq s[P(Q) - P(\hat{Q})] \cdot [Q - \hat{Q}] \\ &+ [P'(Q)Z - P'(\hat{Q})\hat{Z}] \cdot [Q - \hat{Q}] \\ &= s[P(Q) - P(\hat{Q})] \cdot [Q - \hat{Q}] \\ &- [\mathbb{P}'(Z)Z - \mathbb{P}'(\hat{Z})\hat{Z}] \cdot [Z - \hat{Z}] \\ &= (s-1)[P(Q) - P(\hat{Q})] \cdot [Q - \hat{Q}] \\ &- [\mathbb{P}'(Z)Z + \mathbb{P}(Z) - \mathbb{P}'(\hat{Z})\hat{Z} - \mathbb{P}(\hat{Z})] \cdot [Z - \hat{Z}]. \end{aligned}$$

Since $\mathbb{P}(Z) \cdot Z$ is convex, this is

$$\leq \quad (s-1)[P(Q) - P(\hat{Q})] \cdot [Q - \hat{Q}] \leq 0$$

as $P(Q)$ is non-increasing in $Q$ (last bullet of Proposition 1), and $s \geq 1$. Hence $\omega > 0$ is a contradiction. It follows immediately that $\lambda(Q)$ is strictly monotonous in $Q$ when at least one of $(i) - (iii)$ holds. □

With the support of Lemma 5, the issue of uniqueness may now be addressed.

**Theorem 2** (Uniqueness of Cournot-Nash equilibrium) *Suppose there exist two permit profiles $(\tilde{x}_i)_{i \in I}$ and $(\bar{x}_i)_{i \in I}$ that both are interior Cournot-Nash equilibria; that $\lambda(Q)$ is strictly monotonous in $Q$; and that $c_i(x_i)$ are strictly convex for all $i \in F$. Then $(\tilde{x}_i)_{i \in I} = (\bar{x}_i)_{i \in I}$, and hence equilibrium is unique.*

**Proof.** Suppose on the contrary that the profiles $(\tilde{x}_i)_{i \in I}$ and $(\bar{x}_i)_{i \in I}$ are distinct. Denote $\tilde{Q}$ and $\bar{Q}$ as the corresponding amount of permits available to the fringe. Since both are equilibria, we know from Lemma 4 that $\lambda(\tilde{Q}) = \lambda(\bar{Q}) = 0$. As $\lambda$ is strictly monotonous it follows that $\tilde{Q} = \bar{Q}$. As $c_i(x_i)$ is strictly convex for all $i \in F$, the objective of problem (EP) is strictly convex in $(x_i)_{i \in I}$. Since the constraint to that problem is linear, then $(\tilde{x}_i)_{i \in I} \neq (\bar{x}_i)_{i \in I}$ is a contradiction and equilibrium is unique. □

## 5. Equilibrium under Affine Price Expectations

In hindsight we find some results in Section 4, when laden with hard-to verify hypotheses, not very encouraging. To have a more heartening perspective, one conducive to analysis and computations, we decide to weaken next the assumption about perfectly rational expectations. The alternative goes as follows:

**Assumption about affine price expectations:** *Posit henceforth that each strategist behaves as though the price curve were affine. Specifically, each agent $i \in S$ persistently believes that prices be of the form*

$$\hat{Q} \to \mathcal{P}(\hat{Q}) := \mathcal{P}_Q(\hat{Q}) := P(Q) - c_F''(Q)\left[\hat{Q} - Q\right]. \tag{12}$$

*Fringe members, on the other hand, remain price-takers. That is, they maintain (degenerate) point expectations about the price vector.* □

The straightforward interpretation of (12) is that all strategists, in forming their beliefs, use one and the same point $[Q, P(Q)]$ through which passes an imaginary price curve having slope $-c_F''(Q)$. There are two reasons for accommodating these curves. First, upon contending with an affine function $\mathcal{P}(\cdot)$ instead of the possibly more intricate counterpart $P(\cdot)$, players may more easily reason about market interaction. Second, as made clear in Section 4, existence is likely to become a more tractable issue. The appropriate solution concept now assumes a corresponding form:

**Definition 3** (First-order equilibrium) *A feasible emission profile $(\bar{x}_i)_{i\in I}$ constitutes a* ***first-order equilibrium*** *iff $\bar{x}_i$ minimizes agent $i$'s cost plus his proceeds from sale. That is, $\bar{x}_i$ minimizes*

$$\left.\begin{array}{ll} c_i(x_i) + \mathcal{P}(\sum_{i\in I} e_i - \sum_{j\in S\setminus i} \bar{x}_j - x_i)\cdot(x_i - e_i) & \textit{when } i \in S \\ c_i(x_i) + \mathcal{P}(\sum_{i\in I} e_i - \sum_{j\in S} \bar{x}_j)\cdot(x_i - e_i) & \textit{otherwise.} \end{array}\right\} \tag{13}$$

*Moreover, the realized price-quantity pair $p, Q$ must satisfy*

$$p = -\partial c_F(Q) = P(Q) = \mathcal{P}(Q) \textit{and } P'(Q) = -c_F''(Q). \ \square$$

**Theorem 3** (On the existence of first-order equilibrium) *Suppose $c_F$ is twice continuously differentiable. Then, under affine price expectations there exists at least one first-order equilibrium.*

**Proof.** For each $Q$ there is, by the results in Section 4, at least one Nash equilibrium in the noncooperative game that features objectives (13) and price curve $\mathcal{P}_Q(\cdot)$ prescribed in (12). Let $Nash(Q)$ denote the resulting set of Nash equilibria. One may easily argue that the correspondence $Q \rightsquigarrow Nash(Q)$ so defined is upper semicontinuous with convex values. Let

$$\hat{Q}(Q) := \left\{ \sum_{i\in I} e_i - \sum_{i\in S} x_i : (x_i) \in Nash(Q) \right\}$$

be the resulting (possibly set-valued) supply to be allocated among the fringe members. The correspondence $Q \rightsquigarrow \hat{Q}(Q)$ defined in this manner has a fixed point by Kakutani's theorem. Any such point supports a first-order equilibrium. $\square$

A full-fledged Nash equilibrium, as defined in Section 4, is certainly of first order. But conversely, and in principle, a first-order equilibrium need not be Nash. To wit, consider the following one-gas, three-agent example, i.e. $|G| = 1$, $|I| = 3$. Agent 1 is a price-taker endowed with no permits ($e_1 = 0$) and costs

$$c_1(x_1) = \begin{cases} +\infty & \text{when} \quad x_1 \in (-\infty, 0) \\ \frac{110}{19} - 2x_1 + \frac{1}{2}x_1^2 & \text{when} \quad x_1 \in [0, \frac{99}{100}] \\ \frac{163534261}{30852200} - \frac{8219}{8119}x_1 + \frac{1}{2}\frac{19}{8119}x_1^2 & \text{when} \quad x_1 \in \left(\frac{99}{100}, \frac{100}{19}\right] \\ \frac{100380361}{72200} - \frac{10019}{19}x_1 + \frac{1}{2}100x_1^2 & \text{when} \quad x_1 \in \left(\frac{100}{19}, \frac{10019}{1900}\right] \\ 0 & \text{when} \quad x_1 \in \left(\frac{10019}{1900}, +\infty\right). \end{cases} \tag{14}$$

Agents 2 and 3 are strategists with endowments $e_2 = e_3 = \frac{10019}{1900}$ and costs

$$c_i(x_i) = \begin{cases} +\infty & \text{when} \quad x_i \in (-\infty, 0) \\ a_i - b_i x_i & \text{when} \quad x_i \in [0, \frac{10019}{1900}] \\ 0 & \text{when} \quad x_i \in \left(\frac{10019}{1900}, +\infty\right), \end{cases}$$

$i = 2, 3$, where $a_2 = \frac{8826739}{1520000}$, $a_3 = \frac{3817239}{760000}$, $b_2 = \frac{881}{800}$, and $b_3 = \frac{381}{400}$. Clearly, if *either* sort of equilibrium in this example exist, the associated allocation of permits would satisfy $x_i \in (0, \frac{10019}{1900})$ for each $i \in I$. For any such $x_i$, one may verify that the cost function for the price-taker given in (14) is strictly decreasing, strictly convex, and continuously differentiable with marginal costs being piecewise linear. The cost functions for the strategists and the price curve generated by the price-taker, then become exactly as in Novshek [15, Example 3, p. 88]. Hence, as he shows, no full-fledged Nash equilibrium exists.[14] However, it is straightforward to check that the allocation

$$x_1 = \frac{519}{800}, x_2 = \frac{2386}{475}, x_3 = \frac{74091}{15200} \text{ with } p = \frac{1081}{800} \text{ and } p' = 1$$

indeed is a first-order equilibrium. Here, no profitable deviations are available for agents that entertain local perspectives in the form of affine price functions.

[14] In this example, the assumptions of Lemma 1 are violated.

## 6. Concluding Remarks

Our chief errand was to explore existence of equilibrium in imperfect markets with endogenous market curves. Main problems relate to convexity of strategists' preferences. These problems are not of novel notice in the literature on strategic multilateral exchange.[15] For instance, Giraud [7, p. 371] comments on Cournot producers who face competitive consumers, that

"...the best way to model oligopolistic behavior is probably to assume that firms first set quantities, while, second, a price emerges on the consumption markets. Of course, as is well known, one encounters severe difficulties in trying to carry over this program: given the quantities put up for sale in the first-period, there is not necessarily uniqueness of the second-period price equilibrium outcome. As a consequence, the firms' payoff function in the first-period is not well defined—or, at least, is not a function. Moreover, even if uniqueness was guaranteed, this payoff function need not be quasi-concave (so that existence of a subgame-perfect equilibrium would become a problem, unless mixed strategies were allowed)."

In permit markets there is the additional challenge that strategists may reside on either side of the counter. Also, one wonders: what features make some agents qualifying as price takers and others as strategists? This question is particularly pressing since the fringe had to be nonempty here - so as to serve as market clearing device.

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