A Nonnormality and Heteroskedasticity Robust Test for Skewness in Regression Models

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Abstract

In this paper a new asymptotically valid heteroskedasticity and nonnormality robust tests for skewness are proposed. Applying Davidson & Flachaire's (2001) wild bootstrap to the proposed tests is considered. Importantly the proposed tests can provide guidance on the efficacy of this wild bootstrap procedure. The Monte Carlo evidence shows that the proposed tests together with the Davidson & Flachaire's wild bootstrap method perform well.

1 Introduction

In this paper a new asymptotically valid heteroskedasticity and nonnormality robust test for skewness is derived based on standard first order asymptotic theory. A growing body of research has shown that the first order asymptotic theory often provides poor guidance to finite sample behaviour. For example, it is known that the commonly used test for skewness (or kurtosis) is severely undersized even in quite large samples (see, for example, Table 2 in Jarque and Bera (1987). For more general studies, see Orme (1990), Chesher and Spady (1991)). Thus, it is conjectured that rejection probabilities based on standard asymptotic theory may not have good agreement with their desired nominal levels, including cases when heteroskedasticity robust covariance matrices are employed.

However, in such circumstances we might apply the wild bootstrap procedure[1], which is known to have better finite sample performance in general, to the heteroskedasticity robust test for skewness, since it is generally asymptotically justified under unknown heteroskedasticity. Among the various wild bootstrap schemes that have been proposed, Davidson and Flachaire's (2001) wild bootstrap procedure, which is called the DF wild bootstrap hereafter, is the most appropriate one for our test, since, firstly, it enforces the Ordinary Least Square (OLS) residuals in "bootstrap world" to be symmetric, due to the symmetric pick distribution employed, and secondly, the DF wild bootstrap enjoys extra refinement over first order approximation, provided that the bootstrap Data

[1] The wild bootstrap procedre has proposed by Wu (1986) and has developed by Liu (1988) and Mammen (1993).

Generating Process (DGP) and test statistic are asymptotically independent, and that is the case for our test. The evidence of MacKinnon and White (1985) and Davidson and Flachaire (2001) shows that the t-test based either on i) the HCCME using restricted residuals *or* ii) the original form of the HCCME, using the DF wild bootstrap performs well. However, Godfrey and Orme (2002, 2003) have shown that the test for several linear restrictions on regression coefficients with HCCME can give good control over the finite sample significance levels only when both i) *and* ii) are adopted.

The condition for the extra refinement of the DF wild bootstrap DGP and test statistic being asymptotically independent is that the population errors are symmetric. Thus, testing for symmetry under heteroskedastic errors may also provide guidance on the efficacy of the Davidson and Flachaire procedure. In this case, some practitioners may want to implement the proposed skewness test together with an omitted variable test procedure. If so, an investigation into asymptotic insensitivity of the proposed skewness test to omitted variables is of importance, since if it is asymptotically sensitive to omitted variables, inferences based on the proposed skewness test may be misleading.

The test for skewness, which is one part of the popular Jarque and Bera (1987) normality test, could be used independently of that for excess kurtosis. However, Jarque and Bera skewness test is derived under normality assumption, which is too strong to test only for skewness. In view of this, Godfrey and Orme (1991) have derived a nonnormality robust test for skewness, under the assumption of homoskedastic errors. However, they provided no Monte Carlo evidence of the finite sample behaviour of the test. This paper therefore extends their procedure by relaxing the homoskedastic error assumption and by investigating the finite sample behaviour of various tests for skewness.

The plan of this paper is as follows. The model and tests are described in section 2. The asymptotic sensitivity of the proposed test to omitted variables is discussed in section 3. The wild bootstrap and fast double bootstrap schemes and Monte Carlo simulation are described in section 4. Finally, section 5 contains some concluding remarks.

2 The Test Statistic

Consider a linear regression model

$$y_t = \mathbf{x}_t'\boldsymbol{\beta} + u_t,\ t = 1, ..., n \tag{1}$$

where y_t is a random dependent variable, $\mathbf{x}_t$ is a $(k \times 1)$ strictly exogenous regressor vector with its first element equal to unity for all t, $\boldsymbol{\beta}$ is a $(k \times 1)$ parameter coefficient vector, and the error terms u_t are independently but not necessarily identically distributed (*inid*). In general, we can formulate the heteroskedastic errors as $u_t = \sigma(\mathbf{x}_t)\varepsilon_t$, where $\sigma(\mathbf{x}_t)$ is a scaling factor which is some (unknown) function of $\mathbf{x}_t$, and the ε_t are independently and identically distributed (*iid*) random variables, having a zero mean and a unit variance. It is assumed that $0 < E[u_t^2|\mathbf{x}_t] = \sigma_t^2 < \infty$, with $\sigma_t^2 = \sigma^2(\mathbf{x}_t)$.[2] For simplicity, it is also assumed that $p\lim_{n\to\infty}(n^{-1}\sum_{t=1}^{n}\mathbf{x}_t\mathbf{x}_t') = \mathbf{M}$, which is a finite positive

[2] However, note that we do not assume $\sigma_t > 0$, that is, σ_t could be negative. This has some importance on the "potential problem" that will be discussed later.

definite $(k \times k)$ matrix, and all moments of order up to six of u_t exist and are finite. OLS estimation gives $y_t = \mathbf{x}_t'\hat{\boldsymbol{\beta}} + \hat{u}_t$ with an obvious notation. Under heteroskedastic errors, we test the symmetry of *each* u_t, since it has different distribution. That is, the null hypothesis is

$$H_0 : E[u_t^3 | \mathbf{x}_t] = 0.$$

Note that, in this case, if we assume homoskedasticity, such that $\sigma_t^2 = \sigma^2$, the null hypothesis becomes the symmetry of all identically distributed errors, thus, the null hypothesis would be $E[u_t^3] = 0$.

2.1 Testing for Skewness under Homoskedasticity

Jarque and Bera (1987) proposed a score test of normality, which tests for skewness and kurtosis jointly. The test for skewness is

$$JB_n = n^{-1} \left(\sum_{t=1}^{n} \hat{u}_t^3 \right)^2 / 6\hat{\sigma}^6,$$

where $\hat{\sigma}^2 = n^{-1} \sum_{t=1}^{n} \hat{u}_t^2$ is an estimator for the error variance in (1). Under the normality assumption and $\sigma_t^2 = \sigma^2$, $JB_n \xrightarrow{d} \chi^2(1)$.

Godfrey and Orme (1991) have derived a non-normality robust test for skewness, which is

$$GO_n = n^{-1} \left(\sum_{t=1}^{n} \hat{u}_t^3 \right)^2 / \hat{v}_{GO},$$

where $\hat{v}_{GO} = n^{-1} \sum_{t=1}^{n} \hat{u}_t^6 + 9\hat{\sigma}^6 - 6\hat{\sigma}^2 \left(n^{-1} \sum_{t=1}^{n} \hat{u}_t^4 \right)$. Under u_t is $iid(0, \sigma^2)$, $GO_n \xrightarrow{d} \chi^2(1)$. Note, as pointed out by Godfrey and Orme (1991), that under the normality assumption, $n^{-1} \sum_{t=1}^{n} \hat{u}_t^6 \xrightarrow{p} 15\sigma^6$ and $n^{-1} \sum_{t=1}^{n} \hat{u}_t^4 \xrightarrow{p} 3\sigma^4$, so that $\hat{v}_{GO}$ converges to $6\sigma^6$, as it should. Under non-normal error, $6\hat{\sigma}^6$ is inconsistent for the asymptotic variance of $n^{-\frac{1}{2}} \sum_{t=1}^{n} \hat{u}_t^3$, and so JB_n becomes asymptotically invalid, since $JB_n \xrightarrow{d} \lambda\chi^2(1)$, with $\lambda = \text{plim}\, \hat{v}_{GO}/6\hat{\sigma}^6$.[3]

Testing based on White's (1980) HCCME is routinely adopted in empirical work. However, the finite sample behaviour of such procedure can be very poor; Chesher and Jewitt (1987). The recently proposed wild bootstrap procedure by Davidson and Flachaire (2001), the DF wild bootstrap, enjoys a refinement over first order asymptotic theory provided the errors are symmetric. Thus testing for symmetry under heteroskedastic errors may provide guidance on the efficacy of the Davidson and Flachaire procedure. Such a test generalises the statistic proposed by Godfrey and Orme (1991), and is developed in the following two sections.

2.2 Testing for Skewness under Heteroskedasticity

Consider first the asymptotic distribution of $n^{-1/2} \sum_{t=1}^{n} \hat{u}_t^3$ under heteroskedasticity. Define $\mathbf{X} = (\mathbf{x}_1, ..., \mathbf{x}_n)'$ and assume (since $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = o_p(1)$) that the fol-

[3] Godfrey and Orme (1991) provided no Monte Carlo evidence on the finite sample behaviour of their test. We will do so later.

lowing exist and are finite:

$$\begin{aligned}
\operatorname{plim} n^{-1}\sum_{t=1}^{n}\hat{u}_t^j\mathbf{x}_t &= \operatorname{plim} n^{-1}\sum_{t=1}^{n}u_t^j\mathbf{x}_t = \operatorname{plim} n^{-1}\sum_{t=1}^{n}E[u_t^j|\mathbf{x}_t]\mathbf{x}_t \\
&= \mathbf{q}^{(j)} && \text{(A1)}\\
\operatorname{plim} n^{-1}\sum_{t=1}^{n}\hat{u}_t^2\mathbf{x}_t\mathbf{x}_t' &= \operatorname{plim} n^{-1}\sum_{t=1}^{n}u_t^2\mathbf{x}_t\mathbf{x}_t' = \operatorname{plim} n^{-1}\sum_{t=1}^{n}E[u_t^2|\mathbf{x}_t]\mathbf{x}_t\mathbf{x}_t' \\
&= \mathbf{G} && \text{(A2)}\\
\operatorname{plim} n^{-1}\sum_{t=1}^{n}\hat{u}_t^6 &= \operatorname{plim} n^{-1}\sum_{t=1}^{n}u_t^6 = \operatorname{plim} n^{-1}\sum_{t=1}^{n}E[u_t^6|\mathbf{x}_t] \\
&= \bar{\mu}_6, && \text{(A3)}
\end{aligned}$$

with $j = 2, 4$. Dealing initially with $n^{-1/2}\sum_{t=1}^{n}\hat{u}_t^3$, a Taylor series expansion around $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}$ yields

$$\begin{aligned}
n^{-1/2}\sum_{t=1}^{n}\hat{u}_t^3 &= n^{-1/2}\sum_{t=1}^{n}u_t^3 - 3\left(n^{-1}\sum_{t=1}^{n}u_t^2\mathbf{x}_t'\right)n^{1/2}\left(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right) + o_p(1)\\
&= n^{-1/2}\sum_{t=1}^{n}\left[u_t^3 - 3\left(n^{-1}\sum_{t=1}^{n}u_t^2\mathbf{x}_t'\right)\left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{x}_t u_t\right] + o_p(1)\\
&= n^{-1/2}\sum_{t=1}^{n}\left(u_t^2 - 3\mathbf{q}^{(2)\prime}\mathbf{M}^{-1}\mathbf{x}_t\right)u_t + o_p(1)\\
&= n^{-1/2}\sum_{t=1}^{n}w_t + o_p(1). && (2)
\end{aligned}$$

Now $E[w_t|\mathbf{x}_t] = E[u_t^3|\mathbf{x}_t] - 3\mathbf{q}^{(2)\prime}\mathbf{M}^{-1}\mathbf{x}_t E[u_t|\mathbf{x}_t] = 0$ and $E[w_t^2|\mathbf{x}_t] = \eta(\mathbf{x}_t) > 0$ such that $n^{-1}\sum_{t=1}^{n}\eta(\mathbf{x}_t) - n^{-1}\sum_{t=1}^{n}w_t^2 = o_p(1)$ with

$$\operatorname{plim}\left(n^{-1}\sum_{t=1}^{n}w_t^2\right) = \bar{\mu}_6 - 6\mathbf{q}^{(2)\prime}\mathbf{M}^{-1}\mathbf{q}^{(4)} + 9\mathbf{q}^{(2)\prime}\mathbf{M}^{-1}\mathbf{G}\mathbf{M}^{-1}\mathbf{q}^{(2)}.$$

Then a suitable Central Limit Theorem (CLT) ensures that

$$\frac{n^{-1/2}\sum_{t=1}^{n}w_t}{\left(n^{-1}\sum_{t=1}^{n}w_t^2\right)^{\frac{1}{2}}} \xrightarrow{d} N(0,1) \text{ and } \frac{n^{-1}\left(\sum_{t=1}^{n}w_t\right)^2}{n^{-1}\sum_{t=1}^{n}w_t^2} \xrightarrow{d} \chi^2(1). \qquad (3)$$

As $n^{-1/2}\sum_{t=1}^{n}\hat{u}_t^3$ and $n^{-1/2}\sum_{t=1}^{n}w_t$ have the same asymptotic distribution, by (2), we have

$$\frac{n^{-1}\left(\sum_{t=1}^{n}\hat{u}_t^3\right)^2}{\left(n^{-1}\sum_{t=1}^{n}w_t^2\right)} \xrightarrow{d} \chi^2(1).$$

From the assumption (A1)-(A3), we see that $n^{-1}\sum_{t=1}^{n}\hat{w}_t^2 - n^{-1}\sum_{t=1}^{n}w_t^2 = o_p(1)$ where

$$\hat{w}_t = \left[\hat{u}_t^2 - 3\left(n^{-1}\sum_{t=1}^{n}\hat{u}_t^2\mathbf{x}_t'\right)\left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{x}_t\right]\hat{u}_t.$$

Thus, an asymptotically valid non-normality and heteroskedasticity robust test for skewness is obtained as

$$GOh_n = n^{-1}\left(\sum_{t=1}^{n}\hat{u}_t^3\right)^2/\hat{v}_{GOh}, \tag{4}$$

where $\hat{v}_{GOh} = n^{-1}\sum_{t=1}^{n}\hat{w}_t^2$, and above arguments ensure that $GOh_n \xrightarrow{d} \chi^2(1)$, under the null hypothesis, $E[u_t^3|\mathbf{x}_t] = 0$.[4]

Note that $n^{-1}\sum_{t=1}^{n}w_t^2$ can be written as $n^{-1}\mathbf{w}'\mathbf{w}$, where $\mathbf{w}$ is a $(n \times 1)$ vector whose typical element is w_t, and the corresponding consistent variance estimator can be written as $n^{-1}\hat{\mathbf{w}}'\hat{\mathbf{w}}$. Also the test indicator can be written as $\sum_{t=1}^{n}\hat{u}_t^3 = \boldsymbol{\iota}'\hat{\mathbf{w}}$, where $\boldsymbol{\iota}$ is a $(n \times 1)$ vector consisting of 1, because $\sum_{t=1}^{n} 3\left(n^{-1}\sum_{t=1}^{n}\hat{u}_t^2\mathbf{x}_t'\right)\left(n^{-1}\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{x}_t\hat{u}_t = 0$ from the normal equations. Defining a $(n \times n)$ diagonal matrix $\hat{\mathbf{U}}^{(j)}=diag(\hat{u}_t^j)$, a $(n \times 1)$ vector $\hat{\mathbf{u}}^{(j)} = (\hat{u}_1^j, ..., \hat{u}_n^j)$, $j = 1, ..., 6$ and $\mathbf{P_x} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, the asymptotically valid test statistic is readily obtainable as n minus residual sum of squares from

regressing $\boldsymbol{\iota}$ on $\hat{\mathbf{w}}$, where $\hat{\mathbf{w}} = \hat{\mathbf{U}}^{(1)}\hat{\mathbf{u}}^{(2)} - 3\hat{\mathbf{U}}^{(1)}\mathbf{P_x}\hat{\mathbf{u}}^{(2)}$ $(n \times 1)$.

2.3 Potential Inconsistency

The heteroskedasticity robust test, introduced above, has the following potential problem. Recall that the heteroskedastic errors in (1) are $u_t = \sigma(\mathbf{x}_t)\varepsilon_t$, where $\sigma(\mathbf{x}_t)$ is a scaling factor which is some (unknown) function of $\mathbf{x}_t$, and the ε_t are $iid(0,1)$. Theorem 1 of Davidson and Flachaire (2001; p.8) assumes that "the error terms are mutually independent with mean zero and distributions symmetric about the origin". Therefore, under heteroskedasticity, our interest is to test in symmetry of *each* u_t, thus the null hypothesis is $E[u_t^3|\mathbf{x}_t] = E[\sigma(\mathbf{x}_t)^3\varepsilon_t^3|\mathbf{x}_t] = 0$, or equivalently, $E[\varepsilon_t^3] = 0$.

However, the test indicator employed previously has the property that $n^{-1}\sum_{t=1}^{n}\hat{u}_t^3 - n^{-1}\sum_{t=1}^{n}E[u_t^3] = o_p(1)$,[5] and the inconsistency of the test arises because $E[u_t^3] = 0$ need not imply that $E[u_t^3|\mathbf{x}_t] = 0$. For example, suppose that $\sigma(\mathbf{x}_t)$ is a function of $\mathbf{x}_t$ such that $E[\sigma(\mathbf{x}_t)^3] = 0$, and $E[\varepsilon_t^3] = \tau \neq 0$. In this case $E[u_t^3|\mathbf{x}_t] = \sigma(\mathbf{x}_t)^3\tau \neq 0$, however, $E[u_t^3] = E_x[\sigma(\mathbf{x}_t)^3\tau] = 0$. That is, based on (4), we may accept the null hypothesis even though $E[u_t^3|\mathbf{x}_t] \neq 0$, and the test is inconsistent under this alternative.

Note that $E[u_t^3|\mathbf{x}_t]$ can be regarded as a function of $\mathbf{x}_t$, since $E[u_t^3|\mathbf{x}_t] = g(\mathbf{x}_t)E[\varepsilon^3]$ with $g(\mathbf{x}_t) = \sigma(\mathbf{x}_t)^3$. This suggests that a check of whether $E[u_t^3 g(\mathbf{x}_t)] = 0$ is appropriate, since with skewed errors $E[u_t^3 g(\mathbf{x}_t)] = E[g(\mathbf{x}_t)^2\varepsilon^3] \neq 0$ because $E[g(\mathbf{x}_t)^2] > 0$. In general the function $g(\mathbf{x}_t)$ is unknown, so some proxy, say a $(r \times 1)$ vector function $\boldsymbol{\psi}(\cdot)$ can be used, where $\boldsymbol{\psi}(\mathbf{x}_t)$ is correlated to $g(\mathbf{x}_t)$. The test indicator is then such that

$$n^{-1}\sum_{t=1}^{n}\hat{u}_t^3\boldsymbol{\psi}(\mathbf{x}_t) \xrightarrow{p} n^{-1}\sum_{t=1}^{n}E[g(\mathbf{x}_t)\boldsymbol{\psi}(\mathbf{x}_t)\varepsilon_t^3] \neq 0.$$

[4] However, the test is potentially inconsistent under $E(u_t^3|\mathbf{x}_t) \neq 0$; see below.

[5] Standard asymptotic theory tells that $n^{-1}\sum_{t=1}^{n}\hat{u}_t^3 - n^{-1}\sum_{t=1}^{n}u_t^3 = o_p(1)$ and the law of large number ensures that $n^{-1}\sum_{t=1}^{n}u_t^3 - n^{-1}\sum_{t=1}^{n}E[u_t^3] = o_p(1)$.

Following discussion similar to the above, under heteroskedasticity it can be shown by a similar $O_p(1)$ expansion of $n^{-\frac{1}{2}}\sum_{t=1}^{n}\hat{u}_t^3\boldsymbol{\psi}(\mathbf{x}_t)$ that

$$\hat{\mathbf{u}}^{(3)\prime}\boldsymbol{\Psi}\hat{\mathbf{V}}^{-1}\boldsymbol{\Psi}'\hat{\mathbf{u}}^{(3)} \xrightarrow{d} \chi^2(r),$$

where $\boldsymbol{\Psi}$ is $(n\times r)$ matrix whose typical row is $\boldsymbol{\psi}(\mathbf{x}_t)'$, and

$$\hat{\mathbf{V}} = \boldsymbol{\Psi}'\hat{\mathbf{U}}^{(6)}\boldsymbol{\Psi} + 9\boldsymbol{\Psi}'\hat{\mathbf{U}}^{(2)}\mathbf{P}_{\mathbf{x}}\hat{\mathbf{U}}^{(2)}\mathbf{P}_{\mathbf{x}}\hat{\mathbf{U}}^{(2)}\boldsymbol{\Psi} - 6\boldsymbol{\Psi}'\hat{\mathbf{U}}^{(4)}\mathbf{P}_{\mathbf{x}}\hat{\mathbf{U}}^{(2)}\boldsymbol{\Psi}.$$

As a parametric proxy of $\boldsymbol{\psi}(\mathbf{x}_t)$, test regressors used by Ramsey's (1969) RESET test or White's (1980) heteroskedasticity test may be an appropriate choice.

The standard RESET test is based on the result that the conditional expectation of residuals of misspecified linear models, $E[\hat{u}_t|\mathbf{x}_t]$, can be approximated by the polynomials of conditional expectation of dependent variables $E[y_t|\mathbf{x}_t]^j = (\mathbf{x}_t'\boldsymbol{\beta})^j$, $j=2,3$. For the present case we set $\boldsymbol{\psi}(\mathbf{x}_t) = ((\mathbf{x}_t'\boldsymbol{\beta})^2,(\mathbf{x}_t'\boldsymbol{\beta})^3)'$ with $r=3$. Note that $\hat{\boldsymbol{\beta}}$ is used instead of $\boldsymbol{\beta}$ since it is not observable, but this replacement does not affect the asymptotic results (see Chapter 2) under the null. This test statistic is called GOh_R_n.

As in White's (1980) heteroskedasticity test, we choose $\boldsymbol{\psi}(\mathbf{x}_t) = vech(\mathbf{x}_t\mathbf{x}_t')$ with $r = k(k+1)/2$. For both cases, consistency requires non-zero correlation between $\boldsymbol{\psi}(\mathbf{x}_t)$ and $\sigma^3(\mathbf{x}_t)$. This test statistic is called GOh_W_n.

Another interesting testing procedure is that based on Whang's (2000) non-parametric test for misspecified functional form, which is consistent under heteroskedasticity, and generalised in Whang (2001). Whang's test uses a generalised Cramer-Von Mises test, and is powerful against $n^{-1/2}$ local alternatives. (Whang's test does not depend on a smoothing parameter, unlike Zheng's (1996) non-parametric approach.) The asymptotic distribution of Whang's test statistic is case dependent, so the test is not asymptotically pivotal in Beran's (1988) sense. The bootstrap method yields asymptotically valid critical values, but here, the double bootstrapping is also of interest. As Beran (1988) shows, single bootstrapped non-asymptotically pivotal test only yields the same order of the error in rejection probability as the first order approximation, but the double bootstrap yields further refinement.[6]

To explain Whang's approach, consider data $\mathbf{v}_i' = (y_i, \mathbf{x}_i')$ that may be heteroskedastic, and an $(k\times 1)$ unknown parameter vector $\boldsymbol{\theta}$, and $\boldsymbol{\theta}_*$ is a pseudo-true parameter vector such that $\hat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}_*$. We are interested in testing the moment condition $E[\varepsilon(y_i;\boldsymbol{\theta}_*)|\mathbf{x}_i]$

$$\begin{aligned} H_0 &: E[\varepsilon(y_i;\boldsymbol{\theta}_*)|\mathbf{x}_i] = 0 \text{ almost surely for some } \boldsymbol{\theta}_* \in \Theta \\ H_1 &: E[\varepsilon(y_i;\boldsymbol{\theta}_*)|\mathbf{x}_i] \neq 0 \text{ for all } \boldsymbol{\theta} \in \Theta. \end{aligned}$$

Under either the null or alternative, $E[\varepsilon(y_i;\boldsymbol{\theta})|\mathbf{x}_i] = m(\boldsymbol{\theta};\mathbf{x}_i)$ for all $\boldsymbol{\theta}\in\Theta$. Then, under the null hypothesis, there exists a $\boldsymbol{\theta}_*$ such that $m(\boldsymbol{\theta}_*;\mathbf{x}_i) = 0$. Note that under the alternative, no $\boldsymbol{\theta}_*$ satisfies $m(\boldsymbol{\theta}_*;\mathbf{x}_i) = 0$.

[6] Whang (2001) proposed recentering in bootstrap resampling to impose the null hypothesis, however, in our case, the DF wild bootstrap can impose the null hypothesis as shown later, and there is no need of recentering.

Whang (2000) proposes a generalised Cramer-Von Mises statistics.[7] Define

$$\begin{aligned} r_i(\mathbf{z}) &= \mathbb{I}(\mathbf{x}_i \leq \mathbf{z}) \\ &= \mathbf{\Pi}_{m=1}^{k}[I(x_{im} \leq z_m)] \\ H_n(\mathbf{z}) &= n^{-1}\sum_i \varepsilon_i r_i(\mathbf{z}), \end{aligned}$$

where $I(x_{im} \leq z_m)$ is an indicator function and subscript m denotes mth row of the vectors $\mathbf{x}_i$ and $\mathbf{z}$. Let $\mathbb{X} = \{\mathbf{x}_i; i \geq 0\}$ so that

$$\begin{aligned} E[H_n(\mathbf{z})|\mathbb{X}] &= E\left[n^{-1}\sum_i \varepsilon_i r_i(\mathbf{z})|\mathbb{X}\right] \\ &= n^{-1}\sum_i m(\boldsymbol{\theta}_*; \mathbf{x}_i) r_i(\mathbf{z}) \\ &\xrightarrow{p} \int_{\mathbf{x}<\mathbf{z}} m(\boldsymbol{\theta}_*; \mathbf{x}) dG(\mathbf{x}) \begin{cases} = 0, & \text{under } H_0 \\ \neq 0, & \text{under } H_1 \quad \text{for at least one } \mathbf{z} \text{ and } \boldsymbol{\theta}_*, \end{cases} \end{aligned}$$

where $G(\cdot) = \text{plim}_{n\to\infty}\sum_{i=1}^{n} G_i(\cdot)$, and $G_i(\mathbf{x})$ is the empirical distribution of $\{\mathbf{x}_i : i = 1, ..., n\}$. Note that $\int_{\mathbf{x}\in\mathbb{R}^k} m(\boldsymbol{\theta}_*; \mathbf{x}) dG(\mathbf{x})$ could be zero but there may exist some $\mathbf{z}$ such that

$$\int_{\mathbf{x}<\mathbf{z}} m(\boldsymbol{\theta}_*; \mathbf{x}) dG(\mathbf{x}) = \zeta(\boldsymbol{\theta}_*; \mathbf{z}) \neq 0.$$

This "device" prevents Whang test from failing to detect alternatives in any direction, and is therefore useful for our purposes. Now define

$$\hat{H}_n(\mathbf{x}_i) = n^{-\frac{1}{2}}\sum_j \hat{\varepsilon}_j r_j(\mathbf{x}_i).$$

The Whang test statistic is

$$\hat{W}_n = n^{-1}\sum_i \left\{\hat{H}_n(\mathbf{x}_i)\right\}^2$$

where $\hat{W}_n = O_p(1)$ under the null. Its asymptotic distribution is unknown, then, bootstrapping $\hat{W}_n$ yields valid inference; see Whang (2000, 2001). Note that, consistency follows from the fact that[8]

$$n^{-1}\hat{W}_n \xrightarrow{p} \int \xi(\boldsymbol{\theta}_*; \mathbf{x})^2 dG(\mathbf{x}) \geq 0$$

where

$$\begin{cases} \xi(\boldsymbol{\theta}_*; \mathbf{x}) \neq 0 & \text{under } H_1 \quad \text{for some } \mathbf{x} \text{ but} \\ \xi(\boldsymbol{\theta}_*; \mathbf{x}) = 0 & \text{under } H_0 \quad \text{for all } \mathbf{x}, \end{cases}$$

so that $W_n = O_p(n)$ under H_1.

[7] It is based on the work of Andrews (1997).

[8] Note that

$$n^{-1}W_n = n^{-1}\sum_i \left[n^{-1}\sum_j \varepsilon_j r_j(\mathbf{x}_i)\right]^2.$$

For testing the skewness of errors in heteroskedastic models, Whang's test statistic would be

$$Whang_n = n^{-1}\sum_{t=1}^{n}\left\{\hat{H}_n(\mathbf{x}_t^*)\right\}^2$$

with $\mathbf{x}_t^{*\prime} = (x_{t2}, ..., x_{tk})$ is the regressor (with the constant term removed), where

$$\hat{H}_n(\mathbf{x}_t^*) = n^{-1/2}\sum_{s=1}^{n}\hat{u}_s^3 \mathbb{I}(\mathbf{x}_s^* \le \mathbf{x}_t^*)$$

where $\mathbb{I}(\mathbf{x}_s^* \le \mathbf{x}_t^*) = \mathbf{\Pi}_{m=2}^{k}[I(x_{tm} \le x_m)]$ with $I(x_{tm} \le x_m)$ being an indicator function

$$I(x_{tm} \le x_m) = \begin{cases} 1, \text{ if } x_{tm} \le x_m \\ 0, \text{ if } x_{tm} > x_m \end{cases}.$$

In this section, a heteroskedasticity and nonnormality robust test for skewness in linear regression model is derived. Also, the "potential inconsistency" under which GOh_n may have no power is pointed out. To overcome this problem, modified test statistics, GOh_W_n and GOh_W_n that are based on RESET and White approach, as well as Whang's nonparametric test are proposed.

3 Sensitivity of the Heteroskedasticity Robust Skewness Test to Omitted Variables

Here we consider the sensitivity of the skewness test to omitted variables in heteroskedastic models, since one might want to check the skewness of heteroskedastic disturbances before testing for omitted variables in order to check the efficacy of the DF wild bootstrap. In this case, the sensitivity of skewness test is of some importance. Godfrey and Orme (1994) investigated the insensitivity of the skewness test in homoskedastic regression models. Their approach can be used for heteroskedastic models as well, and the discussion follows.

Following Godfrey and Orme (1994), consider a model

$$y_t = \mathbf{x}_t'\boldsymbol{\beta} + \mathbf{z}_t'\boldsymbol{\gamma}_n + u_t, \tag{5}$$

where u_t is homoskedastic $iid(0, \sigma^2)$ or heteroskedastic $inid(0, \sigma_t^2)$, $\boldsymbol{\gamma}_n = n^{-1/2}\boldsymbol{\delta}$, $\boldsymbol{\delta}'\boldsymbol{\delta} < \infty$. The alternative fitted model is denoted as $y_t = \mathbf{x}_t'\bar{\boldsymbol{\beta}} + \mathbf{z}_t'\bar{\boldsymbol{\gamma}} + \tilde{u}_t$. Consider first the unweighted test for skewness. Since $\hat{u}_t = \tilde{u}_t + \mathbf{x}_t'(\bar{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}) + \mathbf{z}_t'\bar{\boldsymbol{\gamma}}$ by standard regression theory,

$$\begin{aligned} n^{-1/2}\sum_{t=1}^{n}\hat{u}_t^3 &= n^{-1/2}\sum_{t=1}^{n}\left\{\tilde{u}_t + \mathbf{x}_t'(\bar{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}) + \mathbf{z}_t'\bar{\boldsymbol{\gamma}}\right\}^3 \\ &= n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^3 + 3n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^2\left\{\mathbf{x}_t'(\bar{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}) + \mathbf{z}_t'\bar{\boldsymbol{\gamma}}\right\} + o_p(1) \\ &= n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^3 + 3n^{-1}\sum_{t=1}^{n}\tilde{u}_t^2\left\{\ddot{\mathbf{z}}_t'\boldsymbol{\delta}\right\} + o_p(1), \end{aligned}$$

with $\ddot{\mathbf{z}}_t$ being a residual vector from regressing $\mathbf{z}_t$ on $\mathbf{x}_t$, since $\bar{\boldsymbol{\gamma}} - \boldsymbol{\delta}/\sqrt{n} = O_p(n^{-1/2})$ and $(\bar{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}) = -(\sum_{t=1}^{n}\mathbf{x}_t\mathbf{x}_t')^{-1}\sum_{t=1}^{n}\mathbf{x}_t\mathbf{z}_t'\bar{\boldsymbol{\gamma}} = O_p(n^{-1/2})$. As $\sum_{t=1}^{n}\ddot{\mathbf{z}} =$

0 because $\mathbf{x}_t$ includes unity,

$$n^{-1/2}\sum_{t=1}^{n}\hat{u}_t^3 = n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^3 + 3n^{-1/2}\left\{n^{-1/2}\sum_{t=1}^{n}\left(\tilde{u}_t^2-\tilde{\sigma}^2\right)\ddot{\mathbf{z}}_t'\right\}\boldsymbol{\delta} + o_p(1) \quad (6)$$

where $\tilde{\sigma}^2 = n^{-1}\sum_{t=1}^{n}\tilde{u}_t^2$. With the symmetric errors, $n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^3 = O_p(1)$. Note that inside of $\{\cdot\}$ of the second term of (6) is Koenker's studentised heteroskedasticity test indicator with test regressors $\ddot{\mathbf{z}}_t$. Under the homoskedasticity, $\{\cdot\}$ is $O_p(1)$, then the second term becomes $O_p(n^{-1/2})$ so

$$n^{-1/2}\sum_{t=1}^{n}\hat{u}_t^3 = n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^3 + o_p(1), \quad (7)$$

which tells that this test for skewness is asymptotically locally insensitive to the omitted variable *under the homoskedasticity.* However, under the heteroskedasticity, the second term in (6) can be $O_p(1)$, and local insensitivity will be lost.

However, under the *local* heteroskedasticity, in a sense that the second term in (6) is still $O_p(n^{-1/2})$, then (7) holds as well as the local insensitivity of the heteroskedasticity robust test for skewness to the omitted variable. To see this, consider heteroskedasticity characterised by

$$\sigma_t^2 = h(\alpha + \mathbf{r}_t'\boldsymbol{\xi}) = h_t(\boldsymbol{\xi})$$

where $h_t(0) = \sigma^2 > 0$, $h'(c_t) = dh(c_t)/dc_t \neq 0$ with $c_t = \alpha + \mathbf{r}_t'\boldsymbol{\xi}$, $h(0) = h'(0) = 1$, $\mathbf{r}_t$ is a vector causing heteroskedasticity, and $\boldsymbol{\xi} = n^{-1/2}\boldsymbol{\zeta}$, $\boldsymbol{\zeta}'\boldsymbol{\zeta} < \infty$. Then, since $\tilde{u}_t = u_t - \mathbf{x}_t'(\bar{\boldsymbol{\beta}}-\boldsymbol{\beta}) - \mathbf{z}_t'(\bar{\boldsymbol{\gamma}}-\boldsymbol{\gamma})$ by standard OLS theory,

$$\begin{aligned}\tilde{u}_t^2 \;=\;& u_t^2 - 2u_t\mathbf{x}_t'(\bar{\boldsymbol{\beta}}-\boldsymbol{\beta}) - 2u_t\mathbf{z}_t'(\bar{\boldsymbol{\gamma}}-\boldsymbol{\gamma}) \\ & +2(\bar{\boldsymbol{\beta}}-\boldsymbol{\beta})'\mathbf{x}_t\mathbf{z}_t'(\bar{\boldsymbol{\gamma}}-\boldsymbol{\gamma}) + (\bar{\boldsymbol{\beta}}-\boldsymbol{\beta})'\mathbf{x}_t\mathbf{x}_t'(\bar{\boldsymbol{\beta}}-\boldsymbol{\beta}) + (\bar{\boldsymbol{\gamma}}-\boldsymbol{\gamma})'\mathbf{z}_t\mathbf{z}_t'(\bar{\boldsymbol{\gamma}}-\boldsymbol{\gamma}).\end{aligned}$$

Thus, inside of $\{\cdot\}$ in (6) is

$$\begin{aligned} n^{-1/2}\sum_{t=1}^{n}\left(\tilde{u}_t^2-\tilde{\sigma}^2\right)\ddot{\mathbf{z}}_t &= n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^2\ddot{\mathbf{z}}_t \\ &= n^{-1/2}\sum_{t=1}^{n}u_t^2\ddot{\mathbf{z}}_t + o_p(1) \\ &= n^{-1/2}\sum_{t=1}^{n}\left(u_t^2-\sigma^2\right)\ddot{\mathbf{z}}_t + o_p(1), \end{aligned}$$

since $n^{-1}\sum_{t=1}^{n}\mathbf{x}_t\mathbf{x}_t'$ and $n^{-1}\sum_{t=1}^{n}\mathbf{z}_t\mathbf{z}_t'$ are $O_p(1)$, and $(\bar{\boldsymbol{\beta}}-\boldsymbol{\beta})$, $(\bar{\boldsymbol{\gamma}}-\boldsymbol{\gamma})$, $n^{-1}\sum_{t=1}^{n}u_t\ddot{\mathbf{z}}_t\mathbf{x}_t'$, and $n^{-1}\sum_{t=1}^{n}u_t\ddot{\mathbf{z}}_t\mathbf{z}_t'$ are $O_p(n^{-1/2})$. Now a mean value expansion of $h_t(\boldsymbol{\xi})$ yields

$$\begin{aligned} E[u_t^2] &= h(\alpha + \mathbf{r}_t'\boldsymbol{\xi}) \\ &= h(\alpha) + h'(\bar{c}_t)\mathbf{r}_t'\boldsymbol{\xi} \\ &= \sigma^2 + h'(\bar{c}_t)\mathbf{r}_t'\boldsymbol{\xi}, \end{aligned}$$

where $\bar{c}_t = \alpha + \mathbf{r}_t'\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\xi}} \in (\boldsymbol{\xi}, \mathbf{0})$. Since $n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(u_t^2 - E(u_t^2)\right)\ddot{\mathbf{z}}_t = O_p(1)$ by a standard CLT, we can write

$$\begin{aligned}
n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(u_t^2-E(u_t^2)\right)\ddot{\mathbf{z}}_t &= n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(u_t^2-\sigma^2-h'(\bar{c}_t)\mathbf{r}_t'\boldsymbol{\xi}\right)\ddot{\mathbf{z}}_t \\
&= n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(u_t^2-\sigma^2\right)\ddot{\mathbf{z}}_t-n^{-\frac{1}{2}}\sum_{t=1}^{n}h'(\bar{c}_t)\mathbf{r}_t'\boldsymbol{\xi}\ddot{\mathbf{z}}_t.
\end{aligned}$$

Now, $\bar{c}_t$ converges to 0 and $h'(\bar{c}_t)$ goes to 1, a Uniform Law of Large Numbers gives

$$\begin{aligned}
n^{-\frac{1}{2}}\sum_{t=1}^{n}h'(\bar{c}_t)\mathbf{r}_t'\boldsymbol{\xi}\ddot{\mathbf{z}}_t &= n^{-1}\sum_{t=1}^{n}h'(\bar{c}_t)\mathbf{r}_t'\boldsymbol{\zeta}\ddot{\mathbf{z}}_t \\
&= n^{-1}\sum_{t=1}^{n}E[\ddot{\mathbf{z}}_t\mathbf{r}_t']\boldsymbol{\zeta}+o_p(1),
\end{aligned}$$

which is $O(1)$. Putting all together, under the local heteroskedasticity,

$$\begin{aligned}
n^{-1/2}\sum_{t=1}^{n}\left(\tilde{u}_t^2-\tilde{\sigma}^2\right)\ddot{\mathbf{z}}_t &= n^{-1/2}\sum_{t=1}^{n}\left(u_t^2-\sigma^2\right)\ddot{\mathbf{z}}_t+o_p(1) \\
&= n^{-1/2}\sum_{t=1}^{n}u_t^2\ddot{\mathbf{z}}_t+o_p(1) \\
&= n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(u_t^2-E(u_t^2)\right)\ddot{\mathbf{z}}_t+o_p(1) \\
&= n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(u_t^2-\sigma^2\right)\ddot{\mathbf{z}}_t-n^{-1}\sum_{t=1}^{n}E[\ddot{\mathbf{z}}_t\mathbf{r}_t']\boldsymbol{\zeta}\ +o_p(1).
\end{aligned}$$

Finally, as $n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(u_t^2-\sigma^2\right)\ddot{\mathbf{z}}_t\xrightarrow{d}N(0,\Xi)$,

$$n^{-1/2}\sum_{t=1}^{n}\left(\tilde{u}_t^2-\tilde{\sigma}^2\right)\ddot{\mathbf{z}}_t\xrightarrow{d}N(\lambda,\Xi),$$

where λ is $-n^{-1}\sum_{t=1}^{n}E[\ddot{\mathbf{z}}_t\mathbf{r}_t']\boldsymbol{\zeta}$ and $\Xi=\operatorname{plim}n^{-1}\sum_{t=1}^{n}\sigma_t^2\ddot{\mathbf{z}}_t\ddot{\mathbf{z}}_t'$.

Therefore, under the local heteroskedasticity, inside of $\{\cdot\}$ of the second term of (6) remains $O_p(1)$ and the second term overall becomes $o_p(1)$, then asymptotic local insensitivity of the skewness test to the omitted variable holds.

Now consider the insensitivity of the modified test to omitted variables. A similar discussion to the above gives

$$\begin{aligned}
n^{-1/2}\sum_{t=1}^{n}\hat{u}_t^3\boldsymbol{\psi}(\mathbf{x}_t) &= n^{-1/2}\sum_{t=1}^{n}\left\{\tilde{u}_t+\mathbf{x}_t'(\bar{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})+\mathbf{z}_t'\bar{\boldsymbol{\gamma}}\right\}^3\boldsymbol{\psi}(\mathbf{x}_t) \\
&= n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^3\boldsymbol{\psi}(\mathbf{x}_t)+3n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^2\boldsymbol{\psi}(\mathbf{x}_t)\left\{\mathbf{x}_t'(\bar{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})+\mathbf{z}_t'\bar{\boldsymbol{\gamma}}\right\}+o_p(1) \\
&= n^{-1/2}\sum_{t=1}^{n}\tilde{u}_t^3\boldsymbol{\psi}(\mathbf{x}_t)+3n^{-1}\sum_{t=1}^{n}\tilde{u}_t^2\left\{\boldsymbol{\psi}(\mathbf{x}_t)\ddot{\mathbf{z}}_t'\boldsymbol{\delta}\right\}+o_p(1),
\end{aligned}$$

and the second term becomes $O_p(1)$ in general, thus the modified indicator is asymptotically sensitive to local omitted variables in homoskedastic or heteroskedastic models.

Therefore, in general, no omitted variable should be assumed before the proposed heteroskedasticity robust skewness tests are implemented. This has an important implication for the testing procedure, as explained below.

When a practitioner employs the DF wild bootstrap procedure for testing omitted variables in heteroskedastic models, he may want to test the efficacy of this procedure by using the proposed skewness test *before* conducting the omitted variable test. In that case, he should use the OLS residuals from the regression model including all the potential regressors for proposed skewness test.[9] After that, he may proceed to the omitted variable test in heteroskedastic models together with DF wild bootstrap procedure, but he should use the residuals of restricted models to obtain reliable test results, as Davidson and Flachaire (2001) and Godfrey and Orme (2002) suggest.

4 Wild Bootstrap Procedure

Bootstrap methods can achieve asymptotic refinements over first order theory by constructing the empirical distribution of the (test) statistic through re-sampling from the null model assuming the original estimates define the "true" data generating process (DGP). However, under unknown heteroskedasticity, the conventional (non-parametric) bootstrap does not work since it cannot mimic the "true" DGP. To overcome this problem, the wild bootstrap is advocated by Wu (1986) and further developed by Liu (1988) and Mammen (1993).[10] Generally, the wild bootstrap resamples the data as $y_t^* = \mathbf{x}_t'\hat{\boldsymbol{\beta}} + u_t^{**}$, where

$$u_t^{**} = f(\hat{u}_t)s_t, \tag{8}$$

in which $f(\hat{u}_t)$ is a transformation of the OLS residual $\hat{u}_t$, and s_t is an *iid* random variable with $E[s_t] = 0$ and $var[s_t] = 1$, drawn from a pick distribution. Under $f(\hat{u}_t) = \hat{u}_t$, $E[u_t^{**}|y_t, \mathbf{x}_t] = \hat{u}_t E[s_t] = 0$ and $Var[u_t^{**}|y_t, \mathbf{x}_t] = \hat{u}_t^2 E[s_t^2] = \hat{u}_t^2$, and this suggests that the wild bootstrap can adequately mimic the unknown heteroskedastic DGP in the sense of providing asymptotically valid inference.

A number of wild bootstrap procedures have been proposed. Liu (1988) proposed a pick distribution

$$s_i = d_i f_i + E[d_i]E[f_i],$$

where d_i and f_i are independent, and

$$\begin{aligned} d_i &\sim N\left(\frac{1}{2}\left(\sqrt{17/6} + \sqrt{1/6}\right), \frac{1}{2}\right) \\ f_i &\sim N\left(\frac{1}{2}\left(\sqrt{17/6} - \sqrt{1/6}\right), \frac{1}{2}\right), \end{aligned}$$

[9] Of course, inclusion of too many irrelevant regressors will cause loss of power of the proposed test.

[10] Freedman's (1981) (y, X) bootstrap ("case resampling"), which is also asymptotically justified under heteroskedasticity, is inferior performance to wild bootstrap, in general. See the evidence in Davidson & Flachaire (2001) and discussion in Horowitz (2001).

s_i satisfies $E[s_i] = 0$ and $var[s_i] = 1$, which is called pick distribution.

Mammen (1993) proposed the following two pick distribution, which satisfy $E[s_i] = 0$, $E[s_i^2] = 1$, and $E[s_i^3] = 1$. The first one, PD_4 is

$$s_i = \begin{cases} -\left(\sqrt{5}-1\right)/2 \text{ with probability } \left(\sqrt{5}+1\right)/2\sqrt{5} \\ \left(\sqrt{5}+1\right)/2 \text{ , otherwise} \end{cases},$$

and the second one, PD_5 is

$$s_i = \left(g_i/\sqrt{2}\right) + \left(h_i^2 - 1\right)/2,$$

where g_i and h_i are $IND(0,1)$.

Recently Davidson and Flachaire (2001) proposed the wild bootstrap, called DF wild bootstrap, which uses $f(\hat{u}_t) = |\hat{u}_t|$ and a pick distribution where

$$s_t = \begin{cases} 1 & \text{with probability } 0.5 \\ -1 & \text{with probability } 0.5. \end{cases}$$

Clearly this satisfies $E[s_t] = 0$, $var[s_t] = 1$, and $E[s_t^3] = 0$.

Confining our interest to testing for skewness under heteroskedasticity, this bootstrap DGP can impose the null hypothesis of symmetry, since $E[u_t^{**3}|y_t, \mathbf{x}_t] = \hat{u}_t^3 E[s_t^3] = 0$. Thus this bootstrap method is suitable for our test. It is also of relevance in the context of the original proposal by Davidson and Flachaire (2001). They analysed the wild bootstrap refinement, when applied to the t-test based on the Heteroskedasticity Consistent Covariance Matrix Estimator (HCCME), analytically using Edgeworth expansion.[11] It was shown that the wild bootstrap, in general, does not possess the full asymptotic refinements that the standard bootstrap procedure enjoys in the *iid* case. Generally the error in rejection probability of the wild bootstrapped t-test statistic is at most $O_p(n^{-\frac{1}{2}})$ which is the same order as that of asymptotic test statistic.[12]

However, as Davidson and MacKinnon (1999) demonstrated more generally, if the test statistic and bootstrap DGP are asymptotically independent, the further refinement is available. In particular, if the error terms have distributions that are symmetric about the mean of zero, all HCCME-based t-test statistics are asymptotically independent of the DF wild bootstrap DGP. According to their analysis, the error in rejection probability becomes at most $O_p(n^{-\frac{3}{2}})$. To show the conditions for asymptotic independence, Lemma 1 of Davidson and Flachaire (2001) is reproduced here:

Lemma 1 *A mean-zero random variable u which has zero probability mass on the origin and the density of which is symmetric about the origin is the product of two independent random variables: the absolute value $|u|$ and the sign $sgn(u)$.*

Because of this lemma, under the null hypothesis, the test statistic is asymptotically independent of the absolute values $|\hat{u}_t|$ of the residuals, and consequently also of the DF wild bootstrap DGP.

[11] For more detail, see Davidson & Flachaire (2001), section 4 and the appendix.

[12] With asymmetric error, if Liu's (1988) condition, $E[s_i^3] = 1$, is satisfied, the ERP becomes at most $O_p(n^{-1})$. We do not consider this variant of wild bootstrap since it does not impose the null hypothesis for the skewness test.

Consider the alternative model to (1)

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{z}\gamma + u_t,\ t = 1, ..., n \tag{9}$$

with $\mathbf{z}$ is a $(n \times 1)$ regressor vector. The null hypothesis is $\gamma = 0$. The estimated alternative model by OLS is $\mathbf{y} = \mathbf{X}\bar{\boldsymbol{\beta}} + \mathbf{z}\tilde{\gamma} + \tilde{u}_t$ with obvious notation. By the standard regression theory,

$$\begin{aligned} \tilde{\gamma} &= (\mathbf{z}'\mathbf{M}_X\mathbf{z})^{-1}\mathbf{z}'\mathbf{M}_X\mathbf{y} \\ &= \gamma + (\mathbf{z}'\mathbf{M}_X\mathbf{z})^{-1}\mathbf{z}'\mathbf{M}_X\mathbf{u}. \end{aligned}$$

Thus the asymptotic variance of $\tilde{\gamma}$ can be consistently estimated as

$$(\mathbf{z}'\mathbf{M}_X\mathbf{z})^{-2}\mathbf{z}'\mathbf{M}_X\hat{\boldsymbol{\Omega}}\mathbf{M}_X\mathbf{z},$$

where $\hat{\boldsymbol{\Omega}} = diag(\hat{u}_t^2)$, based on the restricted residuals. Thus, under the null hypothesis $\gamma = 0$

$$t = \mathbf{z}'\mathbf{M}_X\mathbf{y} / \left(\mathbf{z}'\mathbf{M}_X\hat{\boldsymbol{\Omega}}\mathbf{M}_X\mathbf{z}\right)^{1/2} \xrightarrow{d} N(0,1).$$

As $\hat{\boldsymbol{\Omega}}$ is a diagonal matrix, $\mathbf{z}'\mathbf{M}_X\hat{\boldsymbol{\Omega}}\mathbf{M}_X\mathbf{z}$ can be expressed as

$$\sum_{t=1}^{n} (\mathbf{M}_X\mathbf{z})_t^2\,\hat{u}_t^2$$

with $(\mathbf{M}_X\mathbf{z})_t = \mathbf{m}_t'\mathbf{z}$, where $\mathbf{m}_t'$ is the tth row of $\mathbf{M}_X$. Then, t is asymptotically equal to

$$\frac{\sum_{t=1}^{n} (\mathbf{M}_X\mathbf{z})_t\,u_t}{\left(\sum_{t=1}^{n} (\mathbf{M}_X\mathbf{z})_t^2\,u_t^2\right)^{1/2}}$$

since $\mathbf{M}_X\mathbf{y} = \mathbf{M}_X\hat{\mathbf{u}} = \mathbf{M}_X\mathbf{u}$. Let us write $u_t = |u_t|s_t$, where s_t is the sign of u_t. Also let us write $p_t = (\mathbf{M}_X\mathbf{z})_t\,|u_t|$. Then t is asymptotically equal to

$$\tau = \frac{\sum_{t=1}^{n} p_t s_t}{\left(\sum_{t=1}^{n} p_t^2\right)^{1/2}}.$$

By Lemma 1, the p_t and the s_t are independent. Thus, conditional on the p_t, the s_t are mutually independent and $\tau \xrightarrow{d} N(0,1)$. This asymptotic distribution is independent of p_t and so of $|u_t|$, therefore, t is asymptotically independent of $|u_t|$ and DF wild bootstrap DGP.

The above proof illustrates the importance of the symmetry assumption on the error terms for the refinement of the DF wild bootstrap, since if the population error is skewed, the error in rejection probability reduces to the same order of first order theory. Therefore, testing skewness under heteroskedastic error is of great importance together with the DF wild bootstrap.

Also, observe from the lemma that their "symmetry" means the symmetry in distribution of *each* u_t. If we formulate the heteroskedastic error as $u_t = \sigma(\mathbf{x}_t)\varepsilon_t$, the symmetry of ε_t implies that of $u_t = \sigma(\mathbf{x}_t)\varepsilon_t$. Therefore the efficacy of their procedure depends on whether $E[\varepsilon_t^3] = 0$ or not, and using procedures for this have been discussed previously.

We are therefore interested in testing for skewness in a heteroskedastic model. Since the null imposes symmetry, an obvious wild bootstrap scheme is that of Davidson and Flachaire. We now address the question of whether this wild bootstrap scheme is asymptotically independent of the skewness test statistic, thereby providing an asymptotic refinement over first order asymptotic theory.

We know from (3) that the GOh_n is asymptotically equal to

$$\frac{n^{-1}\left(\sum_{t=1}^{n} w_t\right)^2}{n^{-1}\sum_{t=1}^{n} w_t^2}. \tag{10}$$

Now define

$$q_t = |u_t|^3 - 3\mathbf{q}^{(2)}\mathbf{M}^{-1}\mathbf{x}_t|u_t|,$$
$$s_t = \begin{cases} +1, \text{ if } u_t > 0 \\ -1, \text{ if } u_t < 0 \end{cases}.$$

q_t and s_t are independent from the Lemma 1 of Davidson and Flachaire (2001), because s_t and $|u_t|$ are independent if the distribution of *each* u_t is symmetric. Then we can write

$$\frac{n^{-1}\left(\sum_{t=1}^{n} w_t\right)^2}{n^{-1}\sum_{t=1}^{n} w_t^2} = \frac{n^{-1}\left(\sum_{t=1}^{n} q_t s_t\right)^2}{n^{-1}\sum_{t=1}^{n} q_t^2} + o_p(1). \tag{11}$$

Thus, the asymptotic distribution of (10) conditional on the q_t is chi-squared with 1 degree of freedom, which is independent of the q_t, so of the $|u_t|$. Therefore, if the distribution of *each* u_t is symmetric, the test statistic is asymptotically independent of bootstrap DGP, and the error in rejection probability of the bootstrapped heteroskedasticity robust skewness test has smaller order than the asymptotic one.

Now consider the modified test statistics, GOh_n_R and GOh_n_W. Define a $(n \times r)$ matrix $\mathbf{T}$

$$\mathbf{T} = |\mathbf{U}|^{(3)}\mathbf{\Psi} - 3|\mathbf{U}|^{(1)}\mathbf{P_x}\mathbf{U}^{(2)}\mathbf{\Psi}$$

where $|\mathbf{U}|^{(j)} = diag(|u_t^j|)$, $j = 1, 3$. Also define $\mathbf{A}$, whose typical row is a $(1 \times r)$ vector

$$\mathbf{a}_t' = |u_t|^3 \boldsymbol{\psi}(\mathbf{x}_t)'.$$

It can be shown that

$$\hat{\mathbf{u}}^{(3)\prime}\mathbf{\Psi}\hat{\mathbf{V}}^{-1}\mathbf{\Psi}'\hat{\mathbf{u}}^{(3)} = \mathbf{s}'\mathbf{A}(\mathbf{T}'\mathbf{T})^{-1}\mathbf{A}'\mathbf{s} + o_p(1).$$

Thus, the asymptotic distribution conditional on $\mathbf{A}$ and $\mathbf{T}$ is chi-squared with r degrees of freedom, which is independent of $\mathbf{A}$ and $\mathbf{T}$, so of $|u_t|$. Therefore, if the distribution of *each* u_t is symmetric, the test statistic is asymptotically independent of bootstrap DGP.

4.0.1 Fast Double Bootstrap

As stated above, it is known that the test for skewness (or kurtosis) is severely undersized even in large samples under homoskedasticity. For example, Table 2 of Jarque and Bera (1987) shows that the estimated actual 5% and 10% critical values for their normality test (using 10,000 replications) with $n = 100$, are

4.29 and 3.14, respectively. These values correspond to 11.7% and 20.8% significance levels using $\chi^2(2)$ distribution. Even with $n = 500$, these are 4.82 and 3.91, which correspond to 9.0% and 14.2% significance levels. Thus, it is conjectured that the actual rejection probabilities based on finite sample asymptotic distribution of the proposed heteroskedasticity robust skewness test statistic may not be in good agreement with those predicted by asymptotic theory. One possible reason is that for test for skewness, up to sixth moment of the error term to be estimated, and the estimation of such higher order moments by using the sample counterpart can be very inefficient with finite sample (see Chesher and Spady (1991)).

The use of double bootstrap is of some interest, because its error in rejection probability is smaller order than that of single bootstrap (see Chapter 2 for more detailed discussion). The asymptotic independence of the test statistic and the bootstrap DGP also justifies the use of the fast double bootstrap method, proposed by Davidson and MacKinnon (2000). In general, the full double bootstrap, which is suggested by Beran (1988) in order to obtain further refinement, increases the computational cost drastically. For example, if we generate 499 first stage bootstrap samples and generate 99 second level bootstrap samples, it is necessary to compute $1 + 499 + 99 \times 499 = 49901$ test statistics.[13] As the single bootstrap only requires computation of $1 + 499 = 500$ test statistics, it is a large computational burden. However, the fast double bootstrap requires only $1 + 499 \times 2 = 999$ test statistics. Also the error in rejection probability of fast double bootstrap has smaller order than the single bootstrap, given the asymptotic independence of the test statistic and bootstrap DGP.

The above discussion suggests that under the null of symmetric errors, the DF wild bootstrap and fast double bootstrap may exhibit better finite sample behaviour than that of the asymptotic test. In the next section, we investigate the finite sample behaviour of those tests.

4.1 Monte Carlo Design

To investigate the finite sample behaviour of the proposed skewness tests, size and power, the following Monte Carlo study was implemented.[14] The model is

$$y_t = \beta_1 + \sum_{j=2}^{k} \beta_j x_{tj} + u_t,\ t = 1, ..., n. \tag{12}$$

with $k = 3, 4, 6$. The set of the regressors with $k = 6$ is taken from Godfrey (1998) set 2: x_{t2} is drawn from a uniform distribution with parameters 1 and 31; x_{t3} is drawn from a log-normal distribution with $ln(x_{t3}) \sim N(3, 1)$. x_{t4}, x_{t5}, and x_{t6} are serially correlated such that

$$\begin{aligned} x_{t4} &= 0.9x_{t4} + v_{t4} \\ x_{t5} &= 0.6x_{t5} + v_{t5} \\ x_{t6} &= 0.3x_{t6} + v_{t6}, \end{aligned}$$

[13] We can reduce the number of bootstrap sampling substantially by adapting the stoppage rule advocated by Nankervis (2001), however, the fast double bootstrap still seems much cheaper.

[14] All computations were performed using Gauss 3.2.52 on UNIX (Aptech Systems Inc., 1999).

Table 1: Coefficient of variation of the error variances

k	n	$HET1$	$HET2$	$HET3$
	50	0.795	0.796	0.804
3	80	0.792	0.778	0.806
	100	0.791	0.822	0.883
	50	0.795	0.796	0.802
4	80	0.792	0.778	0.811
	100	0.791	0.822	0.885
	50	0.795	0.796	0.808
5	80	0.792	0.778	0.799
	100	0.791	0.822	0.880

with v_{tr} being independently, normally distributed such that $E[x_{tr}] = 0$ and $var[x_{tr}] = 1$, for $r = 4, 5, 6$. All coefficients are set to 1, namely, $\beta_j = 1$, $j = 1, ..., k$, without loss of generality.[15] The number of the observations were set to $n = 50, 80$, and 100.

As explained before, generally the population error can be written as

$$u_t = \sigma_t \varepsilon_t,$$

where σ_t is the non-stochastic scaling factor and ε_t is an $iid(0,1)$ random variable. Under homoskedasticity, we set $\sigma_t = 1$ without loss of generality, since the invariant results of Breusch (1980) apply. We closely follow the heteroskedastic schemes of Godfrey and Orme (2002). Three distinct heteroskedastic schemes are investigated; the first scheme, called $HET1$, is

$$\begin{aligned} \sigma_t &= (1 - d_t) + \delta d_t, \\ d_t &= \begin{cases} 0, t = 1, ..., n/2 \\ 1, t = 1, ..., n, \end{cases} \end{aligned}$$

which is used by MacKinnon and White (1985), for $\delta = 2.9$. The second heteroskedastic scheme, $HET2$, is

$$\sigma_t = \sqrt{1 + \sum_{j=2}^{k} \gamma_j x_{jt}^2}, \quad t = 1, ..., n$$

where $\gamma_2 = 0.0000775$ and $\gamma_l = 2\gamma_2$, $l = 2, ..., 6$. The final heteroskedastic scheme, $HET3$, is

$$\sigma_t = \exp\left(\lambda\left(\beta_1 + \sum_{j=2}^{k} \beta_j x_{jt}\right)\right), \quad t = 1, ..., n,$$

for $\lambda = 0.0054$. The values of δ, γ_2, and λ are chosen such that the coefficient of variation of the error variance be about 0.8 with $n = 50$ and $k = 3$. These values are fixed for all designs, thus the coefficient of variation of the error variance slightly varies across the designs; see Table 1.

For estimating the actual size of the test, ε_t is drawn from four symmetric distributions; the standard normal distribution, SN; the student's t distribution

[15] Note that for RESET type tests, $\boldsymbol{\beta}$ should not be $\mathbf{0}$, since if $\boldsymbol{\beta} = \mathbf{0}$, then the test variable $\mathbf{x}'\hat{\boldsymbol{\beta}}$ converges to zero.

with 7 degrees of freedom, $t(7)$; the uniform distribution from -1 to 1, UN; and the bimodal mixture normal which drawn from $N(-1.5, 1)$ and $N(1.5, 1)$ with probability 0.5, MN. For estimating the power of the tests under various *iid* errors, ε_t is drawn from three different asymmetric distributions; the log-normal distribution $\ln(N(0, 1))$, LN; the chi-square distribution with 2 degrees of freedom $\chi^2(2)$; and with 8 degrees of freedom, $\chi^2(8)$. All of these errors are standardised.

We used 5000 replications and 499 bootstrap samples to estimate the rejection probabilities of the test. Following Godfrey and Orme (2002), we adopted the following measure for assessing how close the finite sample significance levels are to the nominal value. Suppose that the sample rejection rate is α_n, which satisfies $H_\alpha : \alpha_L \leq \alpha_n \leq \alpha_U$, where $0 < \alpha_L < \alpha_U < 1$ and $\alpha_U - \alpha_L$ is assumed to be $O(1)$, and $\alpha_U + \alpha_L = 2\alpha$, α is the nominal size. The asymptotic significance level of the test H_α is set to 5% here. Then we reject H_α if the estimated size is outside of the interval

$$\left(\alpha_L - 1.645\sqrt{\frac{\alpha_L(1-\alpha_L)}{R}}, \alpha_U + 1.645\sqrt{\frac{\alpha_U(1-\alpha_U)}{R}}\right)$$

as $R \to \infty$, where $R(= 5000)$ is the number of replication.[16] In the experiments, the rejection probabilities is estimated for the nominal levels of 1%, 5%, and 10%. Here, the results for 5% and 10% levels are reported. If the evidence is consistent with the claim that $\alpha - 0.5 \leq \alpha_n \leq \alpha + 0.5$ (resp. $\alpha - 1.0 \leq \alpha_n \leq \alpha + 1.0$), the estimated levels are regarded as "good" (resp. "satisfactory"). We call the former confidence interval as $\pm 0.5\%$ interval and the latter as $\pm 1\%$ interval. With 5% level, $\pm 0.5\%$ and $\pm 1\%$ intervals are $(4.018\%, 6.030\%)$ and $(3.544\%, 6.552\%)$, respectively, and with 10% level, $\pm 0.5\%$ and $\pm 1\%$ intervals are $(8.818\%, 11.213\%)$ and $(8.334\%, 11.728\%)$, respectively.

Furthermore, Chesher and Jewitt (1987) argued that if there are points of high leverage, there can be big downward bias in the HCCME with finite sample. But, if the "hat matrix", $\mathbf{P}_\mathbf{x} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, is well balanced, by chance, such bias does not occur. Thus, it is important that the set of regressors in the experiment is *not* well balanced in order to investigate the finite sample behaviour of the test statistic in general circumstances. Denoting the diagonal elements of $\mathbf{P}_\mathbf{x}$ as h_t, and the average of h_t as $\bar{h}$, following Belsley *et al.* (1980), we call h_t is a high leverage point if $h_t/\bar{h} > 2$. In our design, for $n = 50, 80, 100$ with all k, we have at least 2,4,6 high leverage points respectively, and the maximum $h_t/\bar{h}$ are at least 2.39, 3.56, 3.13, respectively; see Table 2. We did not use transformed OLS residuals $\hat{u}_t$, which are used in MacKinnon and White (1985) and Davidson and Flachaire (2001), because in our preliminary experiment, there was no compelling evidence that the use of, say, $f(\hat{u}_t) = |\hat{u}_t/(1 - \mathbf{x}_t'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_t)|$ with $n = 80$ and $k = 4$ improves the behaviour of the test.

To investigate the finite sample behaviour of the modified skewness tests and the Whang's test under the "potential inconsistency", the following heteroskedastic scheme is used. The model is

$$y_t = \beta_1 + \beta_2 x_{t2} + u_t, \; t = 1, ..., n.$$

[16] It seems reasonable to treat rejection probability as being normally distributed with $R = 5000$.

Table 2: Maximum leverage points and number of leverage points

k	n	Maximum Leverage	Leverage Points
3	50	4.62	3
	80	5.13	4
	100	5.45	6
4	50	3.51	2
	80	3.88	6
	100	4.14	9
6	50	2.39	2
	80	3.56	6
	100	3.13	6

where x_{t2} is drawn from normal distribution mean zero and variance 10. All coefficients are set to 1, namely, $\beta_1 = \beta_2 = 1$ without loss of generality. The heteroskedastic error is formulated as $u_t = \sigma_t \varepsilon_t$, and $HET4$ is set as

$$\sigma_t = 0.8x_{t2}$$

so that $E[\sigma_t^3 \varepsilon_t^3 | x_{t2}] = 0$, even when $E[\varepsilon_t^3] \neq 0$. We set $n = 100$ with 5000 replications, 199 bootstrap sample is used for single bootstrap, and 199 first bootstrap and 99 second bootstrap sample are used for double bootstrap. Homoskedastic errors, and $HET1$, 3, and 4 are considered.[17]

4.2 Simulation Results

First of all, we compare the finite sample behaviour of JB_n, GO_n, GOh_n, and the DF wild bootstrapped GOh_n, and fast double bootstrapped GOh_n are compared.

4.2.1 Size

Under Homoskedasticity Table 3~5 gives the results for the estimated size of the tests. Even under the ideal conditions for JB_n, with homoskedastic normal errors with $n = 100$, tends to be undersized with all k, as illustrated above. Under nonnormal errors, JB_n severely overrejects with $t(7)$ and is undersized with UN errors for all designs which is consistent to the analysis in the previous section. Table 6 gives the number of times in groups of 3 experiments that the rejection frequencies are consistent with the claim that the true significance level is within 0.5% (1.0%) of nominal levels of 5% and 10%, under homoskedasticity. As can be seen, GO_n, which is asymptotically valid under nonnormal homoskedastic errors, gives just 16 out of 72 experiments being "good" and 38 being "satisfactory". GOh_n has better performance, giving 35 out of 72 experiments being "good" and 49 being "satisfactory". In contrast to these asymptotic critical values, the DF wild bootstrap of GOh_n can successfully control the size. 63 out of 72 experiments are "good" and 68 are "satisfactory". Furthermore, for $N = 80$ and 100, all 48 experiments are "good".

[17] As the $HET2$ and $HET3$ exhibited very similar effect on the behaviour of the proposed skewness test, $HET2$ is omitted from the investigation for the modified tests and Whang's test.

Table 3: Estimated size of the tests: $k = 3$

				$n=50$				$n=80$				$n=100$			
				Homo	$HET1$	$HET2$	$HET3$	*Homo*	$HET1$	$HET2$	$HET3$	*Homo*	$HET1$	$HET2$	$HET3$
		JB_n	asy	0.54 **	8.72	3.14	3.26	0.60 **	12.74	5.66	6.02	0.76 **	13.52	8.52	9.64
		GO_n	asy	0.22 *	0.80 **	0.10 *	0.10 *	0.48 **	0.86 **	0.08 *	0.10 *	0.44 **	0.72 **	0.06 *	0.04 *
	1%	GOh_n	asy	0.36 **	1.16 **	0.26 *	0.28 *	0.48 **	1.14 **	0.48 **	0.34 **	0.56 **	1.08 **	0.40 **	0.36 **
		GOh_n	bts	1.32 **	2.16 *	1.46 **	1.52 **	1.26 **	1.48 **	1.26 **	1.40 **	1.04 **	1.44 **	1.22 **	1.18 **
		GOh_n	2bts	1.64 **	2.70	2.08 *	2.14 *	1.28 **	1.54 **	1.46 **	1.52 **	1.02 **	1.54 **	1.68 **	1.48 **
		JB_n	asy	3.02	18.20	8.04	8.18	3.38	23.62	12.50	13.12	3.74 *	25.14	16.36	16.92
		GO_n	asy	3.56 *	5.74 **	2.38	2.28	3.94 *	5.60 **	2.00	1.88	3.94 *	5.62 **	1.40	1.18
SN	5%	GOh_n	asy	4.46 **	7.14	3.86 *	3.74 *	4.34 **	6.68	5.00 **	5.10 **	4.48 **	6.62	3.90 *	4.02 **
		GOh_n	bts	6.46 *	7.40	7.08	6.76	5.50 **	6.28 *	6.26 *	6.44 *	5.42 **	5.70 **	5.42 **	5.48 **
		GOh_n	2bts	6.42 *	7.42	7.96	8.04	5.64 **	6.00 **	6.16 *	6.58	5.50 **	5.62 **	5.78 **	5.40 **
		JB_n	asy	6.36	27.02	12.60	13.18	7.16	31.06	17.98	19.12	7.68	33.76	22.34	23.16
		GO_n	asy	9.16 **	12.88	6.42	6.50	9.46 **	12.62	5.80	5.18	9.68 **	11.84	4.26	3.98
	10%	GOh_n	asy	10.28 **	15.26	9.54 **	9.08 **	10.22 **	13.94	11.44 *	10.96 **	10.34 **	13.00	11.26 *	11.18 **
		GOh_n	bts	11.22 *	12.28	12.16	11.94	10.50 **	11.32 *	12.26	12.10	10.34 **	10.38 **	10.50 **	10.46 **
		GOh_n	2bts	11.42 *	12.34	13.06	12.84	10.78 **	11.12 **	11.84	11.78	10.54 **	10.38 **	10.52 **	10.64 **
		JB_n	asy	8.52	23.44	10.92	11.06	12.96	30.16	17.50	17.80	14.46	33.42	21.26	21.96
		GO_n	asy	0.28 *	0.66 **	0.24 *	0.20 *	0.36 **	0.46 **	0.12 *	0.10 *	0.14 *	0.24 *	0.12 *	0.10 *
	1%	GOh_n	asy	0.34 **	1.04 **	0.62 **	0.42 **	0.48 **	0.90 **	0.54 **	0.48 **	0.34 **	0.40 **	0.48 **	0.40 **
		GOh_n	bts	1.44 **	2.68	2.30 *	2.20 *	1.28 **	1.92 *	1.70 **	1.68 **	1.18 **	1.16 **	1.44 **	1.30 **
		GOh_n	2bts	2.02 *	3.58	3.08	3.10	1.60 **	1.98 *	1.76 **	1.78 **	1.34 **	1.42 **	1.66 **	1.68 **
		JB_n	asy	16.40	35.94	19.56	20.00	21.30	42.56	26.92	27.74	24.50	44.82	31.76	33.36
		GO_n	asy	3.18	5.52 **	2.64	2.58	3.58 *	4.24 **	2.32	1.92	3.56 *	4.12 **	1.94	1.64
$t(7)$	5%	GOh_n	asy	4.36 **	7.08	4.60 **	4.54 **	4.40 **	5.72 **	6.02 **	5.90 **	4.14 **	5.18 **	4.74 **	4.92 **
		GOh_n	bts	6.20 *	8.30	7.98	8.00	5.66 **	6.18 *	6.84	7.02	5.20 **	5.10 **	5.76 **	5.92 **
		GOh_n	2bts	6.54 *	9.04	8.48	8.72	5.78 **	6.22 *	6.46 *	6.76	5.10 **	4.94 **	5.70 **	6.28 *
		JB_n	asy	22.28	44.14	26.38	27.10	28.54	49.78	34.02	34.52	31.54	50.68	39.56	41.32
		GO_n	asy	9.04 **	12.82	7.40	7.14	8.72 *	10.38 **	6.58	6.40	8.72 *	9.98 **	5.32	4.96
	10%	GOh_n	asy	11.36 *	16.36	11.54 *	11.10 **	10.14 **	12.72	12.90	12.72	10.10 **	11.70 *	12.98	13.06
		GOh_n	bts	11.18 **	13.50	13.20	12.88	10.42 **	11.26 *	12.28	12.28	9.88 **	10.44 **	10.70 **	10.78 **
		GOh_n	2bts	11.62 *	13.62	14.02	14.02	10.72 **	11.04 **	11.58 *	11.78	9.82 **	10.44 **	10.66 **	10.68 **
		JB_n	asy	0.00 *	0.02 *	0.42 **	0.44 **	0.00 *	0.04 *	0.82 **	0.98 **	0.00 *	0.04 *	1.24 **	1.50 **
		GO_n	asy	0.64 **	0.70 **	0.20 *	0.20 *	0.68 **	0.86 **	0.14 *	0.20 *	0.88 **	0.56 **	0.14 *	0.12 *
	1%	GOh_n	asy	0.92 **	0.74 **	0.22 *	0.24 *	0.80 **	0.76 **	0.40 **	0.38 **	1.00 **	0.62 **	0.22 *	0.28 *
		GOh_n	bts	1.74 **	1.26 **	1.08 **	1.24 **	1.42 **	1.16 **	1.20 **	1.32 **	1.38 **	0.80 **	1.38 **	1.28 **
		GOh_n	2bts	1.64 **	1.30 **	1.44 **	1.62 **	1.48 **	1.22 **	1.48 **	1.62 **	1.40 **	0.78 **	1.66 **	1.58 **
		JB_n	asy	0.14	0.56	2.00	2.24	0.06	0.46	3.40	3.84 *	0.06	0.28	4.50 **	5.56 **
		GO_n	asy	3.50	4.92 **	1.74	1.90	4.04 **	5.22 **	1.82	2.04	4.34 **	3.76 *	1.28	1.42
UN	5%	GOh_n	asy	3.96 *	4.92 **	3.02	3.22	4.30 **	4.54 **	3.24	3.32	4.70 **	3.86 *	2.76	2.88
		GOh_n	bts	5.88 **	5.38 **	5.54 **	5.72 **	5.48 **	5.04 **	5.12 **	5.44 **	5.76 **	4.06 **	5.04 **	5.14 **
		GOh_n	2bts	6.10 *	5.80 **	6.12 *	6.60	5.64 **	5.40 **	5.40 **	5.78 **	5.86 **	4.32 **	5.58 **	5.46 **
		JB_n	asy	0.56	1.90	4.62	5.08	0.52	1.50	6.56	7.60	0.52	1.04	8.78 *	10.14 **
		GO_n	asy	7.38	10.82 **	5.26	5.90	8.70 *	10.50 **	4.84	5.36	9.60 **	8.72 *	4.10	4.44
	10%	GOh_n	asy	8.42 *	10.36 **	6.52	6.90	9.10 **	9.46 **	7.56	7.90	9.88 **	8.98 **	6.94	7.46
		GOh_n	bts	10.30 **	10.02 **	10.44 **	10.64 **	10.46 **	9.98 **	9.84 **	10.16 **	11.10 **	9.30 **	9.24 **	9.80 **
		GOh_n	2bts	10.70 **	10.44 **	11.72 *	12.00	10.90 **	10.36 **	10.22 **	10.40 **	11.12 **	9.76 **	10.00 **	10.18 **
		JB_n	asy	0.00 *	0.68 **	0.60 **	0.66 **	0.00 *	0.80 **	1.60 **	1.68 **	0.00 *	0.78 **	2.56	3.24
		GO_n	asy	0.22 *	0.48 **	0.14 *	0.16 *	0.64 **	0.82 **	0.26 *	0.22 *	0.54 **	0.62 **	0.18 *	0.16 *
	1%	GOh_n	asy	0.42 **	0.52 **	0.24 *	0.26 *	0.86 **	0.84 **	0.42 **	0.42 **	0.62 **	0.78 **	0.36 **	0.38 **
		GOh_n	bts	1.36 **	1.68 **	1.00 **	1.10 **	1.60 **	1.38 **	1.40 **	1.46 **	1.16 **	1.10 **	1.22 **	1.24 **
		GOh_n	2bts	1.44 **	1.78 **	1.40 **	1.58 **	1.60 **	1.46 **	1.54 **	1.58 **	1.20 **	1.34 **	1.62 **	1.48 **
		JB_n	asy	0.22	3.18	3.12	3.28	0.38	4.02 **	5.20 **	6.08 *	0.20	4.22 **	7.40	8.48
		GO_n	asy	2.52	4.64 **	1.66	1.90	3.86 *	4.68 **	1.84	1.86	3.78 *	4.76 **	1.46	1.50
MN	5%	GOh_n	asy	3.26	4.98 **	2.50	2.60	4.28 **	5.06 **	3.64 *	3.74 *	3.96 *	5.12 **	2.90	3.32
		GOh_n	bts	5.26 **	5.76 **	5.16 **	5.44 **	5.54 **	5.08 **	4.94 **	5.22 **	4.86 **	5.04 **	4.76 **	5.22 **
		GOh_n	2bts	5.46 **	6.10 *	5.76 **	6.38 *	5.74 **	5.10 **	5.04 **	5.28 **	4.78 **	5.18 **	5.10 **	5.36 **
		JB_n	asy	0.88	6.86	5.96	6.42	1.34	8.44 *	9.40 **	10.20 **	1.08	8.50 *	12.16	13.50
		GO_n	asy	7.24	10.46 **	4.84	5.38	8.76 *	11.04 **	5.12	5.26	7.82	10.26 **	4.22	4.24
	10%	GOh_n	asy	8.04	10.72 **	6.90	7.00	9.22 **	11.02 **	7.86	7.86	8.46 *	10.92 **	7.70	8.30
		GOh_n	bts	9.96 **	10.28 **	10.20 **	10.28 **	10.70 **	9.94 **	9.58 **	9.96 **	9.52 **	10.06 **	9.18 **	9.32 **
		GOh_n	2bts	10.26 **	10.74 **	11.30 *	11.46 *	10.70 **	10.06 **	9.66 **	9.98 **	9.74 **	10.36 **	9.46 **	9.88 **

Note:
Homo denotes homoskedastic errors; $HET1,2,3$ denotes heteroskedastic schemes explained section 4.1.
asy denotes asymptotic critical value; bts denotes single DF wild bootstrap; 2bts denotes Davidson & MacKinnon's fast double bootstrap.
* denotes that the estimate is consistent with the true significance level being between -1% and +1% from its nominal level.
** denotes that the estimate is consistent with the true significance level being between -0.5 % and +0.5 % from its nominal level.

Table 4: Estimated size of the tests: $k = 4$

				$n = 50$				$n = 80$				$n = 100$			
				Homo	$HET1$	$HET2$	$HET3$	*Homo*	$HET1$	$HET2$	$HET3$	*Homo*	$HET1$	$HET2$	$HET3$
		JB_n	asy	0.42 **	6.58	2.60	2.80	0.48 **	10.36	5.24	5.70	0.66 **	11.50	7.82	8.84
		GO_n	asy	0.18 *	0.40 **	0.10 *	0.12 *	0.42 **	0.60 **	0.06 *	0.08 *	0.42 **	0.56 **	0.06 *	0.02 *
	1%	GOh_n	asy	0.42 **	0.74 **	0.28 *	0.28 *	0.44 **	0.92 **	0.48 **	0.40 **	0.66 **	1.08 **	0.40 **	0.36 **
		GOh_n	bts	1.24 **	1.88 *	1.38 **	1.26 **	1.04 **	1.94 *	1.10 **	1.20 **	1.12 **	1.70 **	1.16 **	1.12 **
		GOh_n	2bts	1.52 **	2.38	1.80 *	1.88 *	1.26 **	1.88 *	1.54 **	1.56 **	0.98 **	1.68 **	1.54 **	1.48 **
		JB_n	asy	2.20	14.66	6.92	7.10	3.20	20.76	12.22	12.88	3.50	22.94	15.34	16.22
		GO_n	asy	3.24	4.26 **	2.10	2.16	3.90 *	4.50 **	1.92	1.86	4.08 **	4.46 **	1.38	1.20
SN	5%	GOh_n	asy	4.28 **	6.34 *	4.02 **	4.00 *	4.64 **	6.76	5.00 **	5.02 **	4.90 **	6.22 *	4.34 **	4.38 **
		GOh_n	bts	6.14 *	6.86	6.40 *	6.14 *	5.50 **	6.18 *	6.24 *	6.28 *	5.60 **	5.66 **	5.54 **	5.58 **
		GOh_n	2bts	6.26 *	7.30	7.12	7.18	5.76 **	6.16 *	6.12 *	6.42 *	5.70 **	5.34 **	5.94 **	5.66 **
		JB_n	asy	5.32	22.10	11.26 *	11.58 *	6.88	28.46	17.28	18.16	7.32	31.12	21.58	22.18
		GO_n	asy	8.66 *	9.78 **	6.14	6.22	9.58 **	10.24 **	5.90	5.28	9.88 **	10.42 **	4.34	4.02
	10%	GOh_n	asy	10.14 **	13.90	9.50 **	9.12 **	11.06 **	13.52	11.70 *	11.42 *	10.62 **	12.96	12.04	11.78
		GOh_n	bts	11.00 **	12.40	11.78	11.76	11.16 **	11.38 *	12.16	12.20	10.40 **	10.24 **	10.96 **	10.60 **
		GOh_n	2bts	11.26 *	12.56	12.88	12.98	11.22 *	11.32 *	11.68 *	11.78	10.54 **	10.38 **	10.68 **	10.56 **
		JB_n	asy	7.34	20.04	9.42	9.78	12.08	27.08	16.78	17.14	13.82	30.64	20.72	21.50
		GO_n	asy	0.34 **	0.42 **	0.28 *	0.22 *	0.34 **	0.30 *	0.08 *	0.14 *	0.16 *	0.32 *	0.12 *	0.10 *
	1%	GOh_n	asy	0.54 **	1.20 **	0.78 **	0.64 **	0.52 **	0.62 **	0.58 **	0.58 **	0.34 **	0.64 **	0.46 **	0.42 **
		GOh_n	bts	1.60 **	2.88	2.24 *	2.32 *	1.32 **	1.88 *	1.98 *	1.96 *	1.08 **	1.24 **	1.46 **	1.48 **
		GOh_n	2bts	2.38	4.16	2.98	3.08	1.46 **	2.18 *	2.02 *	1.98 *	1.30 **	1.48 **	1.66 **	1.76 **
		JB_n	asy	14.36	31.70	17.76	18.50	20.20	38.84	26.06	27.00	23.40	42.30	31.06	32.38
		GO_n	asy	3.36	4.32 **	2.74	2.58	3.66 *	3.02	2.24	2.06	3.72 *	3.28	1.92	1.62
$t(7)$	5%	GOh_n	asy	5.56 **	7.84	5.56 **	5.20 **	4.84 **	5.54 **	6.64	6.50 *	4.62 **	5.34 **	5.18 **	5.54 **
		GOh_n	bts	6.62	9.40	8.04	8.20	5.64 **	6.30 *	7.08	7.24	5.26 **	5.32 **	5.76 **	5.86 **
		GOh_n	2bts	7.00	9.74	8.92	8.92	5.68 **	6.72	6.78	6.84	5.20 **	5.44 **	6.02 **	6.24 *
		JB_n	asy	20.92	40.18	24.84	25.10	27.32	46.82	32.92	33.60	30.52	48.68	38.76	40.64
		GO_n	asy	9.12 **	10.70 **	7.18	7.08	8.74 *	8.64 *	6.42	6.26	8.44 *	8.68 *	5.46	5.08
	10%	GOh_n	asy	13.56	17.10	13.28	12.64	11.48 *	13.72	13.56	13.34	10.94 **	12.24	14.16	14.50
		GOh_n	bts	11.90	14.80	13.82	13.52	10.52 **	11.64 *	12.18	12.28	9.96 **	10.34 **	10.76 **	10.96 **
		GOh_n	2bts	11.60 *	14.62	14.28	14.14	10.66 **	11.58 *	11.76	11.76	9.92 **	10.16 **	10.96 **	10.76 **
		JB_n	asy	0.00 *	0.04 *	0.30 *	0.36 **	0.00 *	0.02 *	0.78 **	0.94 **	0.00 *	0.04 *	1.02 **	1.30 **
		GO_n	asy	0.48 **	0.44 **	0.16 *	0.20 *	0.72 **	0.68 **	0.16 *	0.16 *	0.82 **	0.84 **	0.14 *	0.14 *
	1%	GOh_n	asy	0.76 **	0.28 *	0.24 *	0.24 *	0.72 **	0.40 **	0.38 **	0.32 *	0.86 **	0.52 **	0.20 *	0.28 *
		GOh_n	bts	1.46 **	1.04 **	0.88 **	0.98 **	1.24 **	0.82 **	1.14 **	1.32 **	1.28 **	0.90 **	1.08 **	1.20 **
		GOh_n	2bts	1.48 **	1.02 **	1.22 **	1.42 **	1.38 **	1.00 **	1.32 **	1.50 **	1.42 **	1.06 **	1.50 **	1.46 **
		JB_n	asy	0.14	0.74	1.66	1.84	0.08	0.52	3.42	3.72 *	0.04	0.48	4.00 *	5.28 **
		GO_n	asy	2.84	4.50 **	1.34	1.54	3.46	5.44 **	1.58	1.82	3.92 *	5.18 **	1.32	1.46
UN	5%	GOh_n	asy	3.28	3.46	2.66	2.68	3.66 *	3.82 *	3.28	3.36	4.20 **	3.80 *	2.86	3.00
		GOh_n	bts	5.30 **	4.96 **	5.08 **	5.32 **	5.26 **	4.88 **	5.00 **	5.56 **	5.76 **	4.28 **	4.64 **	5.22 **
		GOh_n	2bts	5.66 **	5.32 **	5.84 **	6.16 *	5.34 **	5.14 **	5.12 **	5.72 **	5.88 **	4.26 **	5.18 **	5.38 **
		JB_n	asy	0.42	2.18	4.14	4.32	0.54	2.08	6.40	7.40	0.42	1.90	8.24	9.40 **
		GO_n	asy	6.48	10.86 **	4.74	5.24	7.94	11.60 *	4.92	5.20	8.80 *	10.70 **	4.00	4.48
	10%	GOh_n	asy	7.16	8.56 *	6.10	6.48	8.20	8.42 *	7.34	7.66	9.18 **	8.10	6.98	7.50
		GOh_n	bts	9.58 **	9.92 **	9.50 **	9.58 **	10.34 **	9.74 **	10.04 **	10.24 **	10.68 **	8.74 *	9.24 **	9.72 **
		GOh_n	2bts	10.12 **	10.52 **	10.86 **	10.84 **	10.66 **	10.18 **	10.48 **	10.62 **	10.92 **	9.28 **	9.78 **	9.68 **
		JB_n	asy	0.00 *	0.52 **	0.54 **	0.54 **	0.00 *	0.80 **	1.42 **	1.62 **	0.00 *	0.66 **	2.28 *	2.96
		GO_n	asy	0.36 **	0.38 **	0.12 *	0.10 *	0.50 **	0.50 **	0.22 *	0.20 *	0.42 **	0.54 **	0.18 *	0.16 *
	1%	GOh_n	asy	0.46 **	0.26 *	0.24 *	0.24 *	0.58 **	0.52 **	0.30 *	0.34 **	0.52 **	0.44 **	0.40 **	0.44 **
		GOh_n	bts	1.44 **	1.34 **	1.06 **	1.08 **	1.32 **	1.18 **	1.26 **	1.34 **	1.06 **	0.92 **	0.96 **	1.04 **
		GOh_n	2bts	1.48 **	1.50 **	1.36 **	1.40 **	1.52 **	1.28 **	1.30 **	1.38 **	1.22 **	0.92 **	1.14 **	1.32 **
		JB_n	asy	0.22	2.64	2.84	2.90	0.26	3.70 *	4.72 **	5.56 **	0.18	3.92 *	6.84	7.86
		GO_n	asy	2.44	3.82 *	1.58	1.78	3.50	4.36 **	1.86	1.84	3.62 *	4.62 **	1.48	1.58
MN	5%	GOh_n	asy	3.20	3.64 *	2.84	2.92	3.82 *	3.98 *	3.24	3.38	3.92 *	4.38 **	2.96	3.32
		GOh_n	bts	4.98 **	5.28 **	4.64 **	5.18 **	5.36 **	5.00 **	4.96 **	5.24 **	5.02 **	4.82 **	4.88 **	5.10 **
		GOh_n	2bts	5.06 **	5.34 **	5.60 **	5.94 **	5.44 **	5.22 **	5.02 **	5.28 **	5.12 **	5.22 **	4.72 **	5.02 **
		JB_n	asy	0.94	5.50	5.56	5.78	1.14	7.40	9.08 **	9.88 **	0.98	8.32	10.98 **	12.50
		GO_n	asy	6.36	8.96 **	4.72	5.02	8.16	10.26 **	4.78	5.06	7.82	10.60 **	4.38	4.36
	10%	GOh_n	asy	7.60	9.22 **	6.66	6.88	8.84 **	9.64 **	7.66	7.84	8.50 *	9.90 **	7.82	8.34 *
		GOh_n	bts	9.84 **	9.94 **	9.54 **	9.72 **	10.10 **	10.14 **	9.60 **	10.04 **	9.68 **	10.04 **	9.26 **	9.40 **
		GOh_n	2bts	9.92 **	10.62 **	10.64 **	10.88 **	10.22 **	10.40 **	9.72 **	10.14 **	9.90 **	10.10 **	9.32 **	9.62 **

Note:
Homo denotes homoskedastic errors; $HET1$,2,3 denotes heteroskedastic schemes explained section 4.1.
asy denotes asymptotic critical value; bts denotes single DF wild bootstrap; 2bts denotes Davidson & MacKinnon's fast double bootstrap.
* denotes that the estimate is consistent with the true significance level being between -1% and +1% from its nominal level.
** denotes that the estimate is consistent with the true significance level being between -0.5 % and +0.5 % from its nominal level.

Table 5: Estimated size of the tests: $k = 6$

				$n = 50$				$n = 80$				$n = 100$			
				$Homo$	$HET1$	$HET2$	$HET3$	$Homo$	$HET1$	$HET2$	$HET3$	$Homo$	$HET1$	$HET2$	$HET3$
		JB_n	asy	0.34 **	4.56	1.82 *	1.90 *	0.38 **	8.44	3.70	4.14	0.70 **	9.78	7.12	7.70
		GO_n	asy	0.24 *	0.52 **	0.08 *	0.10 *	0.44 **	0.50 **	0.18 *	0.16 *	0.44 **	0.44 **	0.06 *	0.04 *
	1%	GOh_n	asy	0.46 **	1.36 **	0.44 **	0.46 **	0.54 **	1.16 **	0.86 **	0.80 **	0.66 **	0.82 **	0.36 **	0.38 **
		GOh_n	bts	1.18 **	2.24 *	1.26 **	1.28 **	1.34 **	1.80 *	1.46 **	1.54 **	1.12 **	1.64 **	1.14 **	1.08 **
		GOh_n	2bts	1.28 **	2.56	1.74 **	1.74 **	1.56 **	1.74 **	1.46 **	1.56 **	1.10 **	1.82 *	1.48 **	1.48 **
		JB_n	asy	1.62	12.40	5.34 **	5.28 **	2.48	18.70	9.56	9.94	3.10	19.92	14.68	15.02
		GO_n	asy	2.84	4.54 **	1.90	1.92	3.72 *	5.04 **	2.28	2.14	4.04 **	3.96 *	1.52	1.42
SN	5%	GOh_n	asy	4.50 **	8.18	4.14 **	4.00 *	5.16 **	7.70	6.06 *	6.00 **	5.32 **	6.52 *	5.26 **	5.32 **
		GOh_n	bts	5.64 **	7.32	6.10 *	6.00 **	5.64 **	6.40 *	6.54 *	6.66	5.66 **	6.00 **	5.20 **	5.32 **
		GOh_n	2bts	5.74 **	7.30	6.92	6.94	5.92 **	6.40 *	6.34 *	6.68	5.78 **	5.92 **	5.64 **	5.66 **
		JB_n	asy	3.94	18.68	9.26 **	9.88 **	5.84	26.62	15.08	15.92	6.64	27.02	20.32	21.26
		GO_n	asy	8.32	10.64 **	5.90	5.62	9.04 **	11.56 *	5.96	5.74	9.60 **	9.14 **	4.60	4.44
	10%	GOh_n	asy	10.58 **	17.00	10.12 **	9.86 **	11.40 *	16.12	12.54	11.94	11.66 *	13.66	12.36	12.28
		GOh_n	bts	10.64 **	12.64	12.00	11.64 *	10.62 **	11.70 *	12.30	11.80	10.90 **	10.98 **	10.72 **	10.76 **
		GOh_n	2bts	10.92 **	12.62	13.28	12.94	10.68 **	11.40 *	12.22	12.30	10.80 **	10.62 **	10.72 **	10.96 **
		JB_n	asy	5.60	15.76	7.02	7.18	10.32	24.86	13.94	14.36	12.44	27.88	19.28	19.90
		GO_n	asy	0.24 *	0.52 **	0.22 *	0.22 *	0.32 *	0.40 **	0.12 *	0.14 *	0.20 *	0.24 *	0.16 *	0.16 *
	1%	GOh_n	asy	0.84 **	1.94 *	0.96 **	0.80 **	0.68 **	1.32 **	1.18 **	1.08 **	0.60 **	0.88 **	0.72 **	0.68 **
		GOh_n	bts	2.00 *	3.40	2.16 *	2.16 *	1.38 **	2.06 *	2.14 *	2.10 *	1.10 **	1.48 **	1.48 **	1.50 **
		GOh_n	2bts	2.10 *	3.84	2.66	2.50	1.70 **	2.48	2.04 *	2.08 *	1.38 **	1.72 **	1.88 *	1.82 *
		JB_n	asy	11.82	26.84	14.56	14.78	18.30	36.40	23.56	24.14	21.54	38.06	29.18	30.36
		GO_n	asy	2.88	4.52 **	2.64	2.62	3.54	3.68 *	2.30	2.18	3.40	2.64	2.12	1.78
$t(7)$	5%	GOh_n	asy	7.36	11.50	6.90	6.50 *	6.04 *	8.26	7.96	7.52	5.60 **	6.72	6.96	7.22
		GOh_n	bts	6.78	10.00	8.06	8.04	5.86 **	7.08	7.24	7.44	5.38 **	6.08 *	5.74 **	5.90 **
		GOh_n	2bts	6.78	9.90	8.32	8.44	5.86 **	7.22	6.92	6.82	5.52 **	6.20 *	5.92 **	5.82 **
		JB_n	asy	17.28	34.06	20.84	20.92	24.28	44.46	30.16	30.74	28.78	44.98	36.42	37.98
		GO_n	asy	9.30 **	10.76 **	7.14	7.12	9.38 **	9.56 **	7.10	6.46	8.78 *	7.86	5.62	5.28
	10%	GOh_n	asy	15.72	22.34	14.90	15.08	13.44	16.80	15.90	15.54	13.54	14.76	16.38	16.62
		GOh_n	bts	12.12	16.08	13.78	13.72	11.02 **	12.42	13.24	13.10	10.56 **	11.12 **	10.88 **	11.02 **
		GOh_n	2bts	11.96	15.50	14.10	13.98	10.56 **	11.94	12.42	12.20	10.32 **	11.20 **	10.82 **	10.92 **
		JB_n	asy	0.00 *	0.02 *	0.12 *	0.16 *	0.00 *	0.04 *	0.38 **	0.46 **	0.00 *	0.06 *	0.72 **	1.02 **
		GO_n	asy	0.26 *	0.22 *	0.10 *	0.04 *	0.44 **	1.08 **	0.28 *	0.26 *	0.42 **	1.36 **	0.10 *	0.08 *
	1%	GOh_n	asy	0.50 **	0.44 **	0.30 *	0.32 *	0.52 **	0.58 **	0.34 **	0.32 *	0.48 **	0.46 **	0.34 **	0.36 **
		GOh_n	bts	1.24 **	0.74 **	0.82 **	0.96 **	0.98 **	1.10 **	0.76 **	0.94 **	0.98 **	0.74 **	1.00 **	1.12 **
		GOh_n	2bts	1.28 **	1.04 **	1.08 **	1.22 **	1.02 **	1.16 **	0.98 **	1.04 **	1.20 **	0.86 **	1.12 **	1.22 **
		JB_n	asy	0.08	0.38	1.26	1.40	0.06	0.74	2.14	2.36	0.02	1.04	3.36	4.40 **
		GO_n	asy	2.20	3.76 *	1.16	1.36	3.12	5.74 **	1.70	1.82	3.32	7.18	1.12	1.28
UN	5%	GOh_n	asy	3.02	3.42	2.22	2.26	3.48	3.58 *	3.16	3.34	3.22	3.90 *	2.54	2.66
		GOh_n	bts	4.74 **	4.20 **	4.32 **	4.68 **	4.96 **	4.30 **	4.28 **	4.52 **	5.22 **	4.90 **	4.32 **	4.64 **
		GOh_n	2bts	5.02 **	4.64 **	5.24 **	5.42 **	5.16 **	4.80 **	4.66 **	4.94 **	5.70 **	5.06 **	4.86 **	4.98 **
		JB_n	asy	0.34	1.34	3.24	3.44	0.36	2.24	4.48	5.04	0.24	3.32	7.00	8.30
		GO_n	asy	5.52	8.80 *	4.04	4.44	6.68	12.36	4.64	5.30	7.54	13.92	3.74	4.06
	10%	GOh_n	asy	6.34	7.90	6.04	6.26	6.80	8.34 *	6.98	7.16	7.78	8.24	6.58	6.80
		GOh_n	bts	8.74 *	8.94 **	8.82 **	9.20 **	9.90 **	9.34 **	9.20 **	9.58 **	10.04 **	9.52 **	8.56 *	8.72 *
		GOh_n	2bts	9.68 **	9.76 **	10.16 **	10.62 **	10.44 **	9.78 **	9.54 **	9.78 **	10.54 **	10.02 **	9.16 **	9.24 **
		JB_n	asy	0.00 *	0.40 **	0.18 *	0.26 *	0.00 *	0.70 **	1.00 **	1.10 **	0.00 *	0.84 **	2.02 *	2.54
		GO_n	asy	0.04 *	0.22 *	0.00 *	0.00 *	0.38 **	0.70 **	0.16 *	0.18 *	0.32 *	0.44 **	0.14 *	0.12 *
	1%	GOh_n	asy	0.20 *	0.42 **	0.18 *	0.18 *	0.52 **	0.70 **	0.32 *	0.34 **	0.46 **	0.42 **	0.30 *	0.38 **
		GOh_n	bts	0.90 **	0.96 **	0.68 **	0.70 **	1.06 **	1.10 **	0.92 **	0.92 **	1.04 **	1.00 **	0.84 **	1.02 **
		GOh_n	2bts	1.04 **	1.20 **	0.86 **	0.90 **	1.24 **	1.28 **	0.96 **	1.00 **	1.20 **	0.94 **	1.18 **	1.34 **
		JB_n	asy	0.10	1.80	1.84	1.82	0.22	3.48	3.62 *	4.02 **	0.28	3.88 *	5.96 **	7.18
		GO_n	asy	1.96	3.56 *	1.24	1.44	3.10	4.84 **	1.60	1.74	3.02	5.28 **	1.28	1.44
MN	5%	GOh_n	asy	2.98	3.88 *	2.54	2.54	3.86 *	4.22 **	3.60 *	3.48	3.46	4.30 **	3.24	3.48
		GOh_n	bts	4.62 **	4.66 **	4.10 **	4.26 **	5.26 **	4.78 **	4.74 **	4.92 **	4.52 **	5.04 **	4.46 **	4.68 **
		GOh_n	2bts	4.84 **	5.38 **	4.96 **	5.02 **	5.34 **	5.28 **	4.74 **	4.84 **	5.06 **	4.90 **	4.38 **	4.68 **
		JB_n	asy	0.50	4.48	3.84	4.20	1.12	7.12	6.74	7.64	0.92	8.16	10.50 **	11.56 *
		GO_n	asy	5.44	8.54 *	4.10	4.42	7.02	11.04 **	5.08	5.46	7.04	11.28 *	3.82	3.96
	10%	GOh_n	asy	6.94	9.58 **	6.08	6.20	7.94	9.88 **	7.90	7.92	7.64	9.60 **	7.88	8.40 *
		GOh_n	bts	8.90 **	9.22 **	8.90 **	9.02 **	9.50 **	9.32 **	9.40 **	9.70 **	9.16 **	10.00 **	8.94 **	9.16 **
		GOh_n	2bts	9.54 **	9.86 **	9.80 **	10.16 **	10.14 **	9.82 **	9.66 **	10.12 **	9.08 **	9.88 **	9.14 **	9.28 **

Note:
$Homo$ denotes homoskedastic errors; $HET1$,2,3 denotes heteroskedastic schemes explained section 4.1.
asy denotes asymptotic critical value; bts denotes single DF wild bootstrap; 2bts denotes Davidson & MacKinnon's fast double bootstrap.
* denotes that the estimate is consistent with the true significance level being between -1% and +1% from its nominal level.
** denotes that the estimate is consistent with the true significance level being between -0.5 % and +0.5 % from its nominal level.

Table 6: The number of times of experiments under homoskedasticity for 5 and 10 percent nominal levels that the estimated size is marked ** and *

(a) Numbers of times that the estimated size is marked ** of the nominal size 5% and 10%

	SN						$t(7)$						UN						MN						Total	Subtotal					
	$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$			$n-50$		$n-80$		$n-100$	
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%		5%	10%	5%	10%	5%	10%1
JB_n asy	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
GO_n asy	0	1	0	3	2	3	0	3	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0	16	0	4	1	4	3	4
GOh_n asy	3	3	3	2	3	2	2	0	2	1	3	2	0	0	1	1	2	2	0	0	1	2	0	0	35	5	3	7	6	8	6
GOh_n bts	1	2	3	3	3	3	0	1	3	3	3	3	3	2	3	3	3	3	3	3	3	3	3	3	63	7	8	12	12	12	12
GOh_n 2bts	1	1	3	2	3	3	0	0	3	3	3	3	2	3	3	3	3	3	3	3	3	3	3	3	60	6	7	12	11	12	12
out of	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	72	12	12	12	12	12	12

(b) Numbers of times that the estimated size is marked * of the nominal size 5% and 10%

	SN						$t(7)$						UN						MN						Total	Subtotal					
	$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$			$n-50$		$n-80$		$n-100$	
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%		5%	10%	5%	10%	5%	10%
JB_n asy	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0
GO_n asy	1	2	3	3	3	3	0	3	2	3	2	3	0	0	1	1	2	2	0	0	1	1	2	0	38	1	5	7	8	9	8
GOh_n asy	3	3	3	3	3	3	2	1	3	2	3	2	1	1	2	1	2	2	0	0	3	2	2	2	49	6	5	11	8	10	9
GOh_n bts	3	3	3	3	3	3	1	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	68	10	10	12	12	12	12
GOh_n 2bts	3	3	3	3	3	3	1	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	69	10	11	12	12	12	12
out of	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	72	12	12	12	12	12	12

Notes:
** denotes that the estimate is consistent with the true significance level being between -0.5 % and +0.5 % from its nominal level.
* denotes that the estimate is consistent with the true significance level being between -1 % and +1 % from its nominal level.

Table 7: The number of times of experiments for 5 and 10 percent nominal levels that the estimated size is marked ** and *

(a) Numbers of times that the estimated size is marked '' of the nominal size 5% and 10%**

		SN						$t(7)$						UN						MN						Total	Subtotal					
		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$			$n-50$		$n-80$		$n-100$	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%		5%	10%	5%	10%	5%	10%
JB_n	asy	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	2	0	0	5	4	2	2	23	2	2	5	4	6	4
GO_n	asy	2	3	3	4	4	5	3	5	1	3	1	1	2	2	4	1	2	2	1	2	3	3	3	2	62	8	12	11	11	10	10
GOh_n	asy	5	9	8	3	8	3	6	1	6	1	9	2	1	1	2	2	2	3	1	3	3	5	3	3	90	13	14	19	11	22	11
GOh_n	bts	2	2	3	3	12	12	0	1	3	3	11	12	12	11	12	12	12	9	12	12	12	12	12	12	204	26	26	30	30	47	45
GOh_n	2bts	1	1	4	3	12	12	0	0	3	4	9	12	8	10	12	12	12	12	10	10	12	12	12	12	195	19	21	31	31	45	48
out of		12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	288	48	48	48	48	48	48

(b) Numbers of times that the estimated size is marked '*' of the nominal size 5% and 10%

		SN						$t(7)$						UN						MN						Total	Subtotal					
		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$		$n-50$		$n-80$		$n-100$			$n-50$		$n-80$		$n-100$	
		5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%		5%	10%	5%	10%	5%	10%
JB_n	asy	2	4	0	0	1	0	0	0	0	0	0	0	0	0	1	0	5	3	0	0	8	5	4	4	37	2	4	9	5	10	7
GO_n	asy	3	4	6	5	6	5	3	5	4	6	3	5	3	3	4	3	4	4	3	3	4	4	5	3	98	12	15	18	18	18	17
GOh_n	asy	10	9	9	7	11	5	7	3	8	2	9	3	2	3	5	4	5	3	3	3	9	5	4	6	135	22	18	31	18	29	17
GOh_n	bts	7	4	11	6	12	12	1	1	5	5	12	12	12	12	12	12	12	12	12	12	12	12	12	12	232	32	29	40	35	48	48
GOh_n	2bts	3	3	10	7	12	12	1	2	5	6	12	12	11	11	12	12	12	12	12	12	12	12	12	12	227	27	28	39	37	48	48
out of		12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	288	48	48	48	48	48	48

Notes:
** denotes that the estimate is consistent with the true significance level being between -0.5 % and +0.5 % from its nominal level.
* denotes that the estimate is consistent with the true significance level being between -1 % and +1 % from its nominal level.

Performance of GO_n and GOh_n deteriorates as k increases. With $n = 100$ and $k = 3$ when error is $t(7)$, all rejection probabilities for 5% and 10% are "good", and when UN, they fall into the $\pm 0.5\%$ interval. On the other hand, With $n = 100$ and $k = 6$ when errors are $t(7)$, only rejection probability for 10% falls into $\pm 1\%$ interval and when errors are UN, none falls into either intervals. Similar performance is found for GOh_n.

Under Heteroskedasticity With the heteroskedastic normal errors, JB_n severely overrejects the null. For example, with $n = 100$ and $k = 6$, under $HET1$, $HET2$, and $HET3$ schemes, the rejection probabilities for 5% are 19.92%, 14.68%, and 15.02%, respectively. With the heteroskedastic normal or non-normal errors, GO_n behaves relatively well with $HET1$ scheme, however, it is severely undersized systematically with the multiple heteroskedastic schemes, $HET2$ and $HET3$. For example, with $n = 100$ and $k = 6$ and UN errors, under $HET1$, $HET2$, and $HET3$ schemes, the rejection probabilities for 10% are 13.92%, 3.74%, and 4.06%, respectively. These evidences confirm the necessity of controlling the heteroskedasticity for skewness test. However, GOh_n consistently overrejects when errors are SN and $t(7)$, and is undersized when errors are UN. Table 7 gives the number of times in groups of 12 experiments that the rejection frequencies are consistent with the claim that the true significance level is within 0.5% (1.0%) of nominal levels of 5% and 10%. Consequently, the asymptotic critical values of GO_n give only 62 out of 288 being "good" and 98 of 288 being "satisfactory". The asymptotic critical values of GOh_n have better performance, however, they give just 90 out of 288 being "good" and the wild bootstrap critical values give 135 of 288 being "satisfactory".

The DF wild bootstrapped GOh_n enjoys a success in controlling the size with large n. As can be seen from Table 7, the bootstrap critical values of GOh_n give 204 out of 288 being "good" and 232 of 288 being satisfactory. In particular, with $n = 100$, 92 out of 96 experiments are "good", and all 96 are "satisfactory". By contrast, the asymptotic critical values of GOh_n with $n = 100$ give only 33 out of 96 being "good" and 46 of 96 being "satisfactory". In our experiments, there is no evidence that the fast double bootstrap affords further refinement over single DF wild bootstrap. The number of "good" and "satisfactory" of the fast double bootstrapped GOh_n is even slightly fewer than the single DF wild bootstrap. Still, though, DF bootstrap does not behave universally "good", especially when $n = 50$ and 80, since only about 2/3 of the experiments are "good".

4.2.2 Power

Table 8$\sim$10 gives the estimated power of the tests. Overall, the superiority of the DF wild bootstrapped GOh_n over other tests (except Jarque and Bera test) is really remarkable. Of course, it is not possible to compare the power of the tests directly, since the estimated size of asymptotic critical values of the tests had poor agreement with its nominal levels. However, given that the asymptotic critical values of GO_n and GOh_n does not have a very bad agreement with their nominal levels under homoskedastic errors with $n = 100$, the comparison of the powers may be suggestive. With $k = 3$ and $n = 100$ at 5% nominal levels under homoskedastic log-normal errors, the rejection frequencies of asymptotic critical values of GO_n and GOh_n, are 43.78%, 49.54%, while that of DF wild bootstrap critical value of GOh_n is 99.76%. With the same setting but under homoskedastic chi-square errors with 8 degrees of freedom, the rejection frequencies of asymptotic critical values of GO_n and GOh_n, are 73.48%, 76.96%, while that of DF wild bootstrap critical value of GOh_n is 93.70%.

Table 8: Estimated power of the tests: $k = 3$

				$n = 50$				$n = 80$				$n = 100$			
				Homo	*HET1*	*HET2*	*HET3*	*Homo*	*HET1*	*HET2*	*HET3*	*Homo*	*HET1*	*HET2*	*HET3*
		JB_n	asy	97.64	90.16	96.38	96.34	100.00	99.04	99.86	99.84	100.00	99.66	99.98	99.98
		GO_n	asy	11.36	3.88	10.60	10.42	14.58	6.06	12.94	12.58	17.22	7.76	13.56	13.24
	1%	GOh_n	asy	17.06	7.52	15.94	15.62	19.62	11.78	21.62	21.44	21.82	11.80	21.86	21.70
		GOh_n	bts	86.98	51.42	75.62	76.22	97.80	83.46	92.40	92.56	98.92	93.22	94.80	94.66
		GOh_n	2bts	89.92	61.32	76.72	77.42	95.84	84.54	89.08	88.72	96.90	92.76	93.38	93.06
		JB_n	asy	99.46	94.18	98.72	98.70	100.00	99.56	99.96	99.96	100.00	99.90	99.98	99.98
		GO_n	asy	35.94	19.08	32.94	32.48	41.40	27.04	37.92	37.44	43.78	30.96	38.06	37.26
LN	5%	GOh_n	asy	46.00	31.06	45.56	45.00	48.22	36.46	52.14	51.94	49.54	37.70	52.66	52.44
		GOh_n	bts	96.38	74.74	90.34	90.50	99.34	94.92	97.66	97.60	99.76	98.34	98.12	97.98
		GOh_n	2bts	97.44	79.14	90.46	90.70	99.54	95.10	96.90	96.90	99.86	98.84	98.84	98.74
		JB_n	asy	99.74	95.98	99.18	99.24	100.00	99.76	99.96	99.98	100.00	99.94	100.00	99.98
		GO_n	asy	56.10	38.38	51.22	50.86	59.84	47.30	57.14	56.70	62.08	51.54	57.18	56.58
	10%	GOh_n	asy	66.32	52.70	65.22	65.16	66.30	56.36	69.78	69.30	67.74	58.26	71.12	70.84
		GOh_n	bts	98.40	84.46	94.68	94.62	99.74	97.56	98.70	98.62	99.82	99.26	98.98	98.92
		GOh_n	2bts	98.76	86.46	94.38	94.56	99.68	97.64	98.22	98.12	99.98	99.52	99.32	99.32
		JB_n	asy	84.32	75.16	81.06	81.00	99.24	95.48	97.64	97.60	99.86	98.70	99.36	99.32
		GO_n	asy	21.66	5.84	15.94	15.38	35.76	11.12	26.92	25.38	45.68	15.74	30.88	28.94
	1%	GOh_n	asy	27.06	8.82	19.62	19.18	41.54	17.18	37.42	36.34	51.12	21.12	43.54	42.10
		GOh_n	bts	83.94	39.54	65.86	65.74	99.24	75.56	89.34	88.96	99.90	88.34	93.76	93.50
		GOh_n	2bts	84.02	47.88	66.46	67.48	96.42	75.24	84.56	84.10	96.94	86.78	91.74	91.24
		JB_n	asy	95.02	85.54	91.66	91.62	99.94	98.26	99.42	99.38	99.98	99.46	99.80	99.80
		GO_n	asy	55.88	26.68	46.42	45.28	71.42	42.12	59.84	57.86	76.20	51.92	62.12	60.90
$\chi^2(2)$	5%	GOh_n	asy	63.96	35.38	55.26	54.14	76.24	50.56	74.62	73.56	80.12	59.04	79.70	79.16
		GOh_n	bts	96.46	66.54	84.76	84.78	99.94	92.58	96.88	96.60	100.00	97.40	98.26	98.12
		GOh_n	2bts	96.66	70.18	85.24	84.94	99.94	92.56	96.24	96.06	100.00	97.52	98.34	98.30
		JB_n	asy	97.44	89.86	95.16	95.26	99.98	99.06	99.72	99.68	99.98	99.68	99.96	99.98
		GO_n	asy	75.84	47.56	66.02	64.92	86.02	65.48	77.62	76.36	88.68	72.72	78.32	76.98
	10%	GOh_n	asy	82.14	58.78	75.08	74.38	89.00	72.74	88.38	87.64	91.10	77.94	91.76	91.44
		GOh_n	bts	98.48	77.90	90.34	90.48	99.96	96.12	98.32	98.28	100.00	99.02	99.38	99.30
		GOh_n	2bts	98.46	79.94	90.28	90.32	99.96	96.34	98.10	98.04	100.00	98.94	99.42	99.30
		JB_n	asy	29.86	38.48	30.72	31.12	59.36	61.56	56.74	56.82	75.24	74.56	71.24	71.24
		GO_n	asy	7.34	2.00	4.36	4.04	22.06	3.92	10.58	9.60	35.82	6.30	13.92	12.10
	1%	GOh_n	asy	9.72	3.10	5.78	5.18	25.52	5.70	15.80	14.58	39.92	8.78	20.96	19.44
		GOh_n	bts	30.74	8.58	21.20	20.78	62.36	16.54	41.70	40.34	78.26	20.58	51.98	50.36
		GOh_n	2bts	33.48	10.78	23.90	23.28	59.58	17.52	41.38	40.46	74.66	21.54	51.76	50.32
		JB_n	asy	51.46	51.54	49.40	49.50	80.52	74.18	75.08	75.28	90.24	84.68	85.18	84.64
		GO_n	asy	34.82	13.50	24.04	22.94	61.30	22.82	39.02	36.26	73.48	31.60	42.64	39.94
$\chi^2(8)$	5%	GOh_n	asy	40.68	17.70	28.12	26.96	65.98	28.40	51.26	48.48	76.96	37.76	60.14	58.04
		GOh_n	bts	58.16	21.98	44.36	43.66	85.54	37.86	67.44	65.92	93.70	47.96	75.04	73.58
		GOh_n	2bts	58.92	22.76	45.96	45.92	85.14	37.54	67.02	65.92	93.22	47.12	75.58	74.46
		JB_n	asy	63.82	59.22	60.16	60.24	88.00	80.10	82.92	82.82	94.80	88.44	90.54	90.24
		GO_n	asy	56.50	26.74	41.96	40.72	81.92	42.54	59.14	56.76	88.36	53.72	61.40	58.30
	10%	GOh_n	asy	61.94	33.30	47.70	46.78	84.96	49.08	69.50	67.68	90.66	59.22	77.92	75.92
		GOh_n	bts	71.46	32.24	57.34	56.94	92.46	52.68	77.60	76.56	97.22	63.14	83.88	82.64
		GOh_n	2bts	71.74	32.70	58.96	58.56	92.22	51.82	78.00	76.74	97.20	62.34	84.10	82.80

Notes:
Homo denotes homoskedastic errors; $HET1,2,3$ denotes heteroskedastic schemes explained section 4.1.
asy denotes asymptotic critical value; bts denotes single DF wild bootstrap; 2bts denotes fast double bootstrap.

Table 9: Estimated power of the tests: $k = 4$

				$n - 50$				$n - 80$				$n - 100$			
				$Homo$	$HET1$	$HET2$	$HET3$	$Homo$	$HET1$	$HET2$	$HET3$	$Homo$	$HET1$	$HET2$	$HET3$
		JB_n	asy	0.24	2.72	2.06	2.02	0.76	10.06	6.04	6.58	1.48	20.32	10.80	11.90
		GO_n	asy	5.50	6.18	1.72	1.68	17.14	22.96	4.54	4.12	27.50	39.86	5.24	4.90
	1%	GOh_n	asy	6.20	5.26	2.58	2.62	17.22	18.50	6.60	6.32	27.40	34.20	8.10	7.72
		GOh_n	bts	10.86	10.94	6.94	7.00	23.14	27.06	13.14	12.54	33.16	40.18	17.48	16.48
		GOh_n	2bts	10.80	10.98	7.82	8.02	21.20	25.20	13.28	12.68	30.54	37.30	18.36	17.32
		JB_n	asy	2.34	13.46	7.42	7.90	7.22	34.22	16.96	17.98	11.80	50.28	26.18	27.02
		GO_n	asy	20.56	27.28	10.34	10.94	40.64	54.88	18.12	17.88	53.26	70.52	20.32	19.66
LN	5%	GOh_n	asy	21.20	24.46	12.78	12.76	40.64	49.52	21.50	20.46	52.88	66.96	27.36	25.46
		GOh_n	bts	26.12	29.32	18.48	18.50	45.00	53.52	28.20	27.30	56.48	68.06	34.56	32.78
		GOh_n	2bts	26.20	29.16	19.48	19.72	44.26	52.50	28.58	27.92	55.88	67.22	35.20	33.34
		JB_n	asy	6.72	25.08	14.08	14.82	16.54	50.86	26.74	28.24	25.72	66.90	37.22	38.46
		GO_n	asy	32.54	43.26	20.00	20.60	53.86	69.86	30.24	30.74	66.16	82.50	32.66	31.30
	10%	GOh_n	asy	33.44	40.10	21.50	21.34	54.20	65.48	33.38	31.92	65.90	79.58	39.06	37.88
		GOh_n	bts	37.34	41.46	26.74	26.98	57.20	66.88	39.44	38.42	68.58	79.98	44.28	42.78
		GOh_n	2bts	37.58	41.68	28.68	28.44	56.74	66.56	39.94	39.18	68.48	79.32	45.32	44.02
		JB_n	asy	79.56	73.92	76.10	76.56	98.76	95.84	97.22	97.26	99.80	98.66	99.16	99.12
		GO_n	asy	20.36	5.84	14.80	14.22	35.90	11.26	26.46	25.12	45.42	16.00	30.98	28.92
	1%	GOh_n	asy	29.60	12.16	22.18	21.60	46.02	22.16	40.08	38.76	53.90	26.42	46.46	45.20
		GOh_n	bts	79.08	45.84	61.04	61.28	98.64	80.88	88.20	87.18	99.88	91.30	93.54	92.70
		GOh_n	2bts	79.14	54.06	62.36	62.78	95.98	80.38	83.72	83.14	96.76	89.86	91.20	90.74
		JB_n	asy	92.76	85.50	89.66	89.74	99.86	98.52	99.16	99.16	99.96	99.60	99.80	99.78
		GO_n	asy	55.54	27.02	45.40	44.60	71.24	42.58	59.58	57.36	76.18	51.66	62.22	60.72
$\chi^2(2)$	5%	GOh_n	asy	68.70	43.02	59.28	58.50	78.90	57.16	76.44	75.12	81.86	63.78	82.04	81.42
		GOh_n	bts	94.26	72.42	82.16	81.94	99.88	94.88	96.40	96.14	100.00	98.30	98.24	98.08
		GOh_n	2bts	94.80	75.22	82.50	82.74	99.90	95.12	95.94	95.52	99.96	98.40	98.34	98.04
		JB_n	asy	96.12	90.94	93.70	93.80	99.98	99.26	99.62	99.58	99.98	99.78	99.98	99.96
		GO_n	asy	75.32	48.56	65.48	64.78	85.84	66.20	77.18	75.84	88.84	73.10	78.32	76.96
	10%	GOh_n	asy	85.22	65.64	78.60	77.86	90.36	77.44	89.42	88.64	92.26	81.32	92.72	92.56
		GOh_n	bts	97.46	82.68	88.98	88.88	99.96	97.80	98.08	97.98	100.00	99.36	99.34	99.24
		GOh_n	2bts	97.56	83.58	88.96	89.14	99.96	97.84	97.84	97.78	100.00	99.26	99.38	99.28
		JB_n	asy	25.20	35.58	26.52	27.20	55.72	60.72	53.76	54.22	72.40	73.84	69.02	68.78
		GO_n	asy	5.84	1.68	3.80	3.58	20.76	3.42	9.66	8.88	34.36	5.80	13.32	12.04
	1%	GOh_n	asy	9.52	4.02	6.08	5.58	27.40	7.26	16.00	15.08	41.06	10.60	21.48	20.24
		GOh_n	bts	27.92	11.34	18.68	18.26	59.16	20.80	39.24	38.28	74.92	26.24	50.34	48.18
		GOh_n	2bts	29.72	13.88	20.86	21.14	57.12	22.92	38.80	37.48	71.90	27.08	50.22	48.44
		JB_n	asy	46.80	49.52	45.08	45.26	77.52	74.02	71.88	72.36	88.68	84.04	83.44	83.32
		GO_n	asy	32.92	11.92	22.42	21.46	60.28	22.46	37.60	35.36	72.82	31.52	41.74	39.58
$\chi^2(8)$	5%	GOh_n	asy	41.72	20.62	29.54	28.28	67.16	33.20	51.06	48.58	77.74	42.68	61.60	59.50
		GOh_n	bts	54.78	27.08	41.50	40.62	83.40	44.90	65.44	64.06	92.72	54.14	74.40	72.98
		GOh_n	2bts	55.54	28.36	42.92	42.42	83.14	45.12	65.26	64.16	92.46	53.56	74.34	73.04
		JB_n	asy	58.92	57.90	55.80	56.40	86.48	80.22	81.10	81.12	93.68	88.42	89.30	89.02
		GO_n	asy	53.92	25.68	39.82	38.90	80.34	43.24	57.12	54.74	87.56	54.44	60.56	57.32
	10%	GOh_n	asy	62.14	37.42	49.18	48.36	84.76	54.46	69.26	67.48	90.76	63.80	77.90	76.18
		GOh_n	bts	68.48	37.60	54.96	54.44	91.34	58.64	76.50	75.34	96.58	68.52	82.82	81.68
		GOh_n	2bts	68.36	38.26	55.94	55.44	91.00	58.32	76.56	75.14	96.56	67.96	83.14	81.72

Notes:
$Homo$ denotes homoskedastic errors; $HET1, 2, 3$ denotes heteroskedastic schemes explained section 4.1.
asy denotes asymptotic critical value; bts denotes single DF wild bootstrap; 2bts denotes fast double bootstrap.

Table 10: Estimated power of the tests: $k = 6$

				$n - 50$				$n - 80$				$n - 100$			
				Homo	*HET*1	*HET*2	*HET*3	*Homo*	*HET*1	*HET*2	*HET*3	*Homo*	*HET*1	*HET*2	*HET*3
		JB_n	asy	0.12	1.10	1.10	1.28	0.44	7.40	4.00	4.40	1.02	14.02	9.02	9.88
		GO_n	asy	3.22	4.70	1.28	1.26	13.32	18.78	4.02	3.92	22.52	27.42	4.32	3.98
	1%	GOh_n	asy	4.18	4.50	2.14	2.16	13.38	14.06	5.96	5.58	22.06	19.42	7.28	7.00
		GOh_n	bts	7.88	7.96	4.82	4.86	18.74	19.66	10.60	10.44	28.12	26.26	15.36	14.54
		GOh_n	2bts	8.02	8.50	5.38	5.46	17.60	18.40	10.50	10.48	25.96	24.62	16.12	14.90
		JB_n	asy	1.42	8.98	4.82	5.14	4.64	27.16	13.06	14.10	9.16	38.10	22.96	24.06
		GO_n	asy	15.32	21.34	7.84	7.98	34.24	48.96	17.08	17.08	47.20	59.18	18.56	18.30
LN	5%	GOh_n	asy	16.30	20.56	10.16	10.14	33.68	41.46	19.88	19.40	46.82	48.58	25.18	23.94
		GOh_n	bts	21.48	23.16	15.14	14.86	40.12	43.42	25.56	24.92	52.56	51.78	31.84	30.36
		GOh_n	2bts	22.06	23.22	16.38	16.46	40.22	43.18	25.92	25.20	51.78	51.44	32.36	30.76
		JB_n	asy	4.14	18.28	10.04	10.62	13.28	43.88	22.60	23.60	21.46	53.34	33.90	34.78
		GO_n	asy	26.04	36.44	17.12	17.58	47.94	64.60	29.46	29.58	62.34	72.12	30.92	29.92
	10%	GOh_n	asy	27.22	34.56	17.90	17.64	47.36	57.48	30.76	29.70	61.52	62.62	36.74	35.56
		GOh_n	bts	32.50	34.90	23.84	23.62	52.54	57.78	36.18	35.28	65.30	64.26	42.48	41.02
		GOh_n	2bts	33.36	35.86	26.08	25.74	53.16	57.58	37.16	36.20	65.56	64.50	43.30	41.82
		JB_n	asy	67.66	63.52	64.48	64.32	97.28	93.42	95.52	95.56	99.60	98.72	98.68	98.66
		GO_n	asy	18.34	4.98	13.80	13.14	35.52	10.98	26.88	25.16	45.26	15.92	30.74	28.48
	1%	GOh_n	asy	34.40	17.48	26.48	25.84	52.64	28.30	46.54	44.88	59.66	37.98	53.26	52.10
		GOh_n	bts	68.16	37.64	52.32	51.74	95.90	72.82	85.30	84.52	99.44	91.10	92.38	91.86
		GOh_n	2bts	67.88	42.38	52.08	52.32	93.60	74.56	80.76	80.46	96.62	90.20	89.58	89.10
		JB_n	asy	86.12	78.48	82.62	82.64	99.66	97.62	98.70	98.74	99.92	99.50	99.70	99.68
		GO_n	asy	52.34	25.28	42.78	41.96	70.56	42.36	59.96	58.32	75.92	51.72	61.74	60.68
$\chi^2(2)$	5%	GOh_n	asy	73.74	53.32	63.68	63.14	83.28	65.16	80.64	79.54	85.96	72.44	84.92	84.62
		GOh_n	bts	89.76	66.02	76.64	76.24	99.66	92.00	95.72	95.64	99.98	98.62	97.94	97.88
		GOh_n	2bts	89.74	67.02	76.38	76.12	99.62	92.52	95.34	95.10	100.00	99.00	98.00	97.76
		JB_n	asy	92.80	84.96	89.48	89.46	99.82	98.78	99.50	99.42	99.98	99.74	99.88	99.84
		GO_n	asy	72.94	46.52	63.24	62.44	85.74	65.82	77.20	76.34	88.72	73.36	78.04	76.86
	10%	GOh_n	asy	89.04	73.50	81.20	80.54	93.74	83.44	92.56	92.20	94.48	87.04	94.82	94.46
		GOh_n	bts	95.24	77.82	85.90	85.44	99.86	96.48	97.90	97.72	100.00	99.48	99.20	99.02
		GOh_n	2bts	94.86	77.10	85.52	84.74	99.88	96.66	97.78	97.50	100.00	99.52	99.16	99.08
		JB_n	asy	17.74	27.66	19.56	19.90	48.16	54.70	47.28	48.16	66.68	71.04	64.10	64.84
		GO_n	asy	5.00	1.70	3.54	3.30	18.80	3.62	9.06	8.48	32.32	5.20	12.80	11.32
	1%	GOh_n	asy	11.00	5.70	7.60	7.58	27.40	9.74	18.10	16.90	43.02	15.50	24.74	23.66
		GOh_n	bts	22.36	10.10	14.98	14.96	52.64	18.76	35.58	34.80	70.84	31.26	46.84	45.12
		GOh_n	2bts	23.50	11.52	16.40	16.56	51.58	20.52	35.06	34.68	67.96	32.62	47.12	45.60
		JB_n	asy	36.84	41.70	36.58	37.02	71.00	69.46	66.90	67.32	85.16	82.58	80.78	80.88
		GO_n	asy	28.18	11.76	20.02	19.36	57.46	22.58	37.08	35.10	70.74	30.72	40.94	38.56
$\chi^2(8)$	5%	GOh_n	asy	40.94	26.04	29.92	29.42	67.94	38.02	51.82	49.80	79.62	50.82	63.02	60.70
		GOh_n	bts	48.20	25.70	38.00	37.52	79.24	41.48	61.64	60.32	90.46	59.94	71.82	70.60
		GOh_n	2bts	48.86	24.94	38.68	38.40	78.78	41.68	61.96	60.52	90.38	59.44	72.22	71.00
		JB_n	asy	50.02	50.50	47.36	47.66	81.90	76.42	77.00	76.80	91.76	87.92	87.18	87.14
		GO_n	asy	49.00	24.76	36.40	35.44	78.18	42.02	57.22	55.24	86.48	54.42	59.48	56.74
	10%	GOh_n	asy	61.50	43.02	48.04	47.42	85.40	57.38	70.68	68.86	91.70	70.24	78.40	77.12
		GOh_n	bts	62.46	36.64	49.90	49.36	88.50	55.26	74.12	73.02	95.26	72.90	81.26	80.32
		GOh_n	2bts	61.76	35.30	50.38	50.28	88.34	54.72	74.34	73.14	95.14	72.02	81.36	80.38

Notes:
Homo denotes homoskedastic errors; *HET*1,2.,3 denotes heteroskedastic schemes explained section 4.1.
asy denotes asymptotic critical value; bts denotes single *DF* wild bootstrap; 2bts denotes fastbts double bootstrap.

However, with log-normal errors, as k increases from 3 to 4 and 6, the above extremely higher rejection frequencies of DF wild bootstrap over asymptotic critical value disappears very quickly. With $k = 4$ and $n = 100$ at 5% nominal levels under homoskedastic log-normal errors, the rejection frequencies of asymptotic critical values of GO_n and GOh_n, are 53.26%, 52.88%, while that of DF wild bootstrap critical value of GOh_n is 56.48%. On the other hand, with chi-square errors the effect of increase in k on the test statistics is particularly slight.

Heteroskedasticity also affects greatly the power of both conventional tests and DF wild bootstrap test, and it seems that the heteroskedasticity reduces the power. On the whole, $HET1$ scheme reduces the power most, and $HET2$ and $HET3$ have very similar numerical effect on the power. However, the DF wild bootstrapped GOh_n always maintains higher power than the asymptotic GOh_n. For instance, with $k = 4$ and $n = 100$ at 5% nominal levels, under homoskedastic chi-square with 8 degrees of freedom errors, the homoskedastic error gave rejection frequencies of asymptotic critical values and DF wild bootstrap critical values of GOh_n are 77.74% and 92.72%, while under $HET1$ they are 42.68% and 54.14%, under $HET2$ 61.6% and 74.4%, and under $HET3$ 59.50% and 72.98%.

4.2.3 Modified Tests and Whang Test - Under the "Potential Inconsistency"

A Monte Carlo simulation has been implemented to investigate the finite sample behaviour of the proposed tests, under homoskedasticity, $HET1$, $HET3$, and $HET4$; see Table 11 & 12.[18] We compared the DF wild bootstrapped GOh_n, GOh_R_n, GOh_W_n, $Whang_n$, and double bootstrapped $Whang_n$.

Size First of all, the DF wild bootstrapped $Whang_n$ often overrejects the null hypothesis, with $t(7)$ errors. Also there is a tendency that as the degree of heteroskedasticity becomes severer, the overrejection becomes stronger. With $HET1$ at 5% nominal level, whose coefficient of variation is 0.31, the estimated size for 5% nominal level with $t(7)$ errors is 8.48%, but with $HET4$, whose coefficient of variation is 1.33, the estimated size with $t(7)$ errors is 11.22% and here even with standard normal errors the estimated size is 8.20%. On the other hand, with $HET4$ the DF wild bootstrapped GOh_n, GOh_R_n, and GOh_W_n have at least a satisfactory agreement of the estimated frequencies with the nominal size. $Whang_n$'s overrejection may be due to the fact that it is a non-asymptotically pivotal test, thus the single bootstrap just gives the error in rejection probability of the same order of first order approximation. In this situation, double bootstrap can give further refinement (Beran 1988). Now the estimated rejection frequencies of the double bootstrapped $Whang_n$ have much better agreement with the nominal size. For example, the estimated size of double bootstrapped $Whang_n$ with $t(7)$ errors under $HET4$ becomes 7.26% compared with 11.22% using the single bootstrap.

[18] Observe that the rejection frequencies of the same asymptotic test (eg. GOh_n) with the same n and k under homoskedasticity in Table 3~5 & Table 8~10 and Table 11 &12 are not identical, due to the stoppage rule used for single bootstrap.

Table 11: Estimated size and power of the modified tests and Whang test

		$n - 100$ Homo							$n - 100$ HET1 (coefficient of variation of the error variances: 0.313)						
		SN	$t(7)$	UN	MN	LN	$\chi^2(2)$	$\chi^2(8)$	SN	$t(7)$	UN	MN	LN	$\chi^2(2)$	$\chi^2(8)$
JB_n	asy	0.86 **	14.74	0.00 *	0.02 *	100.00	99.98	78.12	2.48	18.78	0.00 *	0.02 *	100.00	99.94	75.90
GO_n	asy	0.36 **	0.34 **	0.70 **	0.70 **	16.64	45.84	36.60	0.36 **	0.32 *	0.90 **	0.46 **	15.28	39.22	28.16
GOh_n	asy	0.42 **	0.40 **	0.78 **	0.84 **	18.58	48.40	39.02	0.40 **	0.32 *	0.96 **	0.54 **	17.10	42.06	30.66
GOh_R_n	asy	0.08 *	0.04 *	0.36 **	0.14 *	6.78	6.80	1.00 **	0.16 *	0.14 *	0.34 **	0.20 *	5.78	5.12	0.88 **
1% GOh_W_n	asy	0.08 *	0.06 *	0.48 **	0.18 *	25.24	44.24	15.02	0.22 *	0.18 *	0.46 **	0.30 *	22.76	38.08	11.00
GOh_n	bts	0.90 **	1.10 **	1.06 **	1.36 **	99.56	99.88	78.92	0.98 **	1.10 **	1.66 **	1.02 **	99.30	99.86	67.48
GOh_R_n	bts	0.92 **	0.90 **	1.42 **	0.92 **	26.96	27.14	7.66	1.14 **	1.12 **	1.30 **	1.44 **	22.92	22.00	6.38
GOh_W_n	bts	0.92 **	0.92 **	1.44 **	1.02 **	69.04	82.20	43.30	1.02 **	1.12 **	1.40 **	1.44 **	65.38	76.24	32.56
$Whang_n$	bts	0.86 **	1.62 **	1.36 **	0.92 **	99.48	98.76	64.46	1.02 **	1.82 *	0.98 **	0.88 **	99.44	98.54	55.76
$Whang_n$	dbts	1.02 **	1.30 **	1.60 **	1.48 **	98.38	97.48	59.42	1.18 **	1.66 **	1.36 **	1.16 **	98.42	96.62	50.68
JB_n	asy	3.92 *	24.86	0.04	0.32	100.00	100.00	91.76	7.80	29.32	0.14	0.62	100.00	99.98	90.14
GO_n	asy	3.84 *	3.26	4.12 **	3.82 *	43.80	76.88	75.72	4.00 *	3.62 *	4.48 **	3.52	42.32	73.10	67.68
GOh_n	asy	4.26 **	3.64 *	4.48 **	4.18 **	46.32	78.98	77.42	4.32 **	3.88 *	4.70 **	3.88 *	45.44	75.50	69.60
GOh_R_n	asy	1.86	1.74	2.88	1.64	27.94	31.18	9.86	1.60	1.96	2.44	2.14	24.88	26.26	8.78
5% GOh_W_n	asy	2.40	2.20	3.52	2.56	58.36	79.24	51.24	2.00	2.32	2.76	2.76	55.58	74.34	41.86
GOh_n	bts	5.40 **	5.48 **	5.12 **	5.28 **	99.84	100.00	94.34	5.12 **	5.38 **	5.84 **	5.22 **	99.76	100.00	89.66
GOh_R_n	bts	4.64 **	4.82 **	5.90 **	4.80 **	48.68	53.42	22.36	4.62 **	5.24 **	5.54 **	5.46 **	43.72	46.94	19.34
GOh_W_n	bts	4.86 **	4.74 **	5.94 **	5.56 **	83.86	93.68	71.50	4.98 **	5.14 **	5.52 **	5.54 **	81.60	90.58	59.90
$Whang_n$	bts	5.40 **	7.38	6.28 *	5.64 **	99.98	100.00	89.64	5.88 **	8.48	5.78 **	4.74 **	99.98	99.98	84.44
$Whang_n$	dbts	5.58 **	5.72 **	6.28 *	6.16 *	99.20	99.76	87.54	5.68 **	6.06 *	5.86 **	5.50 **	99.18	99.54	80.72
JB_n	asy	8.20	32.58	0.44	1.30	100.00	100.00	95.74	13.82	37.08	0.80	2.24	100.00	100.00	94.50
GO_n	asy	9.50 **	9.18 **	9.06 **	9.12 **	62.24	89.36	89.78	9.64 **	9.32 **	9.44 **	8.02	60.42	87.20	84.20
GOh_n	asy	10.14 **	9.82 **	9.62 **	9.52 **	65.20	90.66	90.66	10.04 **	10.20 **	9.94 **	8.50 *	63.38	88.84	85.72
GOh_R_n	asy	5.30	6.08	6.80	4.90	46.96	52.26	22.66	5.34	6.62	6.12	5.70	43.04	46.48	20.14
10% GOh_W_n	asy	7.14	6.76	8.12	7.08	77.30	91.84	71.98	6.58	7.44	7.26	6.50	75.08	88.82	62.10
GOh_n	bts	10.20 **	10.46 **	10.88 **	10.56 **	99.86	100.00	97.58	10.42 **	10.32 **	10.94 **	9.62 **	99.84	100.00	94.48
GOh_R_n	bts	9.38 **	9.88 **	10.78 **	9.06 **	60.82	66.92	33.96	9.32 **	10.38 **	10.56 **	10.58 **	56.40	60.78	29.58
GOh_W_n	bts	9.78 **	10.22 **	11.32 *	10.64 **	89.54	96.70	82.82	9.86 **	10.40 **	11.02 **	10.70 **	88.60	95.74	72.80
$Whang_n$	bts	10.92 **	13.68	11.50 *	11.16 **	100.00	100.00	95.30	11.92	14.88	10.96 **	10.42 **	100.00	100.00	91.72
$Whang_n$	dbts	10.66 **	10.96 **	10.52 **	11.52 *	99.58	100.00	94.32	11.08 **	11.26 *	11.10 **	10.64 **	99.64	99.92	89.76

Notes:
Homo denotes homoskedastic errors; $HET1,2,4$ denotes heteroskedastic schemes explained section 4.1.
asy denotes asymptotic critical value; bts denotes single DF wild bootstrap; dbts denotes full double bootstrap.
* denotes that the estimate is consistent with the true significance level being between -1 % and +1 % from its nominal level.
** denotes that the estimate is consistent with the true significance level being between -0.5 % and +0.5 % from its nominal level.

Table 12: Estimated size and power of the modified tests and Whang's test

		$n - 100$ HET3 (coefficient of variation of the error variances: 0.312)							$n - 100$ HET4 (coefficient of variation of the error variances: 1.33)						
		SN	$t(7)$	UN	MN	LN	$\chi^2(2)$	$\chi^2(8)$	SN	$t(7)$	UN	MN	LN	$\chi^2(2)$	$\chi^2(8)$
JB_n	asy	1.72 **	16.96	0.00 *	0.12 *	100.00	99.90	76.78	41.32	52.92	15.98	22.36	82.44	68.70	52.38
GO_n	asy	0.24 *	0.28 *	0.64 **	0.46 **	15.62	40.76	28.90	0.60 **	0.40 **	0.70 **	0.72 **	0.00 *	0.16 *	0.38 **
GOh_n	asy	0.36 **	0.40 **	0.76 **	0.46 **	18.50	45.50	33.26	0.92 **	0.72 **	0.90 **	0.94 **	0.76 **	1.08 **	1.08 **
GOh_R_n	asy	0.06 *	0.12 *	0.18 *	0.18 *	7.58	8.48	1.56 **	0.14 *	0.48 **	0.16 *	0.24 *	18.42	14.08	2.82
1% GOh_W_n	asy	0.14 *	0.10 *	0.62 **	0.24 *	24.34	43.62	15.00	0.48 **	0.66 **	0.30 *	0.24 *	18.38	10.92	2.16 *
GOh_n	bts	1.00 **	1.06 **	1.32 **	1.02 **	99.50	99.84	72.10	1.28 **	1.38 **	1.38 **	1.56 **	6.86	4.30	2.02 *
GOh_R_n	bts	1.20 **	1.06 **	1.26 **	1.20 **	26.44	30.02	8.88	1.62 **	2.16 *	1.22 **	1.42 **	42.02	37.22	10.40
GOh_W_n	bts	1.06 **	0.86 **	1.36 **	1.28 **	67.34	79.40	41.22	1.42 **	1.94 *	1.28 **	1.32 **	47.14	33.48	8.94
$Whang_n$	bts	1.12 **	1.46 **	0.94 **	0.68 **	99.30	98.78	62.80	1.56 **	3.22	0.66 **	0.60 **	31.04	17.86	5.30
$Whang_n$	dbts	1.18 **	1.42 **	1.32 **	1.10 **	98.06	97.26	58.58	1.90 *	2.68	1.14 **	1.32 **	24.32	14.44	4.68
JB_n	asy	6.08 *	27.16	0.26	0.64	100.00	100.00	90.28	52.84	63.16	29.14	34.84	86.38	75.74	62.86
GO_n	asy	3.14	2.84	5.16 **	4.08 **	41.76	73.62	66.72	4.82 **	3.92 *	5.54 **	4.48 **	1.08	2.00	3.52
GOh_n	asy	3.86 *	3.54	4.64 **	4.04 **	45.68	77.40	71.94	6.64	5.88 **	7.02	5.80 **	4.66 **	7.20	6.74
GOh_R_n	asy	1.72	1.72	2.54	2.32	27.82	33.94	11.26	3.00	3.72 *	3.18	3.16	46.56	45.62	16.20
5% GOh_W_n	asy	2.62	2.62	3.80 *	3.14	57.06	78.88	49.58	3.66 *	4.04 **	2.62	2.42	47.94	40.26	13.58
GOh_n	bts	5.08 **	5.22 **	5.72 **	5.12 **	99.80	100.00	92.00	5.82 **	5.86 **	5.78 **	5.32 **	11.64	8.82	6.28 *
GOh_R_n	bts	4.82 **	4.90 **	5.56 **	5.50 **	47.92	55.90	24.48	5.74 **	6.38 *	6.12 *	5.94 **	62.80	60.12	24.22
GOh_W_n	bts	5.18 **	5.26 **	5.94 **	5.52 **	82.62	92.48	68.06	6.28 *	6.24 *	5.18 **	5.62 **	67.34	56.90	21.02
$Whang_n$	bts	5.16 **	7.22	5.46 **	4.68 **	99.96	99.96	88.62	8.20	11.22	4.26 **	4.44 **	50.78	41.10	18.52
$Whang_n$	dbts	5.10 **	5.52 **	5.30 **	5.04 **	99.04	99.72	87.30	6.48 *	7.26	5.36 **	5.08 **	31.74	27.62	13.16
JB_n	asy	11.04 **	34.26	1.00	2.42	100.00	100.00	95.32	59.60	69.06	37.74	42.70	88.94	79.80	68.44
GO_n	asy	8.02	7.66	10.34 **	9.06 **	61.76	87.12	84.70	11.02 **	10.06 **	12.58	10.36 **	4.82	6.60	8.86 **
GOh_n	asy	9.82 **	9.26 **	8.96 **	8.92 **	65.06	89.20	88.08	15.08	13.36	15.28	13.18	11.44 *	14.56	14.90
GOh_R_n	asy	5.54	5.86	6.34	5.82	46.62	55.16	24.46	9.36 **	10.02 **	9.90 **	8.78 *	64.76	66.14	31.82
10% GOh_W_n	asy	7.62	7.80	9.18 **	7.62	76.24	90.72	70.10	10.24 **	10.72 **	6.88	7.06	66.66	62.08	26.52
GOh_n	bts	9.92 **	9.78 **	9.90 **	9.94 **	99.88	100.00	96.02	11.30 *	10.38 **	10.58 **	9.72 **	15.52	13.72	11.74
GOh_R_n	bts	9.64 **	9.56 **	10.14 **	10.22 **	60.42	68.48	37.06	11.00 **	11.38 *	11.34 *	11.14 **	72.74	71.96	35.54
GOh_W_n	bts	10.12 **	10.14 **	11.08 **	10.68 **	88.96	95.94	79.72	11.72 *	11.24 *	9.78 **	10.96 **	77.34	69.52	30.66
$Whang_n$	bts	10.70 **	13.42	10.94 **	9.84 **	100.00	99.98	94.38	15.26	19.10	10.30 **	10.24 **	60.46	54.98	29.68
$Whang_n$	dbts	10.40 **	10.56 **	10.72 **	10.36 **	99.50	99.94	93.30	11.92	12.44	10.64 **	10.18 **	39.82	39.30	21.70

Notes:
Homo denotes homoskedastic errors; $HET1,2,4$ denotes heteroskedastic schemes explained section 4.1.
asy denotes asymptotic critical value; bts denotes single DF wild bootstrap; dbts denotes full double bootstrap.
* denotes that the estimate is consistent with the true significance level being between -1 % and +1 % from its nominal level.
** denotes that the estimate is consistent with the true significance level being between -0.5 % and +0.5 % from its nominal level.

Power Given the good agreements of the rejection frequencies and the nominal size of the bootstrapped tests, we may be able to rank the tests according to the estimated power from the same table. Under homoskedasticity with $HET1$ and $HET3$, the DF wild bootstrapped GOh_n has the highest power. The single and double bootstrapped $Whang_n$ exhibit almost the same power as GOh_n's. Then, GOh_W_n has slightly lower power than the double bootstrapped $Whang_n$, and GOh_R_n has the lowest power.

Under "potential inconsistency", $HET4$, as we anticipated, the DF wild bootstrapped GOh_n has very little power. On the other hand, the single bootstrapped GOh_R_n, GOh_W_n, $Whang_n$, and the double bootstrapped $Whang_n$ have reasonable power. GOh_R_n has the highest power with chi-squared errors and GOh_W_n has the highest power with log-normal errors. The single bootstrapped $Whang_n$ has lower power than the single bootstrapped GOh_R_n and GOh_W_n, and furthermore, the double bootstrapped $Whang_n$ decreases the power substantially.

On balance, when one is sure that there is no "potential inconsistency", the single DF bootstrapped GOh_n may be the best choice among the tests considered here. However, given a possible existence of "potential inconsistency", the double bootstrapped $Whang_n$ may be the best choice among the tests. Having seen the relatively lower power of the double bootstrapped $Whang_n$ under "potential inconsistency", an employment of it together with the single bootstrapped GOh_R_n, GOh_W_n might be practical.

5 Conclusions

In this paper, new asymptotically valid heteroskedasticity and nonnormality robust tests for skewness has been derived based on standard first order asymptotic theory and the use of the wild bootstrap. The finite sample performances of these skewness tests in the linear regression models have been examined. The evidence shows that the existence of the unknown heteroskedasticity affects the performance of the skewness tests, JB_n test and GO_n test as expected. For the heteroskedasticity robust version of GO_n test, GOh_n, the evidence shows that using the first order asymptotic theory to obtain the critical value does not give a good control over the finite sample significance levels.

The ability of wild bootstrap (Davidson and Flachaire 2000) and the fast double bootstrap (Davidson and MacKinnon 2000) schemes to produce more reliable procedures has been examined. The evidence shows that with a moderately large sample size, their wild bootstrap enjoys a remarkable success in controlling the size, whilst maintaining high power. Indeed, under the null of symmetric errors, the use of the DF wild bootstrap is fully justified and also provides, in theory, an asymptotic refinement over asymptotic critical values. This is because the heteroskedasticity robust tests for skewness and the DF wild bootstrap DGP are asymptotically independent. There is not a strong evidence that the fast double bootstrap yields the better performance over single DF wild bootstrap.

Also, the "potential inconsistency", under which such tests may fail to detect the alternatives, is pointed out. The modified parametric tests and Whang's (2000) non-parametric test are proposed to avoid the "potential inconsistency" of the test. As the Whang test is non-asmptotically pivotal, the double bootstrapped Whang test is proposed to gain further refinements. The evidence shows that the double bootstrapped Whang test works remarkably well, maintaining high power across designs. Interestingly, though, the power of Whang test is reduced when used in conjunction with the double bootstrap under the heteroskedasticity scheme $HET4$, which renders the parametric tests inconsistent. However, when there is a strong evidence of the existence of "potential inconsistency", the DF wild bootstrapped modified parametric tests are recommended. Note that the modified parametric tests do not behave well with the asymptotic critical values, but do much better with the bootstrap.

Finally it is shown that the proposed skewness test and the modified tests are asymptotically sensitive to the omitted variables. Therefore, when one uses the proposed skewness test to check the efficacy of Davidson and Flachaire's wild bootstrap procedure for omitted variable tests, he should use the residuals from the estimated model with a full set of regressors.

Topics for further research includes; 1) further investigation of Whang test in the context of condition moment test under heteroskedasticity; 2) comparison of the wild bootstrap and semiparametric test such as Robinson (1987).

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