Deriving Rank-Dependent Expected Utility Through Probabilistic Consistency

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Abstract. This paper proposes a new preference condition that allows for the separation of probability weights from utility when lotteries over arbitrary sets of outcomes are considered. In the presence of weak order, Jensen-continuity, and stochastic dominance, adding the new "probabilistic consistency" condition to comonotonic independence leads to rank-dependent expected utility.

Keywords: Comonotonic independence, preference foundation, rank-dependent expected utility.

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1 Introduction

Empirical research has shown that expected utility fails to provide a general description of individual behavior under risk. Shortly after its introduction by von Neumann and Morgenstern (1944), the predictive power of expected utility was subject to criticism by thoughtfully designed experiments, the most famous of which were offered by Allais (1953). Since, many new theories have been developed to capture expected utility anomalies. Among those, the so-called rank-dependent utility theories, which allow for a non-linear representation of probabilities, have emerged as an elegant way to capture relevant descriptive generality and to maintain most of the mathematical tractability offered by expected utility (see the recent overview by Starmer 2000). By now these models have established their validity, and it is not surpring that nowadays they have become important tools for general economic analyses.

Most of the provided preference foundations for rank-dependent utility theories require a rich structure on the set of outcomes. Köbberling and Wakker (2003), give a review of such theories, and show that many of the foundations for rank-dependent utilty can be unified under the principle of comonotonic tradeoff consistency. This principle, which implies a separation of utility from attitudes towards probabilities, has lead to more advanced elicitation techniques for utility, firstly by dispensing from parametric specifications for utility, and secondly because of its insensitivity to probability distortions. The requirement of a rich structure on the set of outcomes is, to a large degree, in contrast to the structure needed to establish expected utility, where the richness of the naturally given probability inteval is more relevant. Also, most expected utility paradoxes in fact originate from choice examples that require only a finite number of outcomes (e.g., three outcomes in the Allais-paradoxes). Therefore, preference foundations, which can dispense of a rich structure for outcomes, can be viewed as closer or more natural extensions of expected utility.

Only few such close extensions of expected utility are available in the literature. Loewenstein and Prelec (1991) have considered preference conditions for lotteries with two outcomes one of which is zero. This structure has proven useful in the parametric characterization of inverse S-shaped probability weighting functions (Prelec 1998). Nakamura (1995) is, to my knowledge, the first axiomatization of rank-dependent expected utility that allows for more general probability distributions. In particular, Nakamura shows that, by looking at decumulative probability distributions, new preference conditions can be introduced, which allow for the application of familiar mathematical techniques in the derivation of rank-dependent utility. This has been best illustrated in Abdellaoui (2002), where the probability tradeoff consistency condition has been employed to separate decision weights from the utility of outcomes. From a technical point of view this condition plays the analogue role of comonotonic tradeoff consistency for outcomes when the rich-outcome-set approach is used (see Wakker 1989, 1994, and more recently Köbberling and Wakker 2003). Abdellaoui shows that probability tradoff consistency is a direct implication of von Neumann Morgenstern independence, and that, in the presence of stochastic dominance (a further implication of von Neumann Morgenstern independence) and the additional standard principles of weak ordering and continuity in probabilities, an elegant preference foundation for rank-dependent expected utility results.

The approach in this paper is similar in that it focuses on the structure offered through the probability inteval, leaving aside assumptions on the structure of the outcome set. The goal is to provide a preference characterization of rank-dependent expected utility. Similarly to Abdellaoui (2002), weak ordering, continuity with respect to changes in probability and stochastic dominance are maintained, and supplemented with two further conditions focusing on extreme, that is best or worst, outcomes. The first of these resembles the idea of tail independence (Wakker and Zank 2002) or ordinal independence (Green and Jullien 1988), and is a slight weakening of comonotonic independence (Schmeidler 1989). It means that the preference is independent of common decumulative probabilities for best or worst outcomes. The second condition ensures that probability weights can be separated from the utility of outcomes. It is a novel condition that can be tested empirically, and, on its own, is weaker than probability tradeoff consistency, and consequently also implied by von Neumann Morgenstern independence. The new probabilistic consistency condition is inspired by the fact that under rank-dependent expected utility decision weights are independent of the magnitude of outcomes and only the rank of the latter matters.

The paper continues with some preliminary notation in the next section. In Section 3 tail independence is introduced and some implications are presented. Probabilistic consistency is discussed in Section 4, where also the main result of the paper is presented (Theorem 4). Concluding remarks are provided in Section 5. Most proofs are postponed until the Appendix, unless they are used to supplement the intuition behind some of the axioms.

2 Preliminaries

Let X denote the set of outcomes. A *lottery* is a finite probability distribution over the set X. It is represented by $P = (p_0, x_0; \ldots; p_n, x_n)$ meaning that probability p_j is assigned to outcome $x_j \in X$, for $j = 0, \ldots, n$. Let L denote the set of all lotteries. A preference relation $\succcurlyeq$ is assumed over L, and its restiction to subsets of L (e.g., X) is also denoted by $\succcurlyeq$. The symbol $\succ$ denotes strict preference, $\sim$ denotes indifference, and $\preccurlyeq$ respectively $\prec$ are the usual reversed preferences. In the notation for a lottery we assume that outcomes are ordered from worst to best, i.e., with $P = (p_0, x_0; \ldots; p_n, x_n)$ it holds that $x_0 \preccurlyeq \cdots \preccurlyeq x_n$. It what follows it is convenient to sometimes fix a finite subset of outcomes $Y = \{x_0, \ldots, x_n\} \subseteq X$ and consider the set of lotteries with respect to outcomes in Y, denoted L_Y.

The goal is to provide preference conditions for $\succcurlyeq$ in order to represent the preference relation by a functional V defined over lotteries. That is, V is a mapping from L into the set of real numbers, $I\!R$, such that

$$P \succcurlyeq Q \Leftrightarrow V(P) \geqslant V(Q).$$

This necessarily implies that $\succcurlyeq$ must be a *weak order*, i.e. $\succcurlyeq$ is *complete* ($P \succcurlyeq Q$ or $P \preccurlyeq Q$ for all lotteries P, Q) and *transitive* ($P \succcurlyeq Q$ and $Q \succcurlyeq R$ implies $P \succcurlyeq R$ for all lotteries P, Q, R).

The set of lotteries L is a mixture space endowed with the operation of probability mixing, i.e., for $P, Q \in L$ and $\alpha \in [0,1]$ the mixture $\alpha P + (1-\alpha)Q$ is also a lottery in L. The preference relation $\succcurlyeq$ satisfies *Jensen-continuity* on the set of lotteries L if for all lotteries $P \succ Q$ and R there exist $\rho, \mu \in (0,1)$ such that

$$\rho P + (1-\rho)R \succ Q \text{ and } P \succ \mu R + (1-\mu)Q.$$

The preference relation $\succcurlyeq$ satisfies *vNM-independence* (short for *von Neumann Morgenstern independence*) if for all lotteries $P, Q, R \in L$ and all $\alpha \in (0,1)$ it holds that

$$P \succcurlyeq Q \Leftrightarrow \alpha P + (1-\alpha)R \succcurlyeq \alpha Q + (1-\alpha)R.$$

That is, the preference between P and Q remains unaffected if both, P and Q, are mixed with a common R. Note that in the definition of vNM-independence no restrictions apply to the choice of R.

Given the structure considered here, it is well-known (see Fishburn 1970) that the three axioms above are necessary and sufficient for a representation of $\succcurlyeq$ by expected utility. That is, the representing functional for $\succcurlyeq$ evaluates any lottery $P = (p_0, x_0; \ldots; p_n, x_n)$ by

$$EU(P) = \sum_{j=0}^{n} p_j u(x_j),$$

with utility function $u : X \to I\!R$ unique up to positive affine transformations.

The focus of this paper is on a more general representation, that is, on rank-dependent expected utility, which involves a probability weighting function that transforms decumulative probabilities. Formally, a *probability weighting function* is an increasing continuous mapping $w : [0,1] \to [0,1]$, with $w(0) = 0$ and $w(1) = 1$. *Rank-dependent expected utility* (*RDEU*) represents $\succcurlyeq$ if any lottery $P = (p_0, x_0; \ldots; p_n, x_n)$ is evaluated by

$$RDEU(P) = \sum_{j=1}^{n} [w(p_{j-1} + \cdots + p_n) - w(p_j + \cdots + p_n)] u(x_{j-1}) + w(p_n) u(x_n), \tag{1}$$

with utility function $u : X \to I\!R$ as for expected utility, and unique probability weighting function w. It is convenient for the later exposition to sometimes use the following equivalent formula for RDEU:

$$RDEU(P) = u(x_0) + \sum_{j=1}^{n} w(p_j + \cdots + p_n)[u(x_j) - u(x_{j-1})]. \tag{2}$$

RDEU is more general than expected utility, which can be seen if in the above equations the probability weighting function is assumed linear, i.e., $w(p) = p$ for all $p \in [0,1]$. To allow for the empirically observed deviations from linearity in the weighting functions, vNM-independence needs to be relaxed.

One implication of vNM-independence that is maintained under RDEU is stochastic dominance. For $x \in X$ and a lottery $P \in L$ denote by $P(\{y \in X | y \succcurlyeq x\})$ the probability of receiving an outcome weakly preferred to x under P. *Stochastic dominance* says that lotteries where the probability of receiving weakly preferred outcomes is always higher are preferred. Formally, for all $P, Q \in L$, whenever $P(\{y \in X | y \succcurlyeq x\}) \geqslant Q(\{y \in X | y \succcurlyeq x\})$ for all $x \in X$ and $P \neq Q$, then $P \succ Q$. It can be shown (e.g., Abdellaoui 2002, Proposition 10) that vNM-independence implies stochastic dominance for a weak order $\succcurlyeq$ on L.

Some convenient notation is now introduced. Consider, for a finite set of outcomes $Y = \{x_0, \ldots, x_n\}$, the set of lotteries L_Y. Take any $P \in L_Y$ with $P = (p_0, x_0; \ldots; p_n, x_n)$. Then, the probability p_i^* for $i = 0, \ldots, n$ is equal to

$$p_i^* := P(\{y \in Y | y \succcurlyeq x_i\}) = \sum_{j=k}^{n} p_j, \text{ if } x_{k-1} \prec x_k \sim x_i, \tag{3}$$

where obviously $p_0^* = 1$ and $p_n^* = p_n$. Hence, any lottery $P \in L_Y$ can be identified with the (non-increasing) probability distribution, say $P^* = (1, x_0; p_1^*, x_1; \ldots; p_n^*, x_n)$. Let L_Y^* denote the set of (non-increasing) probability distributions with respect to Y, and more generally let L^* denote the set of (non-increasing) probability distributions with respect to X. If Y is fixed and no confusion arises, then the outcomes in P and P^* are suppressed, i.e., the probability vector $(p_0, \ldots, p_n)$ stands for $P = (p_0, x_0; \ldots; p_n, x_n)$, and $(p_1^*, \ldots, p_n^*)$ stands for $P^* = (1, x_0; \ldots; p_n^*, x_n)$. Note that, in the special case that $x_0 \prec \cdots \prec x_n$ the distribution P^* with respect to Y is in fact a decumulative probability distribution, and therefore L_Y^* is the set of decumulative probability distributions over Y.

The correspondence between a lottery $P = (p_0, x_0; \ldots; p_n, x_n)$ over the outcomes in Y and probability distribution $P^* \in L_Y^*$ through the mapping defined using Equation (3) above

$$\begin{array}{rcl} L_Y & \rightarrow & L_Y^* \\ (p_0, \ldots, p_i, \ldots, p_n) & \mapsto & (p_1^*, \ldots, p_i^*, \ldots, p_n^*), \end{array}$$

with the implicit relation $\sum_{j=0}^{n} p_j = 1$ suppressed, allows the extension of preference conditions defined on L to preference conditions defined on L^*. The fact that L_Y, L_Y^* are mixture spaces, with the operation of probabilistic mixture $\alpha P + (1-\alpha)Q$ on L_Y for $P, Q \in L_Y$ and $\alpha \in [0,1]$ being equivalent to the mixture $\alpha P^* + (1-\alpha)Q^*$ on L_Y^* for $P^*, Q^* \in L_Y^*$ and $\alpha \in [0,1]$, allows to identify the preference relation $\succcurlyeq$ on L_Y with the preference relation $\succcurlyeq$ on L_Y^* through the equivalence

$$P \succcurlyeq Q \Leftrightarrow P^* \succcurlyeq Q^*.$$

Now, the preference conditions of weak ordering and Jensen-continuity, and vNM-independence are extended straightforwardly from L to L^*. Stochastic dominance is reformulated as *monotonicity*: For any $Y = \{x_0, \ldots, x_n\}$, all $P^*, Q^* \in L_Y^*$ with $P^* = (p_1^*, \ldots, p_n^*)$ and $Q^* = (q_1^*, \ldots, q_n^*)$, if $p_j^* \geqslant q_j^*$ for all $j = 1, \ldots, n$ and $P^* \neq Q^*$, then $P^* \succ Q^*$.

As shown in Lemma 18 of Abdellaoui (2002), for any finite subset $Y = \{x_0, \ldots, x_n\} \subseteq X$ the preference relation $\succcurlyeq$ on L_Y^* satisfies (Euclidean) continuity if it is a monotonic and Jensen-continuous weak order. The preference relation $\succcurlyeq$ on L_Y^* is *continuous* if for any $P^* \in L_Y^*$ the sets $\{Q^* \in L_Y^* | Q^* \succ P^*\}$ and $\{Q^* \in L_Y^* | Q^* \prec P^*\}$ are open in L_Y^*.

With the above notation, Equations (1) and (2) can be reformulated in terms of (decumulative) probabilities as follows: A lottery $P = (p_0, x_0; \ldots; p_n, x_n)$ is evaluated by

$$RDEU(P) = \sum_{j=1}^{n} [w(p_{j-1}^*) - w(p_j^*)] u(x_{j-1}) + w(p_n^*) u(x_n), \tag{4}$$

respectively

$$RDEU(P) = u(x_0) + \sum_{j=1}^{n} w(p_j^*)[u(x_j) - u(x_{j-1})]. \tag{5}$$

That this notation makes sense follows from the fact that under RDEU the utility of two indifferent outcomes is equal, i.e., $u(x_i) = u(x_k)$ for $x_i \sim x_k$ (and consequently $p_i^* = p_k^*$).

Two further implications of vNM-independence are considered in the next sections, and as it turns out these suffice for a preference representation by RDEU (see Theorem 4 below).

3 Tail Independence

In this section it is convenient to formulate preference conditions initially for probability distributions in L^*. Before tail independence is introduced some useful notation is presented. Take any finite set of outcomes $Y = \{x_0, \ldots, x_n\}$, such that $x_0 \prec \cdots \prec x_n$, and a subset $I \subseteq \{1, \ldots, n\}$. For $P^*, R^* \in L^*_Y$ define the probability tuple $R^*_I P^* = (t^*_1, \ldots, t^*_n)$ with $t^*_i = r^*_i$ for $i \in I$, and $t^*_i = p^*_i$ for $i \notin I$. Note that, for $R^*_I P^* \in L^*_Y$ to hold, it is necessary that $1 \geqslant t^*_1 \geqslant \cdots \geqslant t^*_n \geqslant 0$.[2]

The preference relation $\succcurlyeq$ satisfies *tail independence* if for any finite set of outcomes $Y = \{x_0, \ldots, x_n\}$, such that $x_0 \prec \cdots \prec x_n$, and any $R^*_I P^*, R^*_I Q^*, S^*_I P^*, S^*_I Q^* \in L^*_Y$ it holds that

$$R^*_I P^* \succcurlyeq R^*_I Q^* \Leftrightarrow S^*_I P^* \succcurlyeq S^*_I Q^*,$$

whenever $I = \{1, \ldots, i\}$ or $I = \{j, \ldots, n\}$ for some $i, j \in \{1, \ldots, n\}$.

Tail independence requires the preference between two probability distributions on any L^*_Y to be invariant to substitution of the common part of the probability distributions when this part refers to the most preferred outcomes or most dispreferred ones. Translating the condition into lottery notation reveals its relationship with vNM-independence. Take any finite set of outcomes $Y = \{x_0, \ldots, x_n\}$, with $x_0 \prec \cdots \prec x_n$. Assume $I = \{1, \ldots, i\}$, for some $0 < i < n$. Suppose $R^*_I P^*, R^*_I Q^*, S^*_I P^*, S^*_I Q^* \in L^*_Y$ with

$$R^*_I P^* = (r^*_1, \ldots, r^*_i, p^*_{i+1}, \ldots, p^*_n),$$

$$R^*_I Q^* = (r^*_1, \ldots, r^*_i, q^*_{i+1}, \ldots, q^*_n),$$

and

$$S^*_I P^* = (s^*_1, \ldots, s^*_i, p^*_{i+1}, \ldots, p^*_n),$$

[2] This notation and the subsequent condition can be formulated for general finite sets of outcomes Y containing indifferent outcomes. Then the additional restriction $t^*_k = t^*_l$ applies if $x_k \sim x_l$.

$$S_I^*Q^* \quad = \quad (s_1^*, \ldots, s_i^*, q_{i+1}^*, \ldots, q_n^*).$$

Then, $R_I^*P^* \succcurlyeq R_I^*Q^* \Leftrightarrow S_I^*P^* \succcurlyeq S_I^*Q^*$ can equivalently be written as

$$\begin{array}{rcl} (r_1^*, \ldots, r_i^*, p_{i+1}^*, \ldots, p_n^*) & \succcurlyeq & (r_1^*, \ldots, r_i^*, q_{i+1}^*, \ldots, q_n^*) \\ & \Leftrightarrow & \\ (s_1^*, \ldots, s_i^*, p_{i+1}^*, \ldots, p_n^*) & \succcurlyeq & (s_1^*, \ldots, s_i^*, q_{i+1}^*, \ldots, q_n^*). \end{array}$$

Hence, the common decumulative probabilities $r_1^*, \ldots, r_i^*$ in the first preference have been replaced in the second preference by the common decumulative probabilities $s_1^*, \ldots, s_i^*$. Translated into lotteries from L_Y the latter equivalence means that

$$\begin{array}{rcl} (1 - r_1^*, \ldots, r_{i-1}^* - r_i^*, r_i^* - p_{i+1}^*, \ldots, p_n^*) & \succcurlyeq & (1 - r_1^*, \ldots, r_{i-1}^* - r_i^*, r_i^* - q_{i+1}^*, \ldots, q_n^*) \\ & \Leftrightarrow & \\ (1 - s_1^*, \ldots, s_{i-1}^* - s_i^*, s_i^* - p_{i+1}^*, \ldots, p_n^*) & \succcurlyeq & (1 - s_1^*, \ldots, s_{i-1}^* - s_i^*, s_i^* - q_{i+1}^*, \ldots, q_n^*). \end{array}$$

Assume now that $1 > q_{i+1}^* \geqslant p_{i+1}^* > 0$. Define the lotteries $R, P, Q, S \in L$ as

$$\begin{array}{rcl} R & = & (\frac{1 - r_1^*}{1 - q_{i+1}^*}, x_0; \ldots; \frac{r_{i-1}^* - r_i^*}{1 - q_{i+1}^*}, x_{i-1}; \frac{r_i^* - q_{i+1}^*}{1 - q_{i+1}^*}, x_i), \\ S & = & (\frac{1 - s_1^*}{1 - q_{i+1}^*}, x_0; \ldots; \frac{s_{i-1}^* - s_i^*}{1 - q_{i+1}^*}, x_{i-1}; \frac{s_i^* - q_{i+1}^*}{1 - q_{i+1}^*}, x_i), \text{ and} \\ P & = & (\frac{q_{i+1}^* - p_{i+1}^*}{q_{i+1}^*}, x_i; \frac{p_{i+1}^* - p_{i+2}^*}{q_{i+1}^*}, x_{i+1}; \ldots; \frac{p_n^*}{q_{i+1}^*}, x_n), \\ Q & = & (\frac{q_{i+1}^* - q_{i+2}^*}{q_{i+1}^*}, x_{i+1}; \ldots; \frac{q_n^*}{q_{i+1}^*}, x_n). \end{array}$$

Note that both, R and S, assign zero probability to $x_{i+1}, \ldots, x_n$, whereas P and Q assign zero probability to $x_0, \ldots, x_{i-1}$, and further Q assigns zero probability to x_i. It is now easily verified that the previous preferences can equivalently be written as preferences between mixtures of lotteries as follows:

$$(1 - q_{i+1}^*)R + q_{i+1}^*P \quad \succcurlyeq \quad (1 - q_{i+1}^*)R + q_{i+1}^*Q$$

$$\Leftrightarrow$$

$$(1 - q^*_{i+1})S + q^*_{i+1}P \quad \succcurlyeq \quad (1 - q^*_{i+1})S + q^*_{i+1}Q.$$

For the case "$0 < q^*_{i+1} \leqslant p^*_{i+1} < 1$" a similar equivalence can be derived, and further the corresponding relations for the case when $I = \{i, \ldots, n\}$, for some $0 < i \leqslant n$, can be derived in an analogous way.

These computations show that indeed tail independence is a weak form of vNM-independence. The outcomes in the common lotteries R and S are either all (weakly) preferred to those of Q and P, or all outcomes of P and Q are preferred to those of R and S. This condition has already appeared in Wakker and Zank (2002) under the same name of tail independence, and further, extended to nonsimple lotteries, in Green and Jullien (1988) as ordinal independence. The comonotonicity requirements introduced by Schmeidler (1989), which underlie all rank-dependent theories, are further weakened in the definition of tail independence, by imposing the condition only for mixtures with lotteries involving best and worst common outcomes.

The next lemma and its elementary proof further clarify the nature of tail independence and its relation to RDEU.

LEMMA 1 *Assume that $\succcurlyeq$ on L is represented by RDEU. Then $\succcurlyeq$ satisfies tail independence.*

PROOF: Take any finite set of outcomes $Y = \{x_0, \ldots, x_n\}$, such that $x_0 \prec \cdots \prec x_n$, and assume $I = \{1, \ldots, i\}$, for some $0 < i < n$. Assume $R^*_I P^*, R^*_I Q^* \in L^*_Y$ with

$$R^*_I P^* \quad = \quad (r^*_1, \ldots, r^*_i, p^*_{i+1}, \ldots, p^*_n),$$

$$R^*_I Q^* \quad = \quad (r^*_1, \ldots, r^*_i, q^*_{i+1}, \ldots, q^*_n),$$

Then, using the RDEU form in Equation (5), it follows that

$$(r^*_1, \ldots, r^*_i, p^*_{i+1}, \ldots, p^*_n) \quad \succcurlyeq \quad (r^*_1, \ldots, r^*_i, q^*_{i+1}, \ldots, q^*_n)$$

$$\Leftrightarrow$$

$$RDEU(r_1^*, \ldots, r_i^*, p_{i+1}^*, \ldots, p_n^*) \quad \geqslant \quad RDEU(r_1^*, \ldots, r_i^*, q_{i+1}^*, \ldots, q_n^*),$$

where on both sides of the latter inequality the common term,

$$u(x_0) + \sum_{j=1}^{i} w(r_j^*)[u(x_j) - u(x_{j-1})],$$

can equivalently be replaced by the common term

$$u(x_0) + \sum_{j=1}^{i} w(s_j^*)[u(x_j) - u(x_{j-1})]$$

for any $1 \geqslant s_1^* \geqslant \cdots \geqslant s_i^* \geqslant \max\{p_{i+1}^*, q_{i+1}^*\}$. Hence, with $S_I^* P^*, S_I^* Q^* \in L_Y^*$ defined as

$$S_I^* P^* \quad = \quad (s_1^*, \ldots, s_i^*, p_{i+1}^*, \ldots, p_n^*),$$

$$S_I^* Q^* \quad = \quad (s_1^*, \ldots, s_i^*, q_{i+1}^*, \ldots, q_n^*),$$

it follows that

$$RDEU(s_1^*, \ldots, s_i^*, p_{i+1}^*, \ldots, p_n^*) \geqslant RDEU(s_1^*, \ldots, s_i^*, q_{i+1}^*, \ldots, q_n^*).$$

In summary, $R_I^* P^* \succcurlyeq R_I^* Q^* \Leftrightarrow S_I^* P^* \succcurlyeq S_I^* Q^*$ follows for the case of common worst outcomes, i.e., $I = \{1, \ldots, i\}$, for some $0 < i < n$. The case of common best outcomes ($I = \{i, \ldots, n\}$, for some $0 < i \leqslant n$) is similar. Hence, as Y is arbitrary, tail independence follows. $\square$

It is useful to recall here an intermediate implication of the preference conditions introduced so far. The next lemma shows that on each set L_Y^* the preference relation is represented by an additive separable functional. One complication that occurs here is the existence of the extreme probabilities 1 respectively 0 in the distributions.

LEMMA 2 *Let $n \geqslant 3$ and let $Y = \{x_0, \ldots, x_n\}$ be a set of outcomes such that $x_0 \prec \cdots \prec x_n$. The following two statements are equivalent for a preference relation $\succcurlyeq$ on the set of lotteries L_Y^*:*

(*i*) *The preference relation* $\succcurlyeq$ *on* L_Y^* *is represented by an additive function*

$$V(p_1^*, \ldots, p_n^*) = \sum_{j=1}^{n} V_j(p_j^*),$$

with continuous strictly monotonic functions $V_1, \ldots, V_n : [0,1] \to I\!R$ *which are bounded except maybe* V_1 *and* V_n *which could be infinite at extreme probabilities* (*i.e., at* $0, 1$).

(*ii*) *The preference relation* $\succcurlyeq$ *is a Jensen-continuous monotonic weak order satisfying tail independence.*

The functions $V_1, \ldots, V_n$ *are jointly cardinal, that is, they are unique up to common scale and location.* □

4 Probabilistic Consistency

The next condition is called probabilistic consistency. It is the main condition of the paper, and it implies the separation of the weighting of the (decumulative) probability from the utility of outcomes. Take a finite set of outcomes $Y = \{x_0, \ldots, x_n\}$, such that $x_0 \prec \cdots \prec x_n$, let $I = \{1, \ldots, i\}$ for some $1 \leqslant i < n$ (or $I = \{i, \ldots, n\}$ for some $1 < i \leqslant n$), and take some $P^* = (p_1^*, \ldots, p_n^*) \in L_Y^*$ and probability $\alpha \in [p_{i+1}^*, 1]$ (or $\alpha \in [0, p_{i-1}^*]$). Then $\alpha_I P^*$ is defined as the probability distribution

$$\alpha_I P^* = \begin{cases} (\alpha, \ldots, \alpha, p_{i+1}^*, \ldots, p_n^*), & \text{if } I = \{1, \ldots, i\}, \\ (p_1^*, \ldots, p_{i-1}^*, \alpha, \ldots, \alpha), & \text{if } I = \{i, \ldots, n\}. \end{cases}$$

The preference relation $\succcurlyeq$ satisfies *probabilistic consistency* if for any finite set of outcomes $Y = \{x_0, \ldots, x_n\}$, such that $x_0 \prec \cdots \prec x_n$, and all probabilities $\alpha, \beta, \gamma, \delta$ with $\alpha_I P^*, \beta_I Q^* \in L_Y^*$ and $\gamma_I P^*, \delta_I Q^* \in L_Y^*$ it holds that

$$\alpha_I P^* \sim \beta_I Q^* \text{ and } \gamma_I P^* \sim \delta_I Q^* \Rightarrow \alpha_J \gamma_I P^* \sim \beta_J \delta_I Q^*$$

for any subset $J \subseteq I$ such that $\alpha_J \gamma_I P^*, \beta_J \delta_I Q^* \in L^*_Y$.

Before the condition is discussed in more detail, it is instructive to consider a specific case for the above implication. Take $Y = \{x_0, \dots, x_n\}$, such that $x_0 \prec \dots \prec x_n$, and probabilities $\alpha > \beta$ and $\gamma > \delta$, and further let $I = \{1, \dots, i\}$ for some $1 \leqslant i < n$, such that the following relationships hold among probability distributions in L^*_Y

$$(\alpha, \dots, \alpha, p^*_{i+1}, \dots, p^*_n) \sim (\beta, \dots, \beta, q^*_{i+1}, \dots, q^*_n)$$

and

$$(\gamma, \dots, \gamma, p^*_{i+1}, \dots, p^*_n) \sim (\delta, \dots, \delta, q^*_{i+1}, \dots, q^*_n).$$

Then, if $\alpha > \gamma$, for $\alpha_J \gamma_I P^*$ to belong to L^*_Y it should hold that $J = \{1, \dots, j\}$ with $1 \leqslant j \leqslant i$. This therefore implies $\beta \geqslant \delta$ (with $\beta > \delta$ in the presence of monotonicity). The above condition then implies that $\alpha_J \gamma_I P^* \sim \beta_J \delta_I Q^*$ or equivalently

$$(\underbrace{\alpha, \dots, \alpha}_{J}, \underbrace{\gamma, \dots, \gamma}_{I \backslash J}, p^*_{i+1}, \dots, p^*_n) \sim (\underbrace{\beta, \dots, \beta}_{J}, \underbrace{\delta, \dots, \delta}_{I \backslash J}, q^*_{i+1}, \dots, q^*_n).$$

So, viewed from the first indifference, probabilistic consistency says that, if initially the (decumulative) probabilities α and β for $x_1, \dots, x_i$ outweigh each other to give indifference $\alpha_I P^* \sim \beta_I Q^*$, and a joint reduction of α to γ and simultaneously of β to δ for $x_1, \dots, x_i$ maintains that indifference (i.e., $\gamma_I P^* \sim \delta_I Q^*$), then a joint reduction of α to γ and simultaneously of β to δ for $x_1, \dots, x_j$ for any $1 \leqslant j \leqslant i$ should also maintain that indifference (i.e., $\alpha_J \gamma_I P^* \sim \beta_J \delta_I Q^*$). Viewed from the second indifference (i.e., $\gamma_I P^* \sim \delta_I Q^*$) joint changes γ to α, respectively δ to β, are improvements in decumulative probabilities, and the condition says that such joint improvements are independent of the outcomes for which they occur, as long as the outcomes are intermediate ones, i.e., $x_0 \prec x_j \prec x_i$.

Above, the case of probabilities for the common worst outcomes has been considered. A similar interpretation applies for the case of probabilities for best outcomes, i.e., $I = \{i, \dots, n\}$

for some $i \in \{1, \ldots, n\}$. Note that this focus on common best or worst outcome is similar to tail independence. Reformulating the previous indifferences in terms of lotteries over $Y = \{x_0, \ldots, x_n\}$ (now including outcomes in the notation) further clarifies this point. The first two indifferences (i.e., $\alpha_I P^* \sim \beta_I Q^*$ and $\gamma_I P^* \sim \delta_I Q^*$) give

$$(1-\alpha, x_0; \alpha - p^*_{i+1}, x_i; \ldots; p^*_n, x_n) \sim (1-\beta, x_0; \beta - q^*_{i+1}, x_i; \ldots; q^*_n, x_n),$$

and

$$(1-\gamma, x_0; \gamma - p^*_{i+1}, x_i; \ldots; p^*_n, x_n) \sim (1-\delta, x_0; \delta - q^*_{i+1}, x_i; \ldots; q^*_n, x_n),$$

and the third indifference (i.e., $\alpha_J \gamma_I P^* \sim \beta_J \delta_I Q^*$) implies that for all $0 < j \leqslant i$ we have

$$(1-\alpha, x_0; \alpha-\gamma, x_j; \gamma - p^*_{i+1}, x_i; \ldots; p^*_n, x_n) \sim (1-\beta, x_0; \beta-\delta, x_j; \gamma - q^*_{i+1}, x_i; \ldots; q^*_n, x_n).$$

Note, that in the lotteries involved in the first two indifferences the outcome x_j has zero probability, whereas in the lotteries involved in the latter indifference it has positive probability $\alpha - \gamma$, respectively $\beta - \delta$. Hence, probabilistic consistency, and similarly tail independence, provide specific rules on how information regarding decumulative probabilities about common best or common worst outcomes is integrated in the evaluation of lotteries. Whereas tail independence refers to invariance of common probabilities, probabilistic consistency indicates how information about a previously impossible outcome is integrated when this outcome becomes possible: $\alpha - \gamma$ outweighs $\beta - \delta$ in the above indifference independently of which of the outcomes $x_1, \ldots, x_{i-1}$ receive this positive probability. All that matters is that this outcome (x_j above) is of common rank 2 in the new pair of lotteries.

The next lemma shows that RDEU requires probabilistic consistency to hold. The proof of the lemma is presented in the main text as it further clarifies the nature of probabilistic consistency.

LEMMA 3 *Assume that $\succcurlyeq$ on L is represented by RDEU. Then $\succcurlyeq$ satisfies probabilistic consistency.*

PROOF: Take $Y = \{x_0, \ldots, x_n\}$, such that $x_0 \prec \cdots \prec x_n$, and probabilities $\alpha > \beta$ and $\gamma > \delta$, and further let $I = \{1, \ldots, i\}$ for some $1 \leqslant i < n$, such that $\alpha_I P^* \sim \beta_I Q^*$ and $\gamma_I P^* \sim \delta_I Q^*$, or equivalently

$$
\begin{aligned}
(\alpha, \ldots, \alpha, p^*_{i+1}, \ldots, p^*_n) &\sim (\beta, \ldots, \beta, q^*_{i+1}, \ldots, q^*_n) \\
\text{and } (\gamma, \ldots, \gamma, p^*_{i+1}, \ldots, p^*_n) &\sim (\delta, \ldots, \delta, q^*_{i+1}, \ldots, q^*_n).
\end{aligned}
$$

Then, using Equation (5), and cancelling the common $u(x_0)$, it follows that

$$
\begin{aligned}
&\sum_{k=1}^{i} w(\alpha)[u(x_k) - u(x_{k-1})] + \sum_{k=i+1}^{n} w(p^*_k)[u(x_k) - u(x_{k-1})] \\
= &\sum_{k=1}^{i} w(\beta)[u(x_k) - u(x_{k-1})] + \sum_{k=i+1}^{n} w(q^*_k)[u(x_k) - u(x_{k-1})]
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^{i} w(\gamma)[u(x_k) - u(x_{k-1})] + \sum_{k=i+1}^{n} w(p^*_k)[u(x_k) - u(x_{k-1})] \\
= &\sum_{k=1}^{i} w(\delta)[u(x_k) - u(x_{k-1})] + \sum_{k=i+1}^{n} w(q^*_k)[u(x_k) - u(x_{k-1})].
\end{aligned}
$$

Subtracting the second equality from the first, and cancelling common terms, gives:

$$
[w(\alpha) - w(\gamma)][u(x_i) - u(x_0)] = [w(\beta) - w(\delta)][u(x_i) - u(x_0)]
$$

(or $[w(\alpha) - w(\gamma)] = [w(\beta) - w(\delta)]$ given that $x_i \succ x_0$) Take any $J = \{1, \ldots, j\}$ with $1 \leqslant j \leqslant i$ and assume that $\alpha_J \gamma_I P^* \nsim \beta_J \delta_I Q^*$ (which implies $\alpha \geqslant \beta, \gamma \geqslant \delta$). This means $RDEU(\alpha_J \gamma_I P^*) \neq RDEU(\beta_J \delta_I Q^*)$ and subtracting that from $RDEU(\alpha_I P^*) = RDEU(\beta_I Q^*)$, and cancelling common terms, implies

$$
[w(\alpha) - w(\gamma)][u(x_j) - u(x_0)] \neq [w(\beta) - w(\delta)][u(x_j) - u(x_0)].
$$

This contradicts the previous equality given that $x_j \succ x_0$. Hence, $\alpha_J \gamma_I P^* \sim \beta_J \delta_I Q^*$ follows for any $J = \{1, \ldots, j\}$ with $1 \leqslant j \leqslant i$. The case $J = \{j, \ldots, i\}$ for some $1 \leqslant j \leqslant i$ follows similarly by jointly reverting the role of α and γ, respectively, β and δ. Further, the case $I = \{i, \ldots, n\}$ for some $1 < i \leqslant n$, is completely analogous. Therefore, probabilistic consistency follows, which completes the proof. □

The following figure helps clarifying the nature of RDEU in relation to EU, and also in relation to Yaari's (1987) dual theory (DT), using probabilistic consistency. Here X is assumed to be the real line $I\!R$, and that $\succcurlyeq$ on X comes down to the natural ordering $\geqslant$ on $I\!R$. Assume $Y = \{x_0, x_1, x_2, x_3\}$ for distict outcomes x_0, x_1, x_2, and x_3, and let $\alpha, \beta, \gamma, \delta$ be decumulative probabilities, such that for $I = \{1, 2\}$ (and $J = \{1\}$), $\alpha_I P^* \sim \beta_I Q^*$ and $\gamma_I P^* \sim \delta_I Q^*$ holds from some $p_3^* = p^*, q_3^* = q^*$. Figure 1(a) depicts the decumulative distribution functions $\alpha_I P^*, \beta_I Q^*$ and $\gamma_I P^*, \delta_I Q^*$, and $\alpha_J \gamma_I P^*, \beta_J \delta_I Q^*$. Under EU, only attitudes towards outcomes matter, and outcomes are translated into utilities, whereas the decumulative probabilities are taken as their objective value. This translates Figure 1(a) to Figure 1(b), where the horizontal axis refers to utilities and on the vertical axis $\alpha - \gamma = \beta - \delta$ is required. Hence, the shaded rectangles in Figure 1(b) are of equal size, i.e.,

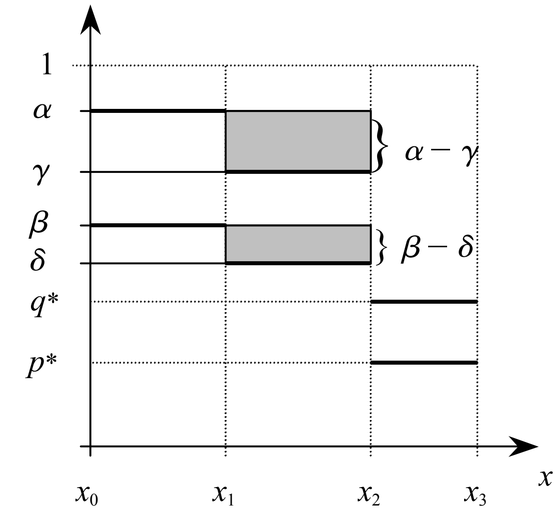
$$(\alpha - \gamma)[u(x_2) - u(x_1)] = (\beta - \delta)[u(x_2) - u(x_1)].$$

Subtracting this equality from $EU(\alpha_I P^*) = EU(\beta_I Q^*)$, and translating the result back into preference notation gives $\alpha_J \gamma_I P^* \sim \beta_J \delta_I Q^*$ under EU.

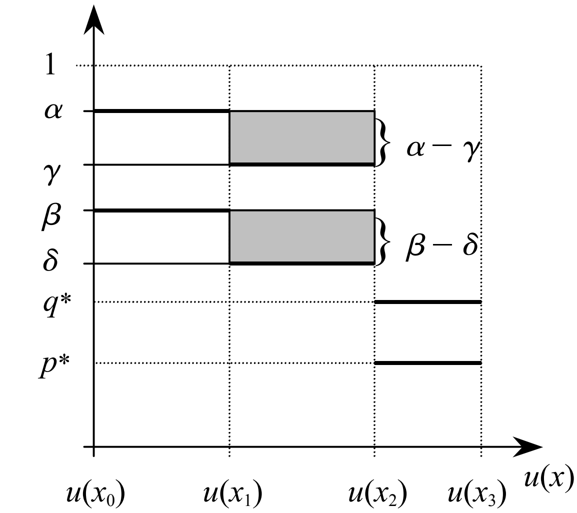
Under DT, only attitudes towards probabilities matter, and decumulative probabilities translate into weighted decumulative probabilities, outcomes however are left unchanged. This translates Figure 1(a) into Figure 1(c), where the vertical axis refers to weighted probabilities and

$$[w(\alpha) - w(\gamma)](x_2 - x_1) = [w(\beta) - w(\delta)](x_2 - x_1)$$

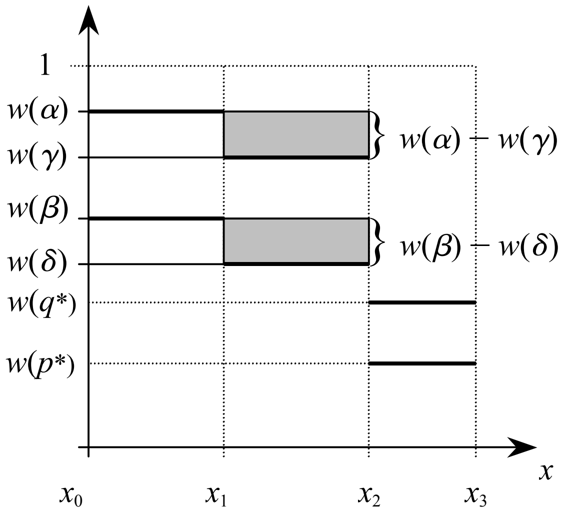
is required. This again implies $\alpha_J \gamma_I P^* \sim \beta_J \delta_I Q^*$.



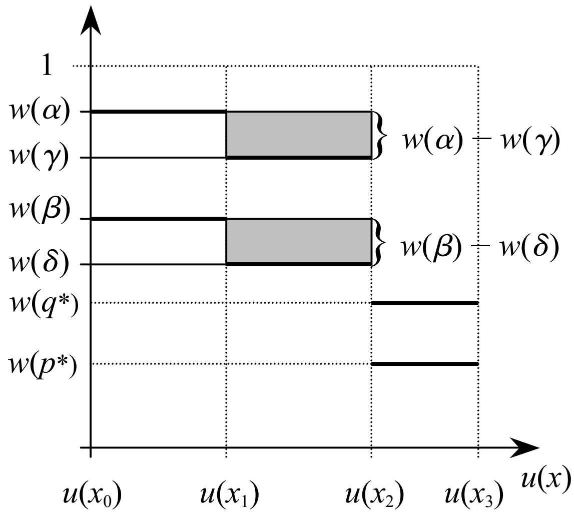
(a) Decumulative probability distributions in transfer invariance.



(b) EU: Only attitudes towards outcomes incorporated.



(c) DT: Only attitudes towards probabilities incorporated.



(d) RDEU: Attitudes towards both probability and outcomes incorporated.

Figure 1: Probabilistic Consistency

Under RDEU, both, attitudes towards outcomes and attitudes towards probabilities matter, so that Figure 1(a) transformes into Figure 1(d). There the horizontal axis refers to utilities and the vertical axis to transformed decumulative probabilities. Now

$$[w(\alpha) - w(\gamma)][u(x_2) - u(x_1)] = [w(\beta) - w(\delta)][u(x_2) - u(x_1)]$$

is required, and this implies $\alpha_J \gamma_I P^* \sim \beta_J \delta_I Q^*$.

Notice that for RDEU and DT, in the above illustration any outcome $x \in X$, such that $x_0 < x < x_2$ holds, can be substituted for x_1. This indicates that RDEU requires that differences of transformed probabilities are independent of the (utility of) outcomes provided that these outcomes are restricted to be of a certain rank, namely, second worst (or similarly second best). This is precisely what probabilistic consistency entails.

The main result of this paper can now be formulated:

THEOREM 4 *Assume the set of outcomes X contains at least four strictly ranked outcomes. Then the following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *Rank-dependent expected utility holds (with a strictly increasing and continuous weighting function w, and a strictly monotone utility function u).*

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous weak order satisfying stochastic dominance, tail independence, and probabilistic consistency.*

The utility function u is unique up to positive affine transformations, and the weighting function w is uniquely determined. □

5 Conclusion

This paper presents a new preference characterization of rank-dependent expected utility using probabilistic consistency. This condition is inspired by two features. First, under rank-dependent expected utility only the rank and not the magnitude of outcomes determines how the probability of an outcome is perceived. Second, the condition entails an aspect relating to the perception of probability changes from impossibility to possibility in aggrement with the many empirical studies that have confirmed that most sensitivity is exhibited near impossibility and certainty. Probabilistic consistency combines this intuition in a weak and natural preference condition that still suffices for a derivation of rank-dependent expected utility. The condition is weaker than the probabilistic tradeoff consistency introduce in Abdellaoui (2002), first because it is separated from comonotonic independence, and second because it is formulated for indifferences only (but see Köbberling and Wakker 2003) and further it applies only to common best or worst outcomes. Therefore, probabilistic consistency can be tested independently of the principle of comonotonic independence, and as such provides a new tool for empirical studies as well as for preference foundations.

6 Appendix

Proof of Lemma 2: It is easy to see that statement (i) implies statement (ii). For the converse, note that a Jensen-continuous monotonic weak order on L_Y^* satisfies continuity on L_Y^*. This is shown in Lemma 18 of Abdellaoui (2002). Then the proof is similar that of Lemma A.2. of Wakker and Zank (2002). The only difference is that in Wakker and Zank (2002) the arguments and preference conditions in their proof apply to the set $\mathbb{R}_\downarrow^n = \{x \in \mathbb{R}^n | x_1 \geqslant \cdots \geqslant x_n\}$ whereas here they apply to the set $[0,1]_\downarrow^n = \{p^* \in [0,1]^n | p_1^* \geqslant \cdots \geqslant p_n^*\}$, which is identified with L_Y^*. This completes the proof of Lemma 2. $\square$

Proof of Theorem 4: In the first step of the proof a similar technique is used to that in the proof of Theorem 2 of Chateauneuf (1999) in order to show that the functions in Lemma 2 are first locally, then globally, proportional. Subsequently, in Step 2 of the proof, Proposition 3.5 of Wakker (1993) applies which shows that all functions $V_1, \ldots, V_n$ can be taken finite at extreme probabilities, and therefore proportional to a function w. Then, a utility function can be defined on Y and further RDEU on L_Y. In Step 3 it is shown that the case when Y contains possibly indifferent outcomes can be inferred from the previous 2 steps in the proof. The remainder of the proof (Step 4) then follows from Abdellaoui (2002).

Step 1: For $n \geqslant 3$ take any fixed set of outcomes $Y = \{x_0, \ldots, x_n\}$, such that $x_0 \prec \cdots \prec x_n$. consider the restriction of $\succcurlyeq$ to the set of lotteries L_Y^*, which is identified here with the set $[0,1]^n_\downarrow = \{(p_1^*, \ldots, p_n^*) \in [0,1]^n | p_1^* \geqslant \cdots \geqslant p_n^*\}$. The preference relation $\succcurlyeq$ inherites weak ordering, monotonicity, Jensen-continuity, tail independence, and probabilistic consistency on L_Y^* from $\succcurlyeq$ on L^*. Then statement (ii) of Lemma 2 is satisfied and this implies the existence of jointly cardinal, continuous strictly monotonic functions $V_1, \ldots, V_n : [0,1] \to I\!R$ which are bounded except maybe V_1 and V_n which could be infinite at extreme probabilities (i.e., $0, 1$), such that $\succcurlyeq$ is represented by

$$V(p_1^*, \ldots, p_n^*) = \sum_{j=1}^{n} V_j(p_j^*).$$

Next the analysis is restricted to the preference over the set $(0,1)^n_\downarrow := \{(p_1^*, \ldots, p_n^*) \in (0,1)^n | 1 > p_1^* \geqslant \cdots \geqslant p_n^* > 0\}$, hence extreme probabilities are excluded. Fix any state $k \in \{2, \ldots, n-1\}$. Take any probability r^* such that there exists an open neighborhood $B(r^*)$ such that for all $\alpha \geqslant \beta \geqslant \gamma$ in that neighborhood with

$$\sum_{j=1}^{k} [V_j(\alpha) - V_j(\beta)] = \sum_{j=1}^{k} [V_j(\beta) - V_j(\gamma)]$$

there exist probabilities $p^* \leqslant q^* \leqslant \gamma$ such that

$$\sum_{j=1}^{k} V_j(\alpha) + \sum_{j=k+1}^{n} V_j(p^*) \;=\; \sum_{j=1}^{k} V_j(\beta) + \sum_{j=k+1}^{n} V_j(q^*)$$

and

$$\sum_{j=1}^{k} V_j(\beta)+\sum_{j=k+1}^{n} V_j(p^*) \quad = \quad \sum_{j=1}^{k} V_j(\gamma)+\sum_{j=k+1}^{n} V_j(q^*).$$

The latter two equalities are equivalent to the following indifferences, where the first dots indicate that the same probability appears in the first k coordinates:

$$(\alpha,\ldots,\alpha,p^*,\ldots,p^*) \quad \sim \quad (\beta,\ldots,\beta,q^*,\ldots,q^*)$$

and

$$(\beta,\ldots,\beta,p^*,\ldots,p^*) \quad \sim \quad (\gamma,\ldots,\gamma,q^*,\ldots,q^*).$$

probabilistic consistency implies that for any $J=\{1,\ldots,i\}$ with $i \leqslant k$

$$\alpha_J(\beta,\ldots,\beta,p^*,\ldots,p^*) \sim \beta_J(\gamma,\ldots,\gamma,q^*,\ldots,q^*).$$

Substituting the representing functional for $\succcurlyeq$ implies that

$$\sum_{j=1}^{i} V_j(\alpha)+\sum_{j=i+1}^{k} V_j(\beta)+\sum_{j=k+1}^{n} V_j(p^*) =\sum_{j=1}^{i} V_j(\beta)+\sum_{j=i+1}^{k} V_j(\gamma)+\sum_{j=k+1}^{n} V_j(q^*).$$

Further, using $(\beta,\ldots,\beta,p^*,\ldots,p^*) \sim (\gamma,\ldots,\gamma,q^*,\ldots,q^*)$ or equivalently

$$\sum_{j=k+1}^{n} V_j(p^*) =\sum_{j=1}^{k} V_j(\gamma)-\sum_{j=1}^{k} V_j(\beta)+\sum_{j=k+1}^{n} V_j(q^*)$$

in the previous equality, and cancelling common terms it follows that

$$\sum_{j=1}^{i} V_j(\alpha)-\sum_{j=1}^{i} V_j(\beta) =\sum_{j=1}^{i} V_j(\beta)-\sum_{j=1}^{i} V_j(\gamma).$$

Hence, it has been shown that

$$\sum_{j=1}^{k} [V_j(\alpha)-V_j(\beta)] \quad = \quad \sum_{j=1}^{k} [V_j(\beta)-V_j(\gamma)]$$

$$\Rightarrow$$

$$\sum_{j=1}^{i} V_j(\alpha)-\sum_{j=1}^{i} V_j(\beta) \quad = \quad \sum_{j=1}^{i} V_j(\beta)-\sum_{j=1}^{i} V_j(\gamma),$$

for any $1 \leqslant i \leqslant k$.

Next the converse is shown. Assume that for some $1 \leqslant i \leqslant k$

$$\sum_{j=1}^{i} V_j(\alpha) - \sum_{j=1}^{i} V_j(\beta) = \sum_{j=1}^{i} V_j(\beta) - \sum_{j=1}^{i} V_j(\gamma)$$

holds, but

$$\sum_{j=1}^{k} [V_j(\alpha) - V_j(\beta)] \neq \sum_{j=1}^{k} [V_j(\beta) - V_j(\gamma)].$$

Consider the case $\sum_{j=1}^{k}[V_j(\alpha) - V_j(\beta)] > \sum_{j=1}^{k}[V_j(\beta) - V_j(\gamma)]$ (and note that the opposite case, i.e., when $<$ holds, is similar). The open neighborhood $B(r^*)$ can be chosen small such that there exist $\alpha' > \alpha$ and $\hat{p}^* \leqslant \hat{q}^* \leqslant \gamma$ such that

$$\sum_{j=1}^{k} [V_j(\alpha') - V_j(\beta)] = \sum_{j=1}^{k} [V_j(\beta) - V_j(\gamma)],$$

and

$$\sum_{j=1}^{k} V_j(\alpha') + \sum_{j=k+1}^{n} V_j(\hat{p}^*) = \sum_{j=1}^{k} V_j(\beta) + \sum_{j=k+1}^{n} V_j(\hat{q}^*)$$

and

$$\sum_{j=1}^{k} V_j(\beta) + \sum_{j=k+1}^{n} V_j(\hat{p}^*) = \sum_{j=1}^{k} V_j(\gamma) + \sum_{j=k+1}^{n} V_j(\hat{q}^*).$$

similar to the analysis above, applying probabilistic consistency, this implies

$$\sum_{j=1}^{i} V_j(\alpha') - \sum_{j=1}^{i} V_j(\beta) = \sum_{j=1}^{i} V_j(\beta) - \sum_{j=1}^{i} V_j(\gamma)$$

for all $1 \leqslant i \leqslant k-1$, which is a contradiction to $\sum_{j=1}^{i}[V_j(\alpha) - V_j(\beta)] = \sum_{j=1}^{i}[V_j(\beta) - V_j(\gamma)]$, given monotonicity (recall $\alpha' > \alpha$).

Hence, we can conclude that for any $1 \leqslant i \leqslant k$ locally $\sum_{j=1}^{k} V_j(\cdot)$ and $\sum_{j=1}^{i} V_j(\cdot)$ order differences the same way, hence the functions are proportional. This statement remains valid for any $k \in \{2, \ldots, n-1\}$.

A similar argument can be used to show that for any $l \in \{2, \ldots, n-1\}$ it follows that for all $l \leqslant i \leqslant n$ locally $\sum_{j=i}^{n} V_j(\cdot)$ and $\sum_{j=l}^{n} V_j(\cdot)$ order differences the same way, hence are proportional.

Next proportionality of the functions V_j, for $j = 1, \ldots, n$ is derived. The proof employs a similar idea as in the proof of Lemma 4 of Chateauneuf (1999). Obviously, for all $1 < l \leqslant k < n$ the identity

$$V(p, \ldots, p) = \sum_{j=1}^{l-1} V_j(p) + \sum_{j=l}^{n} V_j(p) = \sum_{j=1}^{k} V_j(p) + \sum_{j=k+1}^{n} V_j(p)$$

holds for any nonextreme probability p. The above derived proportionality results imply the existence of constants $a_{1,\ldots,k}, a_{l,\ldots,n} > 0$ and $c_{1,\ldots,k}, c_{l,\ldots,n}$ such that

$$\begin{aligned} \sum_{j=l}^{n} V_j(p) &= a_{l,\ldots,n}\{\sum_{j=k+1}^{n} V_j(p)\} + c_{l,\ldots,n} \\ \sum_{j=1}^{k} V_j(p) &= a_{1,\ldots,k}\{\sum_{j=1}^{l-1} V_j(p)\} + c_{1,\ldots,k} \end{aligned}$$

which substituted in the above identity implies

$$(a_{l,\ldots,n} - 1)\{\sum_{j=k+1}^{n} V_j(p)\} + c_{l,\ldots,n} = (a_{1,\ldots,k} - 1)\{\sum_{j=1}^{l-1} V_j(p)\} + c_{1,\ldots,k}.$$

From strict monotonicity it follows that either $(a_{l,\ldots,n} - 1)$ and $(a_{1,\ldots,k} - 1)$ are of the same sign or they are both equal to zero (in which case $a_{l,\ldots,n} = a_{1,\ldots,k} = 1$ follows). The latter case is excluded by strict monotonicity. It follows that

$$\sum_{j=1}^{l-1} V_j(p) = [(a_{l,\ldots,n} - 1)\{\sum_{j=k+1}^{n} V_j(p)\} + c_{l,\ldots,n} - c_{1,\ldots,k}]/(a_{1,\ldots,k} - 1)$$

and hence, inductively, proportionality of all functions V_j, first locally then globally, on the set $(0,1)^n_{\downarrow}$ follows.

<u>Step 2:</u> From Proposition 3.5 of Wakker (1993) it follows that the functions V_j can chosen finite at extreme probabilities, and therefore can continuously be extended to the entire set $[0,1]^n_{\downarrow}$.

Set $\sum_{j=1}^{n} V_j(1) = 1$ and $V_j(0) = 0$ for all $j = 1, \ldots, n$, thereby fixing the location and common scale of the otherwise jointly cardinal V_j. Define $w(\alpha) := \sum_{j=1}^{n} V_j(\alpha)$ then w becomes unique satisfying $w(0) = 0$, $w(1) = 1$. As the functions V_j are proportional to each other, they are also proportional to their sum, w. Hence, there exists positive numbers $\pi_1, \ldots, \pi_n$ summing to one such that

$$V_j(\alpha) = \pi_j w(\alpha).$$

Therefore, on $[0,1]^n_{\downarrow}$ the following functional is a representation for $\succcurlyeq$

$$V(p_1^*, \ldots, p_n^*) = \sum_{j=1}^{n} \pi_j w(p_j^*), \tag{6}$$

where $w : [0,1] \to [0,1]$ is continuous and strictly increasing with $w(0) = 0$ and $w(1) = 1$, and the numbers $\pi_1, \ldots, \pi_n$ are positive summing to one.

Define the utility function as follows:

$$u(x_0) = 0 \text{ and } u(x_j) = \pi_j - \pi_{j-1} \text{ for } j = 1, \ldots, n.$$

Then substitution in Equation (6) above gives the RDEU form similar to Equation (5), namely:

$$V(p_1^*, \ldots, p_n^*) = u(x_0) + \sum_{j=1}^{n} w(p_j^*)[u(x_j) - u(x_{j-1})].$$

This completes the proof for the case that Y contains only strictly ranked outcomes.

Step 3: Thusfar in the analysis the case that Y contains outcomes that are indifferent has been excluded. If Y contains outcomes that are indifferent the proof is of Steps 1 and 2 above is similar. What is essential is that Y contains at least four outcomes that are strictly ordered in terms of $\succcurlyeq$. The proof is then restricted to a maximal subset of Y that contains only strictly ranked outcomes. In a similar fashion as above a representing functional as in Equation (6) can be derived. Then the weights $\pi_i = V_i = 0$ are assigned for outcomes that are indifferent to any of those in the maximal subset of Y having strictly ranked outcomes.

Step 4: The remainder of the proof is identical to the proof of Part II of Theorem 9 in Abdellaoui (2002). Only a brief overview is given here. First, for a fixed set Y of outcomes one can define a utility function u determined by the weights π_j above. It can then be shown that the utility function so defined is affine, and further that w and u are independent of the choice of Y. Because a preference between any two lotteries comes down to a preference relation between those two lotteries as lotteries with respect to a finite set of outcomes the existence of a general RDEU representation is obtained. This completes the proof. $\square$

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