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Abstract

Long-run risk models, a cornerstone in the macro-finance literature for their ability to capture key asset price phenomena, are known to entail implausibly high levels of timing and risk premia. Our paper resolves this puzzle by considering consumption of durable goods in addition to that of non-durable goods. In our estimated model, the timing premium is 11 percent and the risk premium is 16 percent of lifetime consumption. These values are about a third of the previously implied premia and are more consistent with empirical and experimental evidence.

JEL classification: C11, E21, G11, G12.

Keywords: Durable Goods, Long-Run Risk Models, Particle Filtering, Timing and Risk Premium.

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1 Introduction

The Long-Run Risk Model (LRRM), introduced by Bansal and Yaron (2004), is one of the main theoretical pillars in financial macroeconomics. In its original version, the LRRM reconciles several key asset pricing phenomena in a unified framework by combining recursive preferences à la Epstein and Zin (1989) with a model of aggregate consumption growth that exhibits predictable low-frequency movements and time-varying volatility. Despite its success, the LRRM suffers from a quantitative drawback similar to Mehra and Prescott (1985)'s equity premium puzzle.

When calibrated to financial and macroeconomic data, the LRRM implies unrealistically high levels of timing and risk premia, see Epstein, Farhi and Strzalecki (2014). A representative household with recursive preferences, a relative risk aversion of 7.5, and an elasticity of intertemporal substitution of 1.5 would give up around one quarter of lifetime consumption to resolve uncertainty one month earlier, and around half of lifetime consumption to live in a world without consumption risk. Both of these amounts are difficult to reconcile with microeconomic evidence or introspection.

This paper introduces in the standard LRRM consumption of durable goods alongside the consumption of non-durable goods. The main message of our study is that this simple modification can reduce by about two-thirds the timing and risk premia, without compromising (and actually improving) the model's ability to match standard macroeconomic and financial moments. In our benchmark estimation exercise, the extended LRRM can rationalise key asset pricing facts, and deliver a timing premium of 11 percent and a cost of eliminating all consumption uncertainty of 16 percent of lifetime consumption.

Regarding the cost of eliminating total consumption risk, our results are consistent with the empirical evidence provided by Alvarez and Jermann (2004), who put the cost of eliminating consumption risk at around 16 percent of lifetime consumption. In connection to the timing premium, the empirical evidence presented in Schlag, Thimme and Weber (2017) imply a value of 7 percent, while the experimental study of Meissner and Pfeiffer (2018) finds an average timing premium of around 5 percent of lifetime consumption. The timing premium implied by our model is larger, but much closer to the empirical and experimental findings than the original LRRM with only non-durable consumption.

The main driver behind our results is that durable goods yield utility over several periods as their service flow spans over a relatively long time horizon, see for instance Browning and Crossley (2009). In bad times households can cut their expenditure on durable goods, while benefiting from the service flow that their stock of owned durables provides. Therefore, durable consumption supplies partial insurance against future uncertainty, potentially mitigating the timing and risk premia.

Durable consumption is also known to improve substantially the quantitative performance of consumption-based asset pricing models. Yogo (2006) finds that including durable consumption in the standard CCAPM can explain the cross-sectional variation in expected stock returns as well as the time variation in the equity premium. Gomes, Kogan and Yogo (2009) show that durability of output is reflected in stock prices and accounts for differences in risk premia between durable goods producers and service providers. Yang (2011) emphasises the importance of long-run risk in durable consumption risk in understanding asset price phenomena such as pro-cyclical dividend yields, counter-cyclical equity premia and stock return predictability. Eraker, Shaliastovich and Wang (2015) find that the LRRM with durable goods and inflation risk can explain the correlation between expected inflation and future real growth.

Durable consumption makes up a substantial part of household expenditure. According to personal consumption data from the US National Income and Product Accounts, in the past three decades households spent three dollars on durable consumption for each dollar spent on non-durable consumption. Over the last 70 years, on average twice as much was spent on durable than on non-durable consumption.

Conducting the quantitative analysis of our model poses several challenges. First, rather than calibrating the endowment processes for consumption and asset prices, we use a data-driven approach that estimates these processes in a non-linear fashion with a sequential Monte Carlo particle filter as in Schorfheide, Song and Yaron (2018). This approach considerably complicates the evaluation of the likelihood function as well as the implementation of Bayesian inference. However it allows to be less restrictive about the role of the time-varying volatilities in the endowment processes as well as in the long-run components.

Second, we solve and estimate the full non-linear LRRM in the spirit of Chen, Favilukis and Ludvigson (2013). This is particularly challenging from a numerical point of view, due to the presence of durable consumption acting as an extra endogenous state variable. However this technique permits to recover important non-linearities of the LRRM. This is crucial as using Campbell/Schiller linearisation methods can lead to wrong model predictions, see Pohl, Schmedders and Wilms (2018). To the best of our knowledge, our quantitative analysis is the first one where non-linear solution and estimation techniques are applied jointly to the endowment processes and to the LRRM.

The estimated model provides a good fit of the data. The representative household has a risk aversion of 1.86 and its elasticity of intertemporal substitution is 1.18. Crucially, and in line with the existent evidence, e.g. Yogo (2006), we find that durable and non-durable consumption goods are gross complements. We also find that the predictable component of durable consumption growth is more persistent than the predictable component of non-durable consumption growth, as in Yang (2011) and Eraker, Shaliastovich and Wang (2015). Finally, we show that the volatilities of both durable and non-durable long-run components are time-varying and have a strong impact on dividend growth. These results are interesting on their own as they provide further empirical evidence that durable and non-durable consumption do not follow random walk processes. Simulation of the model reveals a mean equity premium of 5.78 percent and an average return volatility of 17 percent. The mean risk-free rate is 1.01 percent. The main achievement of the model however is that these values are obtained with a timing premium of 11 percent and a risk premium of 16 percent.

Related to this paper, Andries, Eisenbach and Schmalz (2018) address the same shortcoming of LRRM studied here. They show that an economy where households have horizon-dependent risk aversion can mitigate (or even reverse) the implied preference for early resolution of uncertainty, thus reducing the term and risk premia of LRRM. This alternative explanation can be seen as complementary to ours based instead on durable consumption. It would be interesting to combine these two approaches in a unified framework but we leave this to future research.

The remainder of the paper is organised as follows. Section 2 presents a LRRM with durable and non-durable consumption. Section 3 presents the estimation of the endowment processes and the LRRM as well as the results. Section 4 concludes.

2 The Model

We consider an infinite-horizon, discrete-time endowment economy à la Lucas (1978) in which in every period t a representative household has rational expectations and derives utility from a bundle of non-durable and durable consumption represented by a Constant Elasticity of Substitution (CES) function

$$u(C_t, D_t) = \left((1-\alpha)C_t^{\frac{\rho-1}{\rho}} + \alpha D_t^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}}. \tag{1}$$

C_t is the non-durable consumption good that is non-storable and is entirely consumed in period t, D_t is the service flow from durable consumption goods, $\alpha \in [0,1]$ is the relative importance of durable consumption, and ρ is the elasticity of substitution between non-durable and durable consumption. When $\rho = 1$, equation (1) collapses to the familiar Cobb-Douglas case, while for $\rho < 1$ ($\rho > 1$) durable and non-durable consumption goods are gross complements (substitutes). As in Yogo (2006), Lustig and Verdelhan (2007) and subsequent contributions, we assume that the service flow from durable consumption good is proportional to the stock of durable goods which evolves according to the law of motion

$$D_t = (1-\delta)D_{t-1} + E_t$$

where $\delta \in (0,1)$ is the depreciation rate and E_t is the expenditure on durable consumption.

The utility function of the household is recursive as in Epstein and Zin (1989, 1991) (see also Kreps and Porteus, 1978 and Weil, 1989), i.e.

$$\mathcal{U}_t = \left\{(1-\beta)u(C_t, D_t)^{\frac{1-\gamma}{\theta}} + \beta\left(\mathbb{E}_t[\mathcal{U}_{t+1}^{1-\gamma}]^{\frac{1}{\theta}}\right)\right\}^{\frac{\theta}{1-\gamma}}. \tag{2}$$

The parameters of the household's utility function are the subjective discount factor $\beta \in (0,1)$, the relative risk aversion coefficient $\gamma > 0$, and the elasticity of intertemporal substitution $\psi \geq 0$ with $\theta \equiv (1-\gamma)/(1-\frac{1}{\psi})$. Recall that the household with utility function in (2) is averse to volatility in future utility, i.e. prefers early resolution of risk, if $\gamma > \psi$, whereas the household loves volatility in future utility, i.e. prefers late resolution of risk, in the opposite case where $\gamma < \psi$. Thus, when $\gamma > \psi$, recursive utility implies a curvature with respect to future risks, a feature that is essential for matching asset-pricing facts.[1]

[1]Note that when $\theta = 1$, i.e. when $\gamma = 1/\psi$, the recursive preferences collapse to a standard Constant Relative Risk Aversion (CRRA) expected utility.

In our endowment economy there are four assets: a non-durable consumption good, a durable consumption good, a stock (in positive net supply), and a risk-free discount bond (in zero net supply). In each period t, the household chooses the level of consumption (both non-durable and durable) and asset holdings to maximize (2) subject to its budget constraint

$$C_t + P_t E_t + B_{b,t} + B_{s,t} = B_{b,t-1} R_{b,t} + B_{s,t-1} R_{s,t} \tag{3}$$

where P_t is the relative price of durable goods in terms of non-durable goods, $B_{b,t}$ is the t-period risk-free bond holdings, $B_{s,t}$ is the t-period stock holdings, $R_{b,t}$ is the return on risk-free bond, and $R_{s,t}$ is the return on stock.

In each period t, a non-durable good C_t, a durable good D_t, and a dividend from stock S_t arrive. As originally introduced by Bansal and Yaron (2004), the growth rate of non-durable consumption, $\Delta C_{t+1} = \log(C_{t+1}/C_t)$, contains a small persistent predictable component x_t,

$$\begin{aligned} \Delta C_{t+1} &= \mu_c + x_t + \sigma_t \varepsilon^c_{t+1} \\ x_{t+1} &= \rho_x x_t + \psi_x \sigma_t \varepsilon^x_{t+1} \end{aligned} \tag{4}$$

where μ_c is the unconditional mean of non-durable consumption growth, ρ_x is the persistence of the predictable component and ψ_x is the loading on the (time-varying) volatility of x_t. As in Eraker, Shaliastovich and Wang (2015), the growth rate of durable consumption, $\Delta D_{t+1} = \log(D_{t+1}/D_t)$, also contains a small persistent predictable component y_t (potentially different from x_t),

$$\begin{aligned} \Delta D_{t+1} &= \mu_d + y_t + \psi_d \sigma_t \varepsilon^d_{t+1} \\ y_{t+1} &= \rho_y y_t + \psi_y \sigma_t \varepsilon^y_{t+1} \end{aligned} \tag{5}$$

where μ_d, ρ_y and ψ_y are defined analogously to (4) but for durable consumption growth. Dividend growth, $\Delta S_{t+1} = \log(S_{t+1}/S_t)$, is exposed to low frequency risks in the aggregate economy, x_t and y_t, and to high frequency shocks from ΔC_{t+1} and ΔD_{t+1},

$$\Delta S_{t+1} = \mu_s + \phi_x x_t + \phi_y y_t + \pi_c \sigma_t \varepsilon^c_{t+1} + \pi_d \sigma_t \varepsilon^d_{t+1} + \psi_s \sigma_t \varepsilon^s_{t+1} \tag{6}$$

where ϕ_x and ϕ_y allow controlling for the correlation of stocks with both non-durable and durable consumption growth. All shock components have a time-varying term, σ_t, whose conditional volatility evolves according to

$$\begin{aligned}\sigma_t &= \bar{\sigma}\exp(h_t)\\ h_{t+1} &= \rho_h h_t + \sigma_h\sqrt{1-\rho_h^2}\,\varepsilon_{t+1}^h\end{aligned} \tag{7}$$

with $\bar{\sigma}$ the unconditional mean of the standard deviation σ_t, ρ_h the persistence parameter of the residual component h_t, and σ_h the (constant) standard deviation of the shock to σ_t. Finally, shocks ε_{t+1}^c, ε_{t+1}^d, ε_{t+1}^x, ε_{t+1}^y, ε_{t+1}^s and ε_{t+1}^h are i.i.d., $\mathcal{N}(0,1)$ and mutually independent.

The solution of the model is characterized by first-order conditions that will be used in the empirical analysis. Let W_t denote the period-t wealth of the household given by

$$W_t = C_t + P_t E_t + B_{b,t} + B_{s,t}$$

while W_{t+1} is given by

$$W_{t+1} = B_{b,t}R_{b,t+1} + B_{s,t}R_{s,t+1}.$$

Total wealth of the household $\widetilde{W}_t$ is defined as the sum of his current wealth and the value of the stock of durable goods

$$\widetilde{W}_t = W_t + (1-\delta)P_t D_{t-1}.$$

Treating the durable consumption good as an asset, the holdings and the return on the durable consumption good are defined as

$$B_{d,t} = P_t D_t, \qquad R_{d,t+1} = (1-\delta)P_{t+1}/P_t.$$

Denoting the share of wealth net of non-durable consumption invested in asset i by

$$\omega_{i,t} = B_{i,t}/(\widetilde{W}_t - C_t)$$

the household's budget constraint can be written in recursive form:

$$\begin{aligned}\widetilde{W}_{t+1} &= (\widetilde{W}_t - C_t)\left(\omega_{b,t}R_{b,t+1} + \omega_{s,t}R_{s,t+1} + \omega_{d,t}R_{d,t+1}\right)\\ \omega_{b,t} &+ \omega_{s,t} + \omega_{d,t} = 1.\end{aligned} \tag{8}$$

The consumption-portfolio choice problem of the household can be expressed as follows. Given current total wealth $\widetilde{W}_t$, it chooses consumption C_t and investment shares $\omega_{b,t}$, $\omega_{s,t}$

and $\omega_{d,t}$ to maximize utility (2) subject to the budget constraint (8). The Bellman equation for the period-t value function of this optimization problem can be written as

$$\mathcal{J}_t(\widetilde{W}_t) = \max_{\{C_t,\omega_{b,t},\omega_{s,t},\omega_{d,t}\}} \left\{ (1-\beta)u(C_t,D_t)^{\frac{1-\gamma}{\theta}} + \beta \left[\mathbb{E}_t \left(\mathcal{J}_{t+1}(\widetilde{W}_{t+1}) \right)^{1-\gamma} \right]^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}. \qquad (9)$$

The solution to this maximization problem yields to two optimality conditions. First, in any given period, the marginal rate of substitution between durable and non-durable consumption good equals their relative prices, i.e.

$$\frac{u_{D,t}}{u_{C,t}} = P_t - (1-\delta)\mathbb{E}_t\left[M_{t+1}P_{t+1}\right] = Q_t \qquad (10)$$

where M_{t+1} is the stochastic discount factor between period t and $t+1$, and Q_t is the user cost of the service flow for the durable good. Second, the intertemporal marginal rate of substitution (IMRS) between any two adjacent periods has to satisfy

$$M_{t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\theta/\psi} \left(\frac{v(D_{t+1}/C_{t+1})}{v(D_t/C_t)} \right)^{\theta(1/\rho-1/\psi)} R_{W,t+1}^{\theta-1} \qquad (11)$$

where the function $v(D_t/C_t)$ is defined as

$$v\left(D_t/C_t\right) = \left[1-\alpha+\alpha\left(D_t/C_t\right)^{1-1/\rho} \right]^{1/(1-1/\rho)} \qquad (12)$$

and $R_{W,t+1} = \widetilde{W}_{t+1}/(\widetilde{W}_t - C_t - Q_t D_t)$ is the return on total consumption, which captures the return on the portfolio. Recall that in the one-good economy ($\alpha = 0$) of Bansal and Yaron (2004), equation (11) reduces to

$$M_{t+1}^{\text{non-durable}} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\theta/\psi} R_{W,t+1}^{\theta-1}. \qquad (13)$$

In contrast to the non-durable consumption case, our model incorporates movements in the relative share of durable and non-durable goods (11) and adds the durable consumption good to the household's portfolio (9). As we will show in detail later, these two ingredients are crucial for the performance of our model.

First-order conditions on non-durable consumption and portfolio choice imply (analogously to the derivation in Epstein and Zin (1989, 1991)) that the return on any tradable asset (risk-free bond b and stock s) in the economy satisfies the Euler equation, i.e.

$$\mathbb{E}_t\left[M_{t+1}R_{i,t+1}\right] = 1, \qquad i \in \{b,s\}. \qquad (14)$$

Similarly, first-order conditions on optimal durable consumption choice imply

$$\mathbb{E}_t\left[M_{t+1}(R_{b,t+1}-R_{d,t+1})\right]=\frac{u_{D,t}}{P_t\,u_{C,t}}. \tag{15}$$

As the Euler equation does not admit an analytical solution, we rely on numerical methods to solve for the asset prices, see Appendix A.2 for a detailed description of our solution algorithm for both the linear and the non-linear case.

2.1 Timing and Risk Premia

Before starting the quantitative assessment of the model, it is useful to define the timing and risk premia, see also Epstein, Farhi and Strzalecki (2014).

Definition of timing and risk premia. Suppose a consumer facing the endowment process described in Section 2, with $t=0,1,2,\ldots$, where consumption and dividend risk is resolved gradually over time (C_t, D_t, S_t, x_t and y_t are realized at time t only). Consider an alternative process in which all the risk is resolved in period 1. The consumer is allowed to chose the alternative endowment process over the original one at the cost of giving up a fraction π of consumption today and in all subsequent periods. The maximum value π^* for which the consumer is willing to accept this offer is defined as the *timing premium*. Formally, let $\mathcal{U}_0$ be the utility with the original endowment process and $\mathcal{U}_0^*$ the utility of the alternative endowment process in which all risk is resolved at time 1. Then, π^* can be written as

$$\pi^*=1-\frac{\mathcal{U}_0}{\mathcal{U}_0^*}.$$

Now, consider another alternative endowment process, in which risk is resolved entirely, and the consumption and dividend processes are deterministic. The maximum fraction of current and future consumption $\bar{\pi}$ which a consumer is willing to give up in favour of this deterministic process is the *risk premium* and is formally defined as

$$\bar{\pi}=1-\frac{\mathcal{U}_0}{\bar{\mathcal{U}}_0}$$

where $\bar{\mathcal{U}}_0$ is the utility associated with the deterministic endowment process.

Calculating timing and risk premia. We rely on numerical methods to calculate the value of $\mathcal{U}_0$. The value function $\mathcal{U}_0(C,D,x,y,\sigma^2)$ is the solution for the recursive functional equation

$$\mathcal{U}_t=\left\{(1-\beta)u(C_t,D_t)^{\frac{1-\gamma}{\theta}}+\beta\left(\mathbb{E}_t[\mathcal{U}_{t+1}^{1-\gamma}]^{\frac{1}{\theta}}\right)\right\}^{\frac{\theta}{1-\gamma}}.$$

Noting that value function $\mathcal{U}$ can be rewritten as $\mathcal{U}(C, D, x, y, \sigma^2) = C\,\mathcal{H}(z, x, y, \sigma^2)$, where $z = D/C$, the functional equation above is

$$\mathcal{H}_t(z_t, x_t, y_t, \sigma_t^2) = \\ \left\{ (1-\beta)\tilde{u}(z_t)^{\frac{1-\gamma}{\theta}} + \beta e^{\left(1-\frac{1}{\psi}\right)\left(\mu_c + x_t + \frac{\theta}{2}\sigma_t^2\right)} \left(\mathbb{E}_t[\mathcal{H}_{t+1}^{1-\gamma}(z_{t+1}, x_{t+1}, y_{t+1}, \sigma_{t+1}^2)]^{\frac{1}{\theta}} \right) \right\}^{\frac{\theta}{1-\gamma}}$$

where $\tilde{u}(z_t) = \tilde{u}(D_t/C_t) = u(C_t, D_t)/C_t$. We approximate $\mathcal{H}$ by Chebyshev polynomials and solve the functional equation using orthogonal collocation method, see Pohl, Schmedders and Wilms (2018) and references therein. The expectation is approximated by Gauss-Hermite quadrature. We then run Monte-Carlo simulations with a fixed time horizon T and pass $\mathcal{U}_0$ as the continuation value at time T. This numerical procedure allows to calculate accurately both $\mathcal{U}_0^*$ and $\bar{\mathcal{U}}_0$.

3 Empirical Analysis

Data. The sample period of our data is 1952:Q1–2014:Q4. Personal consumption data is from the US National Income and Product Accounts Bureau of Economic Analysis (BEA). We measure non-durable consumption as the sum of personal consumption expenditures on non-durable goods and services. This measure includes for instance food, clothing items, housing and utilities, health care services, transportation.

Durable consumption includes for instance motor vehicles and parts, furnishings and durable household equipment, recreational goods and services, jewellery and watches. Since the BEA reports only annual series for consumers stock of durable goods, we interpolate the quarterly series by assuming that the depreciation rate is constant within year, such that the implied value of the depreciation rate is consistent with annual stocks of durable goods both at the beginning and at the end of the year, and with quarterly series of personal consumption expenditure (PCE) on durable goods. This is standard procedure, e.g. Yogo (2006).

Figure 1 plots the durable consumption as a ratio of non-durable consumption (black solid line) from 1952:Q1 to 2014:Q4. The time series exhibits an upward trend during the sample period, with the value of durable consumption relative to non-durable consumption in 2014:Q4 being about 3.5 larger than corresponding value in 1952:Q1. The upward trend in the series is also consistent with the downward trend in price of durable goods relative to non-durable goods (red dashed line in Figure 1).

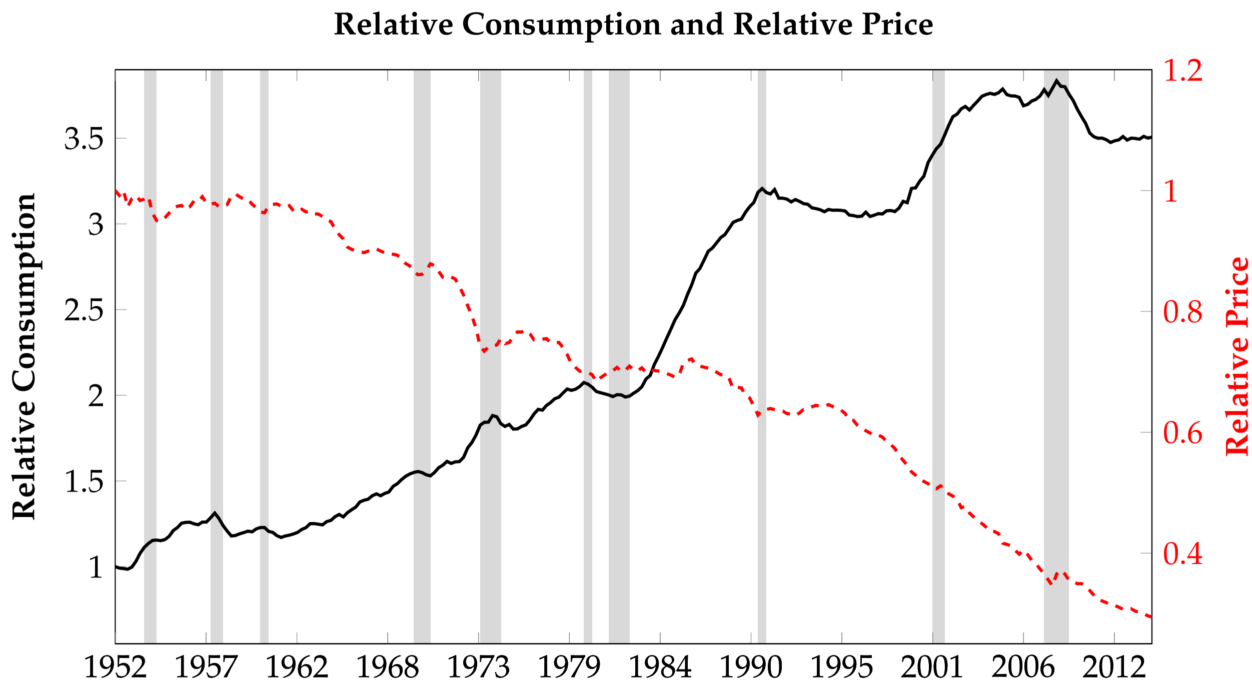

Figure 1: **Relative Consumption and Price.** Time series plot of durable consumption as a ratio of non-durable consumption (black solid line), and relative price of durable to non-durable consumption (red dashed line). The sample period from 1952:Q1 to 2014:Q4, 1952:Q1 values are normalized to 1. The shaded areas indicate NBER recessions.

US Population data are retrieved from the Federal Reserve Bank of St. Louis to obtain the per-capita quantities. The returns on the stock market and the short-term interest rate are from the Center for Research in Security Prices (CRSP). All asset returns are deflated with the PCE price index for non-durable consumption. The real dividend series are from Robert Shiller's website. We construct the ex-ante real risk-free as a fitted value from a projection of ex post real rate on the current nominal yield and inflation over the previous year (nominal yield is the Fama Risk Free Rate and inflation is the CPI rate, both available from CRSP).

3.1 Quantitative Assessment

State-Space Representation and Bayesian Inference. The non-linear state-space system consists of a measurement and a transition equation, determined by (4)–(7). The measurement equation can be written as

$$y_{t+1} = M + Zs_{t+1} \tag{16}$$

with

$$y_{t+1} = \begin{pmatrix} \Delta C_{t+1} \\ \Delta D_{t+1} \\ \Delta S_{t+1} \end{pmatrix}, \; s_{t+1} = \begin{pmatrix} x_t \\ y_t \\ \sigma_t \varepsilon^c_{t+1} \\ \sigma_t \varepsilon^d_{t+1} \\ \sigma_t \varepsilon^s_{t+1} \end{pmatrix}, \; M = \begin{pmatrix} \mu_c \\ \mu_d \\ \mu_s \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \psi_d & 0 \\ \phi_x & \phi_y & \pi_c & \pi_d & \psi_s \end{pmatrix}$$

while the transition equation can be written as

$$\begin{aligned} s_{t+1} &= \Phi s_t + v_{t+1}(h_t) \\ h_{t+1} &= \Psi h_t + \Sigma_h \varepsilon^h_{t+1} \end{aligned} \tag{17}$$

where

$$v_{t+1}(h_t) = \begin{pmatrix} \psi_x \sigma_t \varepsilon^x_{t+1} \\ \psi_y \sigma_t \varepsilon^y_{t+1} \\ \sigma_t \varepsilon^c_{t+1} \\ \sigma_t \varepsilon^d_{t+1} \\ \sigma_t \varepsilon^s_{t+1} \end{pmatrix}, \quad \Phi = \begin{pmatrix} \rho_x & 0 & 0 & 0 & 0 \\ 0 & \rho_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi = \rho_h, \quad \Sigma_h = \sigma_h \sqrt{1 - \rho_h^2}.$$

In order to estimate the parameter vector Θ, with

$$\Theta = \left(\rho_x, \psi_x, \psi_d, \rho_y, \psi_y, \phi_y, \pi_c, \pi_d, \psi_s, \rho_h, \sigma_h\right),$$

we extend the Bayesian, non-linear approach of Schorfheide, Song and Yaron (2018).

This method is conducted as follows. In order to generate draws from the posterior distribution of Θ, $\mathbb{P}(\Theta|Y)$, we specify the prior distribution $\mathbb{P}(\Theta)$ and evaluate numerically the likelihood function $\mathbb{P}(Y|\Theta)$. As the volatility processes affect the conditional mean and the volatility of asset prices, we have to carry out a non-linear estimation of the state space model. Fortunately, one can avoid applying a fully non-linear filter as, conditional on the volatility state h_t, the state-space model can be recast in linear form and as such is Gaussian.

This approximation can be implemented by using a computationally efficient particle filter, where the particle values of the volatilities s_t are replaced by the mean and covariance matrix of the conditional distribution $(s_t|(h_t, Y_{1:t}))$. This latest statistic is obtained by exploiting the Gaussian nature of the state-space system, with linear Kalman filtering.

Then we insert the resulting approximation of the likelihood function, i.e. $\hat{\mathbb{P}}(Y|\Theta)$, into a standard Metropolis-Hastings algorithm. This allows to generate the posterior distribution of the parameter vector Θ, i.e. $\mathbb{P}(\Theta|Y)$.[2]

Parameter Estimates. For the estimation, we use uninformative uniform priors for the majority of our parameters. The only exception is represented by σ_h^2, i.e. the volatility of the volatility parameter, where we impose instead an Inverse-Gamma distribution. For the persistence coefficients we also choose dispersed priors: the 90 percent credible interval for ρ_x and ρ_y ranges from 0.71 to 0.99. These numbers cover the values reported in Bansal and Yaron (2004); Bansal, Kiku and Yaron (2009); Yang (2011); Schorfheide, Song and Yaron (2018); Eraker, Shaliastovich and Wang (2015).

Table 1 reports the 90 percent credible intervals for the priors of the parameters as well as the percentiles of the posterior distribution for the estimated parameters.

Table 1: Estimated Coefficients of the Endowment Process.

Parameter	**Distribution**	**Prior**		**Posterior**		
		5%	95%	5%	50%	95%
ρ_x	Uniform	0.71	0.99	0.79	0.85	0.91
ψ_x	Uniform	0.05	9.95	0.25	0.31	0.38
ψ_d	Uniform	0.05	9.95	0.01	0.05	0.13
ρ_y	Uniform	0.71	0.99	0.88	0.91	0.95
ψ_y	Uniform	0.05	9.95	0.60	0.69	0.79
ϕ_y	Uniform	1	19	0.40	0.69	0.80
π_c	Uniform	1	19	0.00	0.03	0.12
π_d	Uniform	1	19	0.08	0.60	1.22
ψ_s	Uniform	0.05	9.95	0.13	0.83	1.26
ρ_h	Uniform	0.90	0.99	0.93	0.96	0.98
σ_h^2	Inverse-Gamma	0.06	0.37	0.24	0.40	0.67

Estimation results are based on quarterly consumption and dividend data from 1952:Q1 to 2014:Q4. Parameter values $\mu_c = 0.0049, \mu_d = 0.0083$ and $\mu_s = 0.0016$ are set at their sample averages, further $\hat{\sigma} = 0.0096$ and $\phi_x = 4$.

The posterior estimates of the persistence of the long-run components are $\rho_x = 0.85$ and $\rho_y = 0.91$. The long-run component of the durable good is more persistent but also more volatile ($\psi_y = 0.69$) than that of the non-durable good ($\psi_x = 0.31$), in line with Eraker, Shaliastovich and Wang (2015). Moreover, dividends depend on both non-durable and

[2] Andrieu, Doucet and Holenstein (2010) show that use of the approximation of the likelihood function as implemented in this paper delivers draws from the true posterior distribution.

durable long-run components, with the larger effect coming from the non-durable long-run component (setting $\phi_x = 4$, we estimate $\phi_y = 0.69$, similar to Yang, 2011). We also find a non-zero loading on dividend growth from the noise to non-durable and durable consumption, with π_c and π_d both being strictly positive. Finally, the volatility process is highly persistent, with $\rho_h = 0.96$.

3.1.1 Estimating the Elasticity of Substitution

Equation (10) allows estimating the elasticity of substitution ρ directly from the data. Taking logarithms of equation (10) we get

$$\log\left(\frac{\alpha}{1-\alpha}\right) + \frac{1}{\rho}(c_t - d_t) - p_t = q_t - p_t$$

where a lower case variable denotes the logarithm of the corresponding uppercase variable. Assuming that the user cost and the spot price of durable goods are cointegrated (so that $q_t - p_t$ is stationary) implies that $c_t - d_t$ and p_t are cointegrated with the cointegrating vector equal to $(1, -\rho)$. Hence, we can estimate the elasticity of substitution without observing the user cost of durable goods (see Ogaki and Reinhart, 1998, where ρ is estimated by regressing $c_t - d_t$ on p_t). We estimate the elasticity of substitution by a dynamic ordinary least square regression of $c_t - d_t$ on p_t with four leads and lags, as proposed by Stock and Watson (1993):

$$c_t - d_t = const. + \rho\, p_t + \sum_{s=-4}^{4} b_{p,s} \Delta p_{t-s} + \varepsilon_t.$$

For the full sample 1952:Q1 - 2014:Q4, we estimate $\rho = 0.78$ with standard error of 0.03. We test the null hypothesis of no composition $H_0 : \rho = 1$. The t-statistics is $t = -6.85$ and thus we reject the hypothesis of no composition at 1 percent significance level.

3.1.2 Estimating the Linear Model

As a starting point we derive and estimate a linearised version of the model. This is instructive for three reasons. First, the linear model admits an analytical solution of the equilibrium dynamics. Second, it allows to compare the estimates of our model with a large part of the LRR literature that studies linear models. Third, it enables us to assess the relative importance of non-linearities in our class of models.[3]

[3] In a recent contribution, Pohl, Schmedders and Wilms (2018) show the importance of non-linearities for inference in LRR models. Here we extend that result in LRR models with durable consumption.

Technically, we derive an analytical solution through a linear approximation of the conditional volatility process (7). Furthermore, we assume that the volatility is given by a process that has a Gaussian distribution, i.e.

$$
\begin{aligned}
\sigma_{t+1}^2 &\approx \bar{\sigma}^2(1-\rho_h) + \rho_h \sigma_t^2 + 2\bar{\sigma}^2 \sigma_h \sqrt{1-\rho_h^2}\, w_{t+1} \\
&= \hat{\sigma} + \rho_h \sigma_t^2 + \sigma_w w_{t+1}.
\end{aligned}
$$

Next, we derive the asset prices using the standard asset pricing conditions

$$
\mathbb{E}_t \left[\exp(m_{t+1} + r_{i,t+1}) \right] = 1
$$

for any asset return $r_{i,t+1} = \log\left(R_{i,t+1}\right)$. Recall that the log-pricing kernel of the economy can be written as

$$
m_{t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \left(\frac{1}{\rho} - \frac{1}{\psi} \right) \Delta f_{t+1} + (\theta - 1)\, r_{w,t+1}
$$

where $r_{w,t+1}$ is the log return on the consumption claim and $r_{m,t+1}$ is log market return. Then, we use the standard approximation of Campbell and Shiller (1988) for the returns, i.e.

$$
\begin{aligned}
r_{w,t+1} &= z_{w,t+1} - \kappa_0 - \kappa_1 z_{w,t} - z_t + \Delta c_{t+1} \\
r_{m,t+1} &= \kappa_0^m + \kappa_1^m z_{m,t+1} - z_{m,t} + \Delta s_{t+1}
\end{aligned}
$$

where $z_t = \log\left(D_t / C_t\right)$, $z_{w,t}$ is the log wealth-consumption ratio, and $z_{m,t}$ is the log price-dividend ratio. This yields to a solution for the log wealth-consumption ratio and the log price-dividend ratio that is linear in state variables, i.e.

$$
\begin{aligned}
z_{w,t} &= A_0 + A_1 x_t + A_2 y_t + A_3 z_t + A_4 \sigma_t^2 \\
z_{m,t} &= B_0 + B_1 x_t + B_2 y_t + B_3 z_t + B_4 \sigma_t^2
\end{aligned}
\tag{18}
$$

where the functions A_k, B_k, $k = 0, \ldots, 4$, depend on the preference parameters. Given this solution, we can derive analytical expressions for both the market return and for the risk-free rate. Details are provided in Appendix A.1.

Finally, we use the analytical solution of the linear model to estimate the set of preference parameters as defined in Section 2:

$$
\Lambda = \left(\gamma, \psi, \beta, \alpha \right).
$$

We estimate Λ by solving a sample minimum distance problem with the identity weighting matrix, as the problem is exactly identified. To do so, we simulate 100,000 samples of length equal to our sample size and use those to calculate the market and the risk-free returns. Next we estimate Λ to match the first two unconditional moments of the market and risk-free returns.

Table 2 (Panel A, Linear Model) reports the values of the estimated parameters. We find that the subjective discount factor is estimated at $\beta = 0.9985$ and the share of durable consumption in the intraperiod utility function α is about 30 percent. Crucially, the estimate of risk aversion coefficient γ is 2.7 and the estimate of the elasticity of intertemporal substitution ψ is around 1.3. It is interesting to note that while the risk aversion coefficient is relatively low and within the accepted range (less than 10), the elasticity of intertemporal substitution is significantly above 1. This means that in this (linear) economy, the household has a strong preference for early resolution of risk. As we will discuss in details later, these parameters are smaller than those calibrated by Bansal and Yaron (2004) ($\gamma = 10$ and $\psi = 1.5$). The difference is mainly due to the composition risk between durable and non-durable goods: if the two consumption goods are gross complements, and given a level of risk aversion, then marginal utility of consumption rises more sharply when durable consumption falls. Assets deliver unexpectedly low returns when marginal utility rises strongly, that is, during downturns, when durable consumption expenditure falls sharply relative to non-durable consumption. Therefore composition risk implies that the household must be compensated with higher expected returns to hold stocks during recessions.

Table 2 (Panel B, Linear Model) reports the simulated moments. The mean of the risk-free rate and of the risky return is about 1 and 6 percent, which is close to the values observed in the data. The linear model also generates a high volatility of the risky return (about 20 percent compared to the 19 percent that is observed in the data). However the linear model fails to match the standard deviation of the risk free rate, despite it being one of the targeted moments in our estimation exercise. This is mainly due to the assumption of linearity in the variance of the returns. To check the performance of the model, we also report two non-targeted moments, the mean and the variance of the price-dividend ratio. The linear model fails to generate the level and the volatility of these statistics.

3.1.3 Estimating the Non-Linear Model

We now estimate the non-linear version of the model. As we will argue, the numerical errors that are introduced in the LRRM using the Campbell-Shiller linearisation are economically significant and could lead an incorrect inference of the model, see also Pohl, Schmedders and Wilms (2018).

To carry out this task, we employ a semi-parametric estimation methodology similar to that of Chen, Favilukis and Ludvigson (2013). Our procedure consists of two steps. In the first step, for a fixed value of preference parameters, we approximate the unknown wealth-consumption and price-dividend ratios as a series of Chebyshev polynomials, and we estimate non-parametrically these functions by using the intertemporal conditions (i.e. the Euler equations) on the returns of wealth and stock. In the second step, for a given set of fitted functions, we estimate the preference parameters by a sample minimum distance estimator (the analogue of GMM). Details are provided in Appendix A.2.

Table 2 (Panel A, Full Model) reports the values of the estimated parameters of a full model. First, we find that the subjective discount factor is estimated at $\beta = 0.9914$, a value similar to its counterpart in the linear model. Second, and perhaps more strikingly, we estimate the risk aversion coefficient to 1.86 and the elasticity of intertemporal substitution ψ to around 1.18. Both these numbers are substantially smaller than their linear-model counterparts. These differences are due to the importance of non-linearities that our solution fully exploits and which allows to adopt parameters that are more in line with accepted macroeconomic wisdom.

Table 2 (Panel B, Full Model) reports the model-implied simulated moments. The main result from this exercise is that the non-linear model outperforms its linear counterpart in capturing the volatility of the risk free rate as well as both the mean and volatility of the price-dividend ratio. These latter two statistics are quite informative about the performance of the non-linear model since they are not part of the estimation targets and thus represent an external validation of our model.

Table 2: Estimated Preference Parameters and Unconditional Moments of Returns.

A. Estimated Preference Parameters		
	Linear Model	Full Model
Risk aversion	$\gamma = 2.78$	$\gamma = 1.86$
IES	$\psi = 1.29$	$\psi = 1.18$
Subjective discount factor	$\beta = 0.998$	$\beta = 0.991$
Share of durable consumption	$\alpha = 0.30$	$\alpha = 0.15$

B. Unconditional Moments of Returns							
	Data	Linear Model			Full Model		
		5%	50%	95%	5%	50%	95%
Mean of r_f	0.95	0.69	1.08	1.62	-0.59	1.01	8.16
Standard Deviation of r_f	1.61	0.26	0.36	0.49	1.08	1.61	3.40
Mean of r_m	5.57	2.50	6.21	9.84	1.66	5.78	10.89
Standard Deviation of r_m	18.94	16.16	20.52	25.44	13.03	16.85	20.97
Mean of $(p-d)$	4.93	4.20	4.27	4.35	3.83	4.60	5.20
Standard Deviation of $(p-d)$	0.38	0.12	0.17	0.23	0.18	0.31	0.58

C. Timing and Risk Premia	
Timing Premium	$\pi^* = 11\%$
Risk Premium	$\bar{\pi} = 16\%$

Timing and risk premia for different time horizons in the simulated model are presented in Figure 2. $T = 30$ years corresponds to the duration of US Treasury bonds, $T = 63$ years corresponds to the sample size of the data used, and long time horizons are $T = 100, 300, 625$ and $1,000$ years, see also Epstein, Farhi and Strzalecki (2014). The timing premium increases from 5 percent for 30 years to 11 percent for 300 years and remains at that level for all longer time horizons. The risk premium increases from 11 percent for a 30 year time horizon to 16 percent for 100 years and then stays at that level. For $1,000$ years the model generates a timing premium of 11 percent and a risk premium of 16 percent (see also Table 2, Panel C).

The estimated timing and risk premia are 11 and 16 percent, respectively, about a third of what emerge in single consumption good LRR models, e.g. Epstein, Farhi and Strzalecki (2014). This result derives from the fact that durable goods yield utility over several periods as their service flow spans over a relatively long time horizon, see for instance Browning and Crossley (2009). In other words, in bad times households can cut their expenditure on durable goods, while benefiting from the service flow that their stock of owned durables provides. As such, durable consumption supplies partial insurance against future uncertainty, thus mitigating both the timing and risk premia.

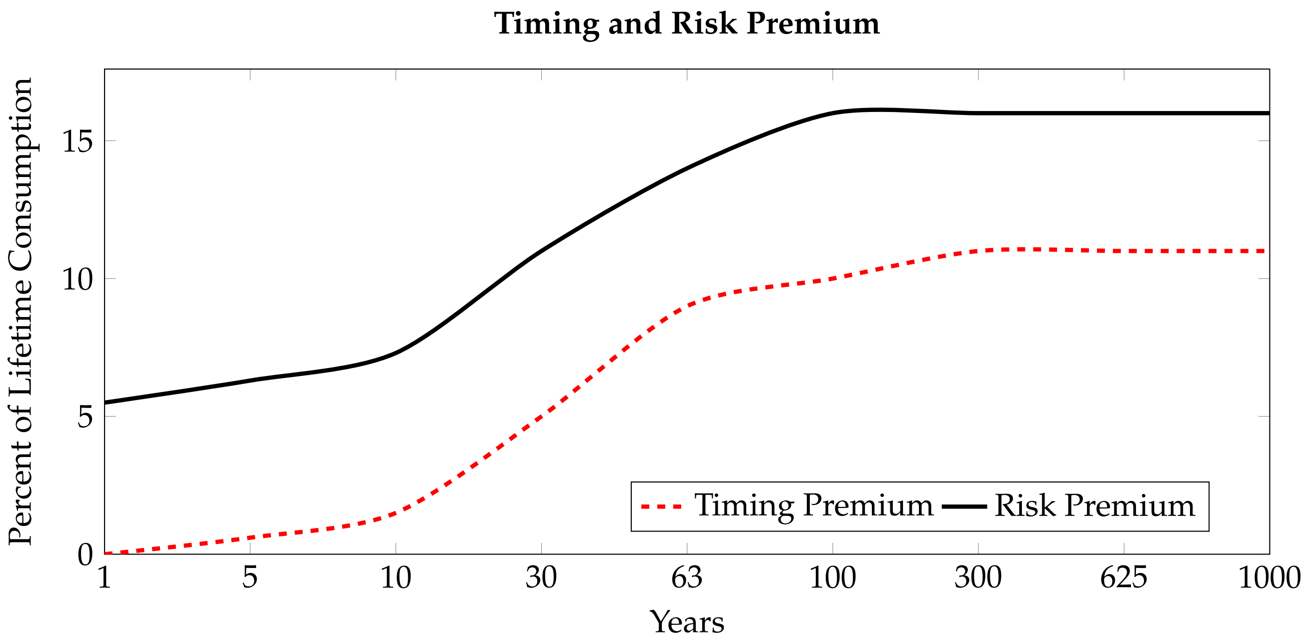

Figure 2: **Timing Premium and Risk Premium.** The figure displays the timing premium (dashed line) and the risk premium (solid line) as functions of time horizon.

To further understand the contribution of composition risk to our results, we analyse the alternative scenario with $\rho = 1$, so that composition risk is completely shut down. In order to do so, we re-estimate both the linear and the fully non-linear model for this case. Table 3 contains the results. Comparing with the benchmark case in Table 2, we find that without the composition effect the values of the estimated parameters as well as the estimated moments change substantially. Risk aversion and intertemporal elasticity of substitution (IES) are much higher, and, moreover, the model fails to match some of the asset pricing moments. There are also substantial differences between the linear and full model, e.g. the linear model fails to match the mean values of the risk-free and equity returns. While the model without the composition effect generates a low value of the timing premium, it generates an unreasonably high value of the risk premium, about twice as high as in Bansal and Yaron (2004). The results in Table 3 suggest that the composition effect plays an important role for the estimated preference parameters and for matching the asset markets' moments.

Table 3: No Composition Risk ($\rho = 1$)

A. Preference Parameters		
	Linear Model	**Full Model**
Risk aversion	$\gamma = 15.35$	$\gamma = 4.83$
IES	$\psi = 1.18$	$\psi = 1.79$
Subjective discount factor	$\beta = 0.997$	$\beta = 0.998$
Share of durable consumption	$\alpha = 0.42$	$\alpha = 0.47$

B. Unconditional Moments of Returns							
	Data	**Linear Model**			**Full Model**		
		5%	50%	95%	5%	50%	95%
Mean of r_f	0.95	-0.87	0.34	1.39	-0.99	0.26	1.31
Standard Deviation of r_f	1.61	0.90	1.27	1.70	0.91	1.27	1.71
Mean of r_m	5.57	-3.09	0.72	4.78	2.67	6.20	10.04
Standard Deviation of r_m	18.94	14.61	18.88	23.41	13.46	17.43	21.61
Mean of $(p - d)$	4.93	8.43	8.49	8.55	4.22	4.28	4.33
Standard Deviation of $(p - d)$	0.38	0.11	0.15	0.21	0.10	0.14	0.19

C. Timing and Risk Premia	
Timing Premium	$\pi^* = 4\%$
Risk Premium	$\bar{\pi} = 52\%$

The intuition for these results is the following. When non-durable consumption growth is low, the household's marginal utility increases. However, if low consumption growth is compounded by a decline in the flow of durable consumption services relative to non-durable consumption, marginal utility increases less when the two goods are complements ($\rho < 1$). Therefore, if consumption of non-durables is declining and durable consumption also declines relative to non-durables, the complementarity between the two types of goods causes a smaller increase in marginal utility compared to the case when composition risk is absent. In other words, in our model, the household's concern with composition risk implies that recessions are not perceived as particularly severe when the expenditure on durable consumption is low and there is complementarity between the two consumption goods. When we eliminate the composition risk, the risk aversion parameter necessary to explain movements in financial markets increases and so does the overall risk premium. However, timing premium decreases compared to the benchmark case. This is because the lack of composition risk increases the relative importance in the utility function of the durable goods ($\alpha = 0.47$). The household will pay less for having all risk resolved in the next period as it can rely on the partial insurance provided by durables more effectively on a longer horizon.

4 Conclusion

We introduce a long-run risk model where durable and non-durable consumption goods are non-separable and gross complements, thus generating the household's concern with short and long run composition risk, i.e. fluctuations in the relative share of durables in their consumption basket. We show that our model matches the key stylized facts of financial markets and at the same time generates levels of timing and risk premia that are consistent with the conventional macroeconomic wisdom. In its benchmark calibration, our model matches financial data well with a risk aversion of 1.86, an elasticity of intertemporal substitution of 1.18, and an elasticity of substitution between durable and non-durable goods of 0.78. With this estimation the timing premium is 11 percent and the risk premium is 16 percent. Compared to the its single consumption good counterpart, our model reduces the timing and risk premia by around two thirds. The paper holds an important message for financial economists: the importance of durable consumption is not limited to explaining financial data but it represents an important factor for obtaining reasonable timing and risk premia in the LRRM.

A Appendix

A.1 Solving the Linear Model

An analytical solution to the linear model is obtained using a linear approximation to the conditional volatility process (7) and expressing volatility as a process that follows a Gaussian distribution:

$$
\begin{aligned}
\sigma_{t+1}^2 &\approx \bar{\sigma}^2(1-\rho_h) + \rho_h \sigma_t^2 + 2\bar{\sigma}^2 \sigma_h \sqrt{1-\rho_h^2}\, w_{t+1} \\
&= \hat{\sigma} + \rho_h \sigma_t^2 + \sigma_w w_{t+1}.
\end{aligned}
$$

The endowment process for the economy is then given by

$$
\begin{aligned}
\Delta C_{t+1} &= \mu_c + x_t + \sigma_t \varepsilon_{t+1}^c \\
\Delta D_{t+1} &= \mu_d + y_t + \psi_d \sigma_t \varepsilon_{t+1}^d \\
\Delta S_{t+1} &= \mu_s + \phi_x x_t + \phi_y y_t + \pi_c \sigma_t \varepsilon_{t+1}^c + \pi_d \sigma_t \varepsilon_{t+1}^d + \psi_s \sigma_t \varepsilon_{t+1}^s \\
x_{t+1} &= \rho_x x_t + \psi_x \sigma_t \varepsilon_{t+1}^x \\
y_{t+1} &= \rho_y y_t + \psi_y \sigma_t \varepsilon_{t+1}^y \\
\sigma_{t+1}^2 &= \hat{\sigma} + \rho_h \sigma_t^2 + \sigma_w w_{t+1} \\
& \varepsilon_{t+1}^c, \varepsilon_{t+1}^d, \varepsilon_{t+1}^s, \varepsilon_{t+1}^x \varepsilon_{t+1}^y, w_{t+1} \sim \mathcal{N}(0,1).
\end{aligned}
$$

We derive the asset prices using the standard asset pricing condition

$$
\mathbb{E}_t \left[e^{m_{t+1} + r_{i,t+1}} \right] = 1
$$

for any asset $r_{i,t+1} = \log\left(R_{i,t+1}\right)$, where the log-pricing kernel of the economy is

$$
m_{t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \left(\frac{1}{\rho} - \frac{1}{\psi} \right) \Delta f_{t+1} + (\theta - 1)\, r_{w,t+1}.
$$

$r_{w,t+1}$ is the log return on the consumption claim, and $r_{m,t+1}$ is log market return. We use the approximation of Campbell and Shiller (1988) for the returns:

$$
\begin{aligned}
r_{w,t+1} &= z_{w,t+1} - \kappa_0 - \kappa_1 z_{w,t} - z_t + \Delta c_{t+1} \\
r_{m,t+1} &= \kappa_0^m + \kappa_1^m z_{m,t+1} - z_{m,t} + \Delta s_{t+1}
\end{aligned}
$$

where $z_t = \log(D_t/C_t)$, $z_{w,t}$ is the log wealth-consumption ratio, and $z_{m,t}$ is the log price-dividend ratio. The approximating constants are given by

$$\kappa_0 = \log\left(e^{\bar{z}_w} - 1 - q(\bar{z})\right) + \frac{1}{e^{\bar{z}_w} - 1 - q(\bar{z})}\left[-e^{\bar{z}_w}\bar{z}_w - \frac{\alpha}{1-\alpha}\left(1-\frac{1}{\rho}\right)e^{\left(1-\frac{1}{\rho}\right)\bar{z}}\bar{z}\right],$$

$$\kappa_0^m = \log\left(1+e^{\bar{z}_m}\right) - \frac{e^{\bar{z}_m}\bar{z}_m}{1+e^{\bar{z}_m}}, \quad \kappa_1^m = \frac{e^{\bar{z}_m}}{1+e^{\bar{z}_m}}.$$

A.1.1 Consumption Claim

We conjecture that the log wealth-consumption ratio $z_{w,t}$ is a linear function of state variables:

$$z_{w,t} = A_0 + A_1 x_t + A_2 y_t + A_3 z_t + A_4 \sigma_t^2.$$

Then

$$\begin{aligned}
r_{w,t+1} =& z_{w,t+1} - \kappa_0 - \kappa_1 z_{w,t} - \kappa_2 z_t + \Delta c_{t+1} \\
=& A_0 + A_1 x_{t+1} + A_2 y_{t+1} + A_3 z_{t+1} + A_4 \sigma_{t+1}^2 \\
& - \kappa_0 - \kappa_1 A_0 - \kappa_1 A_1 x_t - \kappa_1 A_2 y_t - \kappa_1 A_3 z_t - \kappa_1 A_4 \sigma_t^2 - \kappa_2 z_t + \Delta c_{t+1} \\
=& \left\{A_0(1-\kappa_1) - \kappa_0 + A_3(\mu_d - \mu_c) + A_4\hat{\sigma} + \mu_c\right\} \\
& + \left\{A_1\rho_x - A_3 - \kappa_1 A_1 + 1\right\} x_t \\
& + \left\{A_2\rho_y + A_3 - \kappa_1 A_2\right\} y_t \\
& + \left\{A_3 - \kappa_1 A_3 - \kappa_2\right\} z_t \\
& + \left\{A_4\rho_h - \kappa_1 A_4\right\} \sigma_t^2 \\
& + A_1\psi_x\sigma_t\varepsilon_{t+1}^c + A_2\psi_y\sigma_t\varepsilon_{t+1}^y + (1-A_3)\sigma_t\varepsilon_{t+1}^c + A_3\psi_d\sigma_t\varepsilon_{t+1}^d + A_4\sigma_w w_{t+1}.
\end{aligned}$$

Using

$$\begin{aligned}
\Delta f_{t+1} = & \underbrace{\frac{\rho}{\rho-1}\alpha\exp\left(\left(1-\frac{1}{\rho}\right)\bar{z}\right)\left(1-\frac{1}{\rho}\right)}_{K}(z_{t+1}-z_t) \\
& = K\left(\mu_d + y_t + \psi_d\sigma_t\varepsilon_{t+1}^d - \mu_c - x_t - \sigma_t\varepsilon_{t+1}^c\right)
\end{aligned}$$

and

$$m_{t+1} = \theta\log\beta - \frac{\theta}{\psi}\Delta c_{t+1} + \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right)\Delta f_{t+1} + (\theta-1)\, r_{w,t+1}$$

we obtain

$$
\begin{aligned}
m_{t+1} = {} & \theta \log \beta - \frac{\theta}{\psi}\mu_c + \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right) K(\mu_c - \mu_d) \\
& + (\theta - 1)(A_0(1-\kappa_1) - \kappa_0 + A_3(\mu_d - \mu_c) + A_4\hat{\sigma} + \mu_c) \\
& + \Big\{ -\frac{\theta}{\psi} - \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right) K + (\theta-1)\left(A_1\rho_x - A_3 - \kappa_1 A_1 + 1\right) \Big\} x_t \\
& + \Big\{ \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right) K + (\theta-1)(A_2\rho_y + A_3 - \kappa_1 A_2) \Big\} y_t \\
& + \{(\theta-1)(A_3 - \kappa_1 A_3 - \kappa_2)\} z_t \\
& + \{(\theta-1)(A_4\rho_h - \kappa_1 A_4)\} \sigma_t^2 \\
& + (\theta-1)A_1\psi_x\sigma_t\varepsilon_{t+1}^x \\
& + (\theta-1)A_2\psi_y\sigma_t\varepsilon_{t+1}^y \\
& + \left\{ (\theta-1)(1-A_3) - \frac{\theta}{\psi} - \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right) K \right\} \sigma_t\varepsilon_{t+1}^c \\
& + \left\{ (\theta-1)A_3 + \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right) K \right\} \psi_d\sigma_t\varepsilon_{t+1}^d \\
& + (\theta-1)A_4\sigma_w w_{t+1}.
\end{aligned}
$$

Since both m_{t+1} and $r_{w,t+1}$ are conditionally normal, the Euler equation for wealth can be written as

$$
\mathbb{E}_t\left[m_{t+1} + r_{w,t+1}\right] + \frac{1}{2}\mathbb{V}\mathrm{ar}_t\left[m_{t+1} + r_{w,t+1}\right] \approx 0.
$$

We use this equation to solve for the coefficients $A_0, ..., A_4$. These are

$$
\begin{aligned}
A_0 = {} & -\frac{1}{(\kappa_1 - 1)\ (\theta - 1)} \times \Bigg\{ (\theta-1)\ (\kappa_0 - \mu_c - A_4\,\hat{\sigma} + A_3\ (\mu_c - \mu_d)) - \theta\log\beta \\
& \qquad - \frac{A_4^2\,\sigma_w^2\,(\theta-1)^2}{2} + \frac{\mu_c\,\theta}{\psi} + K\,\theta\ \left(\frac{1}{\psi} - \frac{1}{\rho}\right)\ (\mu_c - \mu_d) \Bigg\} \\
A_1 = {} & \frac{1}{(\kappa_1 - \rho_x)\ (\theta - 1)} \times \left\{ \left(\frac{\kappa_2}{\kappa_1 - 1} + 1\right)\ (\theta - 1) + K\,\theta\ \left(\frac{1}{\psi} - \frac{1}{\rho}\right) - \frac{\mu_c\theta}{\psi} \right\} \\
A_2 = {} & -\frac{1}{(\kappa_1 - \rho_y)\ (\theta - 1)} \times \left\{ K\theta\left(\frac{1}{\psi} - \frac{1}{\rho}\right) + \frac{\kappa_2\ (\theta - 1)}{\kappa_1 - 1} \right\} \\
A_3 = {} & -\kappa_2/(\kappa_1 - 1) \\
A_4 = {} & \frac{1}{2\,(\kappa_1 - \rho_h)\ (\theta - 1)} \times \Bigg\{ {\psi_d}^2 \left(A_3\ (\theta - 1) - K\,\theta\ \left(\frac{1}{\Psi} - \frac{1}{\rho}\right)\right)^2
\end{aligned}
$$

$$
+ \left((A_3 - 1)\ (\theta - 1) - K\theta \left(\frac{1}{\Psi} - \frac{1}{\rho} \right) + \frac{\mu_c\,\theta}{\Psi} \right)^2
$$

$$
+ A_1{}^2\,\psi_x{}^2\,(\theta-1)^2 + A_2{}^2\,\psi_y{}^2\,(\theta-1)^2 \Bigg\}.
$$

The innovation to m_{t+1} is given by

$$
m_{t+1} - \mathbb{E}_t[m_{t+1}] = \lambda_x \sigma_t \varepsilon^x_{t+1} + \lambda_y \sigma_t \varepsilon^y_{t+1} + \lambda_c \sigma_t \varepsilon^c_{t+1} + \lambda_d \sigma_t \varepsilon^d_{t+1} + \lambda_w \sigma_w w_{t+1},
$$

where the coefficients $\lambda.$ represent the market price of risk for each source of risk:

$$
\begin{aligned}
\lambda_x =&(\theta-1)A_1\psi_x,\ \lambda_y = (\theta-1)A_2\psi_y, \quad \lambda_c = (\theta-1)(1-A_3) - \frac{\theta}{\psi} - \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right)K,\\
\lambda_d =& \left((\theta-1)A_3 + \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right)K \right)\psi_d, \quad \lambda_w = (\theta-1)A_4.
\end{aligned}
$$

Similarly, the innovation to $r_{w,t+1}$ is given by

$$
r_{w,t+1} - \mathbb{E}_t[r_{w,t+1}] = -\beta_x \sigma_t \varepsilon^x_{t+1} - \beta_y \sigma_t \varepsilon^y_{t+1} - \beta_c \sigma_t \varepsilon^c_{t+1} - \beta_d \sigma_t \varepsilon^d_{t+1} - \beta_w \sigma_w w_{t+1}
$$

where

$$
\beta_x = -A_1\psi_x,\ \beta_y = -A_2\psi_y,\ \beta_c = -(1-A_3),\ \beta_d = -A_3\psi_d,\ \beta_w = -A_4.
$$

The risk premium for the consumption claim is

$$
\begin{aligned}
\mathbb{E}_t[r_{w,t+1} - r_{f,t}] + \frac{1}{2}\mathbb{V}\mathrm{ar}_t[r_{w,t+1}] &= -\mathbb{C}\mathrm{ov}_t[m_{t+1}, r_{w,t+1}]\\
&= \left(\beta_x\lambda_x + \beta_y\lambda_y + \beta_c\lambda_c + \beta_d\lambda_d\right)\sigma_t^2 + \beta_w\lambda_w\sigma_w^2.
\end{aligned}
$$

A.1.2 Market Return

We conjecture that the log price-dividend ratio for the claim on dividends is

$$
z_{m,t} = B_0 + B_1 x_t + B_2 y_t + B_3 z_t + B_4 \sigma_t^2.
$$

Then

$$
\begin{aligned}
r_{m,t+1} &= \kappa_0^m + \kappa_1^m z_{m,t+1} - z_{m,t} + \Delta s_{t+1}\\
&= \kappa_0^m + \kappa_1^m \left(B_0 + B_1 x_{t+1} + B_2 y_{t+1} + B_3 z_{t+1} + B_4 \sigma_{t+1}^2 \right)\\
&\quad - B_0 - B_1 x_t - B_2 y_t - B_3 z_t - B_4 \sigma_t^2
\end{aligned}
$$

$$
\begin{aligned}
&+\mu_s+\phi_x x_t+\phi_y y_t+\pi_c\sigma_t\varepsilon^c_{t+1}+\pi_d\sigma_t\varepsilon^d_{t+1}+\psi_s\sigma_t\varepsilon^s_{t+1}\\
=&\left\{\kappa^m_0+B_0(\kappa^m_1-1)+\kappa^m_1(B_3(\mu_d-\mu_c)+B_4\hat{\sigma})+\mu_s\right\}\\
&+\left\{\kappa^m_1 B_1\rho_x-\kappa^m_1 B_3-B_1+\phi_x\right\}x_t\\
&+\left\{\kappa^m_1 B_2\rho_y+\kappa^m_1 B_3-B_2+\phi_y\right\}y_t\\
&+\left\{\kappa^m_1 B_3-B_3\right\}z_t\\
&+\left\{\kappa^m_1 B_4\rho_h-B_4\right\}\sigma^2_t\\
&+\left\{\kappa^m_1 B_1\psi_x\right\}\sigma_t\varepsilon^x_{t+1}\\
&+\left\{\kappa^m_1 B_2\psi_y\right\}\sigma_t\varepsilon^y_{t+1}\\
&+\left\{-\kappa^m_1 B_3+\pi_c\right\}\sigma_t\varepsilon^c_{t+1}\\
&+\left\{\kappa^m_1 B_3\psi_d+\pi_d\right\}\sigma_t\varepsilon^d_{t+1}\\
&+\left\{\psi_s\right\}\sigma_t\varepsilon^s_{t+1}\\
&+\left\{\kappa^m_1 B_4\right\}\sigma_w w_{t+1}
\end{aligned}
$$

Since both m_{t+1} and $r_{m,t+1}$ are conditionally normal, the Euler equation can be written as

$$
\mathbb{E}_t\left[m_{t+1}+r_{m,t+1}\right]+\frac{1}{2}\mathbb{V}\mathrm{ar}_t\left[m_{t+1}+r_{m,t+1}\right]\approx 0.
$$

We use this equation to solve for the coefficients $B_0, ..., B_4$. They are given by

$$
\begin{aligned}
B_0 &= -\frac{1}{\kappa^m_1-1}\times\left\{\left(\frac{B_4^2\kappa_1^{m2}}{2}+\frac{M_w^{2\,2}}{2}\right)\sigma^2_w+M_0+\kappa^m_0+\mu_s+\kappa^m_1\ \left(B_4\,\hat{\sigma}-B_3\ (\mu_c-\mu_d)\right)\right\}\\
B_1 &= -\frac{M_x+\phi_x-B_3\,\kappa^m_1}{\kappa^m_1\,\rho_x-1}\\
B_2 &= -\frac{M_y+\phi_y+B_3\,\kappa^m_1}{\kappa^m_1\,\rho_y-1}\\
B_3 &= \frac{M_z}{1-\kappa^m_1}\\
B_4 &= -\frac{1}{2\,\kappa^m_1\,\rho_h-2}\times\left\{2M_\sigma+(\pi_c-B_3\kappa^m_1)^2+\left(\pi_y+B_3\kappa^m_1\psi_d\right)^2+M^2_{ec}+M^2_{ed}+M^2_{ex}\right.\\
&\qquad\left.+M^2_{ey}+\psi^2_s+B_1^2\kappa_1^{m2}\psi^2_x+B_2^2\,\kappa_1^{m2}\,\psi^2_y\right\}
\end{aligned}
$$

where

$$
M_0=\left\{\theta\log\beta-\frac{\theta}{\psi}\mu_c+\theta\left(\frac{1}{\rho}-\frac{1}{\psi}\right)K(\mu_c-\mu_d)\right.
$$

$$
\begin{aligned}
&\quad + (\theta-1)(A_0(1-\kappa_1) - \kappa_0 + A_3(\mu_d-\mu_c) + A_4\hat{\sigma} + \mu_c)\Big\} \\
M_x &= -\frac{\theta}{\psi} - \theta\left(\frac{1}{\rho}-\frac{1}{\psi}\right)K + (\theta-1)\left(A_1\rho_x - A_3 - \kappa_1 A_1 + 1\right) \\
M_y &= \theta\left(\frac{1}{\rho}-\frac{1}{\psi}\right)K + (\theta-1)(A_2\rho_y + A_3 - \kappa_1 A_2) \\
M_z &= (\theta-1)(A_3 - \kappa_1 A_3 - \kappa_2) \\
M_\sigma &= (\theta-1)(A_4\rho_h - \kappa_1 A_4) \\
M_{ex} &= (\theta-1)A_1\psi_x \\
M_{ey} &= (\theta-1)A_2\psi_y \\
M_{ec} &= (\theta-1)(1-A_3) - \frac{\theta}{\psi} - \theta\left(\frac{1}{\rho}-\frac{1}{\psi}\right)K \\
M_{ed} &= \left\{(\theta-1)A_3 + \theta\left(\frac{1}{\rho}-\frac{1}{\psi}\right)K\right\}\psi_d \\
M_w &= (\theta-1)A_4.
\end{aligned}
$$

The innovation to $r_{m,t+1}$ is given by

$$
\begin{aligned}
&r_{m,t+1} - \mathbb{E}_t[r_{m,t+1}] = \\
&\quad - \beta_{m,x}\sigma_t\varepsilon^x_{t+1} - \beta_{m,y}\sigma_t\varepsilon^y_{t+1} - \beta_{m,c}\sigma_t\varepsilon^c_{t+1} - \beta_{m,d}\sigma_t\varepsilon^d_{t+1} - \beta_{m,s}\sigma_t\varepsilon^s_{t+1} - \beta_{m,w}\sigma_w w_{t+1}
\end{aligned}
$$

where

$$
\begin{aligned}
&\beta_{m,x} = -\kappa_1^m B_1\psi_x,\ \beta_{m,y} = -\kappa_1^m B_2\psi_y,\ \beta_{m,c} = \kappa_1^m B_3 - \pi_c, \\
&\beta_{m,d} = -\kappa_1^m B_3\psi_d - \pi_d,\ \beta_{m,s} = -\psi_s,\ \beta_{m,w} = -\kappa_1^m B_4.
\end{aligned}
$$

The risk premium for the dividend claim is

$$
\begin{aligned}
\mathbb{E}_t[r_{m,t+1} - r_{f,t}] + \frac{1}{2}\mathbb{V}\mathrm{ar}_t[r_{m,t+1}] &= -\mathbb{C}\mathrm{ov}_t[m_{t+1}, r_{m,t+1}] \\
&= \left(\beta_{m,x}\lambda_x + \beta_{m,y}\lambda_y + \beta_{m,c}\lambda_c + \beta_{m,d}\lambda_d\right)\sigma_t^2 + \beta_{m,w}\lambda_w\sigma_w^2.
\end{aligned}
$$

A.1.3 Risk-Free Rate

Using the Euler equation, the model-implied risk-free rate is given by

$$
r_{f,t} = -\mathbb{E}_t\left[m_{t+1}\right] - \frac{1}{2}\mathbb{V}\mathrm{ar}_t\left[m_{t+1}\right].
$$

Using the derived expression for m_{t+1}, the risk-free rate will be given by

$$
r_{f,t} = C_0 + C_1 x_t + C_2 y_t + C_3 z_t + C_4\sigma_t^2
$$

where

$$
\begin{aligned}
C_0 &= -\left\{\theta \log \beta - \frac{\theta}{\psi}\mu_c + \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right) K(\mu_c - \mu_d)\right. \\
&\qquad \left. + (\theta - 1)(A_0(1-\kappa_1) - \kappa_0 + A_3(\mu_d - \mu_c) + A_4\hat{\sigma} + \mu_c) + \frac{\lambda_w^2 \sigma_w^2}{2}\right\} \\
C_1 &= -\left\{-\frac{\theta}{\psi} - \theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right) K + (\theta - 1)\left(A_1\rho_x - A_3 - \kappa_1 A_1 + 1\right)\right\} \\
C_2 &= -\left\{\theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right) K + (\theta - 1)(A_2\rho_y + A_3 - \kappa_1 A_2)\right\} \\
C_3 &= -\left\{(\theta - 1)(A_3 - \kappa_1 A_3 - \kappa_2)\right\} \\
C_4 &= -\left\{(\theta - 1)(A_4\rho_h - \kappa_1 A_4) + \frac{\lambda_x^2 + \lambda_y^2 + \lambda_c^2 + \lambda_d^2}{2}\right\}.
\end{aligned}
$$

A.2 Solving the Non-Linear Model

In our estimation exercise, we use a Euler equation to back out the asset returns as a function of model parameters and fully estimate these parameters. As the Euler equation does not admit analytical solution, we rely on numerical methods. We proceed in several steps. First, we analytically derive the pricing kernel of the model, as well as price-dividend and wealth-consumption ratios. Second, we express returns in the model as functions of these two ratios, using a simple asset pricing identity. Third, we approximate these ratios (and as a consequence the returns in the model) as a series of Chebyshev polynomials and apply projection methods to the Euler equation to numerically derive the price-dividend and wealth-consumption ratio as a function of model parameters alone. This, in turn, allows us to estimate model parameters using the techniques described in main text.

A.2.1 Pricing Kernel

We first analytically derive the pricing kernel of the economy. Define the return on total consumption as

$$
R_{W,t+1} = \frac{\widetilde{W}_{t+1}}{\widetilde{W}_t - C_t - Q_t D_t}
$$

where total consumption G_t is given by

$$
G_t = C_t + Q_t D_t
$$

and Q_t denotes the user cost of the service flow for the durable good. Following Yogo (2006), Q_t is given as a marginal rate of substitution between non-durable and durable consumption good

$$Q_t = \frac{\partial \mathcal{C}_t}{\partial D_t} \Big/ \frac{\partial \mathcal{C}_t}{\partial C_t}.$$

Given the functional form for $\mathcal{C}_t$, we get

$$Q_t = \frac{\alpha}{1-\alpha} \left(\frac{D_t}{C_t} \right)^{-\frac{1}{\rho}}.$$

Define

$$F_t = \left(1 - \alpha + \alpha \left(\frac{D_t}{C_t} \right)^{1-\frac{1}{\rho}} \right)^{\frac{1}{1-\frac{1}{\rho}}}$$

to write the intertemporal marginal rate of substitution as

$$M_{t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} \left(\frac{F_{t+1}}{F_t} \right)^{\theta\left(\frac{1}{\rho} - \frac{1}{\psi}\right)} R_{W,t+1}^{\theta-1}. \tag{19}$$

Furthermore,

$$R_{W,t+1} = \frac{\widetilde{W}_{t+1}}{\widetilde{W}_t - G_t} = \frac{\frac{\widetilde{W}_{t+1}}{G_{t+1}}}{\frac{\widetilde{W}_t}{G_t} - 1} \frac{G_{t+1}}{G_t}$$

where we can rewrite G_t as

$$G_t = C_t + Q_t D_t = C_t + \frac{\alpha}{1-\alpha} \left(\frac{D_t}{C_t} \right)^{-\frac{1}{\rho}} D_t = C_t \left(1 + \frac{\alpha}{1-\alpha} \left(\frac{D_t}{C_t} \right)^{1-\frac{1}{\rho}} \right)$$

and F_t as

$$F_t = (1-\alpha)^{\frac{1}{1-\frac{1}{\rho}}} \left(1 + \frac{\alpha}{1-\alpha} \left(\frac{D_t}{C_t} \right)^{1-\frac{1}{\rho}} \right)^{\frac{1}{1-\frac{1}{\rho}}}.$$

Substituting the terms in (19) using the above relations, we obtain

$$M_{t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{\theta\left(1-\frac{1}{\psi}\right)-1} \left(\frac{A_{t+1}}{A_t} \right)^{\theta\left(1-\frac{1}{\psi}\right)-1} \left(\frac{\frac{\widetilde{W}_{t+1}}{G_{t+1}}}{\frac{\widetilde{W}_t}{G_t} - 1} \right)^{\theta-1}$$

where

$$A_t = 1 + \frac{\alpha}{1-\alpha} \left(\frac{D_t}{C_t} \right)^{1-\frac{1}{\rho}}.$$

The evolution of D_{t+1}/C_{t+1} (which enters A_{t+1}) can be written as

$$\frac{D_{t+1}}{C_{t+1}} = \frac{D_{t+1}/D_t \cdot D_t}{C_{t+1}/C_t \cdot C_t} = \frac{D_{t+1}}{D_t} \left(\frac{C_{t+1}}{C_t} \right)^{-1} \frac{D_t}{C_t}.$$

Setting $z_t = \log(D_t/C_t)$, we find

$$z_{t+1} = \Delta D_{t+1} - \Delta C_{t+1} + z_t = \mu_d + y_t + \sigma_d \varepsilon_{t+1}^d - \mu_c - x_t - \sigma_x \varepsilon_{t+1}^x + z_t.$$

A.2.2 Application of Projection Method

We now show how the projection method can be applied to the wealth-Euler equation to numerically derive the wealth-consumption ratio as a function of model parameters alone.

We start with the Euler equation for wealth

$$\mathbb{E}_t \left[M_{t+1} R_{W,t+1} \right] = 1$$

where

$$M_{t+1} = \beta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\theta/\psi} \left(\frac{v(D_{t+1}/C_{t+1})}{v(D_t/C_t)} \right)^{\theta(1/\rho - 1/\psi)} R_{W,t+1}^{\theta-1}$$

and

$$v\left(\frac{D_t}{C_t}\right) = F_t = \left[1 - \alpha + \alpha \left(\frac{D_t}{C_t} \right)^{1-1/\rho} \right]^{1/(1-1/\rho)}.$$

Here, $R_{W,t+1}$ is the return on wealth. In logarithms, the Euler equation for wealth becomes

$$\mathbb{E}_t \left[\exp\left(\theta \log\beta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \left(\frac{1}{\rho} - \frac{1}{\psi} \right) \Delta f_{t+1} + (\theta - 1) r_{w,t+1} \right) \right] = 1,$$

where lowercase variables denote the logs of the corresponding uppercase variables, and $\Delta c_{t+1} = c_{t+1} - c_t$ and $\Delta f_{t+1} = f_{t+1} - f_t$. Log-return on wealth $r_{w,t+1}$ can be further written as

$$\begin{aligned} r_{w,t+1} &= \log\left(\frac{\widetilde{W}_{t+1}}{\widetilde{W}_t - C_t - Q_t D_t} \right) = \log\left(\frac{\frac{\widetilde{W}_{t+1}}{C_{t+1}}}{\frac{\widetilde{W}_t}{C_t} - 1 - Q_t \frac{D_t}{C_t}} \cdot \frac{C_{t+1}}{C_t} \right) \\ &= wc_{t+1} - \log\left(wc_t - 1 - Q_t \frac{D_t}{C_t} \right) + \Delta c_{t+1} \end{aligned}$$

where $wc_t = \log(\widetilde{W}_t/C_t)$ is the log wealth-consumption ratio.

We approximate today's and tomorrow's wealth-consumption ratio as a series of Chebyshev polynomials and substitute it back into the log-version of the wealth-Euler equation. We then apply projection methods to numerically solve for the wealth-consumption ratio.

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