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LOG-OPTIMAL AND RAPID PATHS IN VON NEUMANN-GALE DYNAMICAL SYSTEMS

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Abstract: Von Neumann-Gale dynamical systems are defined in terms of multivalued operators in spaces of random vectors, possessing certain properties of convexity and homogeneity. A central role in the theory of such systems is played by a special class of paths (trajectories) called rapid: they grow over each time period $t-1, t$ in a sense faster than others. The paper establishes existence and characterization theorems for such paths showing, in particular, that any trajectory maximizing a logarithmic functional over a finite time horizon is rapid. The proof of this result is based on the methods of convex analysis in spaces of measurable functions. The study is motivated by the applications of the theory of von Neumann-Gale dynamical systems to the modeling of capital growth in financial markets with frictions – transaction costs and portfolio constraints.

Key words and phrases: random dynamical systems, convex multivalued operators, von Neumann-Gale dynamical systems, rapid paths, logarithmically optimal paths, stochastic optimization, convex analysis in L_∞, Yosida-Hewitt decomposition, financial markets, transaction costs, portfolio constraints, capital growth, benchmark strategies.

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_N = \mathcal{F}$ a sequence of σ-algebras containing all sets in $\mathcal{F}$ of measure zero. For each $t = 0, 1, ..., N$, let $X_t(\omega)$ be a closed cone in an m_t-dimensional linear space $\mathbb{R}^{m_t}$ and for each $t = 1, ..., N$, let $(\omega, a) \mapsto A_t(\omega, a)$ be a set-valued operator assigning a non-empty set $A_t(\omega, a) \subseteq X_t(\omega)$ to each $\omega \in \Omega$ and $a \in X_{t-1}(\omega)$. Throughout the paper, the following conditions of homogeneity and convexity will be imposed on the operator $A_t(\omega, \cdot)$. For each ω, we have

$$\lambda A_t(\omega, a) \subseteq A_t(\omega, \lambda a) \tag{1}$$

for all $a \in X_{t-1}(\omega)$, $\lambda \in [0, \infty)$ and

$$\theta A_t(\omega, a) + (1 - \theta) A_t(\omega, a') \subseteq A_t(\omega, \theta a + (1 - \theta) a') \tag{2}$$

for all $a, a' \in X_{t-1}(\omega)$ and $\theta \in [0, 1]$. (A linear combination of two sets in a vector space is the set of pairwise linear combinations of their elements.)

The σ-algebra $\mathcal{F}_t$ $(t = 0, ..., N)$ is interpreted as the class of events occurring prior to time t. Vector functions of $\omega \in \Omega$ measurable with respect to $\mathcal{F}_t$ represent random vectors depending on these events. Denote for shortness by $\mathcal{L}_t^k$ the space $L_\infty(\Omega, \mathcal{F}_t, P, \mathbb{R}^k)$ of essentially bounded $\mathcal{F}_t$-measurable functions of $\omega \in \Omega$ with values in $\mathbb{R}^k$. We say that a vector function $x(\omega)$ is a *random state* of the system at time t and write $x \in \mathcal{X}_t$ if $x \in \mathcal{L}_t^{m_t}$ and $x(\omega) \in X_t(\omega)$ almost surely (a.s.). The mappings $(\omega, a) \mapsto A_t(\omega, a)$ generate a multivalued stochastic dynamical system over the time interval $t = 1, 2, ..., N$. A sequence of random states $x_0 \in \mathcal{X}_0$, $x_1 \in \mathcal{X}_1$, ..., $x_N \in \mathcal{X}_N$ is called a *path* (*trajectory*) of this dynamical system if

$$x_t(\omega) \in A_t(\omega, x_{t-1}(\omega)) \text{ (a.s.)}. \tag{3}$$

Relation (3) can be written in the form

$$(x_{t-1}(\omega), x_t(\omega)) \in Z_t(\omega) \text{ (a.s.)}, \tag{4}$$

where

$$Z_t(\omega) = \{(a, b) \in X_{t-1}(\omega) \times X_t(\omega) : \ b \in A_t(\omega, a)\} \tag{5}$$

is the graph of the set-valued mapping $A_t(\omega, \cdot)$. Clearly conditions (1) and (2) hold if and only if $Z_t(\omega)$ is a cone contained in $X_{t-1}(\omega) \times X_t(\omega)$. Since

$A_t(\omega, a) \neq \emptyset$ for all $a \in X_{t-1}(\omega)$, the projection of $Z_t(\omega)$ on $X_{t-1}(\omega)$ coincides with $X_{t-1}(\omega)$. It is assumed that the cones $X_t(\omega)$ and $Z_t(\omega)$ depend $\mathcal{F}_t$-measurably[4] on ω, which means that they are determined by events occurring prior to time t.

The dynamics of the system under consideration can equivalently be described both in terms of the mappings $A_t(\omega, \cdot)$ and in terms of the cones $Z_t(\omega)$. A sequence $x_0 \in \mathcal{X}_0$, $x_1 \in \mathcal{X}_1$, ..., $x_N \in \mathcal{X}_N$ is a path if and only if

$$(x_{t-1}, x_t) \in \mathcal{Z}_t, \; t = 1, 2, ..., N,$$

where

$$\mathcal{Z}_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \; (x(\omega), y(\omega)) \in Z_t(\omega) \text{ (a.s.)}\}. \tag{6}$$

Such dynamical systems were first considered in the context of the modeling of economic growth by von Neumann [42] and Gale [18]. Important contributions to the field were made by Rockafellar [37], Radner [33], McKenzie [23], Nikaido [24], Makarov and Rubinov [22] and others. For reviews of this field see [22] and [16].

The classical theory of von Neumann-Gale dynamics was purely deterministic. First attempts to build its stochastic generalization were undertaken in the 1970s by Dynkin [9, 10, 11], Radner [34] and their collaborators. However, the initial attack on the problem left many questions unanswered. Substantial progress was made only in the late 1990s, and final solutions to the main open problems were obtained only in the 2000s – see [17].

At about the same time it was observed [7] that stochastic analogues of von Neumann-Gale dynamical systems provide a natural and convenient framework for the modeling of financial markets with frictions – transaction costs and portfolio constraints. This observation not only gave a new momentum to studies in the field and posed new interesting questions, but also made it possible to find a key to the solution of old problems. The new, financial interpretation of the mathematical notions and objects at hand amazingly suggested the way of proofs in [17] that could not be found earlier.

In the present work, we examine the structure of paths of stochastic von Neumann-Gale dynamical systems focusing primarily on questions of their growth. Our main goal is to single out and investigate a class of trajectories which grow faster in a certain sense than other trajectories over each time

[4] A set $A(\omega) \subseteq \mathbb{R}^k$ is said to depend $\mathcal{F}_t$-measurably on ω if the graph $\{(\omega, a) : a \in A(\omega)\}$ of the set-valued mapping $\omega \mapsto A(\omega)$ belongs to the σ-algebra $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^k)$, where $\mathcal{B}(\cdot)$ stands for the Borel σ-algebra.

period $t-1, t$. The central notion here is that of a *rapid path*. Let us give its definition. To this end we will first define the important notion of a *dual path*.

Let $X_t^*(\omega)$ denote the dual cone of $X_t(\omega)$:

$$X_t^*(\omega) = \{p \in \mathbb{R}^{m_t} : \ pa \geq 0, \ a \in X_t(\omega)\},$$

where pa is the scalar product of the vectors p and a in $\mathbb{R}^{m_t}$. For shortness, we will use the notation $\mathcal{P}_t^k = L_1(\Omega, \mathcal{F}_t, P, \mathbb{R}^k)$ for the space of integrable $\mathcal{F}_t$-measurable vector functions with values in $\mathbb{R}^k$. Put

$$\mathcal{F}_{N+1} := \mathcal{F}_N.$$

A *dual path* (*dual trajectory*) is a sequence of vector functions $p_1(\omega), p_2(\omega), ...,$ $p_{N+1}(\omega)$ such that $p_t \in \mathcal{P}_t^{m_{t-1}}$ and for almost all ω,

$$p_t(\omega) \in X_{t-1}^*(\omega), \ t = 1, 2, ..., N+1, \tag{7}$$

and

$$\bar{p}_{t+1}(\omega) b \leq p_t(\omega) a \text{ for all } (a, b) \in Z_t(\omega), \ t = 1, 2, ..., N. \tag{8}$$

Here, $\bar{p}_{t+1}(\omega) := E_t p_{t+1}(\omega)$ and $E_t(\cdot) = E(\cdot|\mathcal{F}_t)$ is the conditional expectation given $\mathcal{F}_t$.

Let us say that a dual path $p_1, p_2, ..., p_{N+1}$ *supports* a path $x_0, x_1, ..., x_N$ if

$$p_{t+1} x_t = 1, \ t = 0, 1, ..., N \text{ (a.s.)}. \tag{9}$$

A trajectory is called *rapid* if there exists a dual trajectory supporting it. What matters in (9) is that $p_{t+1} x_t$ is constant (independent of time and random factors). The value 1 for this constant is chosen only for the sake of convenience.

The term "rapid" is motivated, in particular, by the fact that for each $t = 1, ..., N$,

$$E_t \frac{p_{t+1} y_t}{p_t y_{t-1}} = \frac{\bar{p}_{t+1} y_t}{p_t y_{t-1}} \leq \frac{\bar{p}_{t+1} x_t}{p_t x_{t-1}} = 1 \text{ (a.s.)} \tag{10}$$

for all paths $y_0, y_1, ..., y_N$ with $p_t y_{t-1} > 0$ (a.s.). This means that the path $x_0, x_1, ..., x_N$ maximizes the conditional expectation given $\mathcal{F}_t$ of the *growth rate* $p_{t+1} y_t / p_t y_{t-1}$ over each time period $(t-1, t]$, the maximum being equal to 1. The growth rate is measured in terms of the dual variables p_t, which in economic and financial applications typically represent prices.

In the financial applications (on which we focus in Section 6), paths in the dynamical system at hand represent self-financing *trading strategies*. The cones $X_t(\omega)$ and $Z_t(\omega)$ specify portfolio admissibility constraints and self-financing constraints, respectively. Rapid paths, the main object of our study, are counterparts of *benchmark strategies* (Platen [31], Platen and Heath [32]) or *numeraire portfolios* (Long [21]). For a detailed discussion of these notions in the context of von Neumann-Gale dynamics we refer the reader to [2]. An authoritative reference for various aspects of capital growth theory is Györfi et al. [19].

This paper concentrates on the case of a finite time horizon. It generalizes to general random cones $X_t(\omega)$ the results obtained in [15] (also for a finite-horizon case) in a setting where $X_t(\omega)$ are standard non-negative cones $\mathbb{R}_+^{m_t}$. A central result of this work is Theorem 1 establishing for each N the existence of a path $x_0, x_1, ..., x_N$ maximizing a functional of the form $E \ln \psi(x_N)$ (log-optimal path) and showing that this path is rapid. Extensions to an infinite time horizon, substantially relying upon the results of the present paper, are considered in [2].

The plan of the paper is as follows. The main assumptions and results are formulated in Section 2. In Section 3 we discuss general properties of rapid paths. Section 4 contains some auxiliary results needed for the proof of the main result, which is given in Section 5. Section 6 analyzes a model of a financial market with transaction costs and portfolio constraints which is based on von Neumann-Gale dynamical systems and to which the results of this paper can be applied.

2 Main results

Let $|\cdot|$ denote the norm of a vector in a finite-dimensional space defined as the sum of the absolute values of its coordinates. For a finite-dimensional vector a, we will denote by $\mathbb{B}(a,r)$ the ball $\{b : |b-a| \leq r\}$. Throughout the paper it will be assumed that conditions (**A1**) - (**A4**) we list below hold.

(**A1**) For every $t = 0, 1, ..., N$, there exists an $\mathcal{F}_t$-measurable random vector $q_t(\omega) \in X_t^*(\omega)$ satisfying

$$H_t(\omega)^{-1}|a| \leq q_t(\omega)a \leq H_t(\omega)|a|, \ a \in X_t(\omega), \ \omega \in \Omega, \tag{11}$$

where $H_t(\omega) \geq 1$ is an $\mathcal{F}_t$-measurable function with $E \ln H_t(\omega) < \infty$.

This condition implies, in particular, that the cone $X_t(\omega)$ is *pointed*, i.e., if $a \in X_t(\omega)$ and $-a \in X_t(\omega)$, then $a = 0$.

(**A2**) For every $t = 1, ..., N$, $\omega \in \Omega$ and $a \in X_{t-1}(\omega)$, there exists $b \in X_t(\omega)$ such that $(a, b) \in Z_t(\omega)$.

(**A3**) There exist constants K_t $(t = 1, ..., N)$ such that $|b| \le K_t |a|$ for any $(a, b) \in Z_t(\omega)$ and $\omega \in \Omega$.

(**A4**) For each $t = 1, 2, ..., N$, there exists a bounded $\mathcal{F}_t$-measurable vector function $\mathring{z}_t = (\mathring{x}_t, \mathring{y}_t)$ such that for all $\omega \in \Omega$, we have

$$(\mathring{x}_t(\omega), \mathring{y}_t(\omega)) \in Z_t(\omega), \tag{12}$$

and

$$\mathbb{B}(\mathring{y}_t(\omega), \varepsilon_t) \subseteq X_t(\omega), \tag{13}$$

where $\varepsilon_t > 0$ is some constant.

For a real-valued function $\psi(\omega, a)$ of $\omega \in \Omega$ and $a \in X_t(\omega)$ $(t = 0, 1, ..., N)$, denote by $\bar{\psi}(\omega, a)$ the function of $\omega \in \Omega$ and $a \in \mathbb{R}^{m_t}$ defined by

$$\bar{\psi}(\omega, a) := \begin{cases} \psi(\omega, a) & \text{if } a \in X_t(\omega), \\ \infty & \text{if } a \in \mathbb{R}^{m_t} \setminus X_t(\omega), \end{cases}$$

where "∞" stands for a one-point compactification of $\mathbb{R}$. Denote by Ψ_t the class of real-valued functions $\psi(\omega, a) \ge 0$ of $\omega \in \Omega$ and $a \in X_t(\omega)$ meeting the following requirements:

(ψ.1) The function $\psi(\omega, \cdot)$ is continuous in $a \in X_t(\omega)$ for each ω and $\bar{\psi}_t(\omega, a)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^{m_t})$-measurable in $(\omega, a) \in \Omega \times \mathbb{R}^{m_t}$.

(ψ.2) For all $a, a' \in X_t(\omega)$, we have $\psi(\omega, a + a') \ge \psi(\omega, a) + \psi(\omega, a')$.

(ψ.3) The function $\psi(\omega, a)$ is positively homogeneous (of degree one) in $a \in X_t(\omega)$:

$$\psi(\omega, \lambda a) = \lambda \psi(\omega, a) \text{ for any } \lambda \in [0, \infty) \text{ and } a \in X_t(\omega).$$

(ψ.4) There exists a random variable $H_\psi(\omega) > 0$ such that $E|\ln H_\psi(\omega)| < \infty$ and

$$H_\psi(\omega)^{-1}|a| \le \psi(\omega, a) \le H_\psi(\omega)|a|, \ a \in X_t(\omega). \tag{14}$$

Conditions (ψ.2), (ψ.3) and inequality (14) are supposed to hold for every $\omega \in \Omega$.

Remark 1. From the non-negativity of ψ and requirements (ψ.2), (ψ.3), it follows that the function $\psi(\omega, a)$, $a \in X_t(\omega)$, is concave and *monotone* in a with respect to the partial ordering induced by the cone $X_t(\omega)$:

$$\psi(\omega, a) \le \psi(\omega, a') \text{ if } a' - a \in X_t(\omega). \tag{15}$$

Indeed, we have

$$\psi(\omega, a') = \psi(\omega, (a' - a) + a) \geq \psi(\omega, a' - a) + \psi(\omega, a) \geq \psi(\omega, a).$$

Remark 2. It follows from (ψ.4) that the expectation $E \ln \psi(\omega, x)$ is well-defined and takes values in $[-\infty, \infty)$ for any $x \in \mathcal{X}_t$. Furthermore, we have

$$E\left|\ln \psi(\omega, x(\omega))\right| < \infty \text{ for any } x \in \text{int}\mathcal{X}_t. \tag{16}$$

We write $x \in$int$\mathcal{X}_t$ (the interior of $\mathcal{X}_t$) if $\mathbb{B}(x(\omega), \varepsilon) \subseteq X_t(\omega)$ (a.s.) for some constant $\varepsilon > 0$. If $x \in$int$\mathcal{X}_t$, then $|x(\omega)| \geq \varepsilon$ (a.s.) (since the cone $X_t(\omega)$ is pointed), which yields $E \ln \psi(\omega, x(\omega)) > -\infty$ by virtue of (14). On the other hand, $E \ln \psi(\omega, x(\omega)) < +\infty$, by virtue of (14) and because $x(\omega)$ is essentially bounded.

Remark 3. Observe that the function $\psi(\omega, a) := q_t(\omega) a$, where $q_t(\omega) \in X_t^*(\omega)$ is the random vector described in (**A1**), belongs to the class Ψ_t. Examples of nonlinear functions in Ψ_t can be constructed as follows. Let $\nu(\omega, a)$ be an $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^{m_t})$-measurable function of $(\omega, a) \in \Omega \times \mathbb{R}^{m_t}$ such that $\nu(\omega, \cdot)$ is a norm in $\mathbb{R}^{m_t}$ for each ω. Since all norms in $\mathbb{R}^{m_t}$ are equivalent, there exists an $\mathcal{F}_t$-measurable function $\hat{H}(\omega) \geq 1$ such that

$$\hat{H}(\omega)^{-1}|a| \leq \nu(\omega, a) \leq \hat{H}(\omega)|a| \text{ for all } \omega \in \Omega \text{ and } \mathbb{R}^{m_t}. \tag{17}$$

Assume that $E \ln \hat{H}(\omega) < \infty$. Define

$$\psi(\omega, a) = q_t(\omega) a - \theta(\omega)\nu(\omega, a), \tag{18}$$

where $\theta(\omega) \geq 0$ is some $\mathcal{F}_t$-measurable function. Clearly, the function (18) satisfies (ψ.1) - (ψ.3). Condition (ψ.4) holds if $\theta(\omega)$ is small enough. For example, take some $\mathcal{F}_t$-measurable function $0 < \delta(\omega) \leq 1$ with $E|\ln \delta| < \infty$. If $0 \leq \theta \leq (1-\delta)H_t^{-1}\hat{H}$, where H_t is defined in (**A1**) and $\hat{H}$ in (17), then condition (ψ.4) holds with $H_\psi := \delta^{-1}H_t$. Indeed, we have

$$\delta^{-1}H_t|a| \geq H_t|a| \geq (H_t - \theta\hat{H})|a| \geq q_t a - \theta\nu(a) \geq (H_t^{-1} - \theta\hat{H}^{-1})|a|$$

$$\geq [H_t^{-1} - (1-\delta)H_t^{-1}\hat{H} \cdot \hat{H}^{-1}]|a| = \delta H_t^{-1}|a|.$$

Let x_0 be an $\mathcal{F}_0$-measurable vector function such that $\mathbb{B}(x_0(\omega), \varepsilon_0) \subseteq X_0(\omega)$, where $\varepsilon_0 > 0$ is some constant. The random vector x_0 will be fixed in the remainder of the paper. Denote by $\Pi(x_0, N)$ the set of paths

$\xi = (x_0, ..., x_N)$ starting from the given initial state x_0. Fix some function $\psi_N(\omega, x)$ in Ψ_N and for each path $\xi = (x_0, ..., x_N) \in \Pi(x_0, N)$, define

$$F(\xi) = E \ln \psi_N(\omega, x_N(\omega)). \tag{19}$$

The main result of this paper is as follows.

Theorem 1. *There exists a path $\bar{\xi} = (\bar{x}_0, \bar{x}_1, ..., \bar{x}_N)$ in $\Pi(x_0, N)$ that maximizes the functional $F(\xi)$ over all paths $\xi = (x_0, ..., x_N) \in \Pi(x_0, N)$. This path is rapid.*

Let us say that a path is *log-optimal* if it maximizes a functional of the form (19) with some $\psi_N \in \Psi_N$. Theorem 1 shows that any log-optimal path is rapid and thus provides an efficient method for constructing rapid paths over a finite time horizon.

3 General properties of rapid paths

The first result of this section provides several equivalent definitions of a rapid path.

Fix some $t \geq 1$, $p_t \in \mathcal{P}_t^{m_{t-1}}$, $p_{t+1} \in \mathcal{P}_{t+1}^{m_t}$ and $(x_{t-1}, x_t) \in \mathcal{Z}_t$ satisfying for almost all ω

$$p_t(\omega) \in X_{t-1}^*(\omega),\ p_{t+1}(\omega) \in X_t^*(\omega) \text{ and } p_t x_{t-1} = p_{t+1} x_t = 1. \tag{20}$$

For any pairs (x, y) of functions in $\mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ such that $(x(\omega), y(\omega)) \in Z_t(\omega)$ (a.s.), consider the following four assertions.

(I) If $p_t x > 0$ (a.s.), then

$$E(p_{t+1} y / p_t x) \leq 1. \tag{21}$$

(II) If $p_t x > 0$ (a.s.), then

$$E \ln(p_{t+1} y / p_t x) \leq 0. \tag{22}$$

(This expectation may be a non-positive real number or $-\infty$; the function $\ln r$ is defined as $-\infty$ for $r = 0$.)

(III) The inequality

$$E p_{t+1} y \leq E p_t x \tag{23}$$

holds.

(IV) With probability one, we have

$$E\left(p_{t+1}(\omega) \mid \mathcal{F}_t\right) b \leq p_t(\omega) a \tag{24}$$

for all $(a, b) \in Z_t(\omega)$.

Observe that the inequalities in (21) - (23) hold as equalities if $(x, y) = (x_{t-1}, x_t)$.

Proposition 1. *All assertions (I) - (IV) are equivalent.*

Proof. (I)⇒(II). Let $(x, y) \in \mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ such that $(x(\omega), y(\omega)) \in Z_t(\omega)$ (a.s.). By applying Jensen's inequality to the concave function $\ln r$, $r \geq 0$, and to the integrable non-negative random variable $p_{t+1}y/p_t x$, we find $E \ln\left(p_{t+1}y/p_t x\right) \leq \ln E\left(p_{t+1}y/p_t x\right) \leq 0$, which yields (22).

(II)⇒(III). For any $\theta > 0$, we have $(x_{t-1} + \theta x, x_t + \theta y) \in \mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ and $(x_{t-1}(\omega) + \theta x(\omega), x_t(\omega) + \theta y(\omega)) \in Z_t(\omega)$ (a.s.) because $Z_t(\omega)$ is a convex cone. By (20) and (22),

$$E \ln\left[(1 + \theta p_{t+1} y) / (1 + \theta p_t x)\right] = E \ln\left[p_{t+1}\left(x_t + \theta y\right)/p_t\left(x_{t-1} + \theta x\right)\right] \leq 0.$$

Consequently, $\theta^{-1} E \ln\left(1 + \theta p_{t+1} y\right) \leq \theta^{-1} E \ln\left(1 + \theta p_t x\right)$. In the limit as $\theta \to 0$, we arrive at (23).

(III)⇒(IV). By virtue of the measurable selection theorem (see, e.g., [1], Appendix I), there exists a sequence $((x_n(\omega), y_n(\omega))$, $n = 1, 2, ...$, of measurable vector functions such that $(x_n, y_n) \in \mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ and for all ω the sequence $((x_n(\omega), y_n(\omega))$ forms a dense subset of $Z_t(\omega)$. By applying (23) to (x_n, y_n) for every n, we obtain $E p_{t+1} y_n \leq E p_t x_n$. Since $Z_t(\omega)$ is a cone, this inequality also holds if we replace (x_n, y_n) by $\chi_\Gamma(x_n, y_n)$, where Γ is any set in $\mathcal{F}_t$. Then for all (x_n, y_n), with probability one we obtain

$$E\left(p_{t+1} \mid \mathcal{F}_t\right) y_n \leq p_t x_n \tag{25}$$

Let $(a, b) \in Z_t(\omega)$. Since the sequence $((x_n(\omega), y_n(\omega))$ is dense in $Z_t(\omega)$, it has a subsequence $(x_{n'}(\omega), y_{n'}(\omega))$ converging to (a, b) and satisfying with probability one

$$E\left(p_{t+1} \mid \mathcal{F}_t\right) y_{n'} \leq p_t x_{n'} \tag{26}$$

for each $n' = 1, 2, ...$ By passing to the limit in (26) as $n' \to \infty$ we obtain (24) with probability one.

(IV)⇒(I). By applying (24) to any pairs (x, y) of functions in $\mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ such that $(x(\omega), y(\omega)) \in Z_t(\omega)$ (a.s.) we have

$$E\left(p_{t+1} \mid \mathcal{F}_t\right) y \leq p_t x \text{ (a.s.).}$$

If $p_t x > 0$ (a.s.), we can divide both sides of the above inequality by $p_t x$ and obtain

$$E\left((p_{t+1} y / p_t x) \mid \mathcal{F}_t\right) \leq 1 \text{ (a.s.)}$$

(since $p_t x$ and y are $\mathcal{F}_t$-measurable), which implies (21).

The proof is complete.

Proposition 2. *Replacing (8) by any of the inequalities $E\left(p_{t+1} y / p_t x\right) \leq 1$, $E \ln\left(p_{t+1} y / p_t x\right) \leq 0$ or $E p_{t+1} y \leq E p_t x$ for any pairs (x, y) of functions in $\mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ such that $(x(\omega), y(\omega)) \in Z_t(\omega)$ (a.s.), we obtain an equivalent definition of a rapid trajectory.*

When writing the inequalities $E\left(p_{t+1} y / p_t x\right) \leq 1$ and $E \ln\left(p_{t+1} y / p_t x\right) \leq 0$, we assume that the scalar product $p_t x$ is strictly positive. This assumption is not needed when dealing with the inequality $E p_{t+1} y \leq E p_t x$.

Proof of Proposition 2. The assertion is a direct consequence of Proposition 1.

Proposition 3. *Let $p_1, p_2, ..., p_{N+1}$ be a dual path. For any path $y_0, y_1, ..., y_N$, the random sequence $(p_{t+1} y_t)_{t=0}^N$ is a supermartingale with respect to the filtration $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}_{N+1}$ and the random sequence $(\bar{p}_{t+1} y_t)_{t=0}^N$ is a supermartingale with respect to the filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_N$.*

Proof. This is immediate from the relations:

$$E_t p_{t+1} y_t = \bar{p}_{t+1} y_t \leq p_t y_{t-1} \text{ (a.s.)},\ t = 1, ..., N,$$

and

$$E_{t-1} \bar{p}_{t+1} y_t \leq E_{t-1} p_t y_{t-1} = \bar{p}_t y_{t-1} \text{ (a.s.)},\ t = 1, ..., N,$$

following from (8).

Proposition 4. *A path $(x_t)_{t=0}^N$ is rapid if and only if there exists a sequence $(l_t)_{t=1}^{N+1}$ of random vectors such that*

$$l_{t+1} \in X_t^*(\omega),\ E\left|\ln\left(l_{t+1} x_t\right)\right| < \infty,\ l_{t+1} / l_{t+1} x_t \in \mathcal{P}_{t+1}^{m_t} \tag{27}$$

for all $t = 0, 1, ..., N$, and

$$E \ln\left(l_{t+1} y / l_t x\right) \leq E \ln\left(l_{t+1} x_t / l_t x_{t-1}\right) \tag{28}$$

for any $t = 1, 2, ..., N$ and for any pairs (x, y) of functions in $\mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ such that $(x(\omega), y(\omega)) \in Z_t(\omega)$ (a.s.) with $l_t x > 0$.

This proposition characterizes rapid trajectories as those maximizing *the expectation of the logarithm* of the growth rate. It is important to note that the sequence of random vectors l_t involved in this characterization does not necessarily satisfy the normalization condition $l_{t+1}x_t = 1$ (a.s.). This is in contrast with the original definition of a rapid path, dealing with the maximization of *the expectation* of the growth rate, where the above-mentioned normalization condition is required.

Proof of Proposition 4. If the trajectory $(x_t)_{t=0}^N$ is rapid, then we can set $l_t = p_t$, where $(p)_{t=1}^{N+1}$ is a dual path supporting $(x_t)_{t=0}^N$. The conditions contained in (27) hold since $p_{t+1} \in X_t^*(\omega)$, $p_{t+1}x_t = 1$ and $p_{t+1} \in \mathcal{P}_{t+1}^{m_t}$ for any $t = 0, 1, ..., N$. Relation (28) turns into the inequality $E \ln (p_{t+1}y/p_t x) \leq 0$, which is true by virtue of Proposition 2.

Conversely, suppose conditions (27) and (28) are fulfilled. Put $p_{t+1} = l_{t+1}/l_{t+1}x_t$ $(t = 0, 1, ..., N)$. Note that $l_{t+1}x_t > 0$ (a.s.) because $E|\ln(l_{t+1}x_t)| < \infty$. Then for any $t = 0, 1, ..., N$, we have $p_{t+1} \in X_t^*(\omega)$, $p_{t+1} \in \mathcal{P}_{t+1}^{m_t}$ and $p_{t+1}x_t = 1$. Since $E\,|\ln (l_{t+1}x_t/l_t x_{t-1})| < \infty$, it follows from (28) that

$$E\left[\ln (l_{t+1}y/l_t x) - \ln (l_{t+1}x_t/l_t x_{t-1})\right] \leq 0,$$

which implies $E \ln (p_{t+1}y/p_t x) \leq 0$. This yields (8) by virtue of the implication (II)$\Rightarrow$(IV) proved in Proposition 1.

The proposition is proved.

4 Auxiliary results

Before proving Theorem 1 we establish several lemmas.

Lemma 1. *The functional $F(\xi)$ attains its maximum over the set $\Pi\,(x_0, N)$ of paths $(x_0, ..., x_N)$.*

Proof. Let us regard the class $\Pi\,(x_0, N)$ of paths as a subset of the space $\mathcal{L} := \mathcal{L}_0^{m_0} \bigoplus \mathcal{L}_1^{m_1} \bigoplus ... \bigoplus \mathcal{L}_N^{m_N}$. The subset $\Pi\,(x_0, N)$ is bounded in $\mathcal{L}$ by virtue of (**A3**). Furthermore, $\Pi\,(x_0, N)$ is convex and closed in $\mathcal{L}$ with respect to a.s. convergence because the cones $Z_t\,(\omega)$ and $X_t(\omega)$ are convex and closed for each ω. The functional $F\,(\xi) = E\psi_N\,(\omega, x_N\,(\omega))$, defined for $\xi = (x_0, ..., x_N) \in \Pi\,(x_0, N)$, is concave (which follows from the concavity of $\psi_N\,(\omega, \cdot)$) and upper semicontinuous with respect to a.s. convergence by virtue of condition (ψ.4) and Fatou's lemma. This is sufficient to conclude that F achieves a maximum on $\Pi\,(x_0, N)$; see, e.g., [1], Appendix III, Theorem 5.

The lemma is proved.

Define:

$$\mathcal{U}_t := \{u_t \in \mathcal{L}_t^{m_{t-1}} : \ u_t(\omega) \in X_{t-1}(\omega) \text{ (a.s.)}\}, \ t = 1, 2, ..., N+1; \tag{29}$$

$$\mathcal{V}_t := \{v_t \in \mathcal{L}_t^{m_t} : \ v_t(\omega) \in X_t(\omega) \text{ (a.s.)}\} \ [= \mathcal{X}_t], \ t = 0, 1, ..., N; \tag{30}$$

$$\mathcal{W}_t := \{(u_t, v_t) \in \mathcal{U}_t \times \mathcal{V}_t : (u_t(\omega), v_t(\omega)) \in Z_t(\omega) \text{ (a.s.)}, \ t = 1, ..., N, \tag{31}$$

and denote by $\mathcal{W}$ the set of sequences

$$\zeta = (v_0, u_1, v_1, ..., u_N, v_N, u_{N+1}) \tag{32}$$

with $v_0 = x_0$ such that

$$v_t \in \mathcal{V}_t, \ t = 0, ..., N, \tag{33}$$

$$u_t \in U_t, \ t = 1, 2, ..., N+1, \tag{34}$$

$$(u_t, v_t) \in \mathcal{W}_t \ , t = 1, ..., N, \tag{35}$$

and

$$E \ln \psi_N (\omega, u_{N+1}(\omega)) > -\infty. \tag{36}$$

Observe that the set $\mathcal{U}_t$, as well as $\mathcal{X}_{t-1}$, consists of random vectors $u_t(\omega)$ whose values belong to $X_{t-1}(\omega)$ (a.s.), but these random vectors are measurable with respect to $\mathcal{F}_t$ rather than $\mathcal{F}_{t-1}$, so that $\mathcal{U}_t \supset \mathcal{X}_{t-1} = \mathcal{V}_{t-1}$. Also, note that we defined $\mathcal{F}_{N+1}$ as $\mathcal{F}_N$, therefore $\mathcal{U}_{N+1} = \mathcal{X}_N = \mathcal{V}_N$.

Lemma 2. *For every $u \in \mathcal{U}_t$, there exists $v \in \mathcal{V}_t$ such that $(u, v) \in \mathcal{W}_t$.*

Proof. Consider some $u \in \mathcal{U}_t$. Since $u(\omega) \in X_{t-1}(\omega)$ (a.s.), we can change $u(\omega)$ on a set of measure zero and obtain an $\mathcal{F}_t$-measurable vector function $u'(\omega)$ such that $u'(\omega) = u(\omega)$ (a.s.) and $u'(\omega) \in X_{t-1}(\omega)$ for all ω. It follows from (**A2**) that for each ω, there exists $b \in X_t(\omega)$ for which $(u'(\omega), b) \in Z_t(\omega)$. Therefore we can apply the measurable selection theorem and construct an $\mathcal{F}_t$-measurable vector $v(\omega)$ such that $(u(\omega), v(\omega)) \in Z_t(\omega)$ (a.s.). It follows from (**A3**) that $v(\omega)$ is essentially bounded. Consequently, $v \in \mathcal{V}_t$, $(u', v) \in \mathcal{W}_t$, and therefore $(u, v) \in \mathcal{W}_t$. The proof is complete.

Lemma 3. *Let $\zeta = (v_0, u_1, v_1, ..., u_N, v_N, u_{N+1})$ be a sequence in $\mathcal{W}$ satisfying*

$$v_{t-1} - u_t \in \mathcal{U}_t \ , t = 1, ..., N+1. \tag{37}$$

Then there is a path $(y_0, ..., y_N)$ such that $y_0 = v_0$ and

$$y_t - v_t \in \mathcal{X}_t, \ t = 0, 1, ..., N.$$

Proof. Let us proceed by induction. Put $y_0 = v_0$. Suppose we have constructed $y_0, y_1, ..., y_n$ $(0 \leq n < N)$ satisfying

$$(y_{t-1}, y_t) \in \mathcal{Z}_t,\ 1 \leq t \leq n; \tag{38}$$

$$y_t - v_t \in \mathcal{X}_t,\ 0 \leq t \leq n. \tag{39}$$

(In the case of $n = 0$, the constraint in (38) is absent.) Let us construct y_{n+1} for which the inclusions in (38) and (39) hold for $t = n + 1$. Define $g_{n+1} := y_n - u_{n+1}$. By virtue of (39), we have $y_n - v_n \in \mathcal{X}_n \subseteq \mathcal{U}_{n+1}$. From (37) we get $v_n - u_{n+1} \in \mathcal{U}_{n+1}$. Therefore

$$g_{n+1} = y_n - u_{n+1} = (y_n - v_n) + (v_n - u_{n+1}) \in \mathcal{U}_{n+1}.$$

By applying Lemma 2, we construct $h_{n+1} \in \mathcal{V}_{n+1}$ such that $(g_{n+1}, h_{n+1}) \in \mathcal{W}_{n+1}$.

Put $y_{n+1} := v_{n+1} + h_{n+1}$. We have $(u_{n+1}, v_{n+1}) \in \mathcal{W}_{n+1}$, $(g_{n+1}, h_{n+1}) \in \mathcal{W}_{n+1}$, and so

$$(y_n, y_{n+1}) = (u_{n+1}, v_{n+1}) + (g_{n+1}, h_{n+1}) \in \mathcal{W}_{n+1}.$$

Since y_n is F_n-measurable, this means that $(y_n, y_{n+1}) \in \mathcal{Z}_{n+1}$, i.e., (38) holds for $t = n + 1$. Furthermore

$$y_{n+1} - v_{n+1} = h_{n+1} \in \mathcal{V}_{n+1} = \mathcal{X}_{n+1},$$

which gives (39) for $t = n+1$. Arguing by induction, we construct the desired path $y_0, ..., y_N$.

The lemma is proved.

Lemma 4. *There exists a sequence* $\mathring{\zeta} = (\mathring{v}_0, \mathring{u}_1, \mathring{v}_1, ..., \mathring{u}_N, \mathring{v}_N, \mathring{u}_{N+1}) \in \mathcal{W}$ *such that* $\mathring{v}_0 = x_0$*,*

$$\mathring{v}_{t-1} - \mathring{u}_t \in \mathrm{int}\mathcal{U}_t\ , t = 1, ..., N + 1, \tag{40}$$

and $\mathring{u}_{N+1} \in \mathrm{int}\mathcal{X}_N$*. Furthermore, there exists a path* $\mathring{\xi} = (\mathring{x}_0, \mathring{x}_1, ..., \mathring{x}_N) \in \Pi\,(x_0, N)$ *for which* $\mathring{x}_t \in \mathrm{int}\mathcal{X}_t$*,* $1 \leq t \leq N$*.*

Here we denote by int $\mathcal{U}_t$ the interior of the set $\mathcal{U}_t$ in the topology of the space $\mathcal{L}_t^{m_{t-1}}$. Clearly, a vector function $u \in \mathcal{L}_t^{m_{t-1}}$ belongs to int$\mathcal{U}_t$ if and only if there exists a constant $\varepsilon > 0$ such that $\mathbb{B}(u(\omega), \varepsilon) \in X_{t-1}(\omega)$ (a.s.).

Proof. Let us argue by induction. Put $\mathring{v}_0 = x_0$. Suppose we have constructed random vectors $\mathring{v}_t \in \mathcal{V}_t$, $t = 0, ..., n$ $(0 \le n \le N-1)$ such that

$$\mathring{v}_t \in \mathrm{int}\mathcal{U}_{t+1},\ t = 0, ..., n; \tag{41}$$

and for some $\mathring{u}_t \in \mathcal{U}_t$, $t = 1, ..., n$, we have

$$(\mathring{u}_t, \mathring{v}_t) \in \mathcal{W}_t,\ \mathring{v}_{t-1} - \mathring{u}_t \in \mathrm{int}\mathcal{U}_t,\ t = 1, ..., n. \tag{42}$$

(For $n = 0$, condition (42) does not make sense and is omitted.) Let us construct $\mathring{v}_{n+1} \in \mathcal{V}_{n+1}$ and $\mathring{u}_{n+1} \in \mathcal{U}_{n+1}$ for which the inclusions in (41) and (42) would hold with $t = n+1$.

Consider the pair of random vectors $(\mathring{x}_{n+1}(\omega), \mathring{y}_{n+1}(\omega))$ described in (**A4**). Since $\mathring{x}_{n+1} \in \mathcal{U}_{n+1}$ and $\mathring{v}_n \in$int$\mathcal{U}_{n+1}$, there exists a sufficiently small number $\lambda > 0$, for which $\mathring{v}_n - \lambda\mathring{x}_{n+1} \in$int$\mathcal{U}_{n+1}$. Indeed, if $\mathring{v}_n \in$int$\mathcal{U}_{n+1}$, then $\mathbb{B}(\mathring{v}_n(\omega), \delta) \subseteq X_n(\omega)$ (a.s.) for some $\delta > 0$. By setting $\lambda = \delta/2H$, where H is a constant satisfying $|\mathring{x}_{n+1}| \le H$ (a.s.), we obtain that

$$\mathbb{B}(\mathring{v}_n(\omega) - \lambda\mathring{x}_{n+1}(\omega), \delta/2) \subseteq \mathbb{B}(\mathring{v}_n(\omega), \delta) \subseteq X_n(\omega) \text{ (a.s.)},$$

i.e. $\mathring{v}_n - \lambda\mathring{x}_{n+1} \in$int$\mathcal{U}_{n+1}$. By defining $\mathring{v}_{n+1} := \lambda\mathring{y}_{n+1}$ and $\mathring{u}_{n+1} = \lambda\mathring{x}_{n+1}$, we obtain (41) and (42) for $t = n+1$.

By applying the above induction argument, we construct a sequence $\mathring{v}_0, \mathring{u}_1, \mathring{v}_1, ..., \mathring{u}_N, \mathring{v}_N$ satisfying (41) and (42) with $n = N$. It remains to define $\mathring{u}_{N+1} := \mathring{v}_N/2$. Then $\mathring{u}_{N+1} \in$int$\mathcal{X}_N$ and so $E\ln\psi_{N+1}(\omega, \mathring{u}_{N+1}(\omega)) > -\infty$, see Remark 2. Thus $(\mathring{v}_0, \mathring{u}_1, \mathring{v}_1, ..., \mathring{u}_N, \mathring{v}_N, \mathring{u}_{N+1}) \in \mathcal{W}$.

By virtue of Lemma 3 there exists a path $(\mathring{x}_0, \mathring{x}_1, ..., \mathring{x}_N) \in \Pi(x_0, N)$ for which $\mathring{x}_0 = x_0$ and $\mathring{x}_t - \mathring{v}_t \in \mathcal{X}_t$, $t = 0, 1, ..., N$. Since $\mathring{v}_{t-1} - \mathring{u}_t \in$int $\mathcal{U}_t$ $(t = 1, ..., N+1)$ and $\mathring{u}_t \in \mathcal{U}_t$, we have $\mathring{v}_{t-1} \in$int $\mathcal{U}_t$, and so $\mathring{v}_{t-1} \in$int$\mathcal{X}_{t-1}$. Thus

$$\mathring{x}_t - \mathring{v}_t \in \mathcal{X}_t,\ \mathring{v}_t \in \mathrm{int}\mathcal{X}_t,\ t = 0, 1, ..., N,$$

which yields $\mathring{x}_t = (\ \mathring{x}_t - \mathring{v}_t) + \mathring{v}_t \in$int$\mathcal{X}_t$ for $t = 0, 1, ..., N$.

The proof is complete.

For each sequence $\zeta = (v_0, u_1, v_1, ..., u_N, v_N, u_{N+1}) \in \mathcal{W}$ define

$$G(\zeta) = E\ln\psi_N(\omega, u_{N+1}(\omega))$$

and

$$h(\zeta) := (v_0 - u_1, ..., v_N - u_{N+1}).$$

The mapping h acts from the set $\mathcal{W}$ into the linear space

$$\mathcal{Y} := \mathcal{L}_1^{m_0} \times \mathcal{L}_2^{m_1} \times ... \times \mathcal{L}_{N+1}^{m_N}.$$

Put

$$\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2 \times ... \times \mathcal{U}_{N+1}.$$

We will show that the path $\bar{\xi}$ constructed in Lemma 1 is rapid by analyzing the following stochastic optimization problem:

(**P**) Maximize the functional $G(\zeta)$ over the set of sequences

$$\zeta = (v_0, u_1, v_1, ..., u_N, v_N, u_{N+1}) \in \mathcal{W}$$

satisfying

$$h(\zeta) \in \mathcal{U}. \tag{43}$$

The following lemma shows that the path $\bar{\xi}$ generates a solution to this problem.

Lemma 5. *Let $\bar{\xi} = (\bar{x}_0, \bar{x}_1, ..., \bar{x}_N)$ be a path maximizing the functional $F(\xi) = E \ln \psi_N(\omega, x_N(\omega))$ over the set $\Pi(x_0, N)$ of paths $\xi = (x_0, x_1, ..., x_N)$. Then the sequence $\bar{\zeta} = (\bar{x}_0, \bar{x}_0, \bar{x}_1, \bar{x}_1, ..., \bar{x}_{N-1}, \bar{x}_N, \bar{x}_N) \in \mathcal{W}$ is a solution to the optimization problem (**P**), and $F(\bar{\xi}) = G(\bar{\zeta})$.*

Proof. First of all, $\bar{\zeta} \in \mathcal{W}$ since $(\bar{x}_{t-1}, \bar{x}_t) \in \mathcal{Z}_t$ for $t = 1, 2, ..., N$ and

$$G(\bar{\zeta}) = F(\bar{\xi}) = E \ln \psi_N(\omega, \bar{x}_N(\omega)) \geq F(\mathring{\xi}) = E \ln \psi_N(\omega, \mathring{x}_N(\omega)) > -\infty$$

(see Lemma 4 and (16)). Furthermore, $h\left(\bar{\zeta}\right) = 0 \in \mathcal{U}$, so that the constraint (43) is satisfied. Consider any sequence $\zeta = (v_0, u_1, v_1, ..., u_N, v_N, u_{N+1})$ in $\mathcal{W}$ for which $h(\zeta) \in \mathcal{U}$, i.e. constraints (37) hold. By virtue of Lemma 3, there is a path $\eta = (y_0, ..., y_N)$ such that $y_0 = x_0$ and $y_t - v_t \in \mathcal{X}_t$, $t = 0, 1, ..., N$. For this path, $y_N - v_N \in \mathcal{X}_N$, and so

$$y_N - u_{N+1} \in \mathcal{X}_N \tag{44}$$

because $v_N - u_{N+1} \in \mathcal{X}_N$. Using the monotonicity of $\psi_N(\omega, \cdot)$ (see (15)), we obtain

$$G(\zeta) = E\psi_N(\omega, u_{N+1}(\omega)) \leq E\psi_N(\omega, y_N(\omega)) = F(\eta) \leq F(\bar{\xi}) = G(\bar{\zeta}),$$

which proves the lemma.

5 Existence of rapid paths

Proof of Theorem 1. The existence of the path $\bar{\xi} = (\bar{x}_0, \bar{x}_1, ..., \bar{x}_N)$ maximizing the functional (19) was established in Lemma 1. To show that $\bar{\xi}$ is rapid we will apply to the optimization problem (**P**) a general version of the Kuhn-Tucker theorem established, e.g., in [20], Theorem 5.3.1. The set $\mathcal{W}$ is convex. The set $\mathcal{U}$ is a convex cone. The mapping g is linear. The functional $G(\zeta)$, $\zeta \in \mathcal{W}$, is concave and takes on real values. Thus, in order to justify the use of the Kuhn-Tucker theorem we have to check Slater's condition:

(**S**) There is an element $\mathring{\zeta} \in \mathcal{W}$ such that $g(\mathring{\zeta})$ belongs to the interior int$\mathcal{U}$ of the cone $\mathcal{U}$ in the topology of the space $\mathcal{Y}$.

This condition holds because the sequence $\mathring{\zeta} \in \mathcal{W}$ constructed in Lemma 4 possesses the properties listed in (**S**): relations (42) mean that $g(\mathring{\zeta}) \in$int$\mathcal{U}$.

By the Kuhn-Tucker theorem applied to problem (**P**), there exists a continuous linear functional π on the space $\mathcal{Y}$ such that $\langle \pi, y \rangle \geq 0$ for $y \in \mathcal{U}$ and

$$G(\zeta) + \langle \pi, h(\zeta) \rangle \leq G(\bar{\zeta})$$

for any $\zeta \in \mathcal{W}$. The functional π can be represented in the form $\pi = (\pi_1, ..., \pi_{N+1})$, where π_t is a continuous linear functional on the space $\mathcal{L}_t^{m_{t-1}} = L_\infty(\Omega, \mathcal{F}_t, P, \mathbb{R}^{m_{t-1}})$ such that $\pi_t \in (\mathcal{U}_t)^*$, i.e.,

$$\langle \pi_t, u_t \rangle \geq 0 \text{ for } u_t \in \mathcal{U}_t \ (t = 1, ..., N+1). \tag{45}$$

Thus we have

$$E \ln \psi_N (\omega, u_{N+1}(\omega)) + \sum_{t=1}^{N+1} \langle \pi_t, v_{t-1} - u_t \rangle$$

$$\leq E \ln \psi_N (\omega, \bar{x}_N(\omega)) \ \ [= G(\bar{\zeta}) = F(\bar{\xi})]. \tag{46}$$

for any $\zeta = (v_0, u_1, v_1, ..., u_N, v_N, u_{N+1}) \in \mathcal{W}$.

By virtue of the Yosida-Hewitt theorem [43], each of the functionals π_t can be decomposed into the sum $\pi_t = \pi_t^a + \pi_t^s$ of two functionals π_t^a, $\pi_t^s \in (\mathcal{L}_t^{m_{t-1}})^*$, where π_t^a is *absolutely continuous* and π_t^s is *singular*. According to the definitions of π_t^a and π_t^s, there is a vector function $p_t \in \mathcal{P}_t^{m_{t-1}} = L_1(\Omega, \mathcal{F}_t, P, \mathbb{R}^{m_{t-1}})$ such that

$$\langle \pi_t^a, y \rangle = E p_t y, \ \ y \in \mathcal{L}_t^{m_{t-1}}, \tag{47}$$

and there exist sets $\Gamma_t^1 \supseteq \Gamma_t^2 \supseteq ...$ in $\mathcal{F}_t$ for which $P\left(\Gamma_t^k\right) \to 0$ as $k \to \infty$ and

$$\left\langle \pi_t^s, y\chi_{\Gamma_t^k} \right\rangle = \langle \pi_t^s, y \rangle\,,\ y \in \mathcal{L}_t^{m_{t-1}}, \tag{48}$$

where $\chi_{\Gamma_t^k}$ is the indicator function of the set Γ_t^k.

Observe that relation (45) remains valid if we replace π_t by π_t^a. Indeed, the inclusion $u_t \in \mathcal{U}_t$ means that $u_t \in \mathcal{L}_t^{m_{t-1}}$ and $u_t(\omega) \in X_{t-1}(\omega)$ (a.s.). This implies $u_t\chi_{\Delta_t^k} \in \mathcal{L}_t^{m_{t-1}}$ and $u_t(\omega)\chi_{\Delta_t^k}(\omega) \in X_{t-1}(\omega)$ (a.s.), where $\Delta_t^k := \Omega \setminus \Gamma_t^k$. Consequently, we have

$$0 \le \left\langle \pi_t, u_t\chi_{\Delta_t^k} \right\rangle = \left\langle \pi_t^a, u_t\chi_{\Delta_t^k} \right\rangle + \left\langle \pi_t^s, u_t\chi_{\Delta_t^k} \right\rangle = \left\langle \pi_t^a, u_t\chi_{\Delta_t^k} \right\rangle, \tag{49}$$

where $\left\langle \pi_t^a, u_t\chi_{\Delta_t^k} \right\rangle \to \langle \pi_t^a, u_t \rangle$. By passing to the limit in (49), we obtain that

$$Ep_t u_t = \langle \pi_t^a, u_t \rangle \ge 0,\ u_t \in \mathcal{U}_t. \tag{50}$$

Since $\mathcal{U}_t$ consists of those functions $u_t \in \mathcal{L}_t^{m_{t-1}}$ for which $u_t(\omega) \in X_{t-1}(\omega)$ (a.s.), inequality (50) implies $p_t(\omega) \in X_{t-1}^*(\omega)$ (a.s.), $t = 1, 2, ..., N+1$, i.e. condition (7) holds.

Furthermore, if (45) holds, then

$$\langle \pi_t^s, u_t \rangle \ge 0 \text{ for } u_t \in \mathcal{U}_t\ (t = 1, ..., N+1). \tag{51}$$

Indeed, by virtue of (48) we have

$$0 \le \left\langle \pi_t, u_t\chi_{\Gamma_t^k} \right\rangle = \left\langle \pi_t^a, u_t\chi_{\Gamma_t^k} \right\rangle + \left\langle \pi_t^s, u_t\chi_{\Gamma_t^k} \right\rangle = \left\langle \pi_t^a, u_t\chi_{\Gamma_t^k} \right\rangle + \langle \pi_t^s, u_t \rangle\,,$$

and so

$$\langle \pi_t^s, u_t \rangle \ge -\left\langle \pi_t^a, u_t\chi_{\Gamma_t^k} \right\rangle \to 0,$$

which proves (51).

Let us show that relation (46) remains valid if we replace π_t by π_t^a. We will prove this by way of induction. Fix some $\zeta = (v_0, u_1, v_1, ..., u_N, v_N, u_{N+1}) \in \mathcal{W}$ and consider the inequality

$$E \ln \psi_N\left(\omega, u_{N+1}(\omega)\right) + \sum_{t=1}^{N+1} \langle \pi_t^a, v_{t-1} - u_t \rangle$$

$$+\sum_{t=1}^{M}\langle\pi_t^s, v_{t-1}-u_t\rangle \leq E\ln\psi_N(\omega,\bar{x}_N(\omega)) \tag{52}$$

where $M\in\{0,...,N+1\}$. If $M=0$, then the second sum in (52) is formally defined as 0. For $M=N+1$, relation (52) is equivalent to (46). Suppose inequality (52) is true for some $M\in\{1,...,N+1\}$. Let us show that this inequality is true for $M-1$.

Since $P\left(\Gamma_M^k\right)\to 0$ as $k\to\infty$, we can find a sequence of real numbers $\epsilon_k\in(0,1)$ such that $\epsilon_k\to 0$ and

$$P\left(\Gamma_M^k\right)\ln\epsilon_k\to 0 \tag{53}$$

(e.g., we can define $\epsilon_k:=\exp(-\mu_k^{-1/2})$). Put

$$\Delta_M^k=:\Omega\setminus\Gamma_M^k;\ \gamma_M^k(\omega):=\epsilon_k\chi_{\Gamma_M^k}(\omega)+\chi_{\Delta_M^k}(\omega);$$

$$(u_t^k,v_t^k):=\gamma_M^k(u_t,v_t),\ M\leq t\leq N,\ u_{N+1}:=\gamma_M^k u_{N+1};$$
$$v_0^k=:v_0,\ (u_t^k,v_t^k):=(u_t,v_t),\ 1\leq t<M;$$

and

$$\zeta^k:=(v_0^k,u_1^k,v_1^k,...,u_N^k,v_N^k,u_{N+1}^k)\ (k=1,2,...).$$

We can see that

$$\left(u_t^k(\omega),v_t^k(\omega)\right)\in Z_t(\omega)\ \text{(a.s.)},$$

and so $(u_t^k,v_t^k)\in\mathcal{W}_t$. Furthermore,

$$v_0^k=v_0=x_0,\ u_{N+1}^k=\gamma_M^k u_{N+1}\in\mathcal{U}_{N+1}=\mathcal{X}_N=\mathcal{V}_N,$$

and

$$\begin{aligned}E\ln\psi_N\left(\omega,u_{N+1}^k(\omega)\right)&=E\ln\gamma_M^k(\omega)+E\ln\psi_N(\omega,u_{N+1}(\omega))\\&=P\left(\Gamma_M^k\right)\ln\epsilon_k+E\ln\psi_N(\omega,u_{N+1}(\omega)),\end{aligned} \tag{54}$$

therefore $E\ln\psi_N\left(\omega,u_{N+1}^k(\omega)\right)>-\infty$. Consequently, the sequence ζ^k belongs to $\mathcal{W}$, and we can apply inequality (52) to this sequence.

This yields

$$E\ln\psi_N\left(\omega,u_{N+1}^k(\omega)\right)+\sum_{t=1}^{N+1}\left\langle\pi_t^a,v_{t-1}^k-u_t^k\right\rangle+\sum_{t=0}^{M-1}\langle\pi_t^s,v_{t-1}-u_t\rangle+$$
$$\left\langle\pi_M^s,v_{M-1}-u_M^k\right\rangle\leq E\ln\psi_N(\omega,\bar{x}_N(\omega)). \tag{55}$$

Here, the vectors u_t^k, v_t^k are uniformly bounded, and we have $u_t^k \to u_t$, $v_t^k \to v_t$ (a.s.). Consequently,

$$\left\langle \pi_t^a, v_{t-1}^k - u_t^k \right\rangle = Ep_t \left(v_{t-1}^k - u_t^k \right) \to Ep_t \left(v_{t-1} - u_t \right)$$

for each t. By virtue of (54) and (53), $E \ln \psi_N \left(\omega, u_N^k(\omega) \right) \to E \ln \psi_N \left(\omega, u_N(\omega) \right)$. Furthermore,

$$\left\langle \pi_M^s, v_{M-1} - u_M^k \right\rangle \geq - \left\langle \pi_M^s, u_M^k \right\rangle = - \left\langle \pi_M^s, \chi_{\Gamma_M^k} u_M^k \right\rangle = -\epsilon_k \left\langle \pi_M^s, u_M \right\rangle \to 0,$$

where the inequality in this chain of relations follows from (51) because $v_{M-1} \in V_{M-1} \subseteq U_M$. Thus, by passing to the limit in (55), we conclude that inequality (52) remains true if we replace M by $M-1$.

We have constructed a sequence of functions $p_t \in \mathcal{P}_t$, $t = 1, ..., N+1$, such that

$$E \ln \psi_N \left(\omega, u_{N+1}(\omega) \right) + \sum_{t=1}^{N+1} Ep_t (v_{t-1} - u_t)$$
$$\leq E \ln \psi_N \left(\omega, \bar{x}_N(\omega) \right) \ \ [= G(\bar{\zeta}) = F(\bar{\xi})] \tag{56}$$

for any $\zeta = (v_0, u_1, v_1, ..., u_N, v_N, u_{N+1}) \in \mathcal{W}$. Let us show that $(p_1, ..., p_{N+1})$ is a dual path supporting the path $\bar{\xi}$. We can write the sum in (56) as

$$\sum_{t=1}^{N+1} Ep_t (v_{t-1} - u_t) = -Ep_{N+1} u_{N+1} + \sum_{t=1}^{N} E \left(p_{t+1} v_t - p_t u_t \right) + Ep_1 v_0,$$

where $Ep_1 v_0 = Ep_1 x_0 = Ep_1 \bar{x}_0$, and since

$$-Ep_{N+1} \bar{x}_N + \sum_{t=1}^{N} E \left(p_{t+1} \bar{x}_t - p_t \bar{x}_{t-1} \right) + Ep_1 \bar{x}_0 = 0,$$

we can see that inequality (56) implies

$$E \ln \psi_N \left(\omega, u_{N+1}(\omega) \right) - Ep_{N+1} u_{N+1} + \sum_{t=1}^{N} E \left(p_t v_{t-1} - p_t u_t \right) + Ep_1 x_0$$

$$\leq E \ln \psi_N \left(\omega, \bar{x}_N(\omega) \right) - Ep_{N+1} \bar{x}_N + \sum_{t=1}^{N} E \left(p_{t+1} \bar{x}_t - p_t \bar{x}_{t-1} \right) + Ep_1 x_0.$$

In turn, this yields

$$E\left(p_{t+1}v_t - p_t u_t\right) \leq E\left(p_{t+1}\bar{x}_t - p_t\bar{x}_{t-1}\right),\ (u_t, v_t) \in \mathcal{W}_t,\ t = 1, ..., N, \quad (57)$$

and

$$E \ln \psi_N(u) - Ep_{N+1}u \leq E \ln \psi_N(\bar{x}_N) - Ep_{N+1}\bar{x}_N,\ u \in \mathcal{U}_{N+1} = \mathcal{X}_N. \quad (58)$$

Recall that $\mathcal{W}_t$ consists of $(u_t, v_t) \in \mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ for which $(u_t(\omega), v_t(\omega)) \in Z_t(\omega)$ (a.s.). The set $Z_t(\omega)$ is a cone, consequently, inequality (57) will remain valid if we multiply (u_t, v_t) by any positive constant. This and the fact that $(\bar{x}_{t-1}(\omega), \bar{x}_t(\omega)) \in Z_t(\omega)$ (a.s.), yields

$$Ep_{t+1}v_t - Ep_t u_t \leq 0 = Ep_{t+1}\bar{x}_t - Ep_t\bar{x}_{t-1},\ t = 1, ..., N, \quad (59)$$

for all pairs (u_t, v_t) of functions in $\mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ such that $(u_t(\omega), v_t(\omega)) \in Z_t(\omega)$ (a.s.). By virtue of Proposition 2, this means that $(p_1, ..., p_{N+1})$ is a dual path.

It remains to prove that the dual path we have constructed supports the path $\bar{\xi}$. To this end let us show, by using (58), that $p_{N+1}\bar{x}_N = 1$ (a.s.). Put

$$\gamma := p_{N+1}\bar{x}_N,\ \delta_k := [\gamma + k^{-1}]^{-1},\ u^k := \delta_k \bar{x}_N.$$

The random variable γ is a.s. non-negative, $\mathcal{F}_N$-measurable (because $\mathcal{F}_N = \mathcal{F}_{N+1}$) and integrable because p_{N+1} is integrable and $\bar{x}_N$ is essentially bounded. The random variable δ_k is $\mathcal{F}_N$-measurable and satisfies $0 \leq \delta_k \leq k$ (a.s.). Consequently, $u^k(\omega) \in X_N(\omega)$ (a.s.), and we can apply inequality (58) to $u = u^k$. We have

$$E \ln \psi_N\left(u^k\right) - Ep_{N+1}u^k = E \ln \delta_k + E \ln \psi_N(\bar{x}_N) - E\delta_k\gamma.$$

Since $-\infty < E \ln \psi_N(\bar{x}_N) < +\infty$, it follows from this equality and (58) that

$$E \ln \delta_k \leq E\delta_k\gamma - E\gamma.$$

We have $0 \leq \delta_k\gamma = \gamma[\gamma + k^{-1}]^{-1} \leq 1$ (a.s.), and so $\lim E\delta_k\gamma = 1$. By using Fatou's lemma, we obtain

$$-E \ln \gamma = E \liminf \ln \delta_k \leq \liminf E \ln \delta_k \leq 1 - E\gamma.$$

The use of Fatou's lemma is justified because

$$\ln \delta_k = -\ln(\gamma + k^{-1}) \geq -\ln(\gamma + 1) \geq -\gamma,$$

where γ is integrable. Thus we obtain $-E \ln \gamma \leq 1 - E\gamma$, or equivalently, $E(\gamma - 1 - \ln \gamma) \leq 0$. At the same time, we always have $\gamma - 1 - \ln \gamma \geq 0$. Therefore $p_{N+1}\bar{x}_N = \gamma = 1$ (a.s.).

By virtue of the equality in (59), we have

$$Ep_1\bar{x}_0 = Ep_2\bar{x}_1 = ... = Ep_{N+1}\bar{x}_N. \tag{60}$$

We can replace in (59) (u_t, v_t) by $\chi_\Gamma (u_t, v_t)$, where Γ is any set in $\mathcal{F}_t$. This yields

$$E(p_{t+1}v_t \,|\mathcal{F}_t) - p_t u_t \leq 0 \text{ (a.s.)} \tag{61}$$

for all $(u_t, v_t) \in \mathcal{L}_t^{m_{t-1}} \times \mathcal{L}_t^{m_t}$ such that $(u_t(\omega), v_t(\omega)) \in Z_t(\omega)$ (a.s.). By using (61) and (60), we obtain $E(p_{t+1}\bar{x}_t|\mathcal{F}_t) = p_t\bar{x}_{t-1}$, $t = 1, ..., N$. Since $p_{N+1}\bar{x}_N = 1$, we conclude that $p_{N+1}\bar{x}_N = p_N\bar{x}_{N-1} = ... = p_1\bar{x}_0 = 1$.

Theorem 1 is proved.

In the course of the above proof, we used a procedure for deriving necessary conditions for an extremum based on the Yosida-Hewitt theorem [43]. Apparently the first who applied such methods in optimization (in the context of continuous-time optimal control) were Dubovitskii and Milyutin [8]. Analogous techniques were used in the analysis of discrete-time stochastic models of economic dynamics and related problems of stochastic programming by Radner [35, 36], Evstigneev [12, 13, 14], and others. For a comprehensive review of early literature in the field see the book by Arkin and Evstigneev [1]. In that strand of literature, the results are typically formulated in terms of Lagrange/Kuhn-Tucker multipliers (shadow prices), without explicitly invoking the considerations of duality. Foundations of convex duality theory for discrete-time stochastic dynamic optimization problems were laid in the seminal work of Rockafellar and Wets [38], [39], [40], [41]. For recent developments in the field see Pennanen [25], [26], [27], [28], [29], Pennanen and Perkkiö [30], and references therein. Related questions in continuous-time models were considered by Czichowsky and Schachermayer [4], [5], Czichowsky et al. [6], and others.

6 A financial market model

In this section we give an example of a model for a financial market with transaction costs and portfolio constraints that can be included in the framework of von Neumann-Gale dynamical systems. We provide conditions that guarantee the validity of assumptions (**A1**)-(**A4**) introduced above, which makes it possible to apply in this context the results of the present paper. As regards the aspect of growth-optimal investments, the model under consideration extends the one studied in Bahsoun et al. [3]. In the latter, portfolio constraints are specified by the cones $X_t = \mathbb{R}^n_+$, i.e., short-selling is not allowed. Here, short sales are permitted, but are subject to certain constraints—margin requirements (see below). In this paper, we only briefly discuss the financial aspects, referring the reader for details to [3].

We consider a market where m assets are traded at dates $t = 1, 2, \ldots, N$. Random vectors $a(\omega) \in \mathbb{R}^m$ are interpreted as (contingent) portfolios of assets. Positions $a_i(\omega)$ of the portfolio $a(\omega) = (a_1(\omega), ..., a_m(\omega)) \in \mathbb{R}^m$ are measured in terms of their values in the market prices. For each $t = 0, 1, \ldots, N$ and $i = 1, \ldots, m$ the following $\mathcal{F}_t$-measurable random variables are given: asset prices $S_{t,i}(\omega) > 0$ and transaction cost rates for selling and buying assets $0 \le \lambda^+_{t,i}(\omega) < 1$, $\lambda^-_{t,i}(\omega) \ge 0$. We denote by $R_{t,i} = S_{t,i}/S_{t-1,i}$ the (gross) return on asset i at time t. We omit ω in the notation where it does not lead to ambiguity.

The portfolio constraints in the model are specified by the cones

$$X_t(\omega) = \left\{ a \in \mathbb{R}^m : \sum_{i=1}^m (1 - \lambda^+_{t,i}(\omega)) a^i_+ \ge \mu_t \sum_{i=1}^m (1 + \lambda^-_{t,i}(\omega)) a^i_- \right\}, \qquad (62)$$

where $\mu_t > 1$ are constants (independent of ω). The inequalities in (62) express *margin requirements*: for an admissible portfolio, the total value of its long positions must cover a *margin* μ_t (in the U.S. equity markets $\mu_t = 1.5$) times the total value its short positions. These values are computed taking into account transaction costs for buying and selling assets.

Trading in the market proceeds as follows. At each date $t = 1, 2, \ldots, N$ a trader can rebalance her portfolio $a(\omega) \in X_{t-1}(\omega)$ purchased at the previous date $t-1$ to a new portfolio $b(\omega) \in X_t(\omega)$. The possibilities of rebalancing are specified by the inequality $\psi_t(\omega, a, b) \ge 0$, where

$$\psi_t(a, b) = \sum_{i=1}^m (1 - \lambda^+_{t,i})(R_{t,i} a^i - b^i)_+ - \sum_{i=1}^m (1 + \lambda^-_{t,i})(R_{t,i} a^i - b^i)_-.$$

The first sum represents the amount of money the trader receives for selling assets, the second sum is the amount of money she pays for buying assets, including transaction costs. The inequality $\psi_t(a, b) \ge 0$ means that the trader does not use external funds to rearrange her portfolio, and so it can be regarded as a *self-financing condition.*

Define

$$Z_t(\omega) = \{(a,b) \in X_{t-1}(\omega) \times X_t(\omega) : \psi_t(\omega,a,b) \geq 0\}. \tag{63}$$

Observe that $Z_t(\omega)$ is a cone. Clearly it contains with any vector (a,b) all vectors $\lambda(a,b)$, where $\lambda \geq 0$. Also it is convex, since the function $\psi_t(a,b)$ is concave, which follows from the representation

$$\psi_t(a,b) = \sum_{i=1}^{m}[(1-\lambda^+_{t,i})(R_{t,i}a^i - b^i)] - \sum_{i=1}^{m}[(\lambda^-_{t,i}+\lambda^+_{t,i})(R_{t,i}a^i - b^i)_-],$$

where the first sum is a linear function of (a,b) and the second sum is a convex function of (a,b).

The model of a financial market we deal with corresponds to the von Neumann-Gale dynamical system with the cones $X_t(\omega)$ specified by (62) and the cones $Z_t(\omega)$ given by 63. Paths in this dynamical system are self-financing trading strategies. Rapid paths generalize benchmark strategies [31, 32] and numeraire portfolios [21].

We provide conditions that guarantee that the present model satisfies conditions (**A1**)-(**A4**), and so Theorem 1 is valid for it. Define $\Lambda^+_{t,i}(\omega) = 1-\lambda^+_{t,i}$ and $\Lambda^-_{t,i}(\omega) = 1+\lambda^-_{t,i}$. Let us introduce the following conditions.

(**B1**) For each t, there exist constants $\underline{R}_t$, $\overline{R}_t$, $\underline{\Lambda}_t$, $\overline{\Lambda}_t$ such that $0 < \underline{R}_t \leq R_{t,i}(\omega) \leq \overline{R}_t$, $0 < \underline{\Lambda}_t \leq \Lambda^+_{t,i}(\omega)$, $\Lambda^-_{t,i}(\omega) \leq \overline{\Lambda}_t$ for all i, ω.

(**B2**) For each t, we have $\mu_t > \nu_t$ where

$$\nu_t := \max\{(\overline{\Lambda}_{t+1}\overline{R}_{t+1})/(\underline{\Lambda}_{t+1}\underline{R}_{t+1}); \overline{\Lambda}_t/\underline{\Lambda}_t\}.$$

Proposition 9. *Let conditions (**B1**) and (**B2**) hold. Then the cones $X_t(\omega)$ satisfy condition (**A1**) and the cones $Z_t(\omega)$ satisfy conditions (**A2**)-(**A4**).*

To prove Proposition 9 we need the following auxiliary result.

Lemma 1. *Let conditions (**B1**) and (**B2**) hold. Then*

(a) For each t there exists a constant $C^1_t > 0$ such that for every $a \in X_t(\omega)$ the inequality $|a_+| - \nu_t|a_-| \geq C^1_t|a|$ holds.

(b) For each t there exists a constant C^2_t such that if $a \in X_{t-1}(\omega)$, $b \in X_t(\omega)$ and $|b| \leq C^2_t|a|$, then $(a,b) \in Z_t(\omega)$.

Proof. (a) By virtue of (62), we have $X_t(\omega) \subseteq \tilde{X}_t = \{a \in \mathbb{R}^m : \mu_t|a_-| \leq |a_+|\}$, where $\tilde{X}_t \cap (-\tilde{X}_t) = \{0\}$ since $\mu_t > 1$. Observe that the continuous function $h_t(a) = |a_+| - \nu_t|a_-|$ is strictly positive on the compact set $\hat{X}_t := \tilde{X}_t \cap \{a : |a| = 1\}$. Indeed, since $h_t(a) \geq (\mu_t - \nu_t)|a_-|$ on $\tilde{X}_t$, the equality $h_t(a) = 0$ would imply $|a_-| = 0$, and hence $|a_+| = h_t(a) = 0$, so that $|a| = 0$. Then $h_t(a)$ attains a strictly positive minimum on $\hat{X}_t$, which can be taken as C_t^1.

(b) Let $b \in X_t(\omega)$. It is straightforward to check that for any numbers x, y we have $(x - y)_+ \geq x_+ - y_+$ and $(x - y)_- \leq x_- + y_+$. Using this, for any $a \in X_{t-1}(\omega)$ we obtain

$$\psi_t(a, b) \geq \sum_i (\Lambda_{t,i}^+ R_{t,i} a_+^i - \Lambda_{t,i}^- R_{t,i} a_-^i) - \sum_i (\Lambda_{t,i}^+ + \Lambda_{t,i}^-) b_+^i$$
$$\geq \underline{\Lambda}_t \underline{R}_t |a_+| - \overline{\Lambda}_t \overline{R}_t |a_-| - 2\overline{\Lambda}_t |b_+| \geq \underline{\Lambda}_t \underline{R}_t (|a_+| - \nu_{t-1}|a_-|) - 2\overline{\Lambda}_t |b|$$
$$\geq C_{t-1}^1 \underline{\Lambda}_t \underline{R}_t |a| - 2\overline{\Lambda}_t |b|.$$

Assertion (b) will be valid for the constant $C_t^2 := C_{t-1}^1 \underline{\Lambda}_t \underline{R}_t / (2\overline{\Lambda}_t)$, since if $|b| \leq C_t^2 |a|$, then $\psi_t(a, b) \geq 0$, implying $(a, b) \in Z_t$.

The proof is complete.

Proof of Proposition 9. Let us check (**A1**). Consider the non-random cone $\tilde{X}_t := \{a \in \mathbb{R}^m : \mu_t|a_-| \leq |a_+|\}$, so that $X_t(\omega) \subseteq \tilde{X}_t$. Put $q_t = e$, where $e = (1, ..., 1) \in \mathbb{R}^m$. We can see that $q_t \in X_t^*(\omega)$ since for any $a = (a^1, ..., a^m) \in X_t(\omega)$, we have

$$q_t a = \sum_{i=1}^m a^i = |a_+| - |a_-| \geq (\mu_t - 1)|a_-| \geq 0.$$

Observe that the continuous function $q_t a = \sum_{i=1}^m a^i$ is strictly positive on the compact set $\hat{X}_t = \tilde{X}_t \cap \{a : |a| = 1\}$. Indeed, since $q_t a \geq (\mu_t - 1)|a_-|$ on $\tilde{X}_t$, the equality $q_t a = 0$ would imply $|a| = 0$. Then $q_t a$ attains a strictly positive minimum $Q_t \leq 1$ on $\hat{X}_t$. Define $H_t = Q_t^{-1}$. Hence, for any $a \in X_t(\omega)$ we get

$$H_t^{-1}|a| \leq q_t a \leq H_t |a|,$$

which implies that assumption (**A1**) is satisfied.

Condition (**A2**) follows from statement (b) of Lemma 1 since for any $a \in X_{t-1}(\omega)$, $0 \leq C_t^2|a|$ and so $(a, 0) \in Z_t(\omega)$.

To prove (**A3**), let $(a, b) \in Z_t(\omega)$. Since for any numbers x, y we have $(x - y)_+ \leq x_+ + y_-$ and $(x - y)_- \geq y_+ - x_+$, we obtain

$$0 \le \psi_t(a,b) \le \sum_i (\Lambda^+_{t,i} + \Lambda^-_{t,i}) R_t, i a^i_+ + \sum_i (\Lambda^+_{t,i} b^i_- - \Lambda^-_{t,i} b^i_+)$$

$$\le 2\overline{\Lambda}_t \overline{R}_t |a| + \overline{\Lambda}_t |b_-| - \underline{\Lambda}_t |b_+| \le 2\overline{\Lambda}_t \overline{R}_t |a| - C^1_t \underline{\Lambda}_t |b|, \qquad (64)$$

where in the last inequality, we used that $b \in X_t(\omega)$ and according to statement (a) of Lemma 1, we have $\underline{\Lambda}_t |b_+| - \overline{\Lambda}_t |b_-| \ge \underline{\Lambda}_t (|b_+| - \nu_t |b_-|) \ge C^1_t \underline{\Lambda}_t |b|$. This implies the validity of (**A3**) with the constant $K_t = 2\overline{\Lambda}_t \overline{R}_t / (C^1_t \underline{\Lambda}_t)$.

Now we will prove condition (**A4**). Let $\mathring{x} = (1, ..., 1) \in \mathbb{R}^m$. Put $\mathring{z}_t = (\mathring{x}, \mathring{y}_t)$ with $\mathring{y}_t = (C^2_t/2)\mathring{x}$. Observe that there exists $\delta_t > 0$ such that $\mathbb{B}(\mathring{z}_t, \delta_t) \subset \mathbb{R}^{2m}_+$ and therefore $\mathbb{B}(\mathring{z}_t, \delta_t) \subset X_{t-1} \times X_t$. Since $|\mathring{y}_t| < C^2_t |\mathring{x}|$, statement (b) of Lemma 1 implies $\mathring{z}_t \in Z_t$. Hence, $\mathring{z}_t$ and δ_t satisfy condition (**A4**)

The proof is complete.

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