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Evolutionary Finance: Models with Long-Lived Assets

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Abstract. Evolutionary Finance explores the "survival and extinction" questions of investment strategies (portfolio rules) in the market selection process. It models the stochastic dynamics of financial markets based on behavioral and evolutionary principles, where asset prices are determined endogenously by short-run equilibrium between supply and demand, arising from the interaction of competing portfolio rules. This paper presents a survey of developments in Evolutionary Finance with a focus on long-lived, dividend-paying risky securities, where the budget of each investor comes from asset dividends and capital gains. We review several key models in this area addressing the following problems in order: 1) the most general results under the most general assumptions; 2) global evolutionary stability under restrictive assumptions; 3) viewing the model from a different, game-theoretic, perspective and examining almost sure Nash equilibrium strategies under restrictive assumptions. A central goal of the study is to identify an investment strategy that allows an investor to survive in the market selection process, i.e., to keep with probability one, a strictly positive, bounded away from zero share of market wealth over an infinite time horizon, irrespective of the strategies used by other investors. The main results are under general assumptions, such a survival strategy — an analogue of the famous Kelly rule of "betting one's beliefs" — exists and is asymptotically unique (within a specific class of strategies called basic). Moreover, under the required stronger assumptions, the Kelly rule is globally evolutionarily stable and is the unique investment strategy that forms a symmetric Nash equilibrium almost surely.

Key words: Evolutionary finance, Survival portfolio rules, Long-lived assets, Stochastic dynamic games, Evolutionary game theory.

JEL Classification: C73, D53, G11, C73, D58.

I am grateful to Igor Evstigneev, who introduced me into the fascinating area of Evolutionary Finance and suggested the present work. This paper complements my earlier work (see Chen [29]), which surveys models with short-lived assets, by focusing on the corresponding framework with long-lived, dividend-paying assets.

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1 Introduction

Conventional models of equilibrium and dynamics of asset markets are based on the classical Walrasian general equilibrium theory (see Magill and Quinzii [75]), which relies upon the hypothesis of full rationality of market players, who are assumed to maximize their utilities or preferences subject to budget constraints (i.e., well-defined and precisely stated constrained optimization problems). In contrast, the present work in the field of Evolutionary Finance (EF)¹ relaxes these assumptions and develops an alternative equilibrium concept — *behavioral equilibrium* — admitting that traders/investors may be boundedly rational and have a whole variety of patterns of behavior determined by their individual psychology, not necessarily describable in terms of utility maximization. Strategies may involve, for instance, mimicking, satisficing, rules of thumb (based on experience), etc.; and, might be *interactive*: depending on the behaviour of the others, and *relative*: taking into account the comparative performance of the others. This approach overcomes several drawbacks of traditional theory, particularly by eliminating the need for the “perfect foresight” assumption to establish equilibrium and the reliance on knowledge of unobservable individual utilities and beliefs. In this sense, it opens new possibilities for the modern financial markets modeling, especially on the global level, where the main objectives might be of an evolutionary nature: *domination* in a market segment, fastest capital *growth*, or simply *survival* (especially in crisis environments), etc.

EF models, through a game-theoretic lens, combine elements of the theory of stochastic dynamic games and evolutionary game theory. The former offers the general notion of a strategy and the latter suggests the game solution concept: *survival strategies*. The process of market dynamics is described as a sequence of consecutive short-run equilibria: aggregate market demand of each asset is equal to its supply, where equilibrium asset prices in each period are endogenously determined by the interaction of general, adaptive portfolio rules employed by competing investors, depending on the exogenous random factors and observed history of the game. Uncertainty in asset payoffs at each period is modeled via an exogenous discrete-time stochastic process that governs the evolution of the states of the world, which aims to capture various macroeconomic and business cycle variables that may affect investors’ behavior. There are two fundamental types of models in EF: with *short-lived* (one-period) assets and *long-lived dividend-paying* assets, see, e.g., Amir et al. [6] and [4], respectively. For a recent comprehensive survey of developments in models with short-lived assets, see Chen [29]. This paper focuses on models of the latter type, where assets pay dividends that are random and depend on the process of exogenous “states of the world,” and the budget of each investor comes from the dividends paid by the assets and capital gains. Price changes and dividend

¹Surveys describing the state of the art in EF were given in Evstigneev et al. [39, 41], and an elementary textbook treatment of this subject can be found in Evstigneev et al. [40], Chapter 20. For a most recent review of the development of studies related to this area see Holtfort [58]. General perspectives of a synthesis of behavioral and mainstream economics based on the evolutionary approach are discussed in Aumann [12].

payments of securities induce wealth dynamics among investors using different investment strategies (portfolio rules) in the market. These dynamics act as a natural selection force among the portfolio rules: some prove to be successful and “survive,” accumulating a positive fraction of total market wealth in the limit, while the others fail and “become extinct,” with their share of market wealth tending to zero.

The main focus of the study is to identify investment strategies that guarantee “long-run survival” in the market selection process, i.e., to keep with probability one, a strictly positive, bounded away from zero share of market wealth over an infinite time horizon, irrespective of what strategies adopted by others. Typical results demonstrate that such strategies exist – in particular, a survival strategy can be formulated as the counterpart of the Kelly [60]² portfolio rule in the present framework – and that they are asymptotically unique (within a specific class of strategies called *basic*) as well as easily computable. The computations do not require, in contrast with the classical Walrasian paradigm, the knowledge of hidden agents’ characteristics such as individual utilities and beliefs.

EF integrates modeling principles originating from the following disciplines: (i) behavioral economics and finance — Tversky and Kahneman [99], Shleifer [95], Shiller [94], Thaler [98], Bachmann et al. [14]; (ii) evolutionary economics — Alchian [1], Penrose [82], Nelson and Winter [81]; (iii) evolutionary game theory — Maynard Smith and Price [79], Maynard Smith [78], Schaffer [88, 89], Weibull [101], Vega-Redondo [100], Samuelson [85], Hofbauer and Sigmund [57], Kojima [62], Gintis [46], Sandholm [87]; (iv) games of survival — Milnor and Shapley [80], Shubik and Thompson [96]³; (v) evolutionary econophysics — Arthur et al. [9], Blume and Easley [21], Farmer and Lo [44], LeBaron et al. [67]; (vi) capital growth theory — Shannon [93], Kelly [60], Breiman [26], Algoet and Cover [2], Cover [31], Dempster et al. [33], MacLean et al. [73], and others.

Evolutionary ideas have a long history in the social sciences going back to Malthus, who played an inspirational role for Darwin.⁴ A more recent stage of development of these ideas began in the 1950s with the publications of Alchian [1], Penrose [82] and others. A powerful momentum to work in this area was given by the interdisciplinary research conducted in the 1980s and 1990s under the auspices of the Santa Fe Institute in New Mexico, USA, where researchers of different backgrounds — economists, mathematicians, physicists and biologists — combined their efforts to study evolutionary dynamics in biology, economics and finance; see, e.g., Arthur et al. [10], Farmer and Lo [44], LeBaron et al. [67], Blume and Easley [21], Blume and Durlauf [20].⁵

²The state of the art in this area related to the Kelly investment criterion is surveyed by MacLean et al. [73].

³For a comprehensive discussion of game-theoretic aspects of EF in a different but closely related model (i.e., short-lived assets) see Amir et al. [6], Sections 1 and 6.

⁴For a review of this subject see, e.g., Hodgeson [56].

⁵For other fundamental contributions to the evolutionary modeling of financial markets, see, e.g., Anderson et al. [7], Bottazzi et al. [24, 25], Bottazzi and Dindo [22, 23], Brock et al. [27], Coury and Sciubba [30], Farmer [42], Lo [68, 69, 70, 71], Lo et al. [72], Sciubba [91, 92], and Zhang et al. [103].

The focus on the “survival and extinction” of investment strategies in the market selection process connects the research to evolutionary game theory, which was initially designed for modeling biological systems and later found fruitful applications in economics. The concept of a survival portfolio rule, stable within the market selection process, is akin to the concepts of *evolutionary stable strategies* (ESS) introduced by Maynard Smith and Price [79], and Schaffer [88, 89]. However, there are radical differences of the market selection mechanism in EF from the typical schemes in evolutionary game theory, where species or agents undergo repeated random matchings in large populations, leading to their long-run survival or extinction.

The closest game-theoretic models to the setting of EF are those in capital growth theory considered by Bell and Cover [18, 19], which demonstrate the Kelly portfolio rule as “competitively optimal,” established via an appropriate zero-sum game.⁶ However, there are two crucial differences among them: 1) their models assume exogenous asset prices within a standard framework of capital growth theory, while EF extends that theory to a setting with endogenous price formation mechanism; 2) the fundamental game solution concept adopted in EF is defined in terms of a property holding almost surely, rather than the traditional notion of a Nash equilibrium involving payoff maximization (defined in terms of expectations).

This paper surveys following models with long-lived assets (LLA):

Firstly, we present a model with the most general results under the most general assumptions. The first paper on the LLA model was proposed in Evstigneev et al. [37], and the first substantial results were obtained in Amir et al. [4]. These results are all covered by Evstigneev et al. [35], which will serve as the foundation for the analysis that follows. The main results show that under very general assumptions, a survival strategy exists, is asymptotically unique (within a specific class of strategies called basic), and is easily computed by solving a stochastic equation.

Then, we examine the globally evolutionarily stable strategies under much stronger assumptions, including: independent and identically distributed (i.i.d.) states of the world, a restriction to the simple (fixed-mix) and completely-mixed portfolio rules, the same investment rate for all the assets at all dates, and the supply of each asset is constant over time, etc. In this setting, it is shown that there is a unique portfolio rule with this property (i.e., the Kelly rule) such that those and only those investors who belong to the group of investors adopting this strategy survive, whereas all the others employing distinct rules from the Kelly rule become extinct, regardless of the initial state of the market.

Further, we view the model from a different, game-theoretic, perspective and treat the decision-theoretic framework as a game in which the payoffs of

⁶For related research in capital growth theory, see Kelly [60], Latané [65], Breiman [26], Algoet and Cover [2], Hakanasson and Ziemba [50], MacLean et al. [73], Kuhn and Luenberger [63], Ziemba and Vickson [104], MacLean and Ziemba [74], etc. For the history of these ideas initially expressed by Claude Shannon in his lectures on investment problems [93], and for relevant discussions, see Cover [31]. For textbook treatments towards capital growth theory, see Cover and Thomas [32], Chapter 16; Evstigneev et al. [40], Chapter 17.

the players (investors) are defined in terms of the growth rates of their relative wealth. The analysis is conducted under restrictive assumptions, e.g., i.i.d. states of the world, simple (fixed-mix) portfolio rules, and the same investment rate for all the assets. The results show that in the game under consideration the Kelly rule forms a unique symmetric Nash equilibrium almost surely.

The paper is organized as follows. Section 1 delivers an introduction for this research direction — EF with long-lived assets. Section 2 presents the model under general assumptions. Section 3 studies the main results of this model. Section 4 analyzes the property of global evolutionary stability under more restrictive conditions. Section 5 treats the model from a game-theoretic perspective (Nash equilibrium in a certain dynamic game). Section 6 discusses the modeling approach, characteristic features and applications of EF. And Section 7 concludes.

2 The Model under General Assumptions

In Sections 2 and 3, we review the study by Evstigneev et al. [35], an EF model for long-lived assets with the most general results under the most general assumptions.

Asset Market Settings. We consider a market where $K \geq 2$ risky *assets* (*securities*) are traded. The market is influenced by random factors modeled in terms of an exogenous stochastic process $s_1, s_2, \dots$, where s_t is a random element of a measurable space S_t (“state of the world” at date t). The market opens at date 0 and the assets are traded at all moments of time $t = 0, 1, 2, \dots$. At each date $t = 1, 2, \dots$, assets $k = 1, \dots, K$ pay dividends $D_{t,k}(s^t) \geq 0$ depending on the history $s^t := (s_1, \dots, s_t)$ of states of the world up to date t . The functions $D_{t,k}(s^t)$ (as well as all other functions of s^t we will consider) are assumed to be measurable with respect to the product σ -algebra in the space $S_1 \times \dots \times S_t$ and satisfy

$$\sum_{k=1}^K D_{t,k}(s^t) > 0 \text{ for all } t \geq 1 \text{ and } s^t. \quad (1)$$

This condition means that at each date in each random situation at least one asset yields a strictly positive dividend. The total volume (the number of units) of asset k available in the market at date $t \geq 1$ is $V_{t,k}(s^t) > 0$, where $V_{t,k}(s^t)$ is a measurable function of s^t . For $t = 0$, the number $V_{t,k} = V_{0,k} > 0$ is constant.

We denote by $p_t \in \mathbb{R}_+^K$ the vector of market prices of the assets. For each $k = 1, \dots, K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, \dots, p_{t,K})$ stands for the price of one unit of asset k at date $t \geq 0$. There are $N \geq 2$ *investors* (*traders*) acting in the market. A *portfolio* of investor i at date $t \geq 0$ is specified by a vector $x_t^i = (x_{t,1}^i, \dots, x_{t,K}^i) \in \mathbb{R}_+^K$, where $x_{t,k}^i$ is the amount (“physical units”) of asset k in the portfolio x_t^i . All the coordinates of the vector x_t^i are non-negative: borrowing and short sales are ruled out. The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the value of the investor i ’s portfolio x_t^i at date t in terms of the prices $p_{t,k}$. The *state of the market* at each date t is characterized

by the set of vectors $(p_t, x_t^1, \dots, x_t^N)$, where p_t is the vector of asset prices and $x_t^1, \dots, x_t^N$ are the traders' portfolios.

At date $t = 0$ the investors have *initial endowments* $w_0^i > 0$ ($i = 1, \dots, N$), which form their budgets at date 0. Investor i 's budget at date $t \geq 1$ is

$$w_t^i(s^t) = \langle D_t(s^t) + p_t(s^t), x_{t-1}^i(s^{t-1}) \rangle,$$

where $D_t(s^t) = (D_{t,1}(s^t), \dots, D_{t,K}(s^t))$. It consists of two components: the dividends $\langle D_t, x_{t-1}^i \rangle$ paid by the portfolio x_{t-1}^i and the market value $\langle p_t, x_{t-1}^i \rangle$ of x_{t-1}^i expressed in terms of the prices $p_t = (p_{t,1}, \dots, p_{t,K})$ at date t .

Investment Strategies. For each $t \geq 0$, every trader $i = 1, 2, \dots, N$ selects a vector of *investment proportions* $\lambda_t^i = (\lambda_{t,1}^i, \dots, \lambda_{t,K}^i)$ according to which i plans to distribute the available budget between assets. Vectors λ_t^i belong to the unit simplex

$$\Delta^K := \{(a_1, \dots, a_K) \geq 0 : a_1 + \dots + a_K = 1\}.$$

In terms of the game we are going to describe, the vectors λ_t^i represent the players' (investors') *actions* or *control variables*. The investment proportions at each date $t \geq 0$ are selected by the N traders simultaneously and independently, so that we deal here with a simultaneous-move N -person dynamic game. For $t \geq 1$, players' actions might depend, generally, on the history $s^t = (s_1, \dots, s_t)$ of the realized states of the world and the *history of the game* $(p^{t-1}, x^{t-1}, \lambda^{t-1})$, where $p^{t-1} = (p_0, \dots, p_{t-1})$ is the sequence of asset price vectors up to time $t-1$, and

$$\begin{aligned} x^{t-1} &:= (x_0, x_1, \dots, x_{t-1}), \quad x_l = (x_l^1, \dots, x_l^N), \\ \lambda^{t-1} &= (\lambda_0, \lambda_1, \dots, \lambda_{t-1}), \quad \lambda_l = (\lambda_l^1, \dots, \lambda_l^N), \end{aligned}$$

are the sets of vectors describing the portfolios and the investment proportions of all the players at all the dates up to $t-1$. The history of the game contains information about the *market history* — the sequence $(p_0, x_0), \dots, (p_{t-1}, x_{t-1})$ of the states of the market — and about the actions λ_l^i of all the players (investors) $i = 1, \dots, N$ at all the dates $l = 0, \dots, t-1$. A vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions with values in Δ^K

$$\Lambda_t^i(s^t, p^{t-1}, x^{t-1}, \lambda^{t-1}), \quad t = 1, 2, \dots$$

form an *investment (trading) strategy* Λ^i of trader i , specifying a *portfolio rule* according to which trader i selects investment proportions at each date $t \geq 0$. This is a general game-theoretic definition of a strategy, assuming full information about the history of the game, including the players' previous actions, and the knowledge of all the past and present states of the world.

Among general portfolio rules, we will distinguish those for which Λ_t^i depends only on s^t , rather than on the whole market history $(p^{t-1}, x^{t-1}, \lambda^{t-1})$. We will call such portfolio rules *basic*. They play an important role in the present work: the survival strategy we are going to construct will belong to this class. The essence of the main result (Theorem 2) lies in the fact that it indicates a

relatively simple basic strategy, requiring a very limited volume of information and guaranteeing survival in competition with *any* other strategies which might use all theoretically possible information.

Transaction Costs. For each asset $k = 1, \dots, K$, a sequence of functions $\alpha_{0,k}, \alpha_{1,k}(s^1), \alpha_{2,k}(s^2), \dots$ is given to characterize transaction costs for buying asset k in the market under consideration. It is assumed that $0 < \alpha_{t,k} \leq 1$. If an investor i allocates wealth $w_{t,k}^i$ to asset k at time t , then the value of the k th position of the i 's portfolio will be $p_{t,k}x_{t,k}^i = \alpha_{t,k}w_{t,k}^i$. The amount $(1 - \alpha_{t,k})w_{t,k}^i$ will cover transaction costs.

Consecutive Short-run Equilibria. Suppose that at date 0 each investor i has selected some investment proportions $\lambda_0^i = (\lambda_{0,1}^i, \dots, \lambda_{0,K}^i) \in \Delta^K$. Then the amount allocated to asset k by trader i is $\lambda_{0,k}^i w_0^i$, where $w_0^i > 0$ is the i 's initial endowment, so that the value of the holding of asset k in the i 's portfolio is $\alpha_{0,k} \lambda_{0,k}^i w_0^i$. Thus the value of the total holding of asset k in all the investors' portfolios amounts to $\alpha_{0,k} \sum_{i=1}^N \lambda_{0,k}^i w_0^i$. It is assumed that the market is always in equilibrium (asset supply is equal to asset demand), which makes it possible to determine the equilibrium price $p_{0,k}$ of each asset k from the equations

$$p_{0,k} V_{0,k} = \alpha_{0,k} \sum_{i=1}^N \lambda_{0,k}^i w_0^i, \quad k = 1, \dots, K. \quad (2)$$

On the left-hand side of (2) we have the total value $p_{0,k} V_{0,k}$ of all the assets of type k in the market (recall that the total amount of asset k at date 0 is $V_{0,k}$). The investment proportions $\lambda_0^i = (\lambda_{0,1}^i, \dots, \lambda_{0,K}^i)$ chosen by the traders $i = 1, \dots, N$ at date 0 determine their portfolios $x_0^i = (x_{0,1}^i, \dots, x_{0,K}^i)$ at date 0 by the formula

$$x_{0,k}^i = \frac{\alpha_{0,k} \lambda_{0,k}^i w_0^i}{p_{0,k}}, \quad k = 1, \dots, K, \quad i = 1, \dots, N. \quad (3)$$

Assume now that all the investors have chosen their investment proportion vectors $\lambda_t^i = (\lambda_{t,1}^i, \dots, \lambda_{t,K}^i)$ at date $t \geq 1$. Then the equilibrium of asset supply and demand determines the market clearing prices

$$p_{t,k} V_{t,k} = \alpha_{t,k} \sum_{i=1}^N \lambda_{t,k}^i \langle D_t + p_t, x_{t-1}^i \rangle, \quad k = 1, \dots, K. \quad (4)$$

The investment budgets $\langle D_t + p_t, x_{t-1}^i \rangle$ of the traders $i = 1, \dots, N$ are distributed between assets in the proportions $\lambda_{t,k}^i$, so that the k th position of the trader i 's portfolio $x_t^i = (x_{t,1}^i, \dots, x_{t,K}^i)$ is

$$x_{t,k}^i = \frac{\alpha_{t,k} \lambda_{t,k}^i \langle D_t + p_t, x_{t-1}^i \rangle}{p_{t,k}}, \quad k = 1, \dots, K, \quad i = 1, \dots, N. \quad (5)$$

Note that the price vector p_t is determined implicitly as the solution to the system of equations (4).

Growth of Asset Supply. Define

$$\gamma_{t,k}(s^t) = \frac{V_{t,k}(s^t)}{V_{t-1,k}(s^{t-1})}. \quad (6)$$

The number $\gamma_{t,k}$ characterizes the speed of growth of the total volume $V_{t,k}$ of asset k . It can be shown that a non-negative vector $p_t(s^t)$ satisfying equations (4) exists and is unique (for any s^t and any feasible x_{t-1}^i and λ_t^i) as long as the following condition holds

$$\alpha_{t,k}(s^t) < \gamma_{t,k}(s^t) \text{ for all } t \geq 1 \text{ and all } s^t. \quad (7)$$

This condition is implied by the basic assumptions under which the results of this model are obtained. Note that if there are no transaction costs, i.e., $\alpha_{t,k} = 1$, then (7) means that the total volumes of all the assets grow in time at a strictly positive rate. In another extreme case, when $\gamma_{t,k} = 1$, i.e., $V_{t,k}$ is constant in t , condition (7) requires that $\alpha_{t,k} < 1$, i.e., the transaction cost rate is non-zero. This property — termed in Mathematical Finance "efficient market friction" (see, e.g., Kabanov and Safarian [59], p. 117) — plays an important role in various models with transaction costs, excluding phenomena like the Saint Petersburg paradox. In our context it is indispensable since in those cases when this assumption does not hold, a short-run equilibrium might fail to exist.

Asset Market Dynamics. Given a strategy profile $(\Lambda^1, \dots, \Lambda^N)$ of investors and their initial endowments $w_0^1, \dots, w_0^N$, we can generate a path of the market game by setting

$$\lambda_0^i = \Lambda_0^i, \quad i = 1, \dots, N, \quad (8)$$

$$\lambda_t^i = \Lambda_t^i(s^t, p^{t-1}, x^{t-1}, \lambda^{t-1}), \quad t = 1, 2, \dots, \quad i = 1, \dots, N, \quad (9)$$

and by defining p_t and x_t^i recursively according equations (2)–(5). The random dynamical system described defines step by step the vectors of investment proportions $\lambda_t^i(s^t)$, the equilibrium prices $p_t(s^t)$ and the investors' portfolios $x_t^i(s^t)$ as measurable vector functions of s^t for each moment of time $t \geq 0$. Thus we obtain a random path of the game

$$(p_t(s^t); x_t^1(s^t), \dots, x_t^N(s^t); \lambda_t^1(s^t), \dots, \lambda_t^N(s^t)), \quad t \geq 0, \quad (10)$$

as a vector stochastic process in $\mathbb{R}_+^K \times \mathbb{R}_+^{KN} \times \mathbb{R}_+^{KN}$.

Admissible Strategy Profile. The above description of asset market dynamics requires clarification. Equations (3) and (5) make sense only if $p_{t,k} > 0$, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called *admissible*. In what follows, we will deal only with such strategy profiles. The hypothesis of admissibility guarantees that the

⁷For the proof of this result see Evstigneev et al. [35], Section 5, Proposition 1.

random dynamical system under consideration is well-defined. Under this hypothesis, we obtain by induction that on the equilibrium path all the portfolios $x_t^i = (x_{t,1}^i, \dots, x_{t,K}^i)$ are non-zero and the wealth

$$w_t^i := \langle D_t + p_t, x_{t-1}^i \rangle \quad (11)$$

of each investor is strictly positive. Further, by summing up equations (5) over $i = 1, \dots, N$, we find that

$$\sum_{i=1}^N x_{t,k}^i = \frac{\alpha_{t,k} \sum_{i=1}^N \lambda_{t,k}^i \langle D_t + p_t, x_{t-1}^i \rangle}{p_{t,k}} = \frac{p_{t,k} V_{t,k}}{p_{t,k}} = V_{t,k} \quad (12)$$

(the market clears) for every asset k and each date $t \geq 1$. The analogous relations for $t = 0$ can be obtained by summing up equations (3). Thus for every equilibrium state of the market $(p_t, x_t^1, \dots, x_t^N)$, we have $p_t > 0$, $x_t^i \neq 0$ and (12).

There is a simple sufficient condition for a strategy profile to be admissible. This condition will hold for all the strategy profiles we shall deal with in this model, and in this sense it does not restrict generality. Suppose that some trader, say trader 1, uses a portfolio rule that always prescribes to invest into all the assets in strictly positive proportions $\lambda_{t,k}^1$. Then a strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (2) that $p_{0,k} \geq \alpha_{0,k} V_{0,k}^{-1} \lambda_{0,k}^1 w_0^1 > 0$ and from (3) that $x_0^1 = (x_{0,1}^1, \dots, x_{0,K}^1) > 0$ (coordinatewise). Assuming that $x_{t-1}^1 > 0$ and arguing by induction, we obtain

$$\langle D_t + p_t, x_{t-1}^1 \rangle \geq \langle D_t, x_{t-1}^1 \rangle > 0$$

in view of (1), which in turn yields $p_t > 0$ and $x_t^1 > 0$ by virtue of (4) and (5), as long as $\lambda_{t,k}^1 > 0$.

3 The Main Results on Survival Strategies

The main results of Evstigneev et al. [35], show that under very general assumptions, a survival strategy exists, is asymptotically unique (within a specific class of basic strategies) and easily computable. Now this strategy is defined as a solution to a certain stochastic equation, in contrast with the previous, more specialized, models (e.g., Evstigneev et al. [37], Amir et al. [4], etc.) where it could be represented in an explicit form as the sum of a convergent series. For the proof of the existence and uniqueness of this solution, new mathematical tools related to the ergodic theory of random dynamical systems: non-stationary stochastic Perron-Frobenius theorems⁸ were developed.

These results are collected as follows:

Let $(\Lambda^1, \dots, \Lambda^N)$ be an admissible strategy profile of the investors. Consider the path (10) of the random dynamical system generated by this strategy profile

⁸For stationary versions of these results see, e.g., Babaei et al. [13].

and the given initial endowments w_0^i . We are primarily interested in the long-run behavior of the *market shares* $r_t^i := w_t^i/W_t$ of the traders, where w_t^i is the investor i 's wealth at date $t \geq 0$ and $W_t := \sum_{i=1}^N w_t^i$ is the *total market wealth*. We shall say that a portfolio rule Λ , or an investor i using it, *survives* with probability one if

$$\inf_{t \geq 0} r_t^i > 0 \text{ (a.s.)}.$$

This means that for almost all realizations of the process of states of the world $s_1, s_2, \dots$, the market share of investor i using Λ is bounded away from zero by a strictly positive random variable.

Definition 1 *A portfolio rule Λ is a survival strategy if any investor using it survives with probability one irrespective of what portfolio rules are used by the other investors.*

A strategy Λ^* is constructed, which will be shown to be a survival strategy. Put

$$\rho_{t,k} := \frac{\alpha_{t,k}}{\gamma_{t,k}} = \frac{\alpha_{t,k} V_{t-1,k}}{V_{t,k}}, \quad t \geq 1, \quad k = 1, \dots, K.$$

Define the *relative dividends* of the assets $k = 1, \dots, K$ by

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t) V_{t-1,k}(s^{t-1})}{\sum_{m=1}^K D_{t,m}(s^t) V_{t-1,m}(s^{t-1})}, \quad t \geq 1, \quad k = 1, \dots, K, \quad (13)$$

and put $R_t(s^t) := (R_{t,1}(s^t), \dots, R_{t,K}(s^t))$. The strategy $\Lambda^* = (\lambda_t^*(s^t))_{t \geq 0}$, where $\lambda_t^* = (\lambda_{t,1}^*, \dots, \lambda_{t,K}^*)$, is defined as the basic strategy satisfying the equation

$$E_t[\rho_{t+1,k} \lambda_{t+1,k}^* + (1 - \sum_{m=1}^K \rho_{t+1,m} \lambda_{t+1,m}^*) R_{t+1,k}] = \lambda_{t,k}^* \text{ (a.s.)}, \quad k = 1, \dots, K. \quad (14)$$

Here $E_t(\cdot) = E(\cdot | s^t)$ stands for the conditional expectation given s^t . Conditions are provided below under which the strategy Λ^* exists and is unique up to stochastic equivalence, i.e., if $\Lambda = (\lambda_t(s^t))_{t \geq 0}$ is another solution to (14), then $\lambda_t^* = \lambda_t$ (a.s.) for all t .

We will assume that the following conditions hold:

(A.1) There exist constants $v > 0$ and $l \geq 0$ such that for each t and k , we have

$$\max_{1 \leq m \leq l} R_{t+m,k} \geq v. \quad (15)$$

(A.2) There exist strictly positive constants κ and α such that for all k, t :

$$\alpha \leq \rho_{t,k} \leq 1 - \kappa. \quad (16)$$

Theorem 1 *Under assumptions (A.1) and (A.2), a solution $(\lambda_t^*)_{t \geq 0}$ to equation (14) exists and is unique up to stochastic equivalence. There exists a constant $\delta > 0$ such that $\lambda_{t,k}^* \geq \delta$.*

Let us discuss the meaning of equation (14). Suppose for the moment that the growth rates of all the assets are the same, so that

$$\rho_{t,1} = \rho_{t,2} = \dots = \rho_{t,K} = \rho_t. \quad (17)$$

In this case, equation (14) takes on the following form

$$E_t[\rho_{t+1}\lambda_{t+1,k}^* + (1 - \rho_{t+1})R_{t+1,k}] = \lambda_{t,k}^* \text{ (a.s.)}, \quad (18)$$

and it admits an explicit solution. The k th coordinate $\lambda_{t,k}^*$ of the vector λ_t^* can be represented as the conditional expectation of the sum of the series

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}, \quad (19)$$

where

$$\rho_t^l := \begin{cases} 1 - \rho_{t+1}, & \text{if } l = 1, \\ \rho_{t+1}\rho_{t+2}\dots\rho_{t+l-1}(1 - \rho_{t+l}), & \text{if } l > 1. \end{cases} \quad (20)$$

Note that in view of (16), the series of random variables

$$\sum_{l=1}^{\infty} \rho_t^l = (1 - \rho_{t+1}) + \rho_{t+1}(1 - \rho_{t+2}) + \rho_{t+1}\rho_{t+2}(1 - \rho_{t+3}) + \dots$$

converges uniformly, and its sum is equal to one. Therefore the series of random vectors $\sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}$ in (19) converges uniformly to a random vector belonging to the unit simplex Δ^K , so that the right-hand side of (19) is well-defined.⁹

Assume that $\rho_t = \rho$ is constant. Then formula (19) can be written as

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} [(1 - \rho)\rho^{l-1} R_{t+l,k}]. \quad (21)$$

Further, if the random elements s_t are independent and identically distributed (i.i.d.) and the relative dividends $R_{t,k}(s^t) = R_k(s_t)$ depend only on the current state s_t and do not explicitly depend on t , then $E_t R_k(s_{t+l}) = E R_k(s_t)$ ($l \geq 1$), and so

$$\lambda_{t,k}^* = E R_k(s_t), \quad (22)$$

which means that the strategy Λ^* is formed by the sequence of vectors $(E R_1(s_t), \dots, E R_K(s_t))$ (constant and independent of t and s^t). Note that in this special case, the formula (22) for Λ^* does not involve the factor ρ .

Formulas (19), (21) and (22) reflect two general principles in Financial Economics:

1) The strategy Λ^* prescribes the allocation of wealth among assets in the proportions of their fundamental values — the expectations of the future relative (discounted, weighted) dividends.

⁹The derivation of equation (19) is given in Evstigneev et al. [35], Section 5, Proposition 5.

2) The portfolio rule Λ^* defined in terms of the relative dividends provides an investment recommendation in line with the CAPM principles, emphasizing the role of the market portfolio (see, e.g., Evstigneev et al. [40], Chapter 7).

In this connection it should be emphasized that instead of the traditional weighing assets according to their prices, the weights in the definition of Λ^* are based on fundamentals, so that Λ^* is an example of *fundamental indexing* (Arnott et al. [8]).

As we have already noted, EF can be viewed as an extension of the classical capital growth theory (Kelly [60], Breiman [26], Algoet and Cover [2], and others) to the case of endogenous asset prices and returns. In the classical setting, a central role is played by the famous Kelly portfolio rule [60] guaranteeing the fastest asymptotic growth rate of wealth in the long run. The Kelly rule is obtained by the maximization of the expected logarithm of the portfolio return. It will be shown in Section 6, Proposition 4 that in the present model survival is equivalent to the fastest relative growth of wealth in the long run. Therefore Λ^* may be viewed as a counterpart of the Kelly portfolio rule in the present model. However, in the game-theoretic model at hand, where the performance of a strategy depends not only on the strategy itself but on the whole strategy profile, Λ^* cannot be obtained as a solution to a single-agent optimization problem with a logarithmic or any other objective functional.

It should be noted that in the case of different $\rho_{t,k}$, when condition (17) does not hold, we cannot provide an explicit formula, like (19), for the strategy Λ^* . However, we can suggest an algorithm for computing Λ^* converging at an exponential rate. This algorithm is actually contained in the proof of the existence and uniqueness of a solution to equation (14), see Evstigneev et al. [35], Appendix B, formulas (B.9) and (B.10).

The main results of the study are formulated in Theorems 2 and 3.

Theorem 2 *The portfolio rule Λ^* is a survival strategy.*

As we have already noted, the portfolio rule Λ^* belongs to the class of basic portfolio rules: the investment proportions $\lambda_t^*(s^t)$ depend only on the history s^t of the process of states of the world and do not depend on the market history.

Note that the class of basic strategies is *sufficient* in the following sense. Any sequence of vectors $r_t = (r_t^1, \dots, r_t^N)$ ($r_t = r_t(s^t)$) of market shares generated by some strategy profile $(\Lambda^1, \dots, \Lambda^N)$ can be generated by a strategy profile $(\lambda_t^1(s^t), \dots, \lambda_t^N(s^t))$ consisting of basic portfolio rules. The corresponding vector functions $\lambda_t^i(s^t)$ can be defined recursively by (8) and (9), using (2)–(5). Thus it is sufficient to prove Theorem 2 only for basic portfolio rules; this will imply that the portfolio rule (14) survives in competition with any, not necessarily basic, strategies.

The following result shows that the survival portfolio rule Λ^* is unique in the class of all basic strategies.

Theorem 3 *If there exists another basic survival strategy $\Lambda = (\lambda_t)$, then:*

$$\sum_{t=0}^{\infty} \|\lambda_t^* - \lambda_t\|^2 < \infty \text{ (a.s.)}.$$

It is not known whether this result remains valid for the class of general, not necessarily basic, strategies. This question remains open; it indicates an interesting direction for further research. Some examples pertaining to a different, but closely related, model (i.e., short-lived assets) might suggest a conjecture that the answer to this question is negative (see Amir et al. [6], Section 5).

The proofs of Theorems 1–3 are lengthy, consisting in several steps. We outline these steps in this survey, for the details of the arguments see Evstigneev et al. [35], pp. 126–134.

The plan of the proof of Theorem 1 is as follows.

The purpose of the proof is to show the results formulated in Step 2 below, which implies the existence and uniqueness of the Λ^* strategy. We will deduce the results in Step 2 from Step 1, which represents a non-stationary version of the stochastic Perron-Frobenius theorem. In turn, the results in Step 1 will be obtained as a consequence of a chain of auxiliary results formulated in Lemmas 1–2 and Propositions 1–3 below.¹⁰

Denote by $\mathcal{M}^n$ ($n > 1$) the set of $n \times n$ matrices $B \geq 0$ such that $Bx \neq 0$ for all $x \in Q := \{x : 0 \neq x \geq 0\}$. For $x = (x^1, \dots, x^n) \in R^n$, define $|x| = |x^1| + \dots + |x^n|$, $x^0 = x/|x|$, and, for $B \in \mathcal{M}^n$, put

$$\kappa(B) = \max_{x, y \in Q} |(Bx)^0 - (By)^0|.$$

Let $\phi(B)$ denote the ratio of the smallest and the greatest elements of matrix B .

Lemma 1 *Let $B_1, B_2, \dots, B_k \in \mathcal{M}^n$. If $B_i > 0$ and $n > 1$, then*

$$\kappa(B_k \dots B_1) \leq \rho_1^{-1} \delta_1 \dots \delta_{k-1},$$

where

$$\rho_i = n^{-2} \phi(B_i) \phi(B_{i+1}), \quad \delta_i = (1 - 2\rho_i).$$

Put $\Delta = \{x = (x_1, \dots, x_n) : x_j \geq 0, \sum x_j = 1\}$. Let $\mathcal{D}^n$ denote the set of matrices B in $\mathcal{M}^n$ representing linear transformations of R^n that map Δ into itself. For $\delta > 0$ we will denote by $\mathcal{D}_\delta^n$ the set of matrices $B \in \mathcal{D}^n$ whose elements are not less than δ .

Lemma 2 *Let $B_1, B_2, \dots, B_k \in \mathcal{D}_\delta^n$. Then*

$$\kappa(B_k \dots B_1) < M \rho^{k-1},$$

where $M = n^2 \delta^{-2}$ and $\rho = 1 - n^{-2} \delta^2$.

Let $B_1, B_2, \dots$ be a sequence of matrices in $\mathcal{D}^n$.

¹⁰For the proofs of Lemmas 1–2 and Propositions 1–3, see Evstigneev et al. [35], Appendix B.

Proposition 1 *There exists a sequence $(y_t^*)_{t \geq 0}$ such that $y_t^* \in \Delta$ and*

$$y_t^* = B_{t+1}y_{t+1}^*, \quad t \geq 0. \quad (23)$$

For each $t \geq 1$ and $j \geq 0$ denote $B_t^{t+j} = B_t \dots B_{t+j}$. For any $y = (y_t) \in \Delta^\infty$ denote by $\mathfrak{B}_t^m(y)$ the t th term of the sequence $\mathfrak{B}^m(y) \in \Delta^\infty$, where $\mathfrak{B}^m(y)$ is the m th iterate of the mapping $\mathfrak{B}$. Clearly, if we put $y_t^m = \mathfrak{B}_t^m(y)$ ($t \geq 0$), then

$$y_t^1 = B_{t+1}y_{t+1} = B_{t+1}^{t+1}y_{t+1}, \quad y_t^2 = B_{t+1}y_{t+1}^1 = B_{t+1}B_{t+2}y_{t+2} = B_{t+1}^{t+2}y_{t+2}, \dots,$$

$$y_t^m = B_{t+1}B_{t+2} \dots B_{t+m}y_{t+m} = B_{t+1}^{t+m}y_{t+m}, \quad t \geq 0.$$

Proposition 2 *Suppose there exist an integer $l \geq 0$ and a real number $\delta > 0$ such that for any $t \geq 1$ the matrix B_t^{t+l} belongs to $\mathcal{D}_\delta^n$. Then the solution $y^* = (y_t^*)_{t \geq 0}$ to equation (23) is unique, and for every $t \geq 0$, the sequence $y_t^m = \mathfrak{B}_t^m(y)$ converges to y_t^* uniformly in $y \in \Delta^\infty$.*

Suppose that the matrices $B_t = B_t(\omega) \in \mathcal{D}^n$ are random, i.e., $B_t(\omega)$ for each $t = 1, 2, \dots$ is a measurable matrix function on the probability space $(\Omega, \mathcal{F}, P)$. Assume the following condition holds:

(B) For some $l \geq 0$ and $\delta > 0$, the matrix $B_t^{t+l}(\omega)$ belongs to $\mathcal{D}_\delta^n$ almost surely for all $t \geq 1$.

Proposition 3 *Under assumption (B), there exists a sequence $(y_t^*)_{t \geq 0}$ of measurable vector functions $y_t^*(\omega)$ with values in Δ such that*

$$B_{t+1}y_{t+1}^* = y_t^*, \quad t \geq 0 \quad (\text{a.s.}). \quad (24)$$

The solution $(y_t^)_{t \geq 0}$ to equation (24) is unique, and we have $y_t^*(\omega) \geq \delta e$ (a.s.). There exists a set $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ such that for every $t \geq 0$ and $\omega \in \Omega_1$ the sequence $y_t^m(\omega) = \mathfrak{B}_t^m(y)(\omega)$ converges to $y_t^*(\omega)$ uniformly in $y \in \Delta^\infty$.*

Proof of Theorem 1 (Outline). 1st step. Let $A_1(\omega), A_2(\omega), \dots$ be a sequence of random matrices. Consider the following condition:

(A) For each $t \geq 1$, the matrix $A_t(\omega)$ depends $\mathcal{F}_t$ -measurably on ω , and there exist $l \geq 0$ and $\delta > 0$, such that the matrix $A_t^{t+l}(\omega) := A_t(\omega) \dots A_{t+l}(\omega)$ belongs to $\mathcal{D}_\delta^n$ almost surely for all $t \geq 1$.

By virtue of Lemmas 1–2 and Propositions 1–3, this step aims to establish the following result:¹¹

Under assumption (A), there exists a sequence $(x_t^*(\omega))_{t \geq 0}$ of vector functions with values in Δ such that $x_t^*(\omega)$ is $\mathcal{F}_t$ -measurable and

$$E_t A_{t+1} x_{t+1}^* = x_t^* \quad (\text{a.s.}), \quad t \geq 0. \quad (25)$$

This sequence is unique up to stochastic equivalence, and we have

$$x_t^* \geq \delta e \quad (\text{a.s.}). \quad (26)$$

¹¹For the proof of this result, see Evstigneev et al. [35], Appendix B, Theorem B.1.

2nd step. Let $(\rho_t)_{t \geq 1}$ be a sequence of $\mathcal{F}_t$ -measurable random vectors $\rho_t = (\rho_{t,1}, \dots, \rho_{t,n})$ such that $0 \leq \rho_{t,i} \leq 1$, and $(R_t)_{t \geq 1}$ a sequence of $\mathcal{F}_t$ -measurable random vectors $R_t = (R_{t,1}, \dots, R_{t,n})$ satisfying $R_t \geq 0$, $\sum_{i=1}^n R_{t,i} = 1$. Recall that Λ^* was defined as the solution to equation (14). To prove that this solution exists and is unique let us define for each $t \geq 0$ the linear operator A_{t+1} :

$$\begin{aligned} (A_{t+1}x)_i &= \rho_{t+1,i}x_i + \left(\sum_{m=1}^n x_m - \sum_{m=1}^n \rho_{t+1,m}x_m \right) R_{t+1,i} \\ &= \rho_{t+1,i}x_i + \sum_{m=1}^n (1 - \rho_{t+1,m})x_m R_{t+1,i}. \end{aligned}$$

This operator transforms Δ into itself, and for $x \in \Delta$ we have

$$(A_{t+1}x)_i = \rho_{t+1,i}x_{t+1,i} + \left(1 - \sum_{m=1}^n \rho_{t+1,m}x_{t+1,m} \right) R_{t+1,i}.$$

Consequently, equation (14) can be written in the form (25) (with obvious changes in notation). Let us introduce the following condition:

($\mathcal{R}$) There exist constants $\gamma > 0$ and $l \geq 0$ such that for each t and i , we have

$$\max_{1 \leq m \leq l} R_{t+m,i} \geq \gamma. \quad (27)$$

Based on Step 1, the objective of this step is to conclude that:¹²

Suppose that condition ($\mathcal{R}$) holds and there exists a constant $\theta > 0$ such that $\min\{\rho_{t,i}, 1 - \rho_{t+1,i}\} \geq \theta$ for all t and i . Then a solution $(x_t^*)_{t \geq 0}$ to equation (25) exists, is unique up to stochastic equivalence, and satisfies (26) for some $\delta > 0$. ■

The plan of the proofs of Theorems 2–3 is as follows.

1st step. The system of equations governing the wealth dynamics of investors given their admissible strategy profile $(\Lambda^1, \dots, \Lambda^N)$ can be expressed in the following form:

$$w_{t+1}^i = \sum_{k=1}^K (\rho_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}, \quad i = 1, \dots, N, \quad t \geq 0. \quad (28)$$

2nd step. It is sufficient to consider, without loss of generality, the case of two investors. This reduces the dimension of the original random dynamical system from a general N to $N = 2$. Consider the model with two traders ($N = 2$) using

¹²For the proof of this result, see Evstigneev et al. [35], Appendix B, Theorem B.2.

¹³For the proof of this result, see Evstigneev et al. [35], Section 5, Proposition 2.

strategies $\Lambda^i = (\lambda_{t,k}^i(s^t))$, $i = 1, 2$, and denote by x_t the ratio of their market shares:

$$x_t = \frac{r_t^1}{r_t^2} = \frac{w_t^1}{w_t^2}.$$

Further, let us define for $i = 1, 2$,

$$U_{t+1}^i := 1 - \sum_{k=1}^K \rho_{t+1,k} \lambda_{t+1,k}^i = \sum_{k=1}^K (1 - \rho_{t+1,k}) \lambda_{t+1,k}^i.$$

In the model with two investors $i = 1, 2$ using the strategies Λ and $\tilde{\Lambda}$, respectively, the wealth w_t^1 of the first player coincides with the wealth w_t^1 of the first player in the original model, and the wealth $\tilde{w}_t^2$ of the second "aggregate" investor coincides with the total wealth $w_t^2 + \dots + w_t^N$ of the group of $N - 1$ investors $i = 2, \dots, N$ in the original model.¹⁴

3rd step. The sequence x_t is generated by the following random dynamical system

$$x_{t+1} = x_t \frac{\sum_{k=1}^K [\rho_{t+1,k} \lambda_{t+1,k}^2 + R_{t+1,k} U_{t+1}^2] \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 x_t + \lambda_{t,k}^2}}{\sum_{k=1}^K [\rho_{t+1,k} \lambda_{t+1,k}^1 + R_{t+1,k} U_{t+1}^1] \frac{\lambda_{t,k}^2}{\lambda_{t,k}^1 x_t + \lambda_{t,k}^2}} \quad (t = 0, 1, \dots). \quad (29)$$

It describes a one-dimensional system which governs the evolution of the ratio $x_t = r_t^1/r_t^2$ of the market shares of the two investors, and thus reduces the dimension of the problem to 1.¹⁵

4th step. Our goal is to show that the random sequence $(x_t)_{t \geq 0}$ defined recursively by (29) is bounded away from zero almost surely. To this end it turns out to be convenient to take a "step back" and to increase the dimension to K (the number of assets). Assuming that the first trader uses the investment proportions $\lambda_{t,k}^1 = \lambda_{t,k}^*(s^t)$ prescribed by the portfolio rule Λ^* and the second trader employs investment proportions $\lambda_{t,k}^2 = \lambda_{t,k}(s^t)$ specified by some other portfolio rule Λ , we introduce the following change of variables

$$y_t^k = \lambda_{t,k}/x_t, \quad k = 1, \dots, K,$$

and define $y_t := (y_t^1, \dots, y_t^K)$. We examine the dynamics of the random vectors $y_t = y_t(s^t)$ implied by the system (29). The norm $|y_t| := \sum_k |y_t^k|$ of the vector $y_t \geq 0$ is equal to $\sum_k (\lambda_{t,k}/x_t) = 1/x_t$, and what we need is to show that $1/|y_t|$ is bounded away from zero (a.s.). To prove this, a stochastic Lyapunov function — a function of y_t which forms a non-negative supermartingale (ζ_t) along a path (y_t) of the system at hand (see Lemma 5 below) — is constructed. By using the supermartingale convergence theorem, one can prove that the stochastic process ζ_t converges (a.s.), which implies that it is bounded (a.s.).

¹⁴For the proof of this result, see Evstigneev et al. [35], Section 5, Proposition 4.

¹⁵For the proof of this result, see Evstigneev et al. [35], Section 5, Proposition 3.

5th step. We complete the proof of Theorem 2 by showing that the boundedness of ζ_t implies that $x_t = 1/|y_t|$ is bounded away from zero. By using the above techniques, together with some additional considerations, we complete this section with an outline of the proof of Theorem 3.

The proofs of Theorems 2–3 are based on Lemmas 3–6 below.¹⁶

Let us define the non-negative random variables

$$Y_t := \ln(1 + |y_t|) = -\ln r_t^1,$$

$$Z_{t,k} := \ln\left(1 + \frac{y_t^k}{\lambda_{t,k}^*}\right) = \ln\left(1 + \frac{r_t^2 \lambda_{t,k}}{r_t^1 \lambda_{t,k}^*}\right),$$

$$\gamma_{k,m}^{t+1} = \frac{1 + y_{t+1}^m / \lambda_{t+1,m}^*}{1 + y_t^k / \lambda_{t,k}^*},$$

and a function

$$f(x) = \frac{(x-1)\ln x}{x+2}.$$

Lemma 3 *The function $f(x)$ is non-negative, has a unique root $x = 1$ and satisfies*

$$x - 1 \geq \ln x + f(x), \quad x \in (-\infty, +\infty).$$

Lemma 4 *The following inequality holds:*

$$\begin{aligned} & \sum_{k=1}^K \lambda_{t+1,k}^* Z_{t+1,k} + \sum_{m=1}^K \sum_{k=1}^K R_{t+1,k} (1 - \rho_{t+1,m}) \lambda_{t+1,m}^* f(\gamma_{k,m}^{t+1}) \\ & \leq \sum_{k=1}^K \rho_{t+1,k} \lambda_{t+1,k}^* Z_{t,k} + U_{t+1}^* \sum_{k=1}^K R_{t+1,k} Z_{t,k}. \end{aligned}$$

Put

$$\zeta_t := \sum_{k=1}^K \lambda_{t,k}^* Z_{t,k} + \sum_{m=1}^K \sum_{k=1}^K R_{t,k} (1 - \rho_{t,m}) \lambda_{t,m}^* f(\gamma_{k,m}^t). \quad (30)$$

Lemma 5 *The sequence of random variables ζ_t ($t \geq 1$) is a non-negative supermartingale, and we have*

$$\zeta_t - E_t \zeta_{t+1} \geq \sum_{m=1}^K \sum_{k=1}^K R_{t,k} (1 - \rho_{t,m}) \lambda_{t,m}^* f(\gamma_{k,m}^t) \geq 0 \quad (a.s.). \quad (31)$$

Lemma 6 *Let ζ_t be a supermartingale such that $\inf_t E \zeta_t > -\infty$. Then the series of non-negative random variables $\sum_{t=0}^{\infty} (\zeta_t - E_t \zeta_{t+1})$ converges (a.s.).*

¹⁶For the proofs of Lemmas 3–6, see Evstigneev et al. [35], Appendix A.

Proof of Theorem 2. By Lemma 5, the sequence ζ_t defined in (30) is a non-negative supermartingale. Therefore it converges (a.s.), and hence it is bounded above (a.s.) by some random constant C :

$$\begin{aligned} C \geq \zeta_t &= \sum_{k=1}^K \lambda_{t,k}^* Z_{t,k} + \sum_{m=1}^K \sum_{k=1}^K R_{t,k} (1 - \rho_{t,m}) \lambda_{t,m}^* f(\gamma_{k,m}^t) \\ &\geq \sum_{k=1}^K \lambda_{t,k}^* Z_{t,k} = \sum_{k=1}^K \lambda_{t,k}^* \ln \left(1 + \frac{r_t^2 \lambda_{t,k}}{r_t^1 \lambda_{t,k}^*} \right). \end{aligned}$$

Here, we used the non-negativity of the function f established in Lemma 3 and the non-negativity of the relative dividends $R_{t,k}$, and $\lambda_{t,m}^*$, and assumption (A.2).

Recall that by virtue of Theorem 1, $\lambda_{t,k}^* \geq \delta$ for any t, k . Therefore $C/\delta \geq \ln \left(1 + r_t^2 \lambda_{t,k} / r_t^1 \lambda_{t,k}^* \right)$ for all t, k , and there exists some random variable H such that $H \geq 1 + r_t^2 \lambda_{t,k} / r_t^1 \lambda_{t,k}^*$ for all t, k . Furthermore, there exists some k such that $\lambda_{t,k} \geq 1/K$ (since $\sum_{k=1}^K \lambda_{t,k} = 1$). For this k the following inequality holds:

$$H \geq 1 + \frac{r_t^2 \lambda_{t,k}}{r_t^1 \lambda_{t,k}^*} \geq 1 + \frac{r_t^2}{r_t^1 \lambda_{t,k}^* K} \geq 1 + \frac{r_t^2}{r_t^1 K} = 1 + \frac{(1 - r_t^1)}{r_t^1 K},$$

which implies $r_t^1 \geq (K(H - 1) + 1)^{-1} = \tau$. ■

Proof of Theorem 3 (Outline). *1st step.* Since investor 1 uses the strategy Λ^* , by virtue of Lemma 5 the sequence ζ_t defined by (30) is a non-negative supermartingale. In view of inequality (31) and Lemma 6, we obtain

$$\sum_{t=0}^{\infty} \sum_{m=1}^K \sum_{k=1}^K R_{t,k} (1 - \rho_{t,m}) \lambda_{t,m}^* f(\gamma_{k,m}^t) < \infty,$$

which implies

$$\sum_{t=0}^{\infty} \sum_{m=1}^K \sum_{k=1}^K R_{t,k} f(\gamma_{k,m}^t) < \infty, \quad (32)$$

since we have $(1 - \rho_{t,m}) \geq \varkappa > 0$ by assumption (A.2), and $\lambda_{t,m}^* \geq \delta$. This step aims to show that if (32) converges, then the following series converges as well:

$$\sum_{t=0}^{\infty} \sum_{m=1}^K \sum_{k=1}^K R_{t,k} (\gamma_{k,m}^t - 1)^2 < \infty. \quad (33)$$

2nd step. From (33), the objective of this step is to deduce that

$$\sum_{t=0}^{\infty} \sum_{m=1}^K \sum_{k=1}^K R_{t,k} \left(y_t^m / \lambda_{t,m}^* - y_{t-1}^k / \lambda_{t-1,k}^* \right)^2 < \infty. \quad (34)$$

3rd step. At this step, by using (34), we obtain:

$$\sum_{t=0}^{\infty} \sum_{m=1}^K \sum_{k=1}^K (y_t^m / \lambda_{t,m}^* - y_t^k / \lambda_{t,k}^*)^2 < \infty.$$

This series can be estimated as:

$$\begin{aligned} \infty &> \sum_{t=0}^{\infty} \sum_{m=1}^K \sum_{k=1}^K \left(\frac{y_t^m}{\lambda_{t,m}^*} - \frac{y_t^k}{\lambda_{t,k}^*} \right)^2 \\ &= \sum_{t=0}^{\infty} \left(\frac{r_t^2}{r_t^1} \right)^2 \sum_{m=1}^K \sum_{k=1}^K \left(\frac{\lambda_{t,m}}{\lambda_{t,m}^*} - \frac{\lambda_{t,k}}{\lambda_{t,k}^*} \right)^2 \\ &\geq \sum_{t=0}^{\infty} \left(\frac{r_t^2}{r_t^1} \right)^2 \sum_{k=1}^K \left(\frac{\lambda_{t,m}}{\lambda_{t,m}^*} - \frac{\lambda_{t,k}}{\lambda_{t,k}^*} \right)^2 \end{aligned} \quad (35)$$

for each m . This fact will be used at the next step.

4th step. Next we prove the following estimate for the sum involved in (35):

$$\sum_{k=1}^K (\lambda_{t,m} / \lambda_{t,m}^* - \lambda_{t,k} / \lambda_{t,k}^*)^2 \geq (\lambda_{t,m} / \lambda_{t,m}^* - 1)^2. \quad (36)$$

Finally, by using (35) and inequality (36), we conclude

$$\begin{aligned} \infty &> \sum_{t=0}^{\infty} \left(\frac{r_t^2}{r_t^1} \right)^2 \sum_{m=1}^K \sum_{k=1}^K \left(\frac{\lambda_{t,m}}{\lambda_{t,m}^*} - \frac{\lambda_{t,k}}{\lambda_{t,k}^*} \right)^2 \\ &\geq \sum_{t=0}^{\infty} \left(\frac{r_t^2}{r_t^1} \right)^2 \sum_{m=1}^K \left(\frac{\lambda_{t,m}}{\lambda_{t,m}^*} - 1 \right)^2 \\ &= \sum_{t=0}^{\infty} \left(\frac{r_t^2}{r_t^1} \right)^2 \sum_{m=1}^K \left(\frac{\lambda_{t,m} - \lambda_{t,m}^*}{\lambda_{t,m}^*} \right)^2 \\ &\geq \sum_{t=0}^{\infty} \left(\frac{r_t^2}{r_t^1} \right)^2 \sum_{m=1}^K (\lambda_{t,m} - \lambda_{t,m}^*)^2 \\ &= \sum_{t=0}^{\infty} \left(\frac{r_t^2}{r_t^1} \right)^2 ||\lambda_t - \lambda_t^*||^2 \\ &\geq \sum_{t=0}^{\infty} \phi^2 ||\lambda_t - \lambda_t^*||^2, \end{aligned}$$

where $\phi > 0$ is a random variable such that $r_t^2 / r_t^1 \geq \phi$, which exists because $\Lambda = (\lambda_t)$ is a survival strategy. ■

4 Globally Evolutionarily Stable Strategies

On the basis of the general results obtained, this section examines the work of Evstigneev et al. [38] on globally evolutionarily stable portfolio rules (stronger results) under stronger assumptions, which deals with a special case of the model in which

- (i) the states of the world s_t are independent and identically distributed (i.i.d.), i.e., $s_1, s_2, \dots$ form a sequence of i.i.d. elements in S such that the probability $P\{s_t = s\}$ is strictly positive for each $s \in S$;
- (ii) the relative dividends $R_{t,k}(s^t) = R_k(s_t)$ depend only on the current state s_t and do not explicitly depend on t ;
- (iii) for each $k = 1, \dots, K$, the expectation $ED_{t,k}(s^t)$ is strictly positive, i.e., $ED_{t,k}(s^t) = ED_k(s_t) > 0$. This condition means that for each asset k the probability that it pays a strictly positive dividend is strictly positive;¹⁷
- (iv) all the portfolio rules under consideration are restricted to *simple/fixed-mix* (prescribing to select investment proportions initially and remain them fixed throughout the entire duration, independent of t and s^t) and *completely-mixed* (prescribing to assign a positive percentage of wealth $\lambda_{t,k}^i(s^t)$ to each asset $k = 1, \dots, K$ for all t and s^t) portfolio rules, i.e., $\lambda_{t,k}^i(s^t) = \lambda_k^i > 0$;
- (v) the supply of each asset in each moment of time $V_{t,k}$ is constant and, for simplicity, normalized to 1 (implying $\gamma_{t,k} = V_{t,k}/V_{t-1,k} = 1$);
- (vi) the same investment rate for all the assets at all dates, i.e., $\alpha_{t,k}$ is constant, $\alpha_{t,k} = \rho \in (0, 1)$;
- (vii) the functions $R_1(s), \dots, R_K(s)$ are linearly independent with respect to the probability distribution of s_t (the absence of redundant assets), i.e., the equality $\sum \theta_k R_k(s_t) = 0$ holding almost surely for some constants θ_k implies that $\theta_1 = \dots = \theta_K = 0$.

Model Setup. We follow the same asset market settings as in the general case, except that the stronger assumptions (i)–(vii) are imposed. The short-run (temporary) equilibria that determine the equilibrium asset prices are characterized by the following expressions:

- For date $t = 0$:

$$p_{0,k} = \rho \sum_{i=1}^N \lambda_k^i w_0^i, \quad k = 1, \dots, K. \quad (37)$$

¹⁷Recall that in the general model we assume: $\sum_{k=1}^K D_{t,k}(s^t) > 0$ for all $t \geq 1$ and s^t , which means that at each date in each random situation at least one asset yields a strictly positive dividend. This assumption is fundamental and will also be imposed here: $\sum_{k=1}^K D_k(s) > 0$ for all $s \in S$.

$$x_{0,k}^i = \frac{\rho \lambda_k^i w_0^i}{p_{0,k}}, \quad k = 1, \dots, K, \quad i = 1, \dots, N. \quad (38)$$

- For dates $t \geq 1$:

$$p_{t,k} = \rho \sum_{i=1}^N \lambda_k^i \langle D_t + p_t, x_{t-1}^i \rangle, \quad k = 1, \dots, K. \quad (39)$$

$$x_{t,k}^i = \frac{\rho \lambda_k^i \langle D_t + p_t, x_{t-1}^i \rangle}{p_{t,k}}, \quad k = 1, \dots, K, \quad i = 1, \dots, N. \quad (40)$$

From (39) and (40) we obtain

$$p_{t+1,k} = \rho \sum_{i=1}^N \lambda_k^i \langle p_{t+1} + D_{t+1}, x_t^i \rangle = \rho \sum_{i=1}^N \lambda_k^i w_{t+1}^i = \rho \langle \lambda_k, w_{t+1} \rangle,$$

$$x_{t,k}^i = \frac{\rho \lambda_k^i w_t^i}{p_{t,k}} = \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle},$$

and thus

$$w_{t+1}^i = \sum_{k=1}^K [p_{t+1,k} + D_k(s_{t+1})] x_{t,k}^i = \sum_{k=1}^K [\rho \langle \lambda_k, w_{t+1} \rangle + D_k(s_{t+1})] \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle}. \quad (41)$$

By summing up these equations over $i = 1, \dots, N$, we get

$$W_{t+1} = \sum_{k=1}^K [\rho \langle \lambda_k, w_{t+1} \rangle + D_k(s_{t+1})] \frac{\sum_{i=1}^N \lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle}$$

$$= \rho \sum_{k=1}^K \langle \lambda_k, w_{t+1} \rangle + \sum_{k=1}^K D_k(s_{t+1}) = \rho W_{t+1} + \sum_{k=1}^K D_k(s_{t+1}),$$

which leads to the formula

$$W_{t+1} = \frac{\sum_{k=1}^K D_k(s_{t+1})}{1 - \rho}. \quad (42)$$

From (41) and (42) we find

$$r_{t+1}^i = \sum_{k=1}^K [\rho \langle \lambda_k, r_{t+1} \rangle + (1 - \rho) \frac{D_k(s_{t+1})}{\sum_{k=1}^K D_k(s_{t+1})}] \frac{\lambda_k^i w_t^i / W_t}{\langle \lambda_k, w_t \rangle / W_t}.$$

Consequently, we arrive at the system of equations:

$$r_{t+1}^i = \sum_{k=1}^K [\rho \langle \lambda_k, r_{t+1} \rangle + (1 - \rho) R_k(s_{t+1})] \frac{\lambda_k^i r_t^i}{\langle \lambda_k, r_t \rangle}, \quad i = 1, \dots, N, \quad (43)$$

where

$$R_k(s_{t+1}) = \frac{D_k(s_{t+1})}{\sum_{k=1}^K D_k(s_{t+1})}, \quad k = 1, \dots, K,$$

are the *relative dividends* of the assets $k = 1, \dots, K$.

Consistent with the general model, we are primarily interested in the long-run behavior of the market shares of the investors, i.e., in the asymptotic properties of the sequence of vectors $r_t = (r_t^1, \dots, r_t^N)$ as $t \rightarrow \infty$.

Global Evolutionary Stability. An investment strategy, or a portfolio rule, is called *evolutionarily stable* if the following condition holds. If a group of investors uses this rule, while all the others use different ones, those and only those investors who belong to the former group survive. If this condition holds regardless of the initial state of the market, the investment strategy is called *globally* evolutionarily stable. If it holds under the additional assumption that the group of investors using other portfolio rules (distinct from the one we consider) possesses a sufficiently small initial share of market wealth, then the above property of stability is termed *local*. More precisely, the global evolutionary stability is defined as follows:

Definition 2 A portfolio rule $\lambda = (\lambda_1, \dots, \lambda_K)$ is called globally evolutionarily stable if the following condition holds. Suppose, in a group of investors $i = 1, 2, \dots, J$ ($1 \leq J < N$), all use the portfolio rule λ , while all the others, $i = J + 1, \dots, N$ use portfolio rules $\hat{\lambda}^i$ distinct from λ . Then those investors who belong to the former group ($i = 1, \dots, J$) survive with probability one, whereas those who belong to the latter ($i = J + 1, \dots, N$) become extinct with probability one, regardless of the initial state of the market.

In the above definition, it is supposed that the initial state r_0 in the market selection process governed by the random dynamical system (see equations (43)) on the simplex $\Delta^N = \{r \in \mathbb{R}^N | r^i \geq 0, \sum_i r^i = 1\}$ is *any* strictly positive vector $r_0 \in \Delta^N$, which is reflected in the term “global evolutionary stability.” An analogous local concept¹⁸ is defined similarly, but in the definition of local evolutionary stability, the initial market share $r_0^{J+1} + \dots + r_0^N$ of the group of investors ($i = J + 1, \dots, N$) who use strategies $\hat{\lambda}^i$ distinct from λ is supposed to be small enough.

Our main goal is to identify the portfolio rule which is globally evolutionarily stable. Clearly, if it exists it must be unique. Indeed if there are two such rules, $\lambda \neq \lambda'$, we can divide the population of investors into two groups assuming that the first uses λ and the second λ' . Then, according to the definition of global evolutionary stability, both groups must become extinct with probability one, which is impossible since the sum of the relative wealth of all the investors is equal to one.

Central Result. It can be shown by Theorem 4 that among all fixed-mix and completely-mixed investment strategies considered here, the only globally evolutionarily stable portfolio rule is to invest according to the proportions of

¹⁸See Evstigneev et al. [37], Section 3, Definition 2.

the *expected relative dividends* — analogous to the well-known Kelly’s principle of “betting one’s beliefs.” Define

$$\lambda^* = (\lambda_1^*, \dots, \lambda_K^*), \lambda_k^* = ER_k(s_t), k = 1, \dots, K. \quad (44)$$

Theorem 4 *The Kelly rule (44) is globally evolutionarily stable.*

An earlier version of the proof for Theorem 4 was provided in Evstigneev et al. [38], Appendix. Indeed, this result can be directly derived from the general case (Evstigneev et al. [35]) that we introduced in Sections 2 and 3. The derivation is given as follows:

Proof of Theorem 4. 1st step. Observe that this model (Evstigneev et al. [38]) is a special case of Evstigneev et al. [35], obtained by imposing assumptions (i)–(vii). One can easily verify this relation by substituting these stronger conditions into the original short-run equilibria and random dynamical system. Indeed, since $\alpha_{t,k} = \rho$, $\lambda_{t,k}^i(s^t) = \lambda_k^i$, $V_{t,k} = 1$ for $k = 1, \dots, K$, $i = 1, \dots, N$, and $t \geq 0$, the system of equations (2)–(5) reduces immediately to (37)–(40).

According to assumption (7), formulas (37) and (39) yield a unique equilibrium price vector p_t at each $t \geq 0$ since the condition $\alpha_{t,k}(s^t) = \rho < 1 = \gamma_{t,k}(s^t)$ holds for all $t \geq 1$ and all s^t .¹⁹

The wealth dynamics of traders in the general model is provided in (28), since we have $\rho_{t+1,k} = \alpha_{t+1,k}/\gamma_{t+1,k} = \rho/1 = \rho$, $\lambda_{t,k}^i = \lambda_{t+1,k}^i = \lambda_k^i$, $V_{t,k} = 1$, $D_{t+1,k}(s^{t+1}) = D_k(s_{t+1})$ for each $k = 1, \dots, K$, $i = 1, \dots, N$, and $t \geq 0$, it reduces to (41).

Given the random elements s_t are i.i.d., and the relative dividends $R_{t,k}(s^t) = R_k(s_t)$ depend only on the current state s_t and do not explicitly depend on t , then $E_t R_{t+l,k}(s^{t+l}) = ER_k(s_t)$ ($l \geq 1$), and so the survival strategy (14) can be simplified to the following form:

$$\begin{aligned} \lambda_{t,k}^* &= \lambda_k^* = E_t[\rho_{t+1,k} \lambda_{t+1,k}^* + (1 - \sum_{m=1}^K \rho_{t+1,m} \lambda_{t+1,m}^*) R_{t+1,k}] \\ &= E_t[\rho \lambda_k^* + (1 - \sum_{m=1}^K \rho \lambda_m^*) R_{t+1,k}] = \rho \lambda_k^* + E_t[(1 - \rho \sum_{m=1}^K \lambda_m^*) R_{t+1,k}] \\ &= \rho \lambda_k^* + (1 - \rho) E_t R_{t+1,k} = \rho \lambda_k^* + (1 - \rho) ER_k(s_t) \end{aligned}$$

(a.s.), $k = 1, \dots, K$, which leads to the formula

$$\lambda_k^* = ER_k(s_t), k = 1, \dots, K.$$

It coincides with the survival strategy (44) constructed here.²⁰

Thus, under assumptions (i)–(vii), the setup in Evstigneev et al. [35] corresponds to the special case in Evstigneev et al. [38].

¹⁹In the context of Evstigneev et al. [38], this result is shown in Section 4.1, Proposition 1.

²⁰In the context of Evstigneev et al. [38], it is shown in Section 4.4, Theorem 4 that the Kelly rule (44) survives with probability one.

2nd step. As established in Step 1, Theorem 3 remains valid in this special case. Recall that we now restrict our consideration to the class of fixed-mix and completely-mixed strategies. Assume that one of the investors, say investor 1, adopts the Kelly rule $\lambda_{t,k}^* = \lambda_k^* = ER_k(s_t)$, $k = 1, \dots, K$, while all the other investors $i = 2, \dots, N$ use portfolio rules $\lambda^i \neq \lambda^*$. For each investor i who adopts strategy λ^i , there must exist some asset k' such that $|\lambda_{k'}^* - \lambda_{k'}^i| \geq \delta > 0$, where $\delta := \min_k |\lambda_k^* - \lambda_k^i|$ is a strictly positive constant. (Otherwise, if $|\lambda_k^* - \lambda_k^i| = 0$ for all $k = 1, \dots, K$, then $\lambda^i = \lambda^*$).

Then, we have

$$\|\lambda^* - \lambda^i\|^2 = \sum_{k=1}^K (\lambda_k^* - \lambda_k^i)^2 \geq (\lambda_{k'}^* - \lambda_{k'}^i)^2 \geq \delta^2,$$

for each date $t \geq 0$, and so

$$\sum_{t=0}^{\infty} \|\lambda^* - \lambda^i\|^2 \geq \sum_{t=0}^{\infty} \delta^2 = \delta^2 \cdot \infty = \infty.$$

We apply the contrapositive of Theorem 3, for any other basic (fixed-mix and completely-mixed) strategy $\lambda^i \neq \lambda^*$, $i = 2, \dots, N$, if

$$\sum_{t=0}^{\infty} \|\lambda^* - \lambda^i\|^2 = \infty \text{ (a.s.)},$$

then λ^i becomes extinct (a.s.), i.e.,

$$\lim_{t \rightarrow \infty} r_t^i = 0 \text{ (a.s.)}, \quad i = 2, \dots, N.$$

Since $r \in \Delta^N$, and we know $r_t^i \rightarrow 0$ (a.s.) for $i = 2, \dots, N$, consequently,

$$r_t^1 \rightarrow 1 \text{ (a.s.)},$$

which means that if investor 1 uses the Kelly rule λ^* , while all the others use strategies distinct from λ^* , then investor 1 is almost surely the *single survivor* in the market selection process.

3rd step. In order to obtain the global evolutionary stability result, we have to consider a group of investors $i = 1, \dots, J$ ($1 \leq J < N$) using the portfolio rule λ^* , while all the other investors $i = J+1, \dots, N$ use portfolio rules $\lambda^i \neq \lambda^*$, and show that the former group ($i = 1, \dots, J$) survives, while the latter ($i = J+1, \dots, N$) becomes extinct.

However, it is sufficient to prove it assuming that $J = 1$, in which case the result reduces to the assertion that $r_t^1 \rightarrow 1$ (a.s.). To perform the reduction of the case $J > 1$ to the case $J = 1$, we “aggregate” the group of investors $i = 1, \dots, J$ into one by setting

$$\bar{r}_t^1 = r_t^1 + \dots + r_t^J.$$

By adding up equations (43) over $i = 1, \dots, J$, we obtain

$$\bar{r}_{t+1}^1 = \sum_{k=1}^K [\rho \langle \lambda_k, r_{t+1} \rangle + (1 - \rho) R_k(s_{t+1})] \frac{\lambda_k^* \bar{r}_t^1}{\langle \lambda_k, r_t \rangle},$$

where

$$\langle \lambda_k, r \rangle = \lambda_k^* \bar{r}^1 + \sum_{i=J+1}^N \lambda_k^i r^i.$$

Thus the original model reduces to the analogous one in which there are $N - J + 1$ investors ($i = 1, J + 1, \dots, N$) so that investor 1 uses the Kelly strategy λ^* and all the others, $i = J + 1, \dots, N$, use strategies distinct from λ^* .

According to Step 2, we have shown the global evolutionary stability in the case $J = 1$, that is, $r_t^i \rightarrow 0$ (a.s.) for all $i = J + 1, \dots, N$, and $\bar{r}_t^1 \rightarrow 1$, which means that the group of investors $i = 1, \dots, J$ (which we treat as a single, “aggregate,” investor) accumulates in the limit all market wealth.

It remains to observe that in the original model, the proportions between the market shares of investors i, j who belong to the group $1, \dots, J$ using the Kelly rule do not change in time. This is so because for all such investors, the growth rates of their market shares are the same:

$$\frac{r_{t+1}^i}{r_t^i} = \sum_{k=1}^K [\rho \langle \lambda_k, r_{t+1} \rangle + (1 - \rho) R_k(s_{t+1})] \frac{\lambda_k^*}{\langle \lambda_k, r_t \rangle}, \quad i = 1, \dots, J.$$

Hence,

$$\frac{r_{t+1}^i}{r_t^i} = \frac{r_{t+1}^j}{r_t^j}, \quad i, j = 1, \dots, J,$$

and so

$$\frac{r_{t+1}^i}{r_{t+1}^1} = \frac{r_t^i}{r_t^1} = \frac{r_0^i}{r_0^1}, \quad i = 1, \dots, J.$$

Consequently, $r_t^i = \beta^i r_t^1$ ($i = 1, \dots, J$) for all t , where $\beta^i = r_0^i / r_0^1$ is a strictly positive constant. Since

$$\bar{r}_t^1 = \sum_{i=1}^J r_t^i = \left(\sum_{i=1}^J \beta^i \right) r_t^1 \rightarrow 1 \text{ (a.s.)},$$

we obtain that

$$r_t^i \rightarrow \frac{\beta^i}{\sum_{i=1}^J \beta^i} > 0 \text{ (a.s.)}, \quad i = 1, \dots, J.$$

Therefore, all the investors $i = 1, \dots, J$ adopting the Kelly rule λ^* survive with probability one, while all the other investors $i = J + 1, \dots, N$ using strategies distinct from λ^* become extinct with probability one, regardless of the initial state of the market, which gives the global evolutionary stability. ■

5 Almost Sure Nash Equilibrium Strategies

In this section, we review the study conducted by Bahsoun et al. [15], which views EF from a different perspective and treat its decision-theoretic framework as a game in which the payoffs of the players (investors) are defined in terms of the growth rates of their relative wealth.

Game-theoretic models of asset markets dealing with relative wealth of investors were considered by Bell and Cover [18, 19]. In those models, the objectives of the players were formulated through the expectations of random payoff functions, and the notion of a Nash equilibrium — defined in terms of these expectations — was used as a game solution concept. Bahsoun et al. [15], however, proposed a different (stronger) solution concept: the *almost sure Nash equilibrium*. In such an equilibrium, any unilateral deviation from it leads to a decrease in the random payoff with probability one, not only in its expected payoff. The main result demonstrates that, in the game under consideration, the Kelly portfolio rule of “betting one’s beliefs” forms with probability one a unique symmetric Nash equilibrium strategy.

The analysis is carried out under stronger assumptions than in the general model:

- (i) the states of the world s_t are independent and identically distributed (i.i.d.), i.e., $s_1, s_2, \dots$ form a sequence of i.i.d. elements in S such that the probability $P\{s_t = s\}$ is strictly positive for each $s \in S$;
- (ii) the relative dividends $R_{t,k}(s^t) = R_k(s_t)$ depend only on the current state s_t and do not explicitly depend on t ;
- (iii) for each $k = 1, \dots, K$, the expectation $ED_{t,k}(s^t)$ is strictly positive, i.e., $ED_{t,k}(s^t) = ED_k(s_t) > 0$;²¹
- (iv) all the portfolio rules under consideration are restricted to *simple (fixed-mix)* portfolio rules, prescribing to select investment proportions initially and remain them fixed throughout the entire duration, i.e., $\lambda_{t,k}^i(s^t) = \lambda_k^i$;
- (v) the volume $V_{t,k}$ of each asset k grows (or decreases) at the same rate $\gamma_t = \gamma_t(s^{t-1}) > 0$, with

$$\gamma_t = \frac{V_{t,k}}{V_{t-1,k}} \quad (45)$$

for all $t \geq 1$, where the growth rate process γ_t is *predictable*, depending on the history s^{t-1} of the process (s_t) up to time $t - 1$ (not t);²²

²¹Recall that in the general model we assume: $\sum_{k=1}^K D_{t,k}(s^t) > 0$ for all $t \geq 1$ and s^t . This assumption is fundamental and will also be imposed here: $\sum_{k=1}^K D_k(s) > 0$ for all s .

²²By virtue of (45), we have

$$V_{t,k}(s^{t-1}) = \gamma_t(s^{t-1}) \dots \gamma_2(s_1) \gamma_1 V_k, \quad (46)$$

where $V_k = V_{0,k} > 0$ ($k = 1, \dots, K$) are the initial amounts of the assets. In the case of dividend-paying assets involving investments in the real economy, assumption (45) means that the economic system under consideration is on a *balanced growth path*.

- (vi) the *investment rate* $0 < \alpha_t(s^{t-1}) < 1$ is the same for all traders and assets, and is given by a predictable function $\alpha_t(s^{t-1})$ — depending only on the history s^{t-1} of the states of the world up to time $t-1$ — which is measurable (for $t=0, 1$ it is constant) and satisfies

$$\alpha_t(s^{t-1}) < \frac{V_{t,k}(s^t)}{V_{t-1,k}(s^{t-1})};^{23} \quad (47)$$

- (vii) there exist constants $0 < \rho' < \rho'' < 1$ such that the process

$$\rho_t(s^{t-1}) := \frac{\alpha_t(s^{t-1})}{\gamma_t(s^{t-1})} \quad (48)$$

satisfies $\rho' \leq \rho_t(s^{t-1}) \leq \rho''$,²⁴

- (viii) the functions $R_1(s), \dots, R_K(s)$ are linearly independent with respect to the probability distribution of s_t (the absence of redundant assets), i.e. the equality $\sum \theta_k R_k(s_t) = 0$ holding almost surely for some constants θ_k implies that $\theta_1 = \dots = \theta_K = 0$.

Model Setup. We follow the same asset market settings as in the general case, except that the stronger assumptions (i)–(viii) are imposed. The short-run (temporary) equilibria that determine the equilibrium asset prices are characterized by the following expressions:

- For date $t=0$:

$$p_{0,k} V_{0,k} = \alpha_0 \sum_{i=1}^N \lambda_k^i w_0^i, \quad k = 1, \dots, K. \quad (49)$$

$$x_{0,k}^i = \frac{\alpha_0 \lambda_k^i w_0^i}{p_{0,k}}, \quad k = 1, \dots, K, \quad i = 1, \dots, N. \quad (50)$$

- For dates $t \geq 1$:

$$p_{t,k} V_{t,k} = \alpha_t \sum_{i=1}^N \lambda_k^i \langle D_t + p_t, x_{t-1}^i \rangle, \quad k = 1, \dots, K. \quad (51)$$

$$x_{t,k}^i = \frac{\alpha_t \lambda_k^i \langle D_t + p_t, x_{t-1}^i \rangle}{p_{t,k}}, \quad k = 1, \dots, K, \quad i = 1, \dots, N. \quad (52)$$

²³Recall that in the general model, by formula (6)–(7), we assume $\alpha_{t,k}(s^t) < \gamma_{t,k}(s^t) = V_{t,k}(s^t)/V_{t-1,k}(s^{t-1})$ for all $t \geq 1$ and all s^t . Here $\gamma_t(s^{t-1})$ and $\alpha_t(s^{t-1})$ are assumed to be predictable (depending on s^{t-1} rather than s^t) and same for each asset k .

²⁴Note that here ρ_t does not depend on k , i.e., $\rho_{t,1} = \dots = \rho_{t,K} = \rho_t$. And the inequality follows from Condition (A.2) in the general model (see (16)).

It can be shown that under assumption (47) a non-negative vector p_t satisfying these equations exists and is unique (for any s^t and any feasible x_{t-1}^i and λ^i).²⁵

We will need a description of a system of equations governing the dynamics of the market shares $r_t^i = w_t^i / \sum_j w_t^j$ of the investors given their admissible strategy profile $(\lambda^1, \dots, \lambda^N)$. From (51) and (52), for $t \geq 1$, we have

$$p_{t,k} = V_{t,k}^{-1} \alpha_t \sum_{i=1}^N \lambda_k^i \langle p_t + D_t, x_{t-1}^i \rangle = \alpha_t V_{t,k}^{-1} \sum_{i=1}^N \lambda_k^i w_t^i = \alpha_t V_{t,k}^{-1} \langle \lambda_k, w_t \rangle,$$

$$x_{t,k}^i = \frac{\alpha_t \lambda_k^i w_t^i}{p_{t,k}} = \frac{V_{t,k} \lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle}.^{26}$$

Consequently, for $t \geq 0$, we get

$$\begin{aligned} w_{t+1}^i &= \sum_{k=1}^K (p_{t+1,k} + D_{t+1,k}) x_{t,k}^i \\ &= \sum_{k=1}^K \left(\alpha_{t+1} \frac{\langle \lambda_k, w_{t+1} \rangle}{V_{t+1,k}} + D_{t+1,k} \right) \frac{V_{t,k} \lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle} \\ &= \sum_{k=1}^K \left(\alpha_{t+1} \frac{\langle \lambda_k, w_{t+1} \rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k} V_{t,k} \right) \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle} \\ &= \sum_{k=1}^K (\alpha_{t+1} \gamma_{t+1}^{-1} \langle \lambda_k, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle} \\ &= \sum_{k=1}^K (\rho_{t+1} \langle \lambda_k, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle}. \end{aligned} \quad (53)$$

By summing up these equations over $i = 1, \dots, N$, we obtain

$$\begin{aligned} W_{t+1} &= \sum_{k=1}^K (\rho_{t+1} \langle \lambda_k, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \frac{\sum_{i=1}^N \lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle} \\ &= \sum_{k=1}^K (\rho_{t+1} \langle \lambda_k, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \\ &= \rho_{t+1} W_{t+1} + \sum_{k=1}^K D_{t+1,k} V_{t,k}, \end{aligned}$$

²⁵See Amir et al. [4], Section 4.1, Proposition 1.

²⁶The analogous formulas for $t = 0$,

$$p_{0,k} = \alpha_0 V_{0,k}^{-1} \langle \lambda_k, w_0 \rangle, \quad x_{0,k}^i = \frac{V_{0,k} \lambda_k^i w_0^i}{\langle \lambda_k, w_0 \rangle},$$

follow from (49) and (50).

since ρ_{t+1} does not depend on k (see (48)), which implies

$$W_{t+1} = \frac{1}{1 - \rho_{t+1}} \sum_{m=1}^K D_{t+1,m} V_{t,m}. \quad (54)$$

Dividing both sides of equation (53) by W_{t+1} and using (54), we get

$$\begin{aligned} r_{t+1}^i &= \sum_{k=1}^K \left[\rho_{t+1} \langle \lambda_k, r_{t+1} \rangle + (1 - \rho_{t+1}) \frac{D_{t+1,k} V_{t,k}}{\sum_{m=1}^K D_{t+1,m} V_{t,m}} \right] \frac{\lambda_k^i w_t^i / W_t}{\langle \lambda_k, w_t \rangle / W_t} \\ &= \sum_{k=1}^K [\rho_{t+1} \langle \lambda_k, r_{t+1} \rangle + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^i r_t^i}{\langle \lambda_k, r_t \rangle}, \quad i = 1, \dots, N, \quad t \geq 0. \end{aligned} \quad (55)$$

Almost Sure Nash Equilibrium. We are primarily interested in the long-run behavior of the *relative wealth* of the investors (players). Given a strategy profile $(\lambda^1, \dots, \lambda^N)$, the performance of a strategy λ^i used by investor i will be characterized by the following random variable

$$\xi^i := \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \frac{w_t^i}{\sum_{j \neq i} w_t^j}, \quad (56)$$

generally, taking values in $[-\infty, +\infty]$. The expression $w_t^i / \sum_{j \neq i} w_t^j$ is the relative wealth of player i and the group $\{j : j \neq i\}$ of i 's rivals. The random variable $\xi^i = \xi^i(s^\infty; \lambda^1, \dots, \lambda^N)$ depends on the strategy profile $(\lambda^1, \dots, \lambda^N)$ and on the whole history $s^\infty := (s_1, s_2, \dots)$ of states of the world from time 1 to ∞ . In the game under consideration, ξ^i plays the role of the (random) *payoff function* of player i .

Definition 3 We shall say that a strategy $\bar{\lambda}$ forms a symmetric Nash equilibrium almost surely if

$$\xi^i(s^\infty; \bar{\lambda}, \dots, \bar{\lambda}) \geq \xi^i(s^\infty; \bar{\lambda}, \dots, \lambda, \dots, \bar{\lambda}) \quad (\text{a.s.}) \quad (57)$$

for every i , each strategy λ of investor i and each set of initial endowments $w_0^1 > 0, \dots, w_0^N > 0$. The Nash equilibrium is called strict if the inequality in (57) is strict for any $\lambda \neq \bar{\lambda}$.

Recall that we consider only those strategy profiles which are admissible. If all the players use the same strategy $\bar{\lambda}$, then the strategy profile $(\bar{\lambda}, \dots, \bar{\lambda})$ is admissible if and only if the vector $\bar{\lambda}$ is strictly positive. This is immediate from formula (51).

Central Result. Define the *relative dividends* of the assets $k = 1, \dots, K$ by

$$R_{t,k} = \frac{D_{t,k} V_{t-1,k}}{\sum_{m=1}^K D_{t,m} V_{t-1,m}} = \frac{D_{t,k} V_k}{\sum_{m=1}^K D_{t,m} V_m} \quad (58)$$

(see formula (46)), where $R_{t,k} = R_k(s_t)$ and $D_{t,k} = D_k(s_t)$. Define

$$\lambda^* = (\lambda_1^*, \dots, \lambda_K^*), \quad \lambda_k^* = ER_k(s_t), \quad k = 1, 2, \dots, K. \quad (59)$$

As in formula (44) established in Section 4, the portfolio rule specified by (59) represents a generalization of the Kelly portfolio rule of “betting one’s beliefs”, where the “beliefs” are expressed here in terms of the expected relative dividends. Note that the vector λ^* has strictly positive coordinates, i.e., λ^* is a completely-mixed portfolio rule, since we assume $ED_k(s_t) > 0$ (which does not depend on t because the random elements s_t are i.i.d.).

Theorem 5 *The portfolio rule λ^* is a unique investment strategy forming a symmetric Nash equilibrium almost surely. If an investor i uses any strategy λ distinct from λ^* , then*

$$\xi^i(s^\infty; \lambda^*, \dots, \lambda, \dots, \lambda^*) < \xi^i(s^\infty; \lambda^*, \dots, \lambda^*) = 0 \quad (a.s.), \quad (60)$$

and so the Nash equilibrium formed by the strategy λ^ is strict.*

The result contained in Theorem 5 implies the following property of the portfolio rule λ^* . If all the investors except one, say investor i , use the strategy λ^* and i uses any other strategy λ distinct from λ^* , then the relative wealth $w_t^i / \sum_{j \neq i} w_t^j$ of i tends to zero at the exponential rate $\xi^i < 0$ (a.s.). In other words, the group of the Kelly investors drives the non-Kelly one out of the market, which is interpreted in Evolutionary Finance (EF) as the property of global (holding for all initial states) evolutionary stability of λ^* .²⁷

Remark 1 The result on the global evolutionary stability of λ^* (without an exponential estimate of the convergence rate) follows from Theorem 4 requiring that the state space S is finite and all the strategies under consideration are completely mixed. It is important to note that results of this kind cannot, in general, be extended to settings going beyond the framework of the i.i.d. random elements s_t . The reason for this lies in the fact that in more general settings, the analogues of the Kelly portfolio rule λ^* do not belong to the class of constant proportions strategies. They belong to broader classes of portfolio rules where there might be strategies "coexisting" with the generalized version of λ^* (strategies that cannot be driven by λ^* out of the market); see Amir et al. [4].

Global exponential stability theorems are among the strongest mathematical results in the theory related to EF models. They require strong assumptions and delicate mathematical techniques. The fact of exponential convergence of relative wealth makes it possible to derive the almost sure Nash equilibrium property of the Kelly portfolio rule defined in terms of the Lyapunov exponents (56). For a much simpler model, dealing with short-lived assets, an analogous stability theorem was obtained in Evstigneev et al. [36] under a number of additional assumptions (finite space S , completely mixed strategies, etc.).

²⁷The result is akin to a well known fact of evolutionary game theory: if a Nash equilibrium is strict, then it is evolutionary stable (Weibull [101]).

Remark 2 Recall that the Lyapunov exponent (56) is defined in terms of the variables $w_t^i / \sum_{j \neq i} w_t^j$ expressing the relative wealth of player i and the group $\{j : j \neq i\}$ of i 's rivals. In the EF literature, relative wealth is often defined as $r_t^i = w_t^i / \sum_{j=1}^N w_t^j$ (the market share of investor i). In many cases, results can be equivalently formulated both in terms of relative wealth — as it is defined in this section — and market shares. But this is not the case in the context of the present section. The consideration of the Lyapunov exponent

$$\eta^i := \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \frac{w_t^i}{\sum_{j=1}^N w_t^j}$$

leads to a trivial notion of a Nash equilibrium. With the payoff functions η^i , *any* completely mixed strategy λ forms a symmetric Nash equilibrium. Indeed, $\eta^i(\lambda, \dots, \lambda) = 0$ because if all the investors use the same strategy, their market shares remain constant (which follows from equations (55)). On the other hand, η^i is always non-positive, and so $\eta^i(\lambda^1, \dots, \lambda^K) \leq 0 = \eta^i(\lambda, \dots, \lambda)$, which implies that the strategy profile $(\lambda, \dots, \lambda)$ is a Nash equilibrium.

A key role in the proof of Theorem 5 is played by a lemma which is formulated below. Consider a measurable vector function $R(s) = (R_1(s), \dots, R_K(s))$ on S with values in Δ^K . Assume that the coordinates $R_k(s)$ are linearly independent with respect to the distribution of s_t and $ER_k(s_t) > 0$. For any $\lambda = (\lambda_1, \dots, \lambda_K) \in \Delta^K$, $\kappa \in (0, 1]$ and $0 < \rho < 1$, define

$$F_\rho(\lambda, \kappa; s) := \frac{\sum_{k=1}^K [\rho \lambda_k + (1 - \rho) R_k(s)] \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}}{\sum_{k=1}^K [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}}, \quad (61)$$

where $\lambda_k^* = ER_k(s)$. The function $F_\rho(\lambda, \kappa; s)$ is well-defined and takes on finite non-negative values. Fix some numbers $0 < \rho' < \rho'' < 1$.

Lemma 7 *For any $\lambda \in \Delta^K$ distinct from λ^* there exist constants $H > 0$ and $\delta > 0$ such that*

$$E \min\{H, \ln F_\rho(\lambda, \kappa; s)\} \geq \delta \quad (62)$$

for all $\kappa \in (0, 1]$ and all $\rho \in [\rho', \rho'']$.

The proof of this lemma is routine, but rather lengthy, and we relegate it to the Appendix of this paper.

In fact, we will show that in the proof of Theorem 5 it is sufficient to consider the case of two investors ($N = 2$). In this case, the analysis can be reduced to the consideration of the ratio

$$z_t := \frac{r_t^1}{r_t^2} = \frac{w_t^1}{w_t^2} \quad (63)$$

of the market shares of investors 1 and 2 using strategies $\lambda^* = (\lambda_1^*, \dots, \lambda_K^*)$ and $\lambda = (\lambda_1, \dots, \lambda_K)$. Applying (55) with $N = 2$, we get

$$r_{t+1}^i = \sum_{k=1}^K \left[\rho_{t+1} \left(\lambda_k^i r_{t+1}^i + \lambda_k^j (1 - r_{t+1}^i) \right) + (1 - \rho_{t+1}) R_{t+1,k} \right] \frac{\lambda_k^i r_t^i}{\lambda_k^i r_t^i + \lambda_k^j r_t^j},$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Setting $C_{t,k}^{ij} := \lambda_k^i r_t^i / (\lambda_k^i r_t^i + \lambda_k^j r_t^j)$, we obtain

$$r_{t+1}^i \left[1 + \rho_{t+1} \sum_{k=1}^K (\lambda_k^j - \lambda_k^i) C_{t,k}^{ij} \right] = \sum_{k=1}^K [\rho_{t+1} \lambda_k^j + (1 - \rho_{t+1}) R_{t+1,k}] C_{t,k}^{ij}.$$

Thus

$$\frac{r_{t+1}^i}{r_{t+1}^j} = \frac{A_{t+1}^{ij}/B_{t+1}^{ij}}{A_{t+1}^{ji}/B_{t+1}^{ji}},$$

where

$$\begin{aligned} A_{t+1}^{ij} &:= \sum_{k=1}^K [\rho_{t+1} \lambda_k^j + (1 - \rho_{t+1}) R_{t+1,k}] C_{t,k}^{ij}, \\ B_{t+1}^{ij} &:= 1 + \rho_{t+1} \sum_{k=1}^K (\lambda_k^j - \lambda_k^i) C_{t,k}^{ij}. \end{aligned}$$

Observe that $B_{t+1}^{ji} = B_{t+1}^{ij}$, indeed,

$$\begin{aligned} B_{t+1}^{ij} - B_{t+1}^{ji} &= \rho_{t+1} \sum_{k=1}^K [(\lambda_k^j - \lambda_k^i) C_{t,k}^{ij} - (\lambda_k^i - \lambda_k^j) C_{t,k}^{ji}] \\ &= \rho_{t+1} \sum_{k=1}^K (\lambda_k^j - \lambda_k^i) = \rho_{t+1} \left(\sum_{k=1}^K \lambda_k^j - \sum_{k=1}^K \lambda_k^i \right) = 0 \end{aligned}$$

because $C_{t,k}^{ij} + C_{t,k}^{ji} = 1$. Consequently,

$$\frac{r_{t+1}^1}{r_{t+1}^2} = \frac{A_{t+1}^{12}}{A_{t+1}^{21}} = \frac{r_t^1 \sum_{k=1}^K [\rho_{t+1} \lambda_k^2 + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^1}{\lambda_k^1 r_t^1 / r_t^2 + \lambda_k^2}}{r_t^2 \sum_{k=1}^K [\rho_{t+1} \lambda_k^1 + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^2}{\lambda_k^1 r_t^1 / r_t^2 + \lambda_k^2}},$$

by virtue of (63), we obtain the dynamics of z_t :

$$z_{t+1} = z_t \frac{\sum_{k=1}^K [\rho_{t+1} \lambda_k + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^*}{\lambda_k^* z_t + \lambda_k}}{\sum_{k=1}^K [\rho_{t+1} \lambda_k^* + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k}{\lambda_k^* z_t + \lambda_k}}. \quad (64)$$

Proof of Theorem 5. 1st step. To prove the theorem it is sufficient to consider the case of two investors 1 and 2, using λ^* and λ , and show that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln z_t > 0 \text{ (a.s.)}, \quad (65)$$

where z_t is the ratio of the market shares of 1 and 2. To demonstrate that the problem reduces to the case of two investors, let us first observe that by virtue of symmetry, it is sufficient to verify the property (57) for $i = N$. Suppose

investors $i = 1, 2, \dots, N-1$ use λ^* and investor N uses $\lambda \neq \lambda^*$. Then the total market share $r_t^* := r_t^1 + \dots + r_t^{N-1}$ of $i = 1, 2, \dots, N-1$ satisfies

$$r_{t+1}^* = \sum_{k=1}^K \{ \rho_{t+1} [\lambda_k^* r_{t+1}^* + \lambda_k r_{t+1}^N] + (1 - \rho_{t+1}) R_{t+1,k} \} \frac{\lambda_k^* r_t^*}{\lambda_k^* r_t^* + \lambda_k r_t^N}, \quad (66)$$

which is obtained by summing up equations (55) over $i = 1, 2, \dots, N-1$. At the same time, by virtue of (55), we have

$$r_{t+1}^N = \sum_{k=1}^K \{ \rho_{t+1} [\lambda_k^* r_{t+1}^* + \lambda_k r_{t+1}^N] + (1 - \rho_{t+1}) R_{t+1,k} \} \frac{\lambda_k r_t^N}{\lambda_k^* r_t^* + \lambda_k r_t^N}. \quad (67)$$

Thus the vector (r_t^*, r_t^N) evolves in time as the vector $(\tilde{r}_t^1, \tilde{r}_t^2)$ of market shares of two investors using the strategies λ^* and λ , respectively. If we know that (65) holds, then

$$\begin{aligned} \xi^N(\lambda^*, \dots, \lambda^*, \lambda) &= \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \frac{w_t^N}{w_t^*} = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \frac{r_t^N}{r_t^*} \\ &= \limsup_{t \rightarrow \infty} \left(-\frac{1}{t} \ln z_t \right) = -\liminf_{t \rightarrow \infty} \left(\frac{1}{t} \ln z_t \right) \\ &< 0 = \xi^N(\lambda^*, \dots, \lambda^*, \lambda^*) \text{ (a.s.)}, \end{aligned}$$

where the last equality holds because the market shares of all the investors remain constant, as long as all of them use the same strategy (see (55)). Also, this follows from the fact that if $\lambda_k = \lambda_k^*$ in (64), then z_t is a constant. Thus we have proved (60) and hence showed that λ^* forms a strict symmetric Nash equilibrium.

2nd step. Let us verify (65). Put $G_t = \ln(z_t/z_{t-1})$. Then

$$\sum_{t=1}^T G_t = \sum_{t=1}^T (\ln z_t - \ln z_{t-1}) = \ln z_T - \ln z_0.$$

Therefore it suffices to prove that $\liminf_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T G_t > 0$ (a.s.). For any constant H define $G_t^H := \min\{G_t, H\}$. Since $G_t^H \leq G_t$ it is sufficient to prove that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T G_t^H > 0 \text{ (a.s.)} \quad (68)$$

for some H .

By virtue of (64), we have

$$\begin{aligned} G_{t+1} &= \ln \frac{z_{t+1}}{z_t} = \ln \frac{\sum_{k=1}^K [\rho_{t+1} \lambda_k + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^*}{\lambda_k^* z_t + \lambda_k}}{\sum_{k=1}^K [\rho_{t+1} \lambda_k^* + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k}{\lambda_k^* z_t + \lambda_k}} \\ &= \ln \frac{\sum_{k=1}^K [\rho_{t+1} \lambda_k + (1 - \rho_{t+1}) R_k(s_{t+1})] \frac{\lambda_k^*}{\lambda_k^* r_t^1 + \lambda_k (1 - r_t^1)}}{\sum_{k=1}^K [\rho_{t+1} \lambda_k^* + (1 - \rho_{t+1}) R_k(s_{t+1})] \frac{\lambda_k}{\lambda_k^* r_t^1 + \lambda_k (1 - r_t^1)}} \\ &= \ln F_{\rho_{t+1}}(\lambda, r_t^1; s_{t+1}), \end{aligned}$$

where $r_t^1 = r_t^1(s^t)$ and $\rho_{t+1} = \rho_{t+1}(s^t)$ (recall that the process ρ_t is predictable). By virtue of Lemma 7, there exist $H > 0$ and $\delta > 0$ such that $E_t G_{t+1}^H \geq \delta$, where $E_t(\cdot) = E(\cdot|s^t)$ is the conditional expectation given s^t and

$$G_{t+1}^H(s^{t+1}) = \min\{H, \ln F_{\rho_{t+1}(s^t)}(\lambda, r_t^1(s^t); s_{t+1})\}.$$

When computing $E_t G_{t+1}^H$ we fix s^t and take the unconditional expectation of G_{t+1}^H with respect to s_{t+1} , which is justified because s^t and s_{t+1} are independent.

Finally, we have

$$\frac{1}{T} \sum_{t=1}^T G_t^H = \frac{1}{T} \sum_{t=1}^T E_{t-1} G_t^H + \frac{1}{T} \sum_{t=1}^T (G_t^H - E_{t-1} G_t^H).$$

Since G_t^H is uniformly bounded (see (71) in the Appendix), we can apply to the process $B_t^H := G_t^H - E_{t-1} G_t^H$ the strong law of large numbers for martingale differences (see, e.g., Hall and Heyde [51]), which yields $T^{-1} \sum_{t=1}^T B_t^H \rightarrow 0$ (a.s.).²⁸ Therefore $\liminf_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T G_t^H \geq \delta$, which proves (68).

3rd step. It remains to prove the uniqueness of a symmetric almost sure Nash equilibrium. Suppose a strategy $\lambda \neq \lambda^*$ forms such an equilibrium. Then

$$0 = \xi^N(s^\infty; \lambda, \dots, \lambda) \geq \xi^N(s^\infty; \lambda, \dots, \lambda, \lambda^*) \text{ (a.s.)}, \quad (69)$$

where

$$\xi^N(s^\infty; \lambda, \dots, \lambda, \lambda^*) = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \frac{r_t^N}{1 - r_t^N}.$$

By interchanging λ and λ^* in formulas (66) and (67), we obtain that the vector $(r_t^1 + \dots + r_t^{N-1}, r_t^N)$ evolves in time as the vector $(\hat{r}_t^1, \hat{r}_t^2)$ of market shares of two investors using the strategies λ and λ^* , respectively. As we have proved above, this implies

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \ln \frac{r_t^N}{1 - r_t^N} > 0 \text{ (a.s.)}.$$

Thus $\xi^N(s^\infty; \lambda, \dots, \lambda, \lambda^*) > 0$, which yields the inequality " $<$ " in (69). This is a contradiction. ■

6 Discussion

Based on the discussions in Evstigneev et al. [35], this section presents several key comments on the modeling approach, characteristic features and applications of EF.

1. Marshallian temporary equilibrium. In the general methodological perspective, the modeling framework at hand relies upon the Marshallian [77]

²⁸ As is well known, the finiteness of expectations is not enough for the validity of this version of the law of large numbers, but the boundedness of the random variables is fully sufficient — hence the truncation of F by H .

principle of temporary equilibrium. The dynamics of the asset market in this framework are similar to the dynamics of the commodity market as outlined in the classical treatise by Alfred Marshall [77] (Book V, Chapter II “Temporary Equilibrium of Demand and Supply”). The ideas of Marshall were developed in the framework of mathematical economics by Samuelson [86]. As it was noticed by Samuelson and discussed in detail by Schlicht [90], in order to study the process of market dynamics by using the Marshallian “moving equilibrium method,” one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our case, the set $x_t = (x_t^1, \dots, x_t^N)$ of investors’ portfolios) can be temporarily fixed, while the other (in our case, the asset prices p_t) can be assumed to rapidly reach the unique state of partial equilibrium. Samuelson [86], pp. 321–323, writes about this approach:

I, myself, find it convenient to visualize equilibrium processes of quite different speed, some very slow compared to others. Within each long run there is a shorter run, and within each shorter run there is a still shorter run, and so forth in an infinite regression. For analytic purposes it is often convenient to treat slow processes as data and concentrate upon the processes of interest. For example, in a short run study of the level of investment, income, and employment, it is often convenient to assume that the stock of capital is perfectly or sensibly fixed.

As it follows from the above citation, Samuelson thinks about a hierarchy of various equilibrium processes with different speeds. In our model, it is sufficient to deal with only two levels of such a hierarchy. EF leaves the price adjustment process, which leads to the solution of the partial equilibrium problem (4), beyond the scope of the model. It can be shown, however, that this equilibrium will be reached at an exponential rate in the course of a naturally defined *tâtonnement* procedure. This can be demonstrated by using the contraction property of the operator

$$q_k = \alpha_{t,k} V_{t,k}^{-1} \sum_{i=1}^N \lambda_{t,k}^i \langle D_t + p, x_{t-1}^i \rangle$$

involved in the equilibrium pricing equation (4). Our framework makes it possible to admit a whole spectrum of mechanisms leading to an equilibrium in the short run. In reality, various auction-type mechanisms are used for the purpose of equilibrating bids and offers, resulting in market clearing. An analysis of several types of such mechanisms and their implications for the structure of trading in financial markets has been performed by Bottazzi et al. [25].

A rigorous mathematical treatment of the above multiscale approach, involving “rapid” and “slow” variables, is provided within continuous-time settings in the theory of *singular perturbations*, see e.g., Smith [97], Kevorkian and Cole [61]. In connection with economic modeling, questions of this kind are considered in detail in the monograph by Schlicht [90]. The equations on pp. 29–30 in Schlicht [90] are direct continuous-time (deterministic) counterparts of our equations (4) and (5).

The term "temporary equilibrium" was apparently coined for the first time by Marshall. However, in the last decades this term has been associated basically with a different, non-Marshallian notion, going back to Lindahl [66] and Hicks [54]. This notion was developed in formal settings by Grandmont, Hildenbrand and others [47, 48, 49]. The characteristic feature of the Lindahl-Hicks temporary equilibrium is the idea of *forecasts* or *beliefs* about the future states of the world, which the market participants possess and which are formalized in terms of stochastic kernels (transition functions) conditioning the distributions of future states of the world upon the agents' private information. A comprehensive discussion of this direction of research is provided by Magill and Quinzii [76]. In this work, we pursue a completely different approach. Our model might indirectly take into account agents' forecasts or beliefs, but they can be only implicitly reflected in the agents' investment strategies. We do not need to model in formal terms how the market players form, update and use these beliefs in their investment decisions.

2. In order to survive you have to win! One might think that the focus on survival substantially restricts the scope of the analysis, since "one should care about survival only if things go wrong". It turns out, however, that the class of survival strategies in most of the EF models coincides with the class of unbeatable strategies. A strategy is considered *unbeatable* if, in the long run, it performs no worse in terms of wealth accumulation than any other strategy competing in the market. More precisely, we define unbeatable strategies as follows:

Definition 4 *Suppose investor i employs unbeatable strategy Λ , while all other investors $j \neq i$ employ arbitrary strategies. Then, the wealth process w_t^j of investor j cannot grow asymptotically faster than that of investor i , w_t^i , i.e.,*

$$w_t^j \leq C w_t^i \quad (\text{a.s.}) \quad (70)$$

for each $t \geq 0$, where $C > 0$ is some random constant.

The proposition below demonstrates that survival strategies are those and only those that are unbeatable: in order to survive, you have to win!

Proposition 4 *A portfolio rule is a survival strategy if and only if it is unbeatable.*

Proof of Proposition 4. If the market share r_t^i of investor i satisfies $r_t^i = w_t^i/W_t \geq c$ (a.s.), where c is a strictly positive random variable and $W_t = \sum_{j=1}^N w_t^j$, then $w_t^i \geq c W_t \geq c w_t^j$ (a.s.) for all j . Thus $w_t^j \leq c^{-1} w_t^i$ (a.s.), and so (70) holds. Conversely, if $w_t^j \leq H w_t^i$ (a.s.) for some random constant $H > 0$, then $W_t \leq [(N-1)H + 1] w_t^i$ (a.s.), which yields $r_t^i \geq [(N-1)H + 1]^{-1}$ (a.s.), where $[(N-1)H + 1]^{-1}$ is a strictly positive constant. ■

Indeed, the notion of unbeatable strategies can be formalized as a game-theoretic solution concept, which was introduced in Amir et al. [6] and developed in Amir et al. [5].

Consider an abstract game of N players (in our case, investors) $i = 1, \dots, N$ selecting strategies Λ^i in some set $\mathcal{L}$. Let $w^i = w^i(\Lambda_1, \dots, \Lambda_N) \in W$ be the outcome of the game for player i given the strategy profile $(\Lambda_1, \Lambda_2, \dots, \Lambda_N)$ of the players. Suppose a preference relation

$$w^j \preceq w^i, \quad w^i, w^j \in W$$

is given comparing relative performance of players i and j . A strategy Λ of player i is called *unbeatable* if for any feasible strategy profile $(\Lambda_1, \dots, \Lambda_N)$ in which $\Lambda^i = \Lambda$, we have

$$w^j(\Lambda_1, \dots, \Lambda_N) \preceq w^i(\Lambda_1, \dots, \Lambda_N) \text{ for all } j \neq i.$$

Thus, if player i uses Λ , he/she cannot be outperformed by any other player $j \neq i$, no matter what strategies the rivals use.

In the context of the models considered in EF, the outcome of the game for investor i is the random wealth process $w^i = (w_t^i)$. The preference relation $\preceq$ is defined as follows. For two sequences of positive random numbers $w^i = (w_t^i)$ and $w^j = (w_t^j)$, we define

$$(w_t^j) \preceq (w_t^i) \text{ iff } w_t^j \leq C w_t^i \text{ (a.s.) for all } t = 1, 2, \dots$$

for some random variable $C > 0$. The relation $(w_t^j) \preceq (w_t^i)$ means that w_t^j does not grow asymptotically faster than w_t^i almost surely.

The fundamental solution concepts in evolutionary game theory, known as *evolutionary stable strategies* (ESS) as introduced by Maynard Smith and Price [79], Maynard Smith [78], and Schaffer [88], can be viewed as *conditionally* unbeatable strategies, under the condition that the number of mutants is sufficiently small or they are identical. Unconditional versions of the standard ESS were studied by Kojima [62].

3. Evolutionary portfolio theory. One of the sources of motivation for EF has always been related to quantitative applications of the results to portfolio selection problems. The data of EF models needed for quantitative financial analysis are essentially the same as those needed for the applications of the theory of derivative securities pricing (e.g., the Black-Scholes formula) in Mathematical Finance/Financial Engineering. They do not need the knowledge, or the algorithms for revealing, hidden agents' characteristics such as their utilities and beliefs. The model and the results are described in operational terms and require only statistical estimates of objectively observable asset data.

A crucial role in the applications of EF to portfolio selection is played by the discovery of investment factors that deliver returns in excess of the market. For example, Basu [17] found the so-called value factor, according to which investing into equities with a high book-to-market ratio delivers higher returns than the market. Banz [16] found that the same is true if one invests into equities with small market capitalization. Carhart [28] found the momentum factor according to which investing in equities that have recently gone up delivers

excess returns. Moreover even though by now hundreds of investment factors have been proposed, Harvey et al. [52] have shown that only a few factors are needed to understand the dynamics of equity returns. The current state of these discoveries is summarized in the Fama-French [43] five-factor model. According to these empirical results, the return of every portfolio selection strategy can be decomposed into its allocation to a few investment factors. Thus, it is natural to model the dynamics of equity markets by modeling the dynamic interaction of those investment factors. And this is what EF is perfectly suited for. In the EF framework, an investment factor defines a strategy determining the corresponding investment proportions. Note that investment factors are not based on individuals' utility functions and subjective probabilities! EF can then be used to compute what impact the increase in relative wealth corresponding to one factor has on any other factor. In particular, the impact of a factor on itself gives a model-based measure of the capacity of the factor. This is very practical information since investors should avoid being stuck in crowded strategies. Also, when a certain investment factor gets fashionable this has cross impacts on other factors that one can compute based on the EF model. For example, in recent years investing according to ESG (Environmental, Social, and Corporate Governance) criteria has become fashionable, and the EF approach shows that this has a strong negative impact on the momentum factor. Finally, based on this approach one can compute the dynamics of the relative wealth, so that one can use the EF model to determine which investment factors survive in the long run. A first paper systematically developing these ideas and opening up a new realm of fruitful applications of EF to portfolio selection problems was Hens et al. [53].

7 Conclusion

This paper reviews several key models with long-lived dividend-paying assets in the field of Evolutionary Finance (EF), which focuses on “survival and extinction” questions of investment portfolio rules in the market selection process. The models we examine, through a game-theoretic lens, bridge two fundamental paradigms: stochastic dynamic games (strategic frameworks) and evolutionary game theory (solution concepts).

The conventional theory of asset pricing currently prevailing in Financial Economics is based on the Walrasian equilibrium paradigm going back to Leon Walras, one of the classics of economic thought of the 19th century. Equilibrium models of this kind typically describe the world of small investors who strive to maximize their individual utilities of consumption subject to budget constraints. Market equilibrium is understood as a situation in which the goals and interests of such economic agents are equilibrated by the market clearing prices (see, e.g., Flåm [45]). In contrast with EF, where equilibrium is defined in short-run terms, consecutively from time t to time $t + 1$, in the classical setting one deals with a long-run notion of equilibrium defined for the whole time horizon.

EF depicts a world radically different from the Walrasian one — a world of

large, even super large (primarily institutional) investors who may act on the global level, and whose fundamental objectives are of an evolutionary character: e.g., survival, domination and fastest capital growth. In fact, fastest capital growth is often related, and in our models is equivalent, to survival (see Proposition 4). These factors, rather than the utilities of individual consumption (one gets immeasurably more than one can consume!) come to the fore. Within this framework, investment decisions made by each of the market players might substantially affect the equilibrium prices, in contrast with a variety of classical market models where the influence of every particular individual is negligible.

The primary goal of the studies is to identify investment strategies that guarantee “long-run survival”, i.e., keeping a strictly positive, bounded away from zero, share of market wealth over an infinite time horizon, irrespective of what strategies used by others. It turns out that there always exists a portfolio rule guaranteeing unconditional long-run survival. This rule is an analogue of the well-known Kelly criterion of “betting one’s beliefs” in the present framework and is asymptotically unique within a certain class of basic strategies. This survival strategy is defined as the solution to an easily computable stochastic equation under general assumptions; in more specialized models with stronger assumptions (e.g., Evstigneev et al. [37], Amir et al. [4], etc.), it reduces to an explicit form as the sum of a convergent series; and in models with even more restrictive assumptions (e.g., i.i.d. states of the world, fixed-mix strategies, etc.; see Evstigneev et al. [38], Bahsoun et al. [15], etc.), it further simplifies to a form expressed in terms of the expected relative dividends.

In particular, this paper surveys two stronger solution concepts — globally evolutionarily stable strategies and almost sure Nash equilibrium strategies — developed under stronger assumptions. It is shown that under the assumptions (i)–(vii) we listed in Section 4, if a group of investors adopts the Kelly rule while all the others use different ones, those and only those investors who belong to the former group survive, regardless of the initial state of the market; and under the assumptions (i)–(viii) we listed in Section 5, the Kelly rule is the unique investment strategy forming a symmetric Nash equilibrium almost surely, i.e., if all the investors except one (say, investor i) use the Kelly rule while i adopts any other different strategy, then the relative wealth of investor i converges to zero at an exponential rate almost surely.

Appendix

Proof of Lemma 7. We first observe that the function $F_\rho(\lambda, \kappa; s)$ satisfies

$$F_\rho(\lambda, \kappa; s) > c^2, \quad (71)$$

where $c := \min_k \lambda_k^* (> 0)$. Indeed, $F_\rho(\lambda, \kappa; s) = A/B$, where $A > c$, and

$$\rho'c < B < c^{-1}. \quad (72)$$

Case 1. Assume that at least one of the coordinates of λ is zero, so that $\mathbf{K} := \{k : \lambda_k = 0\} \neq \emptyset$. Then

$$A = (1 - \rho) \frac{1}{\kappa} \sum_{k \in \mathbf{K}} R_k(s) + \sum_{k \notin \mathbf{K}} [\rho \lambda_k + (1 - \rho) R_k(s)] \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}$$

and

$$B = \sum_{k \notin \mathbf{K}} [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}.$$

Since $ER_k(s_t) > 0$, there exists $\theta > 0$ such that $\sum_{k \in \mathbf{K}} R_k(s) \geq \theta$ for all s in a set $\bar{S}$ with $P(\bar{S}) > 0$. Therefore $d_1 \kappa^{-1} \mathbf{1}_{\bar{S}}(s) \leq A < \kappa^{-1} D_1$, where $\mathbf{1}_{\bar{S}}(s)$ is the indicator function of the set $\bar{S}$, $d_1 := (1 - \rho'')\theta$ and $D_1 := 1 + (\min_{k \notin \mathbf{K}} \lambda_k)^{-1}$. Also, we have $d_2 < B < D_2$, where $d_2 := \rho'c \min_{k \notin \mathbf{K}} \lambda_k$ and $D_2 := c^{-1}$ (see (72)). Thus

$$\kappa^{-1} d \mathbf{1}_{\bar{S}}(s) < F_\rho(\lambda, \kappa; s) < \kappa^{-1} D, \quad (73)$$

where $d := d_1/D_2$ and $D := D_1/d_2$. From the first of these inequalities and (71) we obtain $(\ln d - \ln \kappa) \mathbf{1}_{\bar{S}}(s) + (2 \ln c)(1 - \mathbf{1}_{\bar{S}}(s)) < \ln F_\rho(\lambda, \kappa; s)$, and so

$$E \min[H, \ln F_\rho(\lambda, \kappa; s)] \geq 2 \ln c + \min(H, \ln d - \ln \kappa) P(\bar{S}).$$

Define

$$\bar{\kappa} := \exp[\ln d - \frac{1 - 2 \ln c}{P(\bar{S})}], \quad H := \ln D - \ln \bar{\kappa}, \quad (74)$$

and observe that if $0 < \kappa < \bar{\kappa}$, then

$$E \min[H, \ln F_\rho(\lambda, \kappa; s)] \geq 2 \ln c + (\ln d - \ln \bar{\kappa}) P(\bar{S}) = 1 \quad (75)$$

by virtue of the inequality $d < D$. If $\kappa \geq \bar{\kappa}$, then $\ln F_\rho(\lambda, \kappa; s) < \ln D - \ln \kappa \leq \ln D - \ln \bar{\kappa} = H$, and so $\min[H, \ln F_\rho(\lambda, \kappa; s)] = \ln F_\rho(\lambda, \kappa; s)$. Thus in order to complete the proof of the lemma in the case when λ has zero coordinates it remains to show that

$$\inf_{\kappa \in [\bar{\kappa}, 1], \rho \in [\rho', \rho'']} E \ln F_\rho(\lambda, \kappa; s) > 0 \quad (76)$$

for each $\bar{\kappa} \in (0, 1]$. Indeed, then H can be defined by (74) and δ can be defined as the minimum of 1 (see (75)) and the infimum in (76).

By virtue of (71) and (73), the function $E \ln F_\rho(\lambda, \kappa; s)$ is continuous on the compact set $[\rho', \rho''] \times [\bar{\kappa}, 1]$ and hence it attains its minimum on this set. Thus, in order to establish (76) it is sufficient to prove that $E \ln F_\rho(\lambda, \kappa; s) > 0$ for each $\rho \in [0, 1)$ and $\kappa \in (0, 1]$.

By applying Jensen's inequality, we find

$$\begin{aligned} & E \ln \sum_{k=1}^K [\rho \lambda_k + (1 - \rho) R_k(s)] \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} \\ & \geq \rho \ln \sum_{k=1}^K \frac{\lambda_k \lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} + (1 - \rho) E \ln \sum_{k=1}^K R_k(s) \frac{\lambda_k^*}{\lambda_k^* \kappa + (1 - \kappa) \lambda_k} \\ & \geq \rho \ln \sum_{k=1}^K \frac{\lambda_k \lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} + (1 - \rho) \sum_{k=1}^K \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k^* \kappa + (1 - \kappa) \lambda_k}, \end{aligned} \quad (77)$$

and

$$\begin{aligned} & E \ln \sum_{k=1}^K [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} \\ & < \ln \left\{ \sum_{k=1}^K E [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} \right\} = \ln \sum_{k=1}^K \frac{\lambda_k \lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}. \end{aligned} \quad (78)$$

The inequality in (78) is strict because there is no constant τ such that

$$\sum_{k=1}^K [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} = \tau \text{ (a.s.)}. \quad (79)$$

Indeed, if (79) holds, then

$$\sum_{k=1}^K [\rho \lambda_k^* + (1 - \rho) R_k(s)] \varphi_k = 0 \text{ (a.s.)}, \quad (80)$$

where $\varphi_k := \lambda_k [\lambda_k^* \kappa + \lambda_k (1 - \kappa)]^{-1} - \tau$. Observe that at least one of the numbers φ_k is not equal to zero. Otherwise $\lambda_k = \tau [\lambda_k^* \kappa + \lambda_k (1 - \kappa)]$ for all k , and by summing these equalities over k , we get $\tau = 1$, which yields $\lambda_k = \lambda_k^* \kappa + \lambda_k (1 - \kappa)$, $\lambda_k^* \kappa = \lambda_k \kappa$, and $\lambda_k^* = \lambda_k$ (recall that $\kappa \neq 0$). This is a contradiction because $\lambda \neq \lambda^*$. Thus $\varphi = (\varphi_1, \dots, \varphi_K) \neq 0$, and $\sum_{k=1}^K R_k(s) \varphi_k = b$ (a.s.), where b is some constant. This constant is not zero because the functions $R_k(s)$ are linearly independent. By setting $\varphi'_k := \varphi_k / b$, we obtain that the non-zero vector $\varphi' = (\varphi'_1, \dots, \varphi'_K)$ satisfies $\sum_{k=1}^K R_k(s) \varphi'_k = 1$ (a.s.), which yields $\sum_{k=1}^K R_k(s) (\varphi'_k - 1) = 0$ (a.s.). In view of the linear independence of $R_k(s)$, this implies $\varphi'_1 = \dots = \varphi'_K = 1$. Since $\varphi_k = b \varphi'_k = b$, we obtain that the left-hand side of (80) is equal to $b \neq 0$, which is a contradiction.

From (77) and (78) we get

$$E \ln F_\rho(\lambda, \kappa; s) > (1 - \rho) \left[\sum_{k=1}^K \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k^* \kappa + (1 - \kappa) \lambda_k} - \ln \sum_{k=1}^K \frac{\lambda_k \lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} \right]. \quad (81)$$

Denote the expression in the square brackets in (81) by $\Phi_\kappa(\lambda)$. It is proved in Evstigneev et al. [36], Appendix, Lemma 3.1, that

$$\Phi_\kappa(\mu) \geq 0 \text{ for each } \kappa \in [0, 1] \quad (82)$$

for any $\mu = (\mu_k) \in \Delta^K$ with $\mu > 0$. Furthermore, if $\mu \neq \lambda^*$, then $\Phi_\kappa(\mu) > 0$. Therefore $\Phi_\kappa(\lambda(1 - \varepsilon) + \varepsilon \lambda^*) > 0$ for each $\varepsilon \in (0, 1)$. The function $\Phi_\kappa(\lambda)$ is finite and continuous on Δ^K (because $\lambda_k^* > 0$ and $\kappa > 0$). Consequently, $\Phi_\kappa(\lambda) = \lim_{\varepsilon \downarrow 0} \Phi_\kappa(\lambda(1 - \varepsilon) + \varepsilon \lambda^*) \geq 0$. By using (81), we obtain that $E \ln F_\rho(\lambda, \kappa) > 0$ for all $\rho \in [0, 1)$ and $\kappa \in (0, 1]$. This completes the proof of the lemma in the case when the vector λ has zero coordinates.

Case 2. Now assume that $\lambda_k > 0$ for each k . Then the function $\ln F_\rho(\lambda, \kappa; s)$ is well-defined, finite, continuous with respect to (ρ, κ) on the set $[\rho', \rho''] \times [0, 1]$ (including $\kappa = 0$) and uniformly bounded:

$$2 \ln c < \ln F_\rho(\lambda, \kappa; s) < \ln(\min_k \lambda_k)^{-2}.$$

To complete the proof it is sufficient to show that the infimum in (76) with $\bar{\kappa} = 0$ is strictly positive (then δ can be defined as this infimum and H as $2|\ln c| + 2|\ln \min_k \lambda_k|$). In view of the continuity of $E \ln F_\rho(\lambda, \kappa; s)$ this will be proved if we establish the inequality $E \ln F_\rho(\lambda, \kappa; s) > 0$ for each $\rho \in [0, 1)$ and $\kappa \in [0, 1]$. If $\kappa > 0$, this inequality is proved by exactly the same arguments as above — by deriving relations (77), (78), (81) and using (82). If $\kappa = 0$, we change the above arguments as follows: instead of strict, we establish non-strict inequalities in (78) and show that the right-hand side of (81) is strictly positive, because $\Phi_0(\lambda) = \sum_{k=1}^K \lambda_k^* \ln(\lambda_k^* / \lambda_k) > 0$. ■

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²⁹See Cover [31].