MANCHESTER 1824

The University of Manchester

Economics Discussion Paper Series EDP-1802

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January 2018

Economics
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The University of Manchester
Manchester M13 9PL

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This draft: August 2017

Abstract

This paper studies the properties of generalised empirical likelihood (GEL) methods for the estimation of and inference on partially identified parameters in models specified by unconditional moment inequality constraints. The central result is, as in moment equality condition models, a large sample equivalence between the scaled optimised GEL objective function and that for generalised method of moments (GMM) with weight matrix equal to the inverse of the efficient GMM metric for moment equality restrictions. Consequently, the paper provides a generalisation of results in the extant literature for GMM for the non-diagonal GMM weight matrix setting. The paper demonstrates that GMM in such circumstances delivers a consistent estimator of the identified set, i.e., those parameter values that satisfy the moment inequalities, and derives the corresponding rate of convergence. Based on these results the consistency of and rate of convergence for the GEL estimator of the identified set are obtained. A number of alternative equivalent GEL criteria are also considered and discussed. The paper proposes simple conservative consistent confidence regions for the identified set and the true parameter vector based on both GMM with a non-diagonal weight matrix and GEL. A simulation study examines the efficacy of the non-diagonal GMM and GEL procedures proposed in the paper and compares them with the standard diagonal GMM method.

*The authors are grateful for comments on versions of this paper by participants at the 2017 Nemmers Prize Conference, Northwestern University, the 2017 New Zealand Econometrics Study Group Conference, University of Otago, and a Universidade do Porto Econometrics Seminar.

JEL Classification: C12, C13, C14, C19.

Keywords: Moment Inequalities, Generalised Empirical Likelihood, GMM, Set Identification.

1 Introduction

The primary concern of this paper is an examination of the properties of generalised empirical likelihood (GEL) methods for the estimation of and inference on partially identified parameters in models specified by unconditional moment inequality constraints. The central result is, as in moment equality condition models, a large sample equivalence between the scaled optimised GEL objective function and that for generalised method of moments (GMM) with weight matrix equal to the inverse of the sample outer product of the moment indicators, i.e., the efficient GMM metric for moment equality restrictions. Consequently, the paper provides a generalisation of results in the extant GMM literature from the diagonal to the non-diagonal GMM weight matrix setting; see, *inter alia*, Chernozhukov et al. (2007), henceforth CHT. The paper demonstrates that GMM in such circumstances delivers a consistent estimator of the identified set, i.e., those parameter values that satisfy the moment inequalities, and derives the corresponding rate of convergence. Based on these results the consistency of and rate of convergence for the GEL estimator of the identified set are obtained. A number of alternative equivalent GEL criteria are also considered and discussed. The paper proposes simple conservative consistent confidence regions for the identified set and the true parameter vector based on GMM with a non-diagonal weight matrix and GEL. A simulation study corroborates the main theoretical results of this paper and indicates that empirical likelihood and exponential tilting confidence region estimators have favourable coverage properties relative to GMM with a diagonal weight matrix and continuous updating which has very poor coverage outside the identified set.

The econometric literature concerned with partially identified models has grown rapidly in recent years, especially that for models defined by moment inequality restrictions. The impetus for this research originally arose from the recognition that untenable and thus undesirable assumptions may often be imposed in econometric research to achieve point identification of model parameters thereby reducing the credibility of any resultant inference. The analysis of the properties of extremum-type parameter estimators in partially identified models specified by moment inequality restrictions has received particular attention. CHT provides general conditions under which the consistency of estimators for the identified set and the validity of resampling methods to generate consistent confidence regions for either the identified set or the true parameter vector are established. To date much of this literature has concentrated on the GMM criterion and associated GMM estimators. CHT section 4, pp.1261-1267, develops confidence region estimators for the identified set and true vector of parameters based on GMM with a diagonal weight matrix whereas Rosen (2008) does so for the latter based on the GMM criterion with the equality moment constraints efficient metric which avoids the necessity of CHT resampling techniques. An important recent contribution, Chen et al. (2016), develops confidence regions for the identified set based on inverting an optimal sample criterion where cut-off values are computed directly from MCMC simulations of a quasi-posterior distribution of the criterion. However, not unlike CHT for GMM, this method also requires a diagonal variance matrix assumption; see Chen et al. (2016) Assumption 3.2 and Theorem 3.1. Menzel (2014) extends the CHT results for GMM to the case of many moment inequalities; cf. the many moment equalities GMM results of Han and Phillips (2008). Moment inequality selection methods and corresponding methods of inference based on GMM-type estimators are developed in Andrews and Guggenberger (2009), Andrews and Soares (2010) and Andrews and Barwick (2012). Extensions of GMM to conditional moment inequality models have also been considered; see, e.g., Andrews and Shi (2013, 2014), Armstrong (2014, 2015), Armstrong and Chan (2016) and Khan and Tamer (2009). Misspecified moment inequalities are studied in Ponomareva and Tamer (2011) and Bugni et al. (2012).

The criterion function approach of CHT and others, although of general applicability, can be computationally demanding. Another strand of research has focussed on econometric models with compact convex identified sets enabling the identified set to be characterised by its support function which thus provides a computationally tractable representation. See, e.g., Beresteanu and Mollinari (2008), Beresteanu et al. (2011) and Kaido and Santos (2014). Kaido (2016) presents a unification of the two approaches for compact convex identified sets, illustrating the applicability of the results in a number of examples and for models defined by a finite number of moment inequalities.

Despite the many substantial theoretical contributions to research on the estimation of set-identified parameters relatively little is known about the properties of GEL-type estimators. Exceptions are Moon and Schorfheide (2009), which adopts an empirical likelihood approach when parameters are point-identified by over-identifying moment equality restrictions and also subject to moment inequality restrictions, and Canay (2010), which obtains EL-based confidence regions for the true parameter vector when it is partially identified by a set of unconditional moment inequalities. More generally, the asymptotic properties of GEL methods of inference for the identified set and the true parameter vector, the topic of this paper, remain to be developed.

The paper is organised as follows. Section 2 briefly reviews the set-up describing models specified by unconditional moment inequality constraints. Section 3 details GMM and GEL criteria and associated constructs appropriate for estimation and inference in such models. The equivalence of definitions of the identified set based on population GMM and GEL criteria is also discussed and established here. Consistent estimators for the identified set based on GMM and GEL criteria are described in section 4 with, in particular, the asymptotic equivalence of various GEL criteria also shown. Conservative confidence region estimators for the identified set and the true parameter vector based on GMM with a non-diagonal weight matrix and GEL are proposed in section 5. Section 6 provides a simulation study for interval outcomes in a nonlinear regression model to examine the efficacy of GEL procedures proposed in the paper compared with the standard diagonal GMM method. Section 7 summarises and concludes. The appendices contain the technical conditions of CHT, their verification for nondiagonal metric GMM and GEL together with the proofs of results stated in the text.

Throughout the text z_i, $(i = 1, ..., n)$, denotes a random sample of size n on the observation d_z-dimensional vector z. Positive (semi-) definite is abbreviated as p.(s.)d., f.c.r. full column rank and f.o.c. first order condition. The interior of a set A is denoted as $int(A)$. Superscripted vectors denote the requisite element, e.g., a^j is the jth element of vector a; $\|x\|_- = \|[x]_-\|$ with $[x]_- = \min\{x, 0\}$. UWL denotes a uniform weak law of large numbers such as Lemma 2.4 of Newey and McFadden (1994) and CLT is the Lindeberg-Lévy central limit theorem. The symbols "$\Rightarrow$", "$\xrightarrow{p}$" and "$\xrightarrow{d}$" denote weak convergence, convergence in probability and convergence in distribution respectively and "with probability (approaching) 1" written as "w.p.(a.)1". The Hausdorff distance between sets A and B is defined as $d_H(A, B) = \max[\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)]$ where $d(b, A) = \inf_{a\in A} \|b - a\|$ and $d_H(A, B) = \infty$ if either A or B are empty.

2 Moment Inequality Restrictions

Let $m(z, \theta)$ denote a d_m-vector of known functions of the data observation vector z and the d_θ-vector $\theta \in \Theta$ of unknown parameters where $\Theta \subset \mathcal{R}^{d_\theta}$ is the corresponding parameter space. The moment indicator vector $m(z, \theta)$ will form the basis for inference in the following discussion and analysis. Also let $m(\theta) = \mathbb{E}_{P_0}[m(z, \theta)]$ and $\Omega(\theta) = \mathbb{E}_{P_0}[m(z, \theta)m(z, \theta)']$, $\theta \in \Theta$, where $\mathbb{E}_{P_0}[\cdot]$ denotes expectation taken with respect to the true population probability law (P_0) of z.

Assumption A.1. **(a)** *The parameter space* Θ *is a non-empty and compact subset of* $\mathcal{R}^{d_\theta}$*;* **(b)** $m(z,\theta)$ *is continuous at each* $\theta \in \Theta$ *w.p.*1, $\mathbb{E}_{P_0}[\sup_{\theta\in\Theta} \|m(z,\theta)\|^2] < C < \infty$ *for suitably large* $C > 0$*;* **(c)** $\Omega(\theta)$ *is finite and uniformly p.d.* $\theta \in \Theta$*;* **(d)** *the data* z_i*,* $(i = 1, ..., n)$*, are defined on a complete probability space* $(\Omega, \mathcal{F}, P)$*.*

Remark 2.1. Assumption A.1 repeats aspects of CHT Condition M.2(a), p.1265.

It is assumed that the true value θ_0 taken by θ satisfies the population unconditional moment inequality condition under P_0

$$\mathbb{E}_{P_0}[m(z,\theta)] \geq 0. \tag{2.1}$$

Remark 2.2. Moment inequalities such as (2.1) arise in many settings, e.g., interval outcomes in regression models of relevance for empirical models of auctions which forms the basis for the experimental design of the simulation study of section 6. See, *inter alia*, the CHT introduction and CHT section 2.2 which provide several more examples and the associated discussion in Romano and Shaikh (2008) for other common examples.

In many situations the common assumption that θ_0 uniquely satisfies the inequality restrictions (2.1) and is thus point identified would need the, implicit or otherwise, imposition of further potentially stringent and untenable assumptions. In the absence of such additional assumptions a more general and less restrictive approach requires that there exists a subset of Θ, here denoted by Θ_{P_0} and referred to as the *identified set,* for which these inequality constraints hold, i.e., the identified set Θ_{P_0} consists of all those elements $\theta \in \Theta$ that satisfy the moment inequality restrictions (2.1)

$$\Theta_{P_0} = \{\theta \in \Theta : \mathbb{E}_{P_0}[m(z,\theta)] \geq 0\}. \tag{2.2}$$

It is convenient for the following analysis to define a d_m-vector of complementary slackness parameters $t(\theta)$ by the identity

$$t(\theta) = \mathbb{E}_{P_0}[m(z,\theta)] \tag{2.3}$$

with the consequent equivalent re-expression of the moment inequality constraints (2.1) as the equality restrictions $t(\theta) - \mathbb{E}_{P_0}[m(z,\theta)] = 0$ together with the parametric inequality restrictions $t(\theta) \geq 0$. Thus, the identified set Θ_{P_0} may now be re-defined as

$$\Theta_{P_0} = \{\theta \in \Theta : t(\theta) - \mathbb{E}_{P_0}[m(z,\theta)] = 0, t(\theta) \geq 0\}. \tag{2.4}$$

In the following the identified set Θ_{P_0} and the true value θ_0 are of primary inferential interest.

3 GMM and GEL

This section first discusses GMM for models specified by the moment inequality restrictions (2.1). A description of the application of GEL then follows; equivalent GEL variants and their properties are detailed in Appendix E. The section is concluded by an analysis and comparison of the corresponding GMM and GEL definitions of the identified set.

Let $m_i(\theta) = m(z_i, \theta)$, $(i = 1, ..., n)$, $\hat{m}_n(\theta) = \sum_{i=1}^n m_i(\theta)/n$ and $\hat{\Omega}_n(\theta) = \sum_{i=1}^n m_i(\theta)m_i(\theta)'/n$. Assumptions A.1(b) and (c) above ensure $\hat{m}_n(\theta) \xrightarrow{p} m(\theta)$ and $\hat{\Omega}_n(\theta) \xrightarrow{p} \Omega(\theta)$ uniformly $\theta \in \Theta$ by UWL.

3.1 GMM

Define the norm $\|x\|_W^2 = x'Wx$ where W is a p.s.d. matrix. A general formulation for GMM appropriate for the moment inequality constraints (2.1) is based on the objective function

$$\begin{aligned} \hat{Q}_n^W(\theta) &= \inf_{t \geq 0} (\hat{m}_n(\theta) - t)' W_n(\theta) (\hat{m}_n(\theta) - t) \\ &= \inf_{t \geq 0} \|\hat{m}_n(\theta) - t\|_{W_n(\theta)}^2 , \end{aligned} \tag{3.1}$$

where $W_n(\theta)$ is assumed to be uniformly p.s.d. $\theta \in \Theta'$. The solution $\hat{t}_n(\theta)$ to (3.1) satisfies $\hat{t}_n^j(\theta) = 0$ if $\hat{m}_n^j(\theta) < 0$ and $\hat{m}_n^j(\theta)$ if $\hat{m}_n^j(\theta) \geq 0$, $(j = 1, ..., d_m)$. Cf. Rosen (2008); also see CHT and Romano and Shaikh (2008).

Assumption A.2-GMM. **(a)** *The GMM criterion function $\hat{Q}_n^W(\theta)$ is defined on a neighbourhood Θ' of Θ, and is measurable in $\theta \in \Theta'$;* **(b)** *there exists $W(\theta)$ such that* $\sup_{\theta \in \Theta'} |W_n(\theta) - W(\theta)| = o_p(1)$ *where $W(\theta)$ is continuous with finite elements and uniformly p.d. $\theta \in \Theta'$.*

Remark 3.1. Assumption A.2-GMM together with Assumption A.1 reproduces CHT Conditions M.2(a) and M.2(e), p.1265, with an important exception; cf. Rosen (2008) Assumptions A4 and A5, p.110. That is, CHT Assumption M.2(e), p.1265, which imposes diagonality on the asymptotic GMM weight matrix $W(\theta)$, is relaxed here. Consequently, the GMM criterion $\hat{Q}_n^W(\theta)$ in (3.1) may no longer be equivalently expressed asymptotically as the CHT sample criterion $\left\|\hat{m}_n(\theta)' W_n(\theta)^{1/2}\right\|_-^2$ unless $W(\theta)$ is diagonal. Assumption A.2-GMM(b) may be straightforwardly verified by application of UWL.

Remark 3.2. Of particular interest is the GMM objective function with the optimal GMM metric in the unconditional moment equality context, i.e., $W_n(\theta) = \hat{\Omega}_n(\theta)^{-1}$,

$$\begin{aligned}\hat{Q}_n^{\Omega^{-1}}(\theta) &= \inf_{t \geq 0} (\hat{m}_n(\theta) - t)' \hat{\Omega}_n(\theta)^{-1} (\hat{m}_n(\theta) - t) \\ &= \inf_{t \geq 0} \|\hat{m}_n(\theta) - t\|^2_{\hat{\Omega}_n(\theta)^{-1}} . \end{aligned} \tag{3.2}$$

The population counterpart $Q^W(\theta)$ to the GMM criterion (3.1) is defined by

$$\begin{aligned}Q^W(\theta) &= \inf_{t \geq 0} (m(\theta) - t)' W(\theta) (m(\theta) - t) \\ &= \inf_{t \geq 0} \|m(\theta) - t\|^2_{W(\theta)} . \end{aligned} \tag{3.3}$$

3.2 GEL

It is well know that appropriately scaled GEL is first order asymptotically equivalent to optimal GMM in the standard moment equality constraint setting. As is also widely appreciated, GEL includes as special cases empirical likelihood (EL) [Qin and Lawless (1994), Imbens (1997)], exponential tilting (ET) [Kitamura and Stutzer (1997), Imbens et al. (1998)], continuous updating estimation (CUE) [Hansen et al. (1996)] and estimators based on the Cressie-Read power divergence family [Cressie and Read (1984)]. See *inter alia* Newey and Smith (2004) and Smith (1997, 2011). Canay (2010) develops an EL-based confidence region for the true parameter vector θ_0, but does not study the large sample properties of the EL estimator of the identified set.

To describe GEL let $\rho(v)$ be a function of a scalar v that is concave on its domain $\mathcal{V}$, an open interval containing zero. For expositional convenience but without loss of generality $\rho(0)$ is set equal to 0 below. The standard GEL criterion is then defined as

$$\hat{P}_n^{\rho}(\theta, \lambda) = \sum_{i=1}^{n} \rho(\lambda' m_i(\theta))/n, \tag{3.4}$$

in which each element of the auxiliary parameter vector $\lambda \in \mathcal{R}^{d_m}$ is associated with a corresponding element of the moment indicator vector $m_i(\theta)$, $(i = 1, ..., n)$; cf. Newey and Smith (2004) and Smith (1997, 2011).

Let $\hat{\Lambda}_n^{+}(\theta) = \hat{\Lambda}_n(\theta) \cap \{\lambda \geq 0\}$ where $\hat{\Lambda}_n(\theta) = \{\lambda : \lambda' m_i(\theta) \in \mathcal{V}, (i = 1, ..., n)\}$ constrains the domain of $\rho(\cdot)$ to the concavity region $\mathcal{V}$ identically to the standard moment equality restrictions case; see Newey and Smith (2004). Optimization of $\hat{P}_n^{\rho}(\theta, \lambda)$ (3.4) with respect to λ is taken over $\hat{\Lambda}_n^{+}(\theta)$, where the non-negativity restriction $\lambda \geq 0$ reflects the moment inequality constraints (2.1). The profile GEL criterion function $\hat{P}_n^\rho(\theta)$ is then defined by

$$\hat{P}_n^\rho(\theta) = \sup_{\lambda \in \hat{\Lambda}_n^+(\theta)} \hat{P}_n^\rho(\theta, \lambda). \tag{3.5}$$

Let $\rho_1(\cdot)$ and $\rho_2(\cdot)$ denote the first and second derivatives of $\rho(\cdot)$ respectively. The next assumption provides the requisite conditions on the profile GEL criterion $\hat{P}_n^\rho(\theta)$ (3.5) and the function $\rho(\cdot)$.

Assumption A.2-GEL. **(a)** *$\hat{P}_n^\rho(\theta)$ is defined on a neighbourhood Θ' of Θ and is measurable in $\theta \in \Theta'$;* **(b)** *$\rho(\cdot)$ is strictly concave and twice continuously differentiable on an open interval $\mathcal{V}$ that includes 0 such that $\rho(0) = 0$ and $\rho_1(v) < 0$ for all $v \in \mathcal{V}$.*

Remark 3.3. Cf. Assumption A.2-GMM. Assumption A.2-GEL(b) is satisfied by the Cressie-Read (1984) family of divergence measures. In the following, without loss of generality, the first two derivatives of $\rho(\cdot)$ at zero are set to minus unity, i.e., $\rho_1(0) = \rho_2(0) = -1$.

For any $\theta \in \Theta$, define $\hat{\lambda}_n(\theta) = \arg\max_{\lambda \in \hat{\Lambda}_n^+(\theta)} \hat{P}_n^\rho(\theta, \lambda)$ as the solution to the f.o.c. with respect to λ for given θ, i.e.,

$$\sum_{i=1}^n \rho_1(\hat{\lambda}_n(\theta)' m_i(\theta)) m_i(\theta)/n \leq 0, \ \hat{\lambda}_n(\theta) \geq 0. \tag{3.6}$$

In particular $\sum_{i=1}^n \rho_1(\hat{\lambda}_n(\theta)' m_i(\theta)) m_i^j(\theta)/n = 0$ and $\hat{\lambda}_n^j(\theta) > 0$ or $\sum_{i=1}^n \rho_1(\hat{\lambda}_n(\theta)' m_i(\theta))$ $\times m_i^j(\theta)/n < 0$ and $\hat{\lambda}_n^j(\theta) = 0$, $(j = 1, ..., d_m)$, i.e., $\hat{\lambda}_n(\theta)' \sum_{i=1}^n \rho_1(\hat{\lambda}_n(\theta)' m_i(\theta)) m_i(\theta)/n = 0$.

The GEL empirical or implied probabilities are then defined correspondingly by

$$\hat{\pi}_i^\rho(\theta, \lambda) = \frac{\rho_1(\lambda' m_i(\theta))}{\sum_{k=1}^n \rho_1(\lambda' m_k(\theta))}, (i = 1, ..., n); \tag{3.7}$$

cf. Back and Brown (1993), Newey and Smith (2004) and Brown and Newey (1992, 2002).

Remark 3.4. The GEL implied probabilities $\hat{\pi}_i^\rho(\theta) = \hat{\pi}_i^\rho(\theta, \hat{\lambda}_n(\theta))$, $(i = 1, ..., n)$, (3.7), are non-negative by Assumption A.2-GEL(b), sum to unity and satisfy the sample moment inequality condition $\sum_{i=1}^n \hat{\pi}_i^\rho(\theta) m_i(\theta) \geq 0$ (3.6) defining the f.o.c. for $\hat{\lambda}_n(\theta)$.

Remark 3.5. The above optimisation problem may be cast alternatively in terms of the Lagrangean $\tilde{P}_n^\rho(\theta, \lambda, \tau) = \sum_{i=1}^n \rho(\lambda' m_i(\theta))/n + \tau'\lambda$ where τ is the d_m-vector of Lagrange multipliers associated with the inequality constraint $\lambda \geq 0$. Cf. Moon and Schorfheide (2009) (16), p.140. The Lagrange multiplier estimator satisfies $\hat{\tau}_n(\theta) \geq 0$ with $\hat{\lambda}_n(\theta)'\hat{\tau}_n(\theta) = 0$ and, in particular, $\hat{\lambda}_n^j(\theta) = 0$ and $\hat{\tau}_n^j(\theta) > 0$ or $\hat{\lambda}_n^j(\theta) > 0$ and $\hat{\tau}_n^j(\theta) = 0$, $(j = 1, ..., d_m)$. The auxiliary parameter estimator $\hat{\lambda}_n(\theta)$ is the solution to the f.o.c. with respect to λ, i.e., $\sum_{i=1}^n \rho_1(\hat{\lambda}_n(\theta)' m_i(\theta)) m_i(\theta)/n + \hat{\tau}_n(\theta) = 0$. Thus $\hat{\tau}_n(\theta)$ satisfies $\hat{\tau}_n(\theta) = -\sum_{i=1}^n \rho_1(\hat{\lambda}_n(\theta)' m_i(\theta)) m_i(\theta)/n$; cf. (3.6).

Remark 3.6. Appendix E gives alternative equivalent forms of GEL criteria; *viz.* $\tilde{P}_n^{\rho,a}(\theta, \lambda, \tau) = \sum_{i=1}^n \rho(\lambda'(m_i(\theta) - \tau))/n$ (E.1), $\tilde{P}_n^{\rho,b}(\theta, \lambda, \tau) = \sum_{i=1}^n [\rho(\lambda' m_i(\theta)) - \rho(\lambda'\tau)]/n$ (E.3) and $\tilde{P}_n^\rho(\theta, \lambda, \tau) = \sum_{i=1}^n \rho(\lambda' m_i(\theta))/n + \lambda'\tau$ (E.6), cf. Remark 3.5. Lemmas E.1-E.3 in Appendix E.1 provide detailed statements and, in particular, demonstrate that both the solutions to and the optimised values of the corresponding GEL saddle point problems (3.4), (E.6) and (E.1), (E.3) are identical. More specifically, define the slackness parameter space $\mathcal{T} = \{\tau \in \mathcal{R}^{d_m} : \tau \geq 0, \|\tau\| \leq C\}$ with $C > 0$ defined by the boundedness condition in Assumption A.1(b). Then, if $(\tilde{\theta}, \tilde{\lambda}, \tilde{\tau})$, where $\tilde{\tau} \in int(\mathcal{T})$, is a saddlepoint of $\tilde{P}_n^\rho(\theta, \lambda, \tau)$ or $\tilde{P}_n^{\rho,k}(\theta, \lambda, \tau)$, $(k = a, b)$, then $(\tilde{\theta}, \tilde{\lambda})$ is also a saddlepoint of $\hat{P}_n^\rho(\theta, \lambda)$ and, if $(\hat{\theta}, \hat{\lambda})$ is a saddlepoint of $\hat{P}_n^\rho(\theta, \lambda)$ and $\hat{\tau} \in int(\mathcal{T})$ for suitable definitions of the slackness parameter $\hat{\tau}$, then $(\hat{\theta}, \hat{\lambda}, \hat{\tau})$ is also saddlepoint of $\tilde{P}_n^\rho(\theta, \lambda, \tau)$ or $\tilde{P}_n^{\rho,k}(\theta, \lambda, \tau)$, $(k = a, b)$. Cf. Moon and Schorfheide (2009) Lemma A.1, p.150.

3.3 Identified Set

The identified set Θ_{P_0} (2.2) is clearly identical to the GMM population counterpart

$$\Theta_{P_0}^W = \{\theta \in \Theta : \theta = \arg\min_{\theta \in \Theta} Q^W(\theta)\} \tag{3.8}$$

where $Q^W(\theta)$ is defined in (3.3).

Let $\hat{P}^\rho(\theta)$ denote the population counterpart to the profile GEL criterion $\hat{P}_n^\rho(\theta)$ (3.5), i.e., $\hat{P}^\rho(\theta) = \sup_{\lambda \geq 0} \mathbb{E}_{P_0}[\rho(\lambda' m(z, \theta))]$. The GEL population counterpart $\hat{\Theta}_{P_0}^\rho$ to the identified set Θ_{P_0} (2.2) is then defined as

$$\hat{\Theta}_{P_0}^\rho = \{\theta \in \Theta : \theta = \arg\min_{\theta \in \Theta} \hat{P}^\rho(\theta)\} \tag{3.9}$$

which similarly to Canay (2010) for EL may be shown to be identical to the identified set Θ_{P_0} (2.2).

Remark 3.7. Alternative but equivalent population counterparts $\tilde{\Theta}^{\rho}_{P_0}$ (E.8) and $\tilde{\Theta}^{\rho,k}_{P_0}$, $(k = a, b)$, (E.9) may be defined corresponding to the GEL criteria $\tilde{P}^{\rho}_n(\theta, \lambda, \tau)$ (E.6) and $\tilde{P}^{\rho,k}_n(\theta, \lambda, \tau)$, $(k = a, b)$, (E.1) and (E.3). See Appendix E.3 for a detailed description.

Lemma D.1 in Appendix D formally demonstrates the equivalence of $\hat{\Theta}^{\rho}_{P_0}$ (3.9) with the identified set Θ_{P_0} (2.2). Lemmas E.4 and E.5 in Appendix E do likewise for $\tilde{\Theta}^{\rho}_{P_0}$ (E.8) and $\tilde{\Theta}^{\rho,k}_{P_0}$ (E.9), $(k = a, b)$, with $\hat{\Theta}^{\rho}_{P_0}$ (3.9). Theorem 3.1 summarises these results.

Theorem 3.1. *Suppose that Assumptions A.1 and A.2-GEL are satisfied. Then the GEL population counterparts* $\hat{\Theta}^{\rho}_{P_0}$ (3.9), $\tilde{\Theta}^{\rho}_{P_0}$ (E.8) *and* $\tilde{\Theta}^{\rho,k}_{P_0}$ (E.9), $(k = a, b)$, *are identical to the identified set* Θ_{P_0} (2.2).

4 Set Estimation

Let $v_n(\theta) = n^{1/2}(\hat{m}_n(\theta) - \mathbb{E}_{P_0}[m(z, \theta)])$, $\theta \in \Theta'$, $\Omega(\theta_a, \theta_b) = \mathbb{E}_{P_0}[v_n(\theta_a)v_n(\theta_b)']$, $\theta_a, \theta_b \in \Theta'$, and $\Omega(\theta) = \Omega(\theta, \theta)$, $\theta \in \Theta'$, where Θ' is a neighbourhood of Θ. The following assumptions mimic CHT Conditions M.2(c), M.2(d) and M.2(f), pp.1265-1266, respectively.

Assumption A.3. *The collection $\{m_i(\theta) : \theta \in \Theta'\}$ satisfies a P-Donsker property. In particular, $v_n(\cdot) \Rightarrow v(\cdot)$ where $v(\theta)$, $\theta \in \Theta'$, is a zero-mean Gaussian process with almost sure continuous paths and covariance function $\Omega(\theta_a, \theta_b)$, $\theta_a, \theta_b \in \Theta'$, such that $\Omega(\theta)$ is uniformly p.d. $\theta \in \Theta'$.*

Assumption A.4. *There exist positive constants C* and *δ such that $\|\mathbb{E}_{P_0}[m(z, \theta)]\|_- \geq C \cdot (d(\theta, \Theta_{P_0}) \wedge \delta)$ for all $\theta \in \Theta$ with continuous Jacobian $M(\theta) = \partial m(\theta)/\partial\theta'$ for each $\theta \in \Theta'$.*

Assumption A.5. *There exist positive constants (C, M, δ) such that, for all $\theta \in \Theta^{-\epsilon}_{P_0}$, $\min_{1 \leq j \leq d_m} \|\mathbb{E}_{P_0}[m^j(z, \theta)]\|_- \geq C \cdot (\epsilon \wedge \delta)$ and $d_H(\Theta^{-\varepsilon}_{P_0}, \Theta_{P_0}) \leq M\varepsilon$ for all $\varepsilon \in [0, \delta]$ where $\Theta^{-\varepsilon}_{P_0} = \{\theta \in \Theta_{P_0} : d(\theta, \Theta \backslash \Theta_{P_0}) \geq \varepsilon\}$.*

Proofs for the following results are provided in Appendices B and C respectively for GMM and GEL.

4.1 GMM

Let

$$\hat{\Theta}_n^W(c) = \{\theta \in \Theta : n\hat{Q}_n^W(\theta) \leq c\} \tag{4.1}$$

where c denotes a positive scalar. Cf. CHT, eqs. (3.1) and (3.2), p.1253. The GMM estimator of the identified set Θ_{P_0} (2.2) is then defined as the set estimator $\hat{\Theta}_n^W(\hat{c}_W)$ (4.1) for some possibly data dependent level $\hat{c}_W$.

Appendix B establishes the validity for non-diagonal weighted GMM of CHT Conditions C.1, p.1252, C.2, p.1253, and C.3, p.1255, under Assumptions A.1, A.2-GMM and A.3-A.5. These conditions are therefore sufficient for the statement of the following theorem on the consistency and rate of convergence of the GMM set estimator $\hat{\Theta}_n(\hat{c}_W)$ for the identified set Θ_{P_0} given rate restrictions on $\hat{c}_W$ as provided in CHT Theorem 3.2, p.1255.

Theorem 4.1. *Let $\hat{c}_W \geq q_n = \inf_{\theta\in\Theta} n\hat{Q}_n^W(\theta)$ w.p.1 and $\hat{c}_W = O_p(1)$. Then, under Assumptions A.1, A.2-GMM and A.3-A.5,* **(a)** *$\hat{\Theta}_n^W(\hat{c}_W)$ is a consistent estimator of the identified set Θ_{P_0}, i.e., $d_H(\hat{\Theta}_n^W(\hat{c}_W), \Theta_{P_0}) = o_p(1)$;* **(b)** *$d_H(\hat{\Theta}_n^W(\hat{c}_W), \Theta_{P_0}) = O_p(n^{-1/2})$.*

Remark 4.1. Theorem 4.1 is established by verifying the conditions required for CHT Theorem 3.2, p.1255. Unlike CHT Condition M.2(e), p.1265, and the consequent Theorem 4.2, p.1266, Theorem 4.1 does not require the diagonality of the population GMM weight matrix $W(\theta)$, $\theta \in \Theta$, although similarly mild restrictions on the choice of the value $\hat{c}_W$ to those of CHT are imposed. CHT Theorem 4.2, p.1266, also obtains a limiting representation for the statistic $\sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n^W(\theta)$ when the population GMM weight matrix $W(\theta)$, $\theta \in \Theta$, is diagonal. To the best of our knowledge there are as yet no results for the non-diagonal case. Section 5.1 proposes conservative bounds appropriate for GMM criteria with a non-diagonal population weight matrix $W(\theta)$, $\theta \in \Theta$, and, likewise, GEL criteria.

Remark 4.2. Alternatively, cf. CHT Theorem 3.1, p.1254, a similar result holds if $\hat{c}_W \geq \sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n^W(\theta)$ w.p.a.1 and $\hat{c}_W/n = o_p(1)$ with Theorem 4.1(b) restated as $d_H(\hat{\Theta}_n^W(\hat{c}_W), \Theta_{P_0}) = O_p((1 \vee \hat{c}_W)/n)^{1/2}$; cf. Rosen (2008) Proposition 2, p.110, which sets $\hat{c}_W \to \infty$ and $\hat{c}_W/n = o(1)$. Since, in general, $\sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n^W(\theta)$ is unknown, CHT, p.1254, suggests the choice $\hat{c}_W = o(\log(n))$ which yields a rate of convergence of $(\log(n)/n)^{1/2}$.

4.2 GEL

Let

$$\hat{\Theta}_n^{\rho}(\hat{c}_{\rho}) = \{\theta \in \Theta : 2n\hat{P}_n^{\rho}(\theta) \le \hat{c}_{\rho}\} \tag{4.2}$$

where the profile GEL criterion $\hat{P}_n^{\rho}(\theta)$ is defined in (3.5). The GEL estimator of Θ_{P_0} based on (3.4) is the solution to a saddle point problem and is described by the set estimator $\hat{\Theta}_n^{\rho}(\hat{c}_{\rho})$ (4.2) for some possibly data dependent $\hat{c}_{\rho}$.

Remark 4.3. The scale factor 2 introduced in the definition of the GEL set estimator $\hat{\Theta}_n^{\rho}(\hat{c}_{\rho})$ (4.2) reflects the first order asymptotic equivalence between the scaled profile GEL $2n\hat{P}_n^{\rho}(\theta)$ (3.5) and optimal moment equality GMM $n\hat{Q}_n^{\Omega^{-1}}(\theta)$ (3.2) criteria thus ensuring comparability between the respective set estimators. See Appendix C Lemmas C.4 and C.5.

Remark 4.4. Write $\tilde{P}_n^{\rho}(\theta) = \inf_{\tau\in\mathcal{T}} \sup_{\lambda\in\hat{\Lambda}_n(\theta)} \tilde{P}_n^{\rho}(\theta,\lambda,\tau)$ (E.6) and $\tilde{P}_n^{\rho,k}(\theta) = \inf_{\tau\in\mathcal{T}} \sup_{\lambda\in\hat{\Lambda}_n^k(\theta,\tau)} \tilde{P}_n^{\rho,k}(\theta,\lambda,\tau)$, $(k = a, b)$, (E.1, E.3), see Remark 3.6, and define

$$\tilde{\Theta}_n^{\rho}(\hat{c}_{\rho}) = \{\theta \in \Theta : 2n\tilde{P}_n^{\rho}(\theta) \le \hat{c}_{\rho}\} \tag{4.3}$$

and

$$\tilde{\Theta}_n^{\rho,k}(\hat{c}_{\rho}) = \{\theta \in \Theta : 2n\tilde{P}_n^{\rho,k}(\theta) \le \hat{c}_{\rho}\}, (k = a, b). \tag{4.4}$$

Consequently, the set estimators $\hat{\Theta}_n^{\rho}(\hat{c}_{\rho})$ (4.2), $\tilde{\Theta}_n^{\rho}(\hat{c}_{\rho})$ (4.3) and $\tilde{\Theta}_n^{\rho,k}(\hat{c}_{\rho})$, $(k = a, b)$, (4.4) based on the respective GEL criteria (3.4), (E.6), (E.1) and (E.3) evaluated using the same critical value $\hat{c}_{\rho}$ are identical given their numerical equivalence established in Appendix E.1.

Appendix C establishes the corresponding validity of CHT Conditions C.1, p.1252, C.2, p.1253, and C.3, p.1255, under Assumptions A.1, A.2-GEL and A.3-A.5. Hence, a similar result to Theorem 4.1 for GMM may be stated on the consistency and rate of convergence of the GEL set estimator $\hat{\Theta}_n^{\rho}(\hat{c}_{\rho})$ (4.2) for the identified set Θ_{P_0} with some possibly data-dependent $\hat{c}_{\rho}$.

Theorem 4.2. *Let $\hat{c}_{\rho} \ge q_n^{\rho} = \inf_{\theta\in\Theta} 2n\hat{P}_n^{\rho}(\theta)$ w.p.1 and $\hat{c}_{\rho} = O_p(1)$. Then, under Assumptions A.1, A.2–GEL and A.3-A.5,* **(a)** *$\hat{\Theta}_n^{\rho}(\hat{c}_{\rho})$ is a consistent estimator of the* *identified set Θ_{P_0}, i.e., $d_H(\hat{\Theta}_n^\rho(\hat{c}_\rho), \Theta_{P_0})$ is $o_p(1)$;* **(b)** *$d_H(\hat{\Theta}_n^\rho(\hat{c}_\rho), \Theta_{P_0})$ is $O_p(n^{-1/2})$.*

REMARK 4.5. Theorem 4.2 is proved for the alternative GEL criterion $\tilde{P}_n^\rho(\theta, \lambda, \tau)$ (E.6) but given the equivalence established in Appendix E.1 therefore applies to $\hat{P}_n^\rho(\theta, \lambda)$ (3.4). Proofs analogous to those of Newey and Smith (2004) are developed to show these results. In particular, the scaled GEL criterion function is shown to be first-order equivalent to the scaled optimal GMM criterion (3.2) in the unconditional moment equality context and then Theorem 4.1 with population GMM weight matrix $\Omega(\theta)^{-1}$, $\theta \in \Theta$, is invoked.

REMARK 4.6. Similarly to Remark 4.2 above, if $\hat{c}_\rho \geq \sup_{\theta \in \Theta_{P_0}} 2n\hat{P}_n^\rho(\theta)$ w.p.a.1 and $\hat{c}_\rho / n = o_p(1)$, then $d_H(\hat{\Theta}_n^\rho(\hat{c}_\rho), \Theta_{P_0})$ is $O_p((1 \vee \hat{c}_\rho)/n)^{1/2}$.

5 Confidence Region Estimation

Confidence regions for the identified set Θ_{P_0} and the true parameter value θ_0 are of particular interest. Section 5.1 constructs a conservative confidence region for the identified set Θ_{P_0}. Section 5.2 develops conservative GEL-based confidence regions for the true parameter value θ_0 similar to those of Rosen (2008) section 4, pp.111-113.

5.1 Conservative Confidence Regions for Θ_{P_0}

A critical matter is a suitable choice for the possibly data dependent $\hat{c}_W$ or $\hat{c}_\rho$ respectively satisfying the hypotheses of Theorems 4.1 and 4.2 thereby ensuring that the GMM $\hat{\Theta}_n^W(\hat{c}_W) = \{\theta \in \Theta : n\hat{Q}_n^W(\theta) \leq \hat{c}_W\}$, cf. (4.1), or GEL $\hat{\Theta}_n^\rho(\hat{c}_\rho) = \{\theta \in \Theta : 2n\hat{P}_n^\rho(\theta) \leq \hat{c}_\rho\}$, cf. (4.2), estimators possess a confidence region property; see CHT section 3.3, pp.1256-1257. CHT section 4.2, pp.1265-1267, addresses this issue for moment inequalities when the GMM asymptotic weighting matrix $W(\theta)$ is diagonal; see CHT Condition M.2(e), p.1265.

Suppose b moment inequalities bind, i.e., $m^j(\theta) = 0$, $(j = 1, ..., b)$, and the remainder do not, i.e., $m^j(\theta) > 0$, $(j = b + 1, ..., d_m)$, and $c = d_m - b$; note that b and thus c depend on θ. In principle, the critical value $\hat{c}_W$ describing the GMM confidence region estimator $\hat{\Theta}_n^W(\hat{c}_W) = \{\theta \in \Theta : n\hat{Q}_n^W(\theta) \leq \hat{c}_W\}$ would be obtained from consideration of the distribution of the limit quantity $\mathcal{C}^W = \sup_{\theta \in \Theta_{P_0}} \mathcal{C}^W(\theta)$ for the optimised GMM criterion $\sup_{\theta \in \Theta_{P_0}} n\hat{Q}_n^W(\theta)$, where $\mathcal{C}^W(\theta) = (v(\theta) - s(\theta))'W(\theta)(v(\theta) - s(\theta))$, $s(\theta) = \arg\min_{s_b \in \mathcal{R}_+^b, s_c \in \mathcal{R}^c}(v(\theta) - s)'W(\theta)(v(\theta) - s)$, $s = (s_b', s_c')'$ with s_b those b elements of s corresponding to the b binding moment inequalities and s_c the remainder. See Lemmas A.2 and A.3 in Appendix A together with the Proof of CHT Condition C.1(d) in Appendix B. Let $\hat{c}_W(1-\alpha)$ denote a consistent estimator of the $1-\alpha$ quantile $c_W(1-\alpha)$ of the limit quantity $\mathcal{C}^W$. Then the GMM confidence region estimator $\{\Theta_{P_0} \subseteq \hat{\Theta}_n^W(\hat{c}_W(1-\alpha))\}$ defines an asymptotically $(1-\alpha)$ level confidence region for Θ_{P_0} as $\lim_{n\to\infty} \mathcal{P}\{\Theta_{P_0} \subseteq \hat{\Theta}_n^W(\hat{c}_W(1-\alpha))\} = 1-\alpha$ with $\hat{\Theta}_n^W(\hat{c}_W(1-\alpha))$(4.1) a consistent estimator of Θ_{P_0} in Hausdorff distance at rate $n^{-1/2}$; see Theorem 4.1 and CHT, p.1266. Similar results may be stated for the GEL confidence region estimator $\{\Theta_{P_0} \subseteq \hat{\Theta}_n^\rho(\hat{c}_\rho(1-\alpha))\}$ and $\hat{\Theta}_n^\rho(\hat{c}_\rho(1-\alpha))$ (4.2) given the limiting relationship of the scaled GEL criterion (3.5) to the GMM criterion $n\hat{Q}_n^W(\theta)$ (3.1) when $W_n(\theta) = \hat{\Omega}_n(\theta)^{-1}$; see Theorem 4.2 and Appendix C.

Remark 5.1. To the best of our knowledge, no formal results yet exist establishing the asymptotic validity of sub-sampling methods for approximating the distribution of the limit GMM quantity $\mathcal{C}^W$ with a non-diagonal GMM weight matrix $W(\theta)$, in particular, $\Omega(\theta)^{-1}$, required for simulating the GEL confidence region quantile estimator $\hat{c}_\rho(1-\alpha)$. Cf. CHT section 3.4, pp.1257-1258.

To deal with the difficulty outlined in Remark 5.1, a simple valid but conservative confidence region estimator for the identified set Θ_{P_0} is now described. The difficulty is easily circumvented by replacing the optimal GMM slackness parameter estimator $\hat{t}_n(\theta)$ by $[\hat{m}_n(\theta)]_-$, i.e., the estimator that solves a GMM criterion with diagonal weight-matrix as metric, thereby bounding the GMM criterion $n\hat{Q}_n^W(\theta)$ (3.1) above; cf. CHT Condition M.2(e), p.1265. Let

$$\begin{aligned}\underline{\hat{Q}}_n^W(\theta) &= [\hat{m}_n(\theta)]_-' W_n(\theta)[\hat{m}_n(\theta)]_- \\ &= \left\|[\hat{m}_n(\theta)]_-' W_n(\theta)^{1/2}\right\|^2. \end{aligned} \tag{5.1}$$

Then, by definition,

$$\hat{Q}_n^W(\theta) \leq \underline{\hat{Q}}_n^W(\theta)$$

for all n and $\theta \in \Theta$.

Remark 5.2. The population counterpart $\underline{Q}^W(\theta)$ to the bounding GMM criterion $\underline{\hat{Q}}_n^W(\theta)$ (5.1) is defined by $\underline{Q}^W(\theta) = [m(\theta)]_-' W(\theta)[m(\theta)]_- = \left\|[m(\theta)]_-' W(\theta)^{1/2}\right\|^2$; cf. $Q^W(\theta)$ (3.3).

The Proofs of CHT Conditions C.4, p.1256, and C.5, p.1257, in Appendix B establish the limiting behaviour of the scaled bounding GMM criterion $n\underline{\hat{Q}}_n^W(\theta)$ (5.1); cf. CHT Proof of Theorem 4.2 Steps 4 and 5, pp.1279-1280. The Proof of CHT Condition C.4 in Appendix B, in particular, see (B.2) and (B.3) of Appendix B, establishes that the limit $\underline{\mathcal{C}}^W$ of $\underline{\mathcal{C}}_n^W = \sup_{\theta\in\Theta_{P_0}} n\underline{\hat{Q}}_n^W(\theta)$ is described by

$$\underline{\mathcal{C}}^W = \sup_{\theta\in\Theta_{P_0}} \|[v(\theta)+\xi(\theta)]_-\|^2_{W(\theta)}$$

where $\xi^j(\theta)=0$ if $m^j(\theta)=0$, $(j=1,...,b)$, and $\xi^j(\theta)=\infty$ if $m^j(\theta)>0$, $(j=b+1,...,d_m)$, for $\theta\in\Theta_{P_0}$.

Correspondingly the $1-\alpha$ quantile $\underline{c}_W(1-\alpha)$ of the limit $\underline{\mathcal{C}}^W$ of the scaled bounding GMM criterion $n\underline{\hat{Q}}_n^W(\theta)$ (5.1) satisfies

$$\mathcal{P}\{\underline{\mathcal{C}}^W \leq \underline{c}_W(1-\alpha)\} = 1-\alpha,$$

i.e., $\lim_{n\to\infty}\mathcal{P}\{\sup_{\theta\in\Theta_{P_0}} n\underline{Q}_n^W(\theta) \leq \underline{c}_W(1-\alpha)\} = 1-\alpha$. It is then immediate that $\lim_{n\to\infty}\mathcal{P}\{\sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n^W(\theta) \leq \underline{c}_W(1-\alpha)\} \geq \lim_{n\to\infty}\mathcal{P}\{\sup_{\theta\in\Theta_{P_0}} n\underline{\hat{Q}}_n^W(\theta) \leq \underline{c}_W(1-\alpha)\}$. Hence the asymptotic level of the GMM confidence region estimator $\{\Theta_{P_0}\subseteq\hat{\Theta}_n^W(\underline{c}_W(1-\alpha))\}$ is bounded below by $1-\alpha$, i.e.,

$$\lim_{n\to\infty}\mathcal{P}\{\Theta_{P_0}\subseteq\hat{\Theta}_n^W(\underline{c}_W(1-\alpha))\} \geq 1-\alpha. \tag{5.2}$$

Remark 5.3. To implement the confidence region estimator $\{\Theta_{P_0}\subseteq\hat{\Theta}_n^W(\underline{c}_W(1-\alpha))\}$ requires a consistent estimate of the quantile $\underline{c}_W(1-\alpha)$ of the limit $\underline{\mathcal{C}}^W$ of $\underline{\mathcal{C}}_n^W = \sup_{\theta\in\Theta_{P_0}} n\underline{\hat{Q}}_n^W(\theta)$. A simulation procedure similar to that outlined in CHT Remarks 4.2, pp.1263-1264, and 4.5, p.1267, suffices. In particular, let z_i^*, $(i=1,...,n)$, denote n `i.i.d.` draws from the standard normal $N(0,1)$ distribution. Thus, the process $v_n^*(\theta) = n^{-1/2}\sum_{i=1}^n m_i(\theta)z_i^*$ is zero-mean Gaussian with covariance function $\sum_{i=1}^n m_i(\theta_a)m_i(\theta_b)'/n$. Define $\hat{\xi}_n^j(\theta)=0$ if $\hat{m}_n^j(\theta)\leq c_j((\log n)/n)^{1/2}$ and ∞ if $\hat{m}_n^j(\theta)> c_j((\log n)/n)^{1/2}$ for some positive constants $c_j>0$, $(j=1,...,d_m)$. Also let $\hat{\Theta}_n$ denote a consistent estimator of Θ_{P_0}; see section 4. Quantiles of the limit $\underline{\mathcal{C}}^W$ can then be estimated by simulation from the distribution of $\underline{\hat{C}}_n^{W*} = \sup_{\theta\in\hat{\Theta}_n}\underline{\hat{Q}}_n^{W*}(\theta)$ where $\underline{\hat{Q}}_n^{W*}(\theta) = \|[v_n^*(\theta)+\hat{\xi}_n(\theta)]_-\|_{W_n(\theta)}$.

5.2 Confidence Regions for θ_0

This section is concerned with GMM and GEL estimation of confidence regions for the true parameter value θ_0. Of central interest here is the optimal GMM criterion in the unconditional moment equality context, i.e., $\hat{Q}_n^{\Omega^{-1}}(\theta)$ (3.2) when the GMM metric

$W_n(\theta) = \hat{\Omega}_n(\theta)^{-1}$. CHT section 5, pp.1267-1270, analyses the issue with an asymptotically diagonal GMM weight matrix whereas Rosen (2008) deals with the optimal GMM criterion.To ease the notational burden the optimal GMM metric Ω^{-1} is omitted in the following discussion.

Let $b(\theta)$ denote the number of binding moments for $\theta \in \Theta_{P_0}$. Define $c(\theta) = d_m - b(\theta)$, $\theta \in \Theta_{P_0}$. Without loss of generality also let $m^j(\theta) = 0$, $(j = 1, ..., b(\theta))$, and $m^j(\theta) > 0$, $(j = b(\theta) + 1, ..., d_m)$, $\theta \in \Theta_{P_0}$.

By Lemma A.3 in Appendix A

$$\begin{aligned} n\hat{Q}_n(\theta) &= \inf_{s_b \in \mathcal{R}^b_+, s_c \in \mathcal{R}^c} (v(\theta) - s)'\Omega(\theta)^{-1}(v(\theta) - s) + o_p(1) \\ &= (v(\theta) - s(\theta))'\Omega(\theta)^{-1}(v(\theta) - s(\theta)) + o_p(1) \end{aligned}$$

uniformly $\theta \in \Theta_{P_0}$. Therefore, cf. Rosen (2008) Proposition 3, p.110, uniformly $\theta \in \Theta_{P_0}$,

$$\lim_{n \to \infty} \mathcal{P}\{n\hat{Q}_n(\theta) > c\} = \sum_{j=1}^{b(\theta)} w(b(\theta), b(\theta) - j), \Omega(\theta))\mathcal{P}\{\chi^2_j > c\}, \tag{5.3}$$

a weighted chi-bar square distribution, where χ^2_j, $(j = 1, ..., b(\theta))$, denote independent chi-square random variates with j degrees of freedom respectively. The weights $w(b(\theta), b(\theta) - j), \Omega(\theta))$, $(j = 1, ..., b(\theta))$, in (5.3) are defined in Kudo (1963) and Wolak (1987) and correspond to the probability that exactly j of the $b(\theta)$ binding inequality constraints bind, i.e., $\mathcal{P}\{s(\theta)$ has j zero components$\}$, $(j = 1, ..., b(\theta))$; e.g., ${}^{b(\theta)}C_j/2^{b(\theta)}$ if $\Omega(\theta)$ is diagonal. See the discussion in Rosen (2008), p.111.

Clearly the GMM statistic (3.2) $n\hat{Q}_n(\theta)$ (3.2) is asymptotically non-pivotal. As noted in Rosen (2008), if both $b(\theta)$ and $\Omega(\theta)$ were known, the limiting distribution (5.3) could easily be simulated with a valid confidence region for the true value θ_0 obtained by inversion of the non-rejection region $\{n\hat{Q}_n(\theta) \leq c\}$ with c determined to deliver the desired confidence level from (5.3). The limiting distribution (5.3), however, is discontinuous in $b(\theta)$ rendering an estimator for this limiting distribution based on simulation after substitution of consistent estimators $\hat{b}_n(\theta)$ and $\hat{\Omega}_n(\theta)$ for $b_n(\theta)$ and $\Omega(\theta)$ respectively inconsistent. Consequently, Rosen (2008), p.111, suggests using a least favourable asymptotic distribution approach based on an estimated upper bound for $b(\theta)$. In particular, define $\hat{b}_n(\theta) = \sum_{j=1}^{d_m} 1[\hat{m}^j_n(\theta) < C((\log n)/n)^{1/2}]$ for some constant $C > 0$. Then, since $\lim_{n\to\infty} \mathcal{P}\{\hat{b}_n(\theta) = b(\theta)\} = 1$, uniformly $\theta \in \Theta_{P_0}$, see CHT Remark 4.2, p.1267, Rosen (2008) proposes the upper bound estimator $\hat{b}^{\sup}_n = \sup_{\theta \in \hat{\Theta}_n(\hat{c})} \hat{b}_n(\theta)$ where $\hat{\Theta}_n(\hat{c})$ is the consistent GMM identified set estimator (4.1) with level $\hat{c}$ satisfying Theorem 4.1.

Let $b^{\sup} = \sup_{\theta\in\Theta_{P_0}} b(\theta)$. Then

$$\sup_{\theta\in\Theta_{P_0}} \lim_{n\to\infty} \mathcal{P}\{n\hat{Q}_n(\theta) > c\} \leq \frac{1}{2}\mathcal{P}\{\chi^2_{b^{\sup}} > c\} + \frac{1}{2}\mathcal{P}\{\chi^2_{b^{\sup}-1} > c\};$$

see Rosen (2008) Corollary 1, p.113. Therefore, setting c such that

$$\alpha = \frac{1}{2}\mathcal{P}\{\chi^2_{b^{\sup}} > c\} + \frac{1}{2}\mathcal{P}\{\chi^2_{b^{\sup}-1} > c\},$$

a conservative $1-\alpha$ level confidence region for θ_0 is given by

$$\begin{aligned}\inf_{\theta\in\Theta_{P_0}} \lim_{n\to\infty} \mathcal{P}\{n\hat{Q}_n(\theta) &\leq c\} = 1 - \sup_{\theta\in\Theta_{P_0}} \lim_{n\to\infty} \mathcal{P}\{n\hat{Q}_n(\theta) > c\} \\ &\geq 1-\alpha.\end{aligned}$$

See the associated discussion in Rosen (2008) section 4, pp.111-113.

Remark 5.4. The scaled optimised GEL criterion $2n\hat{P}^{\rho}_n(\theta)$ (3.4) is asymptotically equivalent to the GMM criterion $n\hat{Q}_n(\theta)$ (3.2), uniformly $\theta \in \Theta_{P_0}$; see Lemma C.4 in Appendix C. Therefore, a valid conservative GEL confidence region for θ_0 asymptotically equivalent to that defined in (5.4) is given by substitution of this GEL criterion for $n\hat{Q}_n(\theta)$ in (5.4) based on the respective consistent GEL identified set estimator $\hat{\Theta}^{\rho}_n(\hat{c}_\rho)$ (4.2) in place of the GMM identified set estimator $\hat{\Theta}_n(\hat{c})$ (4.1).

6 Simulation Evidence

This section reports the results from a simulation study for interval outcomes in a nonlinear conditional mean regression model to assess the performance of some GMM and GEL confidence region estimators for the identified set Θ_{P_0}. Cf. CHT Example 2, p.1248.

6.1 Experimental Design

The nonlinear conditional mean regression for the latent scalar variable y given the scalar covariate x is described by

$$y = x^{\theta_0} + u$$

where $u|x \sim N(0,1)$, x is uniformly distributed on the unit interval $[0,1]$ and $\theta_0 = 0$ is the true value of the scalar parameter θ.

The regressand y is only partially observed according to the interval observation rule

$$y_1 \leq y \leq y_2$$

with $y_1 = y - \omega_1 x^2$ and $y_2 = y + \omega_2 x$ observed where $\omega_1, \omega_2 \geq 0$. Hence

$$\mathbb{E}_{P_0}[y_1|x] \leq x^{\theta_0} \leq \mathbb{E}_{P_0}[y_2|x] \text{ a.s. } x.$$

and thus

$$\mathbb{E}_{P_0}[y_1 x] \leq \mathbb{E}_{P_0}[x^{\theta_0+1}] \leq \mathbb{E}_{P_0}[y_2 x]$$

Defining the moment indicator vector $m(z,\theta) = \big(-(y_1 - x^\theta)x, (y_2 - x^\theta)x\big)'$,

$$\begin{aligned}
\mathbb{E}_{P_0}[m(z,\theta)] &= \begin{pmatrix} -(E[x] - \omega_1 E[x^3] - E[x^{\theta+1}]) \\ E[x] + \omega_2 E[x^2] - E[x^{\theta+1}] \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2} + \frac{\omega_1}{4} + \frac{1}{\theta+2} \\ \frac{1}{2} + \frac{\omega_2}{3} - \frac{1}{\theta+2} \end{pmatrix}.
\end{aligned}$$

Therefore, with the moment inequality contraint $\mathbb{E}_{P_0}[m(z,\theta)] \geq 0$, the identified set Θ_{P_0} is given by the interval

$$\Theta_{P_0} = \left[-\frac{4\omega_2}{3 + 2\omega_2}, \frac{\omega_1}{1 - \omega_1/2} \right].$$

To obtain the moment matrix $\Omega(\theta_0) = \mathbb{E}_{P_0}[m(z,\theta_0)m(z,\theta_0)']$, note that $m^1(z,\theta_0) = -ux + \omega_1 x^3$ and $m^2(z,\theta_0) = ux + \omega_2 x^2$. Hence

$$\Omega(\theta_0) = \begin{pmatrix} \frac{1}{3} + \frac{\omega_1^2}{7} & \frac{\omega_1\omega_2}{6} - \frac{1}{3} \\ \frac{\omega_1\omega_2}{6} - \frac{1}{3} & \frac{1}{3} + \frac{\omega_2^2}{5} \end{pmatrix}$$

which is diagonal when $\omega_1\omega_2 = 2$.

The experiments consider two designs.

- Design 1.

  $$\omega_1 = \frac{2}{3}, \omega_2 = 3.$$

  In this case, $\Omega(\theta_0)$ is diagonal, the identified set

  $$\Theta_{P_0} = [-\frac{4}{3}, 1]$$

  and moment matrix

  $$\Omega(\theta_0) = \begin{pmatrix} \frac{25}{63} & 0 \\ 0 & \frac{32}{15} \end{pmatrix}.$$

- Design 2.

  $$\omega_1 = \frac{2}{5}, \omega_2 = \frac{1}{2}.$$

  Now $\Omega(\theta_0)$ is non-diagonal, the identified set

  $$\Theta_{P_0} = [-\frac{1}{2}, \frac{1}{2}]$$

and moment matrix

$$\Omega(\theta_0) = \begin{pmatrix} \frac{187}{515} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{23}{60} \end{pmatrix}.$$

REMARK 6.1. Design 1 studies GMM with a diagonal metric under somewhat favourable conditions since $\Omega(\theta_0)$ is diagonal. In this design, though, the moment matrix $\Omega(\theta)$ cannot be diagonal everywhere on Θ_{P_0}, as is likely to be the case in practice. When $\theta = \theta_0$ diagonal weighted GMM and GEL criteria are asymptotically equivalent but, importantly, this is not generally so for $\theta \in \Theta_{P_0} \backslash \{\theta_0\}$. Indeed, as the width of Θ_{P_0} decreases, corresponding to a lessening failure of point identification, the moment matrix $\Omega(\theta_0)$ approaches singularity. Intuitively, as ω_1 and ω_2 approach 0 and, thus, y_1 and y_2 both approach y, the difference between the moment indicators decreases and their correlation tends to minus unity as Θ_{P_0} approaches θ_0. Indeed, for Design 2, in which the identified set is less than half the width of that in Design 1, the correlation between elements of the moment indicator vector $m(z, \theta_0)$ is now -0.894.

Experimental data are generated as random samples of size $n = 50$, 100, 500 and 1000 from the joint distribution of (y, x). Each simulation experiment comprises 500 replications.

6.2 Criteria

The criteria $n\hat{Q}_n^j(\theta)$, ($j =$EL, ET, CUE, GMM), are considered where $\hat{Q}_n^j(\theta) = 2\hat{P}_n^\rho(\theta)$ (3.5), ($j =$EL, ET, CUE), with $\rho(v) = \log(1 - v)$ [EL] empirical likelihood, $\rho(v) = -\exp(v) + 1$ [ET] exponential tilting and $\rho(v) = -v^2/2 - v$ [CUE] continuous updating estimator respectively and $\hat{Q}_n^{\text{GMM}}(\theta) = \hat{Q}_n^{\Omega_D^{-1}}(\theta)$ [GMM] the GMM objective function with diagonal metric $W_n(\theta) = \hat{\Omega}_n^D(\theta)^{-1}$, where $\hat{\Omega}_n^D(\theta) = \texttt{diag}(\hat{\Omega}_n(\theta))$, i.e., the diagonal elements of `the efficient` moment equality metric $\hat{\Omega}_n(\theta)^{-1}$, thus mimicing the approach in CHT section 4.2, pp.1265-67.

Each criterion is evaluated across the grids $\Theta_n = \{-8.\dot{3}, ..., 0, ..., 8\}$ and $\Theta_n = \{-6, 5, ..., 0, ..., 6.5\}$ for Designs 1 and 2 respectively.[1]

[1]To reduce computer processing time the grid spacing was increased for values of θ in Θ_n further away from the bounds defining Θ_{P_0}, though always maintaining $d_H(\Theta_{P_0}, \Theta_n) = O(1/n)$.

6.3 Level

The definitions of the level c_n^j used to evaluate the coverage probability $\mathcal{P}\{n\hat{Q}_n^j(\theta) \leq c_n^j\}$ are

$$c_n^j \in \{\hat{c}_n^j, \log\log n/2, \log n/2, \sqrt{n}/2\},$$

where $\hat{c}_n^j = \inf_{\Theta_n} n\hat{Q}_n^j(\theta) + 0.001$, ($j$ =EL, ET, CUE, GMM), with the grids Θ_n given in section 6.2.

Remark 6.2. Hence, the moment function satisfies the degeneracy condition CHT Condition C.3, p.1255, see Appendix B, rendering each set estimator a consistent estimator of the identified set Θ_{P_0}; see section 4.

6.4 Results

Figures 1 and 2 about here

Figures 1 and 2 for Designs 1 and 2 respectively indicate that the coverage probability $\mathcal{P}\{n\hat{Q}_n^j(\theta) \leq c_n^j\}$ converges to unity for $\theta \in \Theta_{P_0}$ for $\hat{c}_n^j$ providing empirical support for the GMM and GEL consistency results of Theorems 4.1 and 4.2 respectively and with $\hat{c}_n^j/n \to 0$; cf. Remarks 4.2 and 4.6. With faster rates of growth for the level c_n^j the coverage probability $\mathcal{P}\{n\hat{Q}_n^j(\theta) \leq c_n^j\}$ is closer to 1 for a higher proportion of $\theta \in \Theta_{P_0}$. However, there is also a tendency for an increase in the coverage probability $\mathcal{P}\{n\hat{Q}_n^j(\theta) \leq c_n^j\}$ for $\theta \in \Theta_n \backslash \Theta_{P_0}$; this is especially marked in the smaller samples, e.g., see Figures 1 and 2 with $n = 50$ and 100 when $\theta > 1$ and $\theta > 1/2$, the respective upper bounds of the identified set Θ_{P_0}. There are major differences between the criteria for $\theta < -4/3$ and $\theta < -1/2$, the lower bounds of Θ_{P_0}. Both CUE and, to a slightly lesser extent, GMM exhibit very high coverage probabilities whereas those for both EL and ET (and CUE and GMM with level $\hat{c}_n^j$) are close to zero. Overall, there is very little difference in coverage probabilities when $\theta > 1$ and $\theta > 1/2$ between all criteria; the differences between EL and ET as compared to CUE and GMM are far less pronounced than those for $\theta < -4/3$ and $\theta < -1/2$. For the larger sample sizes $n = 500$ and 1000 the coverage probabilities are very similar for EL, ET, CUE and GMM for both $\theta \in \Theta_{P_0}$ and $\theta > 1$ and $\theta > 1/2$ but CUE and GMM (except with level $\hat{c}_n^j$) continue to display very poor properties, especially CUE, for $\theta < -4/3$ and $\theta < -1/2$.

Tables 1 and 2 about here

To cast further light on these results Tables 1 and 2 provide some summary statistics on the coverage properties of $\hat{\Theta}_n^j(c_n^j)$, (j =EL, ET, CUE, GMM), in Designs 1 and 2 respectively for levels $c_n^j \in \{\hat{c}_n^j, \log\log n/2, \log n/2\}$.[2] Let $\theta_{0L} = \min\{\theta \in \Theta_{P_0}\}$ and $\theta_{0U} = \max\{\theta \in \Theta_{P_0}\}$. Likewise define $\hat{\theta}_{0L}^j = \min\{\theta \in \hat{\Theta}_n^j(c_n^j)\}$ and $\hat{\theta}_{0U}^j = \max\{\theta \in \hat{\Theta}_n^j(c_n^j)\}$, ($j$ =EL, ET, CUE, GMM). By Theorems 4.1(b) and 4.2(b) $\hat{\theta}_{0L}^j - \theta_{0L} = O_p(n^{-1/2})$ and $\hat{\theta}_{0U}^j - \theta_{0U} = O_p(n^{-1/2})$, ($j$ =EL, ET, CUE, GMM). The "Bounds" columns provide intervals whose lower and upper bounds are the empirical means of respectively $\hat{\theta}_{0L}^j$ and $\hat{\theta}_{0U}^j$ whereas those headed "MSE" are the empirical means of $[(\hat{\theta}_{0L}^j - \theta_{0L})^2 + (\hat{\theta}_{0U}^j - \theta_{0U})^2]^{1/2}$, these last columns providing an indication of the precision of the identified set estimators. The final columns headed "$p_{\Theta_{P_0}}$" record the average percentage of the identified set Θ_{P_0} covered by the identified set estimator; from Theorems 4.1 and 4.2 the expectation is that $p_{\Theta_{P_0}}$ should approach 100 as sample size n increases.

With $c_n^j = \hat{c}_n^j$, in both designs, the performances of EL, ET and GMM are more or less identical and are relatively superior to CUE according to all comparators. Similarly to Figures 1 and 2, there is a marked deterioration for GMM and especially CUE as the level c_n^j increases from $\log\log n/2$ to $\log n/2$ with the lower bound of the identified set severely underestimated and consequential and substantial rises in MSE. In contradistinction, both EL and ET perform well although less so than with the level $c_n^j = \hat{c}_n^j$.

To study the properties of the conservative inferential procedures described in section 5.1 based on the GEL confidence region estimator $\hat{\Theta}_n^\rho(\hat{c}_\rho(1-\alpha))$ (4.2), the quantiles of the bounding statistic $\sup_{\theta\in\Theta_{P_0}} \underline{\hat{Q}}_n^{\Omega^{-1}}(\theta)$ (5.1) are required where $\underline{\hat{Q}}_n^{\Omega^{-1}}(\theta) = [\hat{m}_n(\theta)]'_- \hat{\Omega}_n(\theta)^{-1}[\hat{m}_n(\theta)]_-$. As described in Remark 5.3 of section 5.1 suitable estimates are provided by simulation of the quantiles from $\underline{\hat{C}}_n^{\Omega^{-1}*} = \sup_{\theta\in\hat{\Theta}_n} \underline{\hat{Q}}_n^{\Omega^{-1}*}(\theta)$ where $\underline{\hat{Q}}_n^{\Omega^{-1}*}(\theta) = \|[v_n^*(\theta) + \hat{\xi}_n(\theta)]_-\|_{\hat{\Omega}_n(\theta)^{-1}}$ setting $c_1 = c_2 = 1/5$. Likewise, for the GMM confidence region estimator $\hat{\Theta}_n(\hat{c}_{\Omega_D^{-1}}(1-\alpha))$ (4.1) based on the GMM criterion $\hat{Q}_n^{\mathrm{GMM}}(\theta) = \hat{Q}_n^{\Omega_D^{-1}}(\theta)$, the quantiles from $\hat{C}_n^{\Omega_D^{-1}*} = \sup_{\theta\in\hat{\Theta}_n} \hat{Q}_n^{\Omega_D^{-1}*}(\theta)$ where $\hat{Q}_n^{\Omega_D^{-1}*}(\theta) = \left\|(v_n^*(\theta) + \hat{\xi}_n(\theta))'\hat{\Omega}_n^D(\theta)^{-1/2}\right\|_-^2$ need to be simulated. Note that if interest is in ascertaining whether a hypothesised identified set $\Theta_{P_0}^0$ is credible by examining whether $\Theta_{P_0}^0 \subseteq \hat{\Theta}_n(\hat{c}^*_{\Omega_D^{-1}}(1-\alpha))$ or $\Theta_{P_0}^0 \subseteq \hat{\Theta}_n^\rho(\underline{\hat{c}}^*_{\Omega^{-1}}(1-\alpha))$, where $\hat{c}^*_{\Omega_D^{-1}}(1-\alpha)$ and $\underline{\hat{c}}^*_{\Omega^{-1}}(1-\alpha)$ are the $1-\alpha$ quantiles of $\underline{\hat{C}}_n^{\Omega_D^{-1}*}$ and $\underline{\hat{C}}_n^{\Omega^{-1}*}$ respectively, it is only necessary to take the suprema of $\hat{Q}_n^{\Omega_D^{-1}*}(\theta)$ and $\underline{\hat{Q}}_n^{\Omega^{-1}*}(\theta)$ over $\Theta_{P_0}^0$ rather than an estimator $\hat{\Theta}_n$. As a consequence, inference is greatly simplified and no further sampling uncertainty is added to the estimate of the distribution of the limit vari-

[2]Results for $c_n^j = \sqrt{n}/2$ are omitted for brevity being less likely to be used in practice to form an estimator of Θ_{P_0} but are available from the authors upon request.

ates $\underline{\mathcal{C}}^{\Omega^{-1}}$ and $\mathcal{C}^{\Omega_D^{-1}} = \sup_{\theta\in\Theta_{P_0}} \left\|(v(\theta)+\xi(\theta))'\Omega^D(\theta)^{-1/2}\right\|_-^2$ where $\Omega^D(\theta) = \mathtt{diag}(\Omega(\theta))$; see below Remark 5.2.

Figure 3 about here

Figure 3 displays for Designs 1 and 2 the estimated quantiles of $\sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n^j(\theta)$, ($j$ =ET, EL, CUE, GMM), together with those of the GEL bounding variate $\underline{\hat{C}}_n^{\Omega^{-1}*} = \sup_{\theta\in\Theta_{P_0}} n\underline{\hat{Q}}_n^{\Omega^{-1}*}(\theta)$ (5.1), and $\hat{C}_n^{\Omega_D^{-1}*} = \sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n^{\Omega_D^{-1}*}(\theta)$. As expected the quantiles of the GEL variates $\sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n(\theta)^j$, ($j$ =ET, EL, CUE), are bounded below by those of $\underline{\hat{C}}_n^{\Omega^{-1}*}$ with the bound rather conservative for the range $1-\alpha \in (0.9, 1.0)$ typically used for inference in practice; this is especially the case in Design 2 with the more highly correlated moments and thus larger elements of the inverse $\hat{\Omega}_n(\theta)^{-1}$. The ET, EL and CUE quantiles are almost identical at all sample sizes which corroborates empirically their first order equivalence on Θ_{P_0} detailed in Theorem 4.2. Rather surprisingly, the quantiles of the diagonally weighted GMM criterion $\sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n^{\mathrm{GMM}}(\theta)$ provides a close approximation to those for EL, ET and CUE as do the simulated quantiles of $\hat{C}_n^{\Omega_D^{-1}*}$.

Tables 3 and 4 about here

Finally, Tables 3 and 4 detail the empirical volume and coverage of the confidence regions in both designs. As expected from Figure 3, the empirical coverage of the confidence regions based on the bounding conservative $\underline{\hat{C}}_n^{\Omega^{-1}*}$ quantiles exceeds the nominal coverage probabilities in almost all cases no matter the sample size; this finding is especially the case in Design 2. The confidence regions for the identified set Θ_{P_0} are correspondingly wide although those for EL and ET are substantially smaller than those of CUE. Although the simulated quantiles of $\hat{C}_n^{\Omega_D^{-1}*}$ are appropriate for diagonally weighted GMM, the empirical coverage probabilities still deviate substantially from nominal values with correspondingly wide confidence regions. Again, surprisingly, the simulated quantiles of $\hat{C}_n^{\Omega_D^{-1}*}$ provide quite close approximations to those for EL and ET with the corresponding EL and ET confidence regions substantially smaller than those of GMM and CUE.

7 Concluding Remarks

This paper examines the properties of GEL methods for the estimation of the identified set in models specified by unconditional moment inequality constraints.

The paper extends the results for GMM estimation in CHT section 4, pp.1261-1267, to permit a non-diagonal weight matrix in the GMM criterion, in particular, the inverse of the moment variance matrix, the optimal GMM metric appropriate for moment equality conditions. Unlike the moment equality context, this extension of GMM to GEL is relatively non-trivial. Analogously to moment equality condition models, an asymptotic equivalence exists between various scaled optimised GEL criteria and that for GMM with optimal moment equality weight matrix. Consequently, similarly to CHT, conditions are provided for consistent GEL estimation of the identified set at the parametric rate $n^{1/2}$. When the moment matrix is non-diagonal on the identified set the limit of the scaled optimised GEL statistic differs from that for GMM with diagonal weight matrix which the case studied in CHT section 4, pp.1261-1267. To the best of our knowledge there are, as yet, no results for the asymptotic validity of a bootstrap or sub-sampling approximation to the limiting distribution of these statistics; cf. the application of CHT section 3.4, pp.1257-1258, to GMM with a diagonal weight matrix given in CHT section 4, pp.1261-1267. A conservative confidence region estimator for the identified set is therefore developed. The GMM criterion with non-diagonal weight matrix may be bounded above by a statistic the limit of which can be approximated using a resampling method similar to that described in CHT Remarks 4.2, pp.1263-1264, and 4.5, p.1267, for GMM with a diagonal weight matrix. Conservative GMM and GEL confidence region estimators for the true parameter, cf. Rosen (2008), are also described.

A simulation study for interval outcomes in a nonlinear conditional mean regression model corroborates the main theoretical results of the paper with favourable small sample properties for EL and ET estimators of the identified set and conservative EL and ET confidence region estimators for the identified set. Somewhat surprisingly the simulated quantiles associated with diagonally weighted GMM provide rather better approximations to those for EL and ET than the simulated quantiles based on the conservative approach.

Appendix

The argument θ is suppressed for expositional simplicity throughout the Appendices where there is no possibility of confusion.

Throughout the Appendices, C will denote a generic positive constant that may be different in different uses with CS, M and T the Cauchy-Schwarz, Markov and triangle inequalities respectively. In addition CMT is the continuous mapping theorem and $\lambda_{\min}(\cdot)$

and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues respectively of $\cdot$.

The following convention is employed. $\mathbb{E}_{P_0}[m^j(z,\theta)] < 0$, $(j = 1, ..., a)$, $\mathbb{E}_{P_0}[m^j(z,\theta)] = 0$, $(j = a+1, ..., a+b)$, and $\mathbb{E}_{P_0}[m^j(z,\theta)] > 0$, $(j = a+b+1, ..., d_m)$. Defining $c = d_m - a - b$, a, b and thus c depend on θ. Vectors are correspondingly partitioned, e.g., $s = (s_a', s_b', s_c')'$ such that s_a corresponds to $\mathbb{E}_{P_0}[m^j(z,\theta)] < 0$, $(j = 1, ..., a)$, i.e., those a elements of s for which (2.1) is false, s_b to $\mathbb{E}_{P_0}[m^j(z,\theta)] = 0$, $(j = a+1, ..., a+b)$, i.e., those b elements of s corresponding to the b binding moment inequalities and s_c to $\mathbb{E}_{P_0}[m^j(z,\theta)] > 0$, $(j = a+b+1, ..., d_m)$, i.e., the remainder.

Let $\Lambda_n = \{\lambda \in \mathcal{R}^{d_m} : \|\lambda\| \leq C n^{-1/2}\}$. Also let $\Theta_{P_0}^{-\epsilon} = \{\theta \in \Theta_{P_0} : d(\theta, \Theta \backslash \Theta_{P_0}) \geq \epsilon\}$ where $\epsilon > 0$. A closed ball of radius $\delta > 0$ is denoted by $\mathcal{B}_\delta = \{\theta \in \mathcal{R}^{d_\theta} : \|\theta\| \leq \delta\}$. Recall Θ' is a neighbourhood of Θ in $\mathcal{R}^{d_\theta}$.

Recall the GMM sample criterion (3.1) $\hat{Q}_n^W(\theta) = \inf_{t\geq 0} \|\hat{m}_n(\theta) - t\|^2_{W_n(\theta)}$. Also define $\bar{Q}_n^W(\theta) = \inf_{t\geq 0} \|\hat{m}_n(\theta) - t\|^2_{W(\theta)}$ and $nQ_n^W(\theta) = \inf_{s_b\in\mathcal{R}^b_+, s_c\in\mathcal{R}^c} \|v_n(\theta) - s\|^2_{W_n(\theta)}$. Recall the corresponding GMM population criterion $Q^W(\theta) = \inf_{t\geq 0} \|m(\theta) - t\|^2_{W(\theta)}$, $\theta \in \Theta$, where $m(\theta) = \mathbb{E}_{P_0}[m(z,\theta)]$.

Define

$$\hat{t}_n(\theta) = \arg\min_{t\geq 0} \|\hat{m}_n(\theta) - t\|^2_{W_n(\theta)},$$

i.e., $\hat{Q}_n^W(\theta) = \left\|\hat{m}_n(\theta) - \hat{t}_n(\theta)\right\|^2_{W_n(\theta)}$.

Appendix A: Preliminary Lemmas

To simplify the Proofs for CHT Conditions C.1, p.1252, and C.2, p.1253, Lemmas A.1 and A.2 show that the weighting matrix $W_n(\theta)$ in the GMM criterion $\hat{Q}_n^W(\theta)$ (3.1) may be replaced by $W(\theta)$ w.p.a.1 uniformly $\theta \in \Theta$, i.e., $n\hat{Q}_n^W(\theta) = nQ_n^W(\theta) + o_p(1)$ uniformly $\theta \in \Theta$.

Lemma A.1. *Let Assumptions A.1, A.2-GMM and A.3 be satisfied. Then*

$$\sup_{\theta\in\Theta_{P_0}} \left| n\hat{Q}_n^W(\theta) - nQ_n^W(\theta)\right| = o_p(1).$$

Proof. The proof closely follows that for Rosen (2008) Proposition 3, pp.115-116. Consider

$$\begin{aligned} n\hat{Q}_n^W(\theta) &= \inf_{t\geq 0} n \|\hat{m}_n(\theta) - t\|^2_{W_n(\theta)} \\ &= \inf_{s\geq -n^{1/2}m(\theta)} \|v_n(\theta) - s\|^2_{W_n(\theta)} \end{aligned}$$

where $s = n^{1/2}(t - m(\theta))$. Write $\hat{s}_n(\theta) = \underset{s \geq -n^{1/2}m(\theta)}{\arg\min} \|v_n(\theta) - s\|^2_{W_n(\theta)}$.

Suppose $\theta \in \Theta_{P_0}$. Thus, s_a is empty, i.e., $a = 0$; also $c = d_m - b$ and $m(\theta) \geq 0$, $\theta \in \Theta_{P_0}$. Let $m_c(\theta) = (m^{b+1}(\theta), ..., m^{d_m}(\theta))'$. In this case

$$\begin{aligned} n\hat{Q}^W_n(\theta) &= \inf_{s_b \geq 0, s_c \geq -n^{1/2}m_c(\theta)} \|v_n(\theta) - s\|^2_{W_n(\theta)} \\ &= \|v_n(\theta) - \hat{s}_n(\theta)\|^2_{W_n(\theta)} \end{aligned}$$

with solution $\hat{s}_n(\theta) = (\hat{s}_{bn}(\theta)', \hat{s}_{cn}(\theta)')' = \arg\min_{s_b \geq 0, s_c \geq -n^{1/2}m_c(\theta)} \|v_n(\theta) - s\|^2_{W_n(\theta)}$. Now

$$\begin{aligned} nQ^W_n(\theta) &= \inf_{s_b \in \mathcal{R}^b_+, s_c \in \mathcal{R}^c} \|v_n(\theta) - s\|^2_{W_n(\theta)} \\ &= \|v_n(\theta) - s_n(\theta)\|^2_{W_n(\theta)} \end{aligned}$$

with solution $s_n(\theta) = (s_{bn}(\theta)', s_{cn}(\theta)')' = \arg\min_{s_b \in \mathcal{R}^b_+, s_c \in \mathcal{R}^c} \|v_n(\theta) - s\|^2_{W_n(\theta)}$. Note that $s_{cn}(\theta) = v_{cn}(\theta)$, $\theta \in \Theta_{P_0}$, and thus, from Rosen (2008) Lemma 1, p.115, w.p.a.1 $nQ^W_n(\theta) = v_{bn}(\theta)'(W^{bb}_n(\theta))^{-1}v_{bn}(\theta)$ where $W^{bb}_n(\theta)$ denotes the top left hand $b \times b$ sub-matrix of $W_n(\theta)^{-1}$.

To show that $\sup_{\theta \in \Theta_{P_0}} \left| n\hat{Q}^W_n(\theta) - nQ^W_n(\theta) \right| = o_p(1)$, i.e., $\sup_{\theta \in \Theta_{P_0}} \|\hat{s}_n(\theta) - s_n(\theta)\| = o_p(1)$, it is only necessary to demonstrate

$$\sup_{\theta \in \Theta_{P_0}} \|\hat{s}_{cn}(\theta) - s_{cn}(\theta)\| = o_p(1)$$

or $\sup_{\theta \in \Theta_{P_0}} \|\hat{s}_{cn}(\theta) - v_{cn}(\theta)\| = o_p(1)$. Now, since v_n is P-Donsker, $\sup_{\theta \in \Theta} \|v_n(\theta)\| = O_p(1)$ by Assumption A.3, i.e., for any ε, $\delta > 0$, there exists $N(\varepsilon, \delta)$ such that, for all $n > N(\varepsilon, \delta)$, $\mathcal{P}\{\sup_{\theta \in \Theta_{P_0}} \|v_n(\theta)\| < \varepsilon\} > 1 - \delta$. Choose $\varepsilon = \max_j \sup_{\theta \in \Theta_{P_0}} m^j(\theta)$ such that for all $n > N(\varepsilon, \delta)$

$$\mathcal{P}\{\sup_{\theta \in \Theta_{P_0}} \|v_n(\theta)\| < \max_j \sup_{\theta \in \Theta_{P_0}} m^j(\theta)\} > 1 - \delta.$$

In particular, for all $n > N(\varepsilon, \delta)$, with probability at least $1 - \delta$, $\sup_{\theta \in \Theta_{P_0}} |v^j_n(\theta)| < \max_j \sup_{\theta \in \Theta} m^j(\theta)$ and, thus, $\hat{s}^j_n(\theta) = v^j_n(\theta)$, $(j = b + 1, ..., d_m)$, uniformly $\theta \in \Theta_{P_0}$, i.e.,

$$\mathcal{P}\{\sup_{\theta \in \Theta_{P_0}} \|\hat{s}_{cn}(\theta) - v_{cn}(\theta)\| = 0\} > 1 - \delta.$$

Therefore,

$$n\hat{Q}^W_n(\theta) = nQ^W_n(\theta) + o_p(1)$$

uniformly $\theta \in \Theta_{P_0}$.■

Lemma A.2. *Let Assumptions A.1, A.2-GMM and A.3 be satisfied. Then*

$$\inf_{\theta\in\Theta\backslash\Theta_{P_0}} n\hat{Q}_n^W(\theta) \xrightarrow{p} \infty.$$

Proof. Let $\theta \in \Theta\backslash\Theta_{P_0}$. In this case, since s_a is no longer empty, define $m_a(\theta) = (m^1(\theta), ..., m^a(\theta))'$. Hence,

$$\begin{aligned} n\hat{Q}_n^W(\theta) &= \inf_{s\geq -n^{1/2}m(\theta)} \|v_n(\theta)-s\|^2_{W_n(\theta)} \\ &= \inf_{s_a\geq -n^{1/2}m_a(\theta), s_b\geq 0, s_c\geq -n^{1/2}m_c(\theta)} \|v_n(\theta)-s\|^2_{W_n(\theta)} \\ &\geq \inf_{s_a\geq -n^{1/2}m_a(\theta), s_b\in\mathcal{R}^b_+, s_c\in\mathcal{R}^c} \|v_n(\theta)-s\|^2_{W_n(\theta)} \\ &\geq \inf_{s_a\geq -n^{1/2}m_a(\theta)} \|v_{an}(\theta)-s_a\|^2_{(W_n^{aa}(\theta))^{-1}} \end{aligned}$$

w.p.a.1 where $W_n^{aa}(\theta)$ denotes the $a\times a$ top left hand sub matrix of $W_n(\theta)^{-1}$ corresponding to $m_a(\theta)$; see Rosen (2008) Lemma 1, p.115. Now $\sup_{\theta\in\Theta}\|v_n(\theta)\| = O_p(1)$ by Assumption A.3. Thus, since $-n^{1/2}m_a(\theta)\to\infty$ if $\theta\in\Theta\backslash\Theta_{P_0}$ and $\sup_{\theta\in\Theta}\|W_n(\theta)-W(\theta)\| = o_p(1)$ with $W(\theta)$ uniformly p.d. from Assumption A.2-GMM(b), the statistic $n\hat{Q}_n^W(\theta)$ diverges, i.e., $n\hat{Q}_n^W(\theta)\xrightarrow{p}\infty$, uniformly $\theta\in\Theta\backslash\Theta_{P_0}$.■

Lemma A.3. *Let Assumptions A.1, A.2-GMM and A.3 be satisfied. Then*

$$n\hat{Q}_n^W(\theta) = \inf_{s_b\in\mathcal{R}^b_+, s_c\in\mathcal{R}^c} \|v(\theta)-s\|^2_{W(\theta)} + o_p(1)$$

uniformly $\theta\in\Theta_{P_0}$.

Proof. Now $\sup_{\theta\in\Theta}\|W_n(\theta)-W(\theta)\| = o_p(1)$ by Assumption A.2-GMM(b). Thus, $nQ_n^W(\theta) = v_n^b(\theta)'(W^{bb}(\theta))^{-1}v_n^b(\theta) + o_p(1)\left\|v_n^b(\theta)\right\|^2_-$ from Rosen (2008) Lemma 1, p.115, as $s_n^c(\theta) = v_n^c(\theta)$, $\theta\in\Theta_{P_0}$, where $W^{bb}(\theta)$ denotes the top left hand $b\times b$ sub-matrix of $W(\theta)^{-1}$. Then, from Lemma A.1,

$$n\hat{Q}_n^W(\theta) = \inf_{s_b\in\mathcal{R}^b_+, s_c\in\mathcal{R}^c} \|v_n(\theta)-s\|^2_{W(\theta)} + o_p(1)$$

uniformly $\theta\in\Theta_{P_0}$, noting $\sup_{\theta\in\Theta}\left\|v_n^b(\theta)\right\| = O_p(1)$. Now $\sup_{\theta\in\Theta}\|v_n(\theta)-v(\theta)\| = o_p(1)$ by v_n P-Donsker from Assumption A.3 yielding

$$n\hat{Q}_n^W(\theta) = \inf_{s_b\in\mathcal{R}^b_+, s_c\in\mathcal{R}^c} \|v(\theta)-s\|^2_{W(\theta)} + o_p(1)$$

uniformly $\theta \in \Theta_{P_0}$.$\blacksquare$

Define

$$s(\theta) = \arg \min_{s_b \in \mathcal{R}^b_+, s_c \in \mathcal{R}^{q-b}} \|v(\theta) - s\|^2_{W(\theta)}.$$

LEMMA A4. *Suppose that Assumptions A.1, A.2-GMM and A.3 hold. Then*

$$\sup_{\theta \in \Theta_{P_0}} \|s(\theta)\| = O_p(1).$$

PROOF. The dependence on θ is ignored for ease of exposition.

Now,

$$\inf_{s_b \in \mathcal{R}^b_+, s_c \in \mathcal{R}^c} \|v - s\|^2_W = \inf_{s_b \in \mathcal{R}^b_+} \|(v - s)_b\|^2_{W_{bb}}$$

where $(\cdot)_b$ denotes the first b elements of $(\cdot)$; see Rosen (2008) Lemma 1, p.115.

Therefore, from the first order conditions, either (a) $[-W_{bb}(v-s)_b]^j = 0$ and $s^j > 0$ or (b) $[-W_{bb}(v-s)_b]^j > 0$ and $s^j = 0$, $(j = 1, ..., b)$. Define $\mathcal{J} = \{j : [-W_{bb}(v-s)_b]^j = 0$ and $s^j > 0$, $(j = 1, ..., b)\}$. Now, from (a) and (b), $\sum_{k \in \mathcal{J}} W_{bbjk}(v - s)^k = -\sum_{k \notin \mathcal{J}} W_{bbjk} v^k = O_p(1)$, $j \in \mathcal{J}$, uniformly $\theta \in \Theta_{P_0}$, since $\sup_{\theta \in \Theta} \|v(\theta)\| = O_p(1)$ and $\sup_{\theta \in \Theta} \|W(\theta)\| = O(1)$ by Assumptions A.2-GMM(b) and A.3. Hence, $s^j = O_p(1)$, $j \in \mathcal{J}$, uniformly $\theta \in \Theta_{P_0}$ and $s^j = 0$, $j \in \mathcal{J}^c$. Hence the result follows because $(v-s)_c = -W_{cc}W_{cb}(v-s)_b$; see Rosen (2008) eq. (22), p.115.$\blacksquare$

Appendix B: Proofs for GMM

Appendix B establishes the validity of CHT Conditions C.1-C.3 for the GMM criterion $n\hat{Q}^W_n(\theta)$ (3.1) under Assumptions A.1, A.2-GMM and A.3-A.5. CHT Conditions C.4 and C.5 are established for the bounding GMM statistic $n\underline{\hat{Q}}^W_n(\theta)$ (5.1). The relevant CHT constants and sequences are defined as $\gamma = 2$, $a_n = n$ and $b_n = n^{1/2}$. See CHT Theorem 4.2, p.1266.

CHT CONDITION C.1. Consistency: (**a**) *The parameter space* Θ *is a nonempty compact subset of* $\mathcal{R}^{d_\theta}$. (**b**) *There is a lower semi-continuous population criterion function* $Q : \Theta \to \mathcal{R}_+$ *such that* $\inf_\Theta Q = 0$. *Let* $\Theta_{P_0} = \arg\inf_\Theta Q$ *be the set of its minimisers, called the identified set.* (**c**) *There is a sample criterion function* $\hat{Q}_n(\theta) = \hat{Q}_n(\theta, \{z_i\}^n_{i=1})$ *that takes values in* $\mathcal{R}_+$ *and is jointly measurable in the parameter* $\theta \in \Theta$ *and the data* z_i, $(i = 1, ..., n)$ *defined on a complete probability space* $(\Omega, \mathcal{F}, P)$. (**d**) *The sample criterion function is uniformly no smaller than the population function in large samples, that is,*

$\sup_{\Theta}(Q - \hat{Q}_n)_+ = O_p(n^{-1/2})$. (**e**) *The sample criterion converges to the limit criterion function over the identified set* Θ_{P_0} *at the rate* $1/n$*, that is,* $\sup_{\Theta_{P_0}} \hat{Q}_n = O_p(n^{-1})$.

PROOF. **(a)** Holds by Assumption A.1(a). **(b)** Recall the population GMM criterion function $Q^W(\theta) = \inf_{t \geq 0} \|m(\theta) - t\|^2_{W(\theta)} \geq 0$; see (3.3). Now, $t^j(\theta) = m^j(\theta)$ if $m^j(\theta) > 0$ and 0 if $m^j(\theta) = 0$, $(j = 1, ..., d_m)$, $\theta \in \Theta_{P_0}$. Hence, $Q^W(\theta)$ takes a zero value on Θ_{P_0}, i.e., $\inf_{\theta \in \Theta} Q^W(\theta) = 0$. **(c)** Holds by Assumptions A.1(b) and A.1(d). **(d)** Lemmas A.1 and A.3 establish that $n\hat{Q}^W_n(\theta) = \inf_{s_b \in \mathcal{R}^b_+, s_c \in \mathcal{R}^{d_m - b}} \|v(\theta) - s\|^2_{W(\theta)} + o_p(1)$ uniformly $\theta \in \Theta_{P_0}$ and Lemma A.2 that $n\hat{Q}^W_n(\theta) \xrightarrow{p} \infty$ uniformly $\theta \in \Theta \backslash \Theta_{P_0}$. **(e)** By (b) $Q^W(\theta) = 0$ uniformly $\theta \in \Theta_{P_0}$. Thus $\sup_{\theta \in \Theta_{P_0}} \left|\hat{Q}^W_n(\theta) - Q^W(\theta)\right| = \sup_{\theta \in \Theta_{P_0}} \left|\hat{Q}^W_n(\theta)\right| = O_p(n^{-1})$ again using Lemmas A.1 and A.3.■

CHT CONDITION C.2. Existence of a Polynomial Minorant: *There exist positive constants* (δ, κ) *such that for an* $\varepsilon \in (0, 1)$ *there are* $(\kappa_\varepsilon, n_\varepsilon)$ *such that for all* $n \geq n_\varepsilon$, $\hat{Q}_n(\theta) \geq \kappa \cdot [d(\theta, \Theta_{P_0}) \wedge \delta]^2$ *uniformly on* $\{\theta \in \Theta : d(\theta, \Theta_{P_0}) \geq (\kappa_\varepsilon/n)^{1/2}\}$ *with probability at least* $1 - \varepsilon$.

PROOF. Write $W(\theta) = X(\theta)\Lambda(\theta)X(\theta)'$, $\theta \in \Theta$, where the matrix of eigenvectors $X(\theta)$ is orthonormal, i.e., $X(\theta)^{-1} = X(\theta)'$, and eigenvalue matrix $\Lambda(\theta)$ diagonal, $\theta \in \Theta$. Hence, since $X(\theta)X(\theta)' = I_{d_m}$, as $\sup_{\theta \in \Theta} \|W_n(\theta) - W(\theta)\| = o_p(1)$ from Assumption A.2-GMM(b), w.p.a.1 uniformly $\theta \in \Theta$,

$$\begin{aligned}
n\hat{Q}^W_n(\theta) &\geq \inf_{\theta \in \Theta} \lambda_{\min}(W_n(\theta)) \cdot n \min_{t \geq 0} \|\hat{m}_n(\theta) - t\|^2 \\
&= \inf_{\theta \in \Theta} \lambda_{\min}(W(\theta)) \cdot \left\|n^{1/2}\hat{m}_n(\theta)\right\|^2_- \\
&= \inf_{\theta \in \Theta} \lambda_{\min}(W(\theta)) \cdot \left\|v_n(\theta) + n^{1/2}m(\theta)\right\|^2_- \\
&= \inf_{\theta \in \Theta} \lambda_{\min}(W(\theta)) \cdot \left\|n^{1/2}m(\theta)\right\|^2_- \\
&\quad \times \left\|v_n(\theta) + n^{1/2}m(\theta)\right\|^2_- / \left\|n^{1/2}m(\theta)\right\|^2_-
\end{aligned}$$

where the inequality follows from Assumption A.2-GMM(b) since $\inf_{\theta \in \Theta} \lambda_{\min}(W(\theta)) > 0$ as $W(\theta)$ is uniformly p.d. $\theta \in \Theta$. Now, by Assumption A.4, $\left\|n^{1/2}m(\theta)\right\|^2_- \geq C \cdot n \cdot (d(\theta, \Theta_{P_0}) \wedge \delta)^2$ for some $C > 0$ and $\delta > 0$. Therefore, as in CHT Proof of Theorem 4.2 Step 1, p.1278, for any $\varepsilon > 0$, with probability at least $1 - \varepsilon$,

$$n\hat{Q}^W_n(\theta) \geq \frac{1}{2} \inf_{\theta \in \Theta} \lambda_{\min}(W(\theta)) \cdot C \cdot n \cdot (d(\theta, \Theta_{P_0}) \wedge \delta)^2$$

uniformly $\{\theta \in \Theta : d(\theta, \Theta_{P_0}) \geq (\kappa_\varepsilon/n)^{1/2}\}$, $n > n_\varepsilon$, for some $(\kappa_\varepsilon, n_\varepsilon)$, from $\sup_{\theta\in\Theta} \|v_n(\theta)\| = O_p(1)$ by the P-Donsker property of Assumption A.3 and $\|y+x\|_- / \|x\|_- \to 1$ as $\|x\|_- \to \infty$ for any $y \in \mathcal{R}^{d_m}$.■

CHT Condition C.3. Degeneracy: *There is a sequence of subsets Θ_n of Θ, which could be data dependent, such that $\hat{Q}_n$ vanishes on these subsets, that is, $\hat{Q}_n(\theta) - \inf_{\theta\in\Theta} \hat{Q}_n(\theta) = 0$ for each $\theta \in \Theta_n$, for each n, and these sets can approximate the identified set arbitrarily well in the Hausdorff distance, that is, $d_H(\Theta_n, \Theta_{P_0}) \leq \epsilon_n$ for some sequence $\epsilon_n = O_p(n^{-1/2})$.*

Proof. Similarly to the Proof of CHT Condition C.2 above, w.p.a.1 uniformly $\theta \in \Theta_{P_0}$,

$$
\begin{aligned}
n\hat{Q}_n^W(\theta) &\leq \sup_{\theta\in\Theta} \lambda_{\max}(W(\theta)) \cdot n \min_{t\geq 0} \|\hat{m}_n(\theta) - t\|^2 \\
&= \sup_{\theta\in\Theta} \lambda_{\max}(W(\theta)) \cdot \left\|v_n(\theta) + n^{1/2} m(\theta)\right\|_-^2 \\
&\leq \sup_{\theta\in\Theta} \lambda_{\max}(W(\theta)) \cdot \sum_{j=1}^{d_m} [v_n^j(\theta) + n^{1/2} m^j(\theta)]_-^2 \\
&\leq \sup_{\theta\in\Theta} \lambda_{\max}(W(\theta)) \cdot d_m \cdot [O_p(1) + n^{1/2} \cdot C \cdot (d(\theta, \Theta \backslash \Theta_{P_0}) \wedge \delta)]_-^2
\end{aligned}
$$

where the first inequality follows from $W(\theta)$ uniformly p.d. $\theta \in \Theta$ and bounded by Assumption A.2-GMM(b), the second by T and the third inequality by Assumption A.5. The conclusion follows as in CHT Proof of Theorem 4.2 Step 2, p.1278, since, with $\epsilon_n = O_p(n^{-1/2})$, $\hat{Q}_n^W(\theta) = 0$ for $\theta \in \Theta_{P_0}^{-\epsilon_n}$.■

The Proofs of CHT Conditions C.4, p.1256, and C.5, p.1257, given below concern the bounding statistic $n\underline{\hat{Q}}_n^W(\theta)$ (5.1) for the GMM criterion $n\hat{Q}_n^W(\theta)$ (3.1). These results establish the validity of the asymptotically conservative inference procedure for Θ_{P_0} described in section 5.1.

Define $\underline{\mathcal{C}}_n^W = \sup_{\theta\in\Theta_{P_0}} \underline{\hat{Q}}_n^W(\theta)$ and $\underline{\mathcal{C}}^W = \sup_{\theta\in\Theta_{P_0}} \|[v(\theta) + \xi(\theta)]_-\|_{W(\theta)}^2$ where $\xi^j(\theta) = 0$ if $m^j(\theta) = 0$, $(j = 1, ..., b)$, and $\xi^j(\theta) = \infty$ if $m^j(\theta) > 0$, $(j = b+1, ..., d_m)$, $\theta \in \Theta_{P_0}$.

CHT Condition C.4. Convergence of $\underline{\mathcal{C}}_n^W$: $\mathcal{P}[\underline{\mathcal{C}}_n^W \leq \underline{c}_W] \to \mathcal{P}[\underline{\mathcal{C}}^W \leq \underline{c}_W]$ *for each $\underline{c}_W \in [0, \infty)$, where the distribution function of $\underline{\mathcal{C}}^W$ is non-degenerate and continuous on $[0, \infty)$.*

Proof. Define $\theta_n(\lambda) = \theta + n^{-1/2}\lambda$ and $l_n^W(\theta, \lambda) = n\hat{\underline{Q}}_n^W(\theta_n(\lambda))$. Then, for $(\theta, \lambda) \in \Theta \times \mathcal{B}_\delta$,

$$\begin{aligned} l_n^W(\theta,\lambda) &= [n^{1/2}\hat{m}_n(\theta_n(\lambda))]_-' W_n(\theta_n(\lambda))[n^{1/2}\hat{m}_n(\theta_n(\lambda))]_- \\ &= [v_n(\theta_n(\lambda)) + n^{1/2}m(\theta_n(\lambda))]_-' W_n(\theta_n(\lambda))[v_n(\theta_n(\lambda)) + n^{1/2}m(\theta_n(\lambda))]_- \\ &= \left\|[v_n(\theta_n(\lambda)) + n^{1/2}m(\theta_n(\lambda))]_-\right\|^2_{W_n(\theta_n(\lambda))} \end{aligned}$$

First, by the P-Donsker property of $v_n(\theta)$ of Assumption A.3, $v_n(\theta) \Rightarrow v(\theta)$ and $v(\theta)$ stochastically equicontinuous. Hence, $v_n(\theta_n(\lambda)) \Rightarrow v(\theta)$ uniformly $(\theta, \lambda) \in \Theta \times \mathcal{B}_\delta$. Secondly, from Assumption A.2-GMM(b), $\sup_{\theta\in\Theta} |W_n(\theta) - W(\theta)| = o_p(1)$ and $W(\theta)$ continuous, thus $W_n(\theta_n(\lambda)) \xrightarrow{p} W(\theta)$ uniformly $(\theta, \lambda) \in \Theta \times \mathcal{B}_\delta$. Therefore, uniformly $(\theta, \lambda) \in \Theta \times \mathcal{B}_\delta$,

$$l_n^W(\theta,\lambda) = \left\|[v(\theta) + n^{1/2}m(\theta_n(\lambda))]_-\right\|^2_{W(\theta)} + o_p(1). \tag{B.1}$$

Next define

$$\begin{aligned} l_\infty^W(\theta,\lambda) &= [v(\theta) + M(\theta)\lambda + \xi(\theta)]_-' W(\theta)[v(\theta) + M(\theta)\lambda + \xi(\theta)]_- \\ &= \|[v(\theta) + M(\theta)\lambda + \xi(\theta)]_-\|^2_{W(\theta)}. \end{aligned}$$

By Assumption A.4 $n^{1/2}m(\theta_n(\lambda)) = M(\theta)\lambda + \xi(\theta) + o(1)$ uniformly $(\theta, \lambda) \in \Theta_{P_0} \times \mathcal{B}_\delta$. Therefore, from (B.1),

$$l_n^W(\theta,\lambda) - l_\infty^W(\theta,\lambda) = o_p(1) \tag{B.2}$$

uniformly $L^\infty(\Theta_{P_0} \times \mathcal{B}_\delta)$.

Now, by definition, $\underline{\mathcal{C}}_n^W = \sup_{\theta\in\Theta_{P_0}} l_n^W(\theta, 0)$ and $\underline{\mathcal{C}}^W = \sup_{\theta\in\Theta_{P_0}} l_\infty^W(\theta, 0)$. Therefore, by (B.2),

$$\underline{\mathcal{C}}_n^W \xrightarrow{d} \underline{\mathcal{C}}^W. \blacksquare \tag{B.3}$$

CHT Condition C.5. Approximability of $\underline{\mathcal{C}}_n^W$: *Let Θ_n be any sequence of subsets of Θ such that $d_H(\Theta_n, \Theta_{P_0}) = o_p(n^{-1/2})$ and define $\underline{\mathcal{C}}_n^{W\prime} = \sup_{\theta\in\Theta_n} n\hat{\underline{Q}}_n^W(\theta)$. Then for any $\underline{c}_W \geq 0$, we have that $\mathcal{P}[\underline{\mathcal{C}}_n^{W\prime} \leq \underline{c}_W] = \mathcal{P}[\underline{\mathcal{C}}_n^W \leq \underline{c}_W] + o(1)$.*

Proof. By arguments similar to those in CHT Proof of Theorem 4.2 Step 4, pp.1279-80,

$$\begin{aligned} \underline{\mathcal{C}}_n^{W\prime} &= \sup_{\theta\in\Theta_n} n\hat{\underline{Q}}_n^W(\theta) \\ &= \sup_{\theta\in\Theta_n} \left\|[v(\theta) + n^{1/2}m(\theta) + o_p(1)]_-\right\|^2_{W(\theta)} \\ &= \sup_{\theta\in\Theta_{P_0}} \left\|[v(\theta) + n^{1/2}m(\theta) + o_p(1)]_-\right\|^2_{W(\theta)} \end{aligned}$$

using the approximation device in the Proof of CHT Condition C.4 above, cf. CHT Proof of Theorem 4.2 Step 2, p.1278, the stochastic equicontinuity of $\theta \to (v(\theta), W(\theta))$ and $\left\| n^{1/2}(m(\theta) - m(\theta')) \right\| = o(1)$ uniformly on $\{\theta, \theta' \in \Theta : \|\theta - \theta'\| \leq o_p(n^{-1/2})\}$. The conclusion then follows as in CHT Proof of Theorem 4.2 Step 3, p.1279.■

Appendix C: Proofs for GEL

In the following $\hat{Q}_n(\theta)$ and $Q(\theta)$ refer to sample and population GMM criteria that respectively employ the efficient metrics $\hat{\Omega}_n(\theta)^{-1}$ and $\Omega(\theta)^{-1}$ appropriate for unconditional moment equaility restrictions.

Lemma C.1. *If Assumptions A.1 and A.2-GEL hold then* **(a)** $\max_{1\leq i\leq n}\sup_{\theta\in\Theta,\lambda\in\Lambda_n} |\lambda' m_i(\theta)| \xrightarrow{p} 0$; **(b)** *w.p.a.1,* $\Lambda_n \subseteq \hat{\Lambda}_n(\theta)$ *for all* $\theta \in \Theta$.

Proof. Follows directly from Newey and Smith (2004, Lemma A1, p.239) and the extension Parente and Smith (2011, Lemma A.1, p.101).■

Statements and proofs are given for the alternative GEL criterion $\tilde{P}^{\rho}_n(\theta)$ (E.6) defined in Appendix E; those for the GEL criterion $\hat{P}^{\rho}_n(\theta)$ (3.4) and alternative GEL criteria $\tilde{P}^{\rho,k}_n(\theta)$, $(k = a, b)$, (E.1) and (E.3), follow similarly.

Recall $\hat{\Omega}_n(\theta) = \sum_{i=1}^n m_i(\theta)m_i(\theta)'/n$. The next Lemma and its proof mirror Newey and Smith (2004, Lemma A2, p.239) for the moment equality case.

Lemma C.2. *Let* $\theta \in \Theta_{P_0}$. *Let the arbitrary sequence* $\tau_n(\theta)$ *obey* $\|\hat{m}_n(\theta) - \tau_n(\theta)\| = O_p(n^{-1/2})$ *uniformly* $\theta \in \Theta_{P_0}$. *If Assumptions A.1 and A.2-GEL are satisfied, then* $\tilde{\lambda}_n(\theta) = \arg\max_{\lambda\in\hat{\Lambda}_n(\theta)} \tilde{P}^{\rho}_n(\theta, \lambda, \tau_n(\theta))$ *exists w.p.a.1,* $\tilde{\lambda}_n(\theta) = O_p(n^{-1/2})$ *and* $\sup_{\lambda\in\hat{\Lambda}_n(\theta)} \tilde{P}^{\rho}_n(\theta, \lambda, \tau_n(\theta)) \leq O_p(n^{-1})$.

Proof. By Assumption A.2-GEL and UWL $\hat{\Omega}_n(\theta) \xrightarrow{p} \Omega(\theta)$ uniformly $\theta \in \Theta$. Then, by $\Omega(\theta)$ p.d. uniformly $\theta \in \Theta$ from Assumption A.1(c), the smallest eigenvalue of $\hat{\Omega}_n(\theta)$ is bounded away from zero w.p.a.1. By Lemma C.1 and twice continuous differentiability of $\rho(\cdot)$ in a neighborhood of zero from Assumption A.2-GEL(b), $\tilde{P}^{\rho}_n(\theta, \lambda, \tau_n(\theta))$ is twice continuously differentiable on Λ_n w.p.a.1 uniformly $\theta \in \Theta$. Write $\lambda_n = \lambda_n(\theta)$. Then, $\lambda_n = \arg\max_{\lambda\in\Lambda_n} \tilde{P}^{\rho}_n(\theta, \lambda, \tau_n(\theta))$ exists w.p.a.1. Furthermore, for any $\dot{\lambda}$ on the line segment joining λ_n and 0, by Lemma C.1 and $\rho_2(0) = -1$, $\max_{1\leq i\leq n} \rho_2(\dot{\lambda}' m_i(\theta)) < -1/2$

w.p.a.1. Hence, by a Taylor expansion around $\lambda = 0$ with Lagrange remainder, there is $\dot{\lambda}$ on the line joining λ_n and 0 such that

$$\begin{aligned} 0 &= \tilde{P}_n^\rho(\theta, 0, \tau_n(\theta)) \\ &\leq \tilde{P}_n^\rho(\theta, \lambda_n, \tau_n(\theta)) = -(\hat{m}_n(\theta) - \tau_n(\theta))'\lambda_n + \frac{1}{2}\lambda_n'[\sum_{i=1}^n \rho_2(\dot{\lambda}'m_i(\theta))m_i(\theta)m_i(\theta)'/n]\lambda_n \\ &\leq -(\hat{m}_n(\theta) - \tau_n(\theta))'\lambda_n - \frac{1}{4}\lambda_n'\hat{\Omega}_n(\theta)\lambda_n \leq \|\lambda_n\|\|\hat{m}_n(\theta) - \tau(\theta)\| - C\|\lambda_n\|^2 \end{aligned}$$

w.p.a.1 uniformly $\theta \in \Theta_{P_0}$. Adding $C\|\lambda_n\|^2$ to both sides and dividing by $\|\lambda_n\|$ yields $C\|\lambda_n\| \leq \|\hat{m}_n(\theta) - \tau_n(\theta)\|$ w.p.a.1. By hypothesis, $\hat{m}_n(\theta) - \tau_n(\theta) = O_p(n^{-1/2})$, $\theta \in \Theta_{P_0}$, and, thus, $\|\lambda_n\| = O_p(n^{-1/2})$. Therefore, w.p.a.1 $\lambda_n \in int(\Lambda_n)$ and hence $\partial\tilde{P}_n^\rho(\theta, \lambda_n, \tau_n(\theta))/\partial\lambda = 0$, the first order conditions for an interior maximum. By Lemma C.1, w.p.a.1 $\lambda_n \in \hat{\Lambda}_n(\theta)$, so by the concavity of $\tilde{P}_n^\rho(\theta, \lambda, \tau_n(\theta))$ and convexity of $\hat{\Lambda}_n(\theta)$ it follows that $\tilde{P}_n^\rho(\theta, \lambda_n, \tau_n(\theta)) = \sup_{\lambda \in \hat{\Lambda}_n(\theta)} \tilde{P}_n^\rho(\theta, \lambda, \tau_n(\theta))$, giving the first and second conclusions with $\lambda_n = \tilde{\lambda}_n$. Then, by the last inequality of the above equation, $\|\hat{m}_n(\theta) - \tau_n(\theta)\| = O_p(n^{-1/2})$, and $\|\lambda_n\| = O_p(n^{-1/2})$, we obtain $\tilde{P}_n^\rho(\theta, \tilde{\lambda}_n, \tau_n(\theta)) \leq \|\tilde{\lambda}_n\|\|\hat{m}_n(\theta) - \tau_n(\theta)\| - C\|\tilde{\lambda}_n\|^2 = O_p(n^{-1})$ uniformly $\theta \in \Theta_{P_0}$.■

Lemma C.3. *Let $\theta \in \Theta_{P_0}$. If Assumptions A.1 and A.2-GEL are satisfied, then $\tilde{\lambda}_n(\theta) = \arg\max_{\lambda \in \hat{\Lambda}_n(\theta)} \tilde{P}_n^\rho(\theta, \lambda, \tilde{\tau}_n(\theta))$ exists w.p.a.1, $\tilde{\lambda}_n(\theta) = O_p(n^{-1/2})$, $\sup_{\theta \in \Theta_{P_0}} \|\hat{m}_n(\theta) - \tilde{\tau}_n(\theta)\| \leq O_p(n^{-1/2})$ and $\sup_{\lambda \in \hat{\Lambda}_n(\theta)} \tilde{P}_n^\rho(\theta, \lambda, \tilde{\tau}_n(\theta)) \leq O_p(n^{-1})$.*

Proof. From the Proofs of Lemmas E.1 and E.3 below the population auxiliary paramater $\lambda(\theta) = 0$, $\theta \in \Theta_{P_0}$. Thus, the population slackness parameter $\tau(\theta) = \mathbb{E}_{P_0}[m(z, \theta)] \geq 0$. In particular, $\tau^j(\theta) > [=]0$ if and only if $m^j(\theta) > [=]0$, $(j = 1, ..., d_m)$. Let $\tilde{\lambda}_n$ satisfy $\tilde{P}_n^\rho(\theta, \tilde{\lambda}_n, \tau(\theta)) = \sup_{\lambda \in \hat{\Lambda}_n(\theta)} \tilde{P}_n^\rho(\theta, \lambda, \tau(\theta))$. Then, $\tilde{P}_n^\rho(\theta, \tilde{\lambda}_n(\theta), \tilde{\tau}_n(\theta)) \leq \tilde{P}_n^\rho(\theta, \tilde{\lambda}_n(\theta), \tau(\theta)) \leq \tilde{P}_n^\rho(\theta, \tilde{\lambda}_n, \tau(\theta))$ uniformly $\theta \in \Theta_{P_0}$. Therefore, from the Proof of Lemma C.2, $\tilde{\lambda}_n(\theta) = O_p(n^{-1/2})$ and $\tilde{P}_n^\rho(\theta, \tilde{\lambda}_n(\theta), \tilde{\tau}_n(\theta)) \leq O_p(n^{-1})$, i.e.,

$$\inf_{\tau \in \mathcal{T}} \sup_{\lambda \in \hat{\Lambda}_n(\theta)} \tilde{P}_n^\rho(\theta, \lambda, \tau) \leq O_p(n^{-1})$$

uniformly $\theta \in \Theta_{P_0}$.

Let $\bar{\lambda}_n = -n^{-1/2}(\hat{m}_n(\theta) - \tilde{\tau}_n(\theta))/\|\hat{m}_n(\theta) - \tilde{\tau}_n(\theta)\|$ and, thus, $\bar{\lambda}_n \in \Lambda_n$, $\theta \in \Theta_{P_0}$. By Lemma C.1, $\max_{1 \leq i \leq n} |\bar{\lambda}_n' m_i(\theta)| \xrightarrow{p} 0$ and $\bar{\lambda}_n \in \hat{\Lambda}_n(\theta)$ w.p.a.1. Thus, for any $\dot{\lambda}$ on the line joining $\bar{\lambda}_n$ and 0, w.p.a.1 $\rho_2(\dot{\lambda}' m_i(\theta)) \geq -C$, $(i = 1, ..., n)$. Also, by UWL and Assumption A.2, the largest eigenvalue of $\sum_{i=1}^n m_i(\theta)m_i(\theta)'/n$ is bounded above w.p.a.1.

An expansion then gives

$$\begin{aligned}\tilde{P}^{\rho}_{n}(\theta, \bar{\lambda}_{n}, \tilde{\tau}_{n}(\theta)) &= -(\hat{m}_{n}(\theta) - \tilde{\tau}_{n}(\theta))'\bar{\lambda}_{n} + \frac{1}{2}\bar{\lambda}_{n}'[\sum_{i=1}^{n}\rho_{2}(\dot{\lambda}'m_{i}(\theta))m_{i}(\theta)m_{i}(\theta)'/n]\bar{\lambda}_{n} \\ &\geq n^{-1/2}\left\|\hat{m}_{n}(\theta) - \tilde{\tau}_{n}(\theta)\right\| - C\frac{1}{2}\bar{\lambda}_{n}'\hat{\Omega}_{n}(\theta)\bar{\lambda}_{n} \\ &\geq n^{-1/2}\left\|\hat{m}_{n}(\theta) - \tilde{\tau}_{n}(\theta)\right\| - Cn^{-1}\end{aligned}$$

w.p.a.1. uniformly $\theta \in \Theta_{P_0}$. Hence,

$$n^{-1/2}\left\|\hat{m}_{n}(\theta) - \tilde{\tau}_{n}(\theta)\right\| - Cn^{-1} \leq \tilde{P}^{\rho}_{n}(\theta, \bar{\lambda}, \tilde{\tau}_{n}(\theta)) \leq \tilde{P}^{\rho}_{n}(\theta, \tilde{\lambda}_{n}(\theta), \tilde{\tau}_{n}(\theta)) \leq O_{p}(n^{-1}). \tag{C.1}$$

Solving eq. (C.1) for $\|\hat{m}_{n}(\theta) - \tilde{\tau}_{n}(\theta)\|$ then gives

$$\|\hat{m}_{n}(\theta) - \tilde{\tau}_{n}(\theta)\| \leq O_{p}(n^{-1/2}). \tag{C.2}$$

uniformly $\theta \in \Theta_{P_0}$.■

Recall $\tilde{P}^{\rho}_{n}(\theta) = \inf_{\tau\in\mathcal{T}}\sup_{\lambda\in\hat{\Lambda}_{n}(\theta)}\tilde{P}^{\rho}_{n}(\theta, \lambda, \tau)$.

LEMMA C.4. *Under Assumptions A.1 and A.2-GEL*

$$\begin{aligned}2n\tilde{P}^{\rho}_{n}(\theta) &= n\hat{Q}_{n}(\theta) + o_{p}(1) \\ &= \inf_{s_{b}\in\mathcal{R}^{b}_{+}, s_{c}\in\mathcal{R}^{d_{m}-b}}\|v(\theta) - s\|^{2}_{\Omega(\theta)^{-1}} + o_{p}(1),\end{aligned}$$

uniformly $\theta \in \Theta_{P_0}$.

PROOF. Cf. Canay (2010) Proof of Theorem 3.1, pp.418-419. Let the arbitrary sequence τ_n obey $\|\hat{m}_{n}(\theta) - \tau_{n}\| = O_{p}(n^{-1/2})$; cf. Lemmas C.1 and C.2 above. Define $\tilde{\lambda}_{n}(\tau_{n}) = \arg\max_{\lambda\in\hat{\Lambda}_{n}(\theta)}\tilde{P}^{\rho}_{n}(\theta, \lambda, \tau_{n})$. Therefore, cf. the Proof of Lemma C.2 above, w.p.a.1, $\tilde{\lambda}_{n}(\tau_{n}) \in int(\hat{\Lambda}_{n}(\theta))$ and $\tilde{\lambda}_{n}(\tau_{n})$ satisfies the first order conditions for an interior maximum $\partial\tilde{P}^{\rho}_{n}(\theta, \lambda, \tau_{n})/\partial\lambda = 0$, i.e., $\tilde{\lambda}_{n}(\tau_{n}) = -\hat{\Omega}_{n}^{-1}(\hat{m}_{n}(\theta) - \tau_{n}) + o_{p}(n^{-1/2})$ uniformly

$\theta \in \Theta_{P_0}$. Hence, defining $\dot{\lambda}(\tau)$ on the line joining $\tilde{\lambda}_n(\tau)$ and 0,

$$
\begin{aligned}
2n\tilde{P}_n^\rho(\theta, \tilde{\lambda}_n(\tilde{\tau}_n), \tilde{\tau}_n) &= 2 \inf_{\tau \in \mathcal{T}} \sum_{i=1}^n \rho(\tilde{\lambda}_n(\tau)' m_i(\theta)) + \tilde{\lambda}_n(\tau)'\tau \\
&= 2 \inf_{\tau \in \mathcal{T}} -n(\hat{m}_n(\theta) - \tau)'\tilde{\lambda}_n(\tau) \\
&\quad + \frac{1}{2} n \tilde{\lambda}_n(\tau)' [\sum_{i=1}^n \rho_2(\dot{\lambda}(\tau)' m_i(\theta)) m_i(\theta) m_i(\theta)'/n] \tilde{\lambda}_n(\tau) \\
&= 2 \inf_{\tau \in \mathcal{T}} -n(\hat{m}_n(\theta) - \tau)'\tilde{\lambda}_n(\tau) - \frac{1}{2} n \tilde{\lambda}_n(\tau)' \hat{\Omega}_n(\theta) \tilde{\lambda}_n(\tau) + o_p(1) \\
&= \inf_{\tau \in \mathcal{T}} n \left\| \hat{m}_n(\theta) - \tau \right\|^2_{\hat{\Omega}_n^{-1}} + o_p(1) \\
&= n\hat{Q}_n(\theta) + o_p(1),
\end{aligned}
$$

uniformly $\theta \in \Theta_{P_0}$, using Lemmas C.1 and C.3.

It then follows by Lemma A.3 in Appendix A that, uniformly $\theta \in \Theta_{P_0}$,

$$
\begin{aligned}
2n\tilde{P}_n^\rho(\theta, \tilde{\lambda}_n, \tilde{\tau}_n) &= \inf_{t \geq 0} n \left\| \hat{m}_n(\theta) - t \right\|^2_{\Omega(\theta)^{-1}} + o_p(1) \\
&= \inf_{s_b \in \mathcal{R}^b_+, s_c \in \mathcal{R}^{d_m - b}} \left\| v(\theta) - s \right\|^2_{\Omega(\theta)^{-1}} + o_p(1). \blacksquare
\end{aligned}
$$

Lemma C.5. *Under Assumptions A.1-A.2-GEL,* $2n\tilde{P}_n^\rho(\theta) \xrightarrow{p} \infty$ *uniformly* $\theta \in \Theta \backslash \Theta_{P_0}$.

Proof. The structure of the following argument closely resembles that of Smith (2007) Proof of Theorem 4.1, pp.112-114; cf. Kitamura et al. (2004), Proof of Theorem 3.1, pp.1686-1688, for EL.

Let $c > 0$ such that $(-c, c) \in \mathcal{V}$. Define $C_n = \{z \in \mathbb{R}^{d_z} : \sup_{\theta \in \Theta} \|m(z, \theta)\| \leq cn^{1/2}\}$ and $m_{ni}(\theta) = I_i m_i(\theta)$, where $I_i = I\{z_i \in C_n\}$. Let $\bar{\lambda}(\theta, \tau) = -(m(\theta) - \tau)/(1 + \|m(\theta) - \tau\|)$; note that $n^{-1/2}\bar{\lambda}(\theta, \tau) \in \Lambda_n$.

Then,

$$
\begin{aligned}
\sup_{\lambda \in \hat{\Lambda}_n(\theta)} \tilde{P}_n^\rho(\theta, \lambda, \tau) &\geq \tilde{Q}_n^\rho(\theta, \tau) \qquad \text{(C.3)} \\
&= \sum_{i=1}^n \rho(n^{-1/2}\bar{\lambda}(\theta, \tau)' m_{ni}(\theta))/n + n^{-1/2}\bar{\lambda}(\theta, \tau)'\tau.
\end{aligned}
$$

Now

$$
\rho(n^{-1/2}\bar{\lambda}(\theta, \tau)' m_{ni}(\theta)) + n^{-1/2}\bar{\lambda}(\theta, \tau)'\tau = -n^{-1/2}\bar{\lambda}(\theta, \tau)'(m_i(\theta) - \tau) + r_{ni}(t),
$$

for some $t \in (0,1)$ and remainder

$$r_{ni}(t) = n^{-1/2}\bar{\lambda}(\theta,\tau)'m_i(\theta)(1-I_i) + n^{-1/2}\bar{\lambda}(\theta,\tau)'m_{ni}(\theta)[\rho_1(tn^{-1/2}\bar{\lambda}(\theta,\tau)'m_{ni}(\theta)) - \rho_1(0)]. \tag{C.4}$$

From Lemma C.4 $\sup_{\theta\in\Theta,n^{-1/2}\lambda\in\Lambda_n,1\le i\le n}\left|\rho_1(n^{-1/2}\lambda'm_i(\theta)) - \rho_1(0)\right| \xrightarrow{p} 0$. Also $\max_{1\le i\le n}(1-I_i) = o_p(1)$. Hence, from eq. (C.4),

$$\begin{aligned} n^{1/2}\sum_{i=1}^n r_{ni}(t)/n &= o_p(1)\bar{\lambda}(\theta,\tau)'\hat{m}_n(\theta) + o_p(1)\bar{\lambda}(\theta,\tau)'\hat{m}_n(\theta) \\ &\quad -o_p(1)\bar{\lambda}(\theta,\tau)'\sum_{i=1}^n m_i(\theta)(1-I_i)/n \\ &= o_p(1)\bar{\lambda}(\theta,\tau)'\hat{m}_n(\theta) \end{aligned}$$

uniformly $\theta \in \Theta$ and $\tau \in \mathcal{T}$. Thus,

$$\begin{aligned} n^{1/2}\sup_{\theta\in\Theta,\tau\in\mathcal{T}}\left|\sum_{i=1}^n r_{ni}(t)/n\right| &\le o_p(1)\sup_{\theta\in\Theta}\|\hat{m}_n(\theta)\| \\ &= o_p(1)O_p(1) = o_p(1) \end{aligned}$$

as $\sup_{\theta\in\Theta}\|\hat{m}_n(\theta)\| \le \sup_{\theta\in\Theta}\|m(\theta)\| + o_p(1)$ by T and UWL. Therefore, substituting eq. (C.3), $n^{1/2}\tilde{Q}_n^\rho(\theta,\tau) = -\bar{\lambda}(\theta,\tau)'(\hat{m}_n(\theta)-\tau) + o_p(1)$ uniformly $\theta \in \Theta$ and $\tau \in \mathcal{T}$. By UWL

$$n^{1/2}\sup_{\theta\in\Theta,\tau\in\mathcal{T}}\left|\tilde{Q}_n(\theta,\tau) - \tilde{Q}(\theta,\tau)\right| = o_p(1), \tag{C.5}$$

where

$$\begin{aligned} n^{1/2}\tilde{Q}(\theta,\tau) &= -\bar{\lambda}(\theta,\tau)'(m(\theta)-\tau) \\ &= \frac{\|m(\theta)-\tau\|^2}{1+\|m(\theta)-\tau\|}. \end{aligned}$$

Thus, from eqs. (C.3) and (C.5), cf. Kitamura et al. (2004) eqs. (A.6) and (A.7), p.1687,

$$n^{1/2}\inf_{\tau\in\mathcal{T}}\sup_{\lambda\in\hat{\Lambda}_n(\theta)}\tilde{P}_n^\rho(\theta,\lambda,\tau) \ge n^{1/2}\inf_{\tau\in\mathcal{T}}\tilde{Q}(\theta,\tau) + o_p(1) \tag{C.6}$$

uniformly $\theta \in \Theta$.

The function $\|m(\theta)-\tau\|^2/(1+\|m(\theta)-\tau\|)$ is continuous in θ and τ. By definition of the identified set Θ_{P_0}, $\inf_{\tau\in\mathcal{T}}\|m(\theta)-\tau\|^2/(1+\|m(\theta)-\tau\|)$ takes the value zero for all $\theta \in \Theta_{P_0}$ and is strictly positive for all $\theta \in \Theta\backslash\Theta_{P_0}$, i.e.,

$$\inf_{\tau\in\mathcal{T}}\tilde{Q}(\theta,\tau) = 0 \Longleftrightarrow \theta \in \Theta_{P_0}$$

and

$$\inf_{\tau\in\mathcal{T}}\tilde{Q}(\theta,\tau)>0\Longleftrightarrow\theta\notin\Theta_{P_0}.$$

Therefore, from eq. (C.6), uniformly $\theta\in\Theta\backslash\Theta_{P_0}$, $n^{1/2}\inf_{\tau\in\mathcal{T}}\sup_{\lambda\in\hat{\Lambda}_n(\theta)}\tilde{P}^{\rho}_n(\theta,\lambda,\tau)\xrightarrow{p}\infty$.

Similarly to the Proof of Condition C.1(d) for GMM, $2n\tilde{P}^{\rho}_n(\theta)=2n\tilde{P}^{\rho}_n(\theta,\tilde{\lambda}_n(\theta),\tilde{\tau}_n(\theta))\xrightarrow{p}\infty$ uniformly $\theta\in\Theta\backslash\Theta_{P_0}$.■

Recall from section 3.4 above the GEL population criterion defined by $\tilde{P}^{\rho}(\theta)=\inf_{\tau\in\mathcal{T}}\sup_{\lambda\in\mathcal{R}^{d_m}}\tilde{P}^{\rho}(\theta,\lambda,\tau)$ with $\tilde{P}^{\rho}(\theta,\lambda,\tau)=\mathbb{E}_{P_0}[\rho(\lambda'm(z,\theta))-\rho_0]+\lambda'\tau$ corresponding to the alternative GEL criterion $\tilde{P}^{\rho}_n(\theta)$ (E.6). Proofs are presented for $\tilde{P}^{\rho}_n(\theta)$ (E.6); those for $\hat{P}^{\rho}_n(\theta)$ (3.4) and $\tilde{P}^{\rho,k}_n(\theta)$, $(k=a,b)$, (E.1) and (E.3), follow similarly. In the following discussion $\tilde{P}^{\rho}(\theta)$ substitutes for $\hat{Q}_n(\theta)$ in the statements of CHT Conditions C.1-C.3 in Appendix B.

Proof of CHT Condition C.1. **(a)** Holds by Assumption A.1. **(b)** Follows for $\tilde{P}^{\rho}(\theta)$ from Lemmas E.1 and E.3 below as $\tilde{P}^{\rho}(\theta)=\hat{P}^{\rho}(\theta)=0$ for all $\theta\in\Theta_{P_0}$, i.e., $\inf_{\theta\in\Theta}\tilde{P}^{\rho}(\theta)=0$. **(c)** Holds by Assumption A.2-GEL(a). **(d)** Lemma C.4 establishes that $2n\tilde{P}^{\rho}_n(\theta,\tilde{\lambda}_n(\theta),\tilde{\tau}_n(\theta))=n\hat{Q}_n(\theta)+o_p(1)$ uniformly $\theta\in\Theta_{P_0}$ and Lemma C.5 that $2n\tilde{P}^{\rho}_n(\theta,\tilde{\lambda}_n(\theta),\tilde{\tau}_n(\theta))\xrightarrow{p}\infty$ uniformly $\theta\in\Theta/\Theta_{P_0}$. **(e)** By (b) $\tilde{P}^{\rho}(\theta)=0$ uniformly $\theta\in\Theta_{P_0}$. Thus $\sup_{\theta\in\Theta_{P_0}}\left|\tilde{P}^{\rho}_n(\theta)-\tilde{P}^{\rho}(\theta)\right|=\sup_{\theta\in\Theta_{P_0}}\left|\tilde{P}^{\rho}_n(\theta)\right|=O_p(n^{-1})$ using Lemma C.3.■

Proof of CHT Condition C.2. Similarly to the Proof of Condition C.2 for GMM in Appendix B, from Lemmas C.4 and C.5, w.p.a.1 uniformly $\theta\in\Theta$,

$$\begin{aligned}2n\tilde{P}^{\rho}_n(\theta) &= n\hat{Q}_n(\theta)\\ &\geq \inf_{\theta\in\Theta}\left\|n^{1/2}m(\theta)\right\|_-^2\\ &\quad\times\left\|v_n(\theta)+n^{1/2}m(\theta)\right\|_-^2/\lambda_{\max}(\Omega(\theta))\left\|n^{1/2}m(\theta)\right\|_-^2.\end{aligned}$$

Therefore, for any $\varepsilon>0$, with probability at least $1-\varepsilon$,

$$2n\tilde{P}^{\rho}_n(\theta)\geq\frac{1}{2}\inf_{\theta\in\Theta}C\cdot n\cdot(d(\theta,\Theta_{P_0})\wedge\delta)^2/\lambda_{\max}(\Omega(\theta))$$

uniformly $\{\theta\in\Theta:d(\theta,\Theta_{P_0})\geq(\kappa_\varepsilon/n)^{1/2}\}$, $n>n_\varepsilon$, for some $(\kappa_\varepsilon,n_\varepsilon)$, from $\sup_{\theta\in\Theta}\left\|v_n(\theta)\right\|=O_p(1)$ by the P-Donsker property of Assumption A.3, Assumption A.4 and $\left\|y+x\right\|_-/\left\|x\right\|_-\to$

1 as $\|x\|_- \to \infty$ for any $y \in \mathcal{R}^{d_m}$.■

Proof of CHT Condition C.3. Similarly to the Proof of Condition C.3 for GMM in Appendix B, w.p.a.1 uniformly $\theta \in \Theta_{P_0}$,

$$\begin{aligned} 2n\tilde{P}_n^\rho(\theta) &= n\hat{Q}_n(\theta) \\ &\leq \sup_{\theta\in\Theta} d_m \cdot [O_p(1) + n^{1/2} \cdot C \cdot (d(\theta, \Theta\backslash\Theta_{P_0}) \wedge \delta)]_-^2 / \lambda_{\min}(\Omega(\theta)) \end{aligned}$$

where the inequality follows from $\Omega(\theta)$ uniformly p.d. $\theta \in \Theta$ and bounded by Assumption A.2-GMM(b) and Assumption A.5. The conclusion follows as in the Proof of Condition C.3 for GMM, since with $\epsilon_n = O_p(n^{-1/2})$, $\tilde{P}_n^\rho(\theta) = 0$ on $\Theta_{P_0}^{-\epsilon_n}$.■

Appendix D: Identified Set

Recall the partition of the index set $\{1, ..., d_m\}$ according to $m^j(\theta) < 0$, $(j = 1, ..., a)$, $m^j(\theta) = 0$, $(j = a+1, ..., a+b)$ and $m^j(\theta) > 0$, $(j = a+b+1, ..., d_m)$. Let $c = d_m - a - b$. Note again that a, b and thus c depend on θ. Also recall the notation $m(\theta) = \mathbb{E}_{P_0}[m(z, \theta)]$.

Recall $\hat{\Theta}_{P_0}^\rho = \{\theta \in \Theta : \theta = \arg\min_{\theta\in\Theta} \hat{P}^\rho(\theta)\}$ (3.9).

Lemma D.1. *Suppose that Assumptions A.1 and A.2-GEL are satisfied. Then* $\hat{\Theta}_{P_0}^\rho = \Theta_{P_0}$.

Proof. Now

$$\begin{aligned} \hat{P}^\rho(\theta) &= \sup_{\lambda\geq 0} \mathbb{E}_{P_0}[\rho(\lambda(\theta)' m(z, \theta))] \\ &= \mathbb{E}_{P_0}[\rho(\lambda(\theta)' m(z, \theta))] \geq 0 \end{aligned}$$

since $\rho(\lambda' m(z, \theta)) = 0$ at $\lambda = 0$.

Fix $\theta \in \Theta_{P_0}$; thus $a = 0$. Consider $j \in \{b+1, ..., d_m\}$, i.e., $m^j(\theta) > 0$. Suppose that the associated auxiliary parameter $\lambda^j(\theta) > 0$. Now

$$\begin{aligned} \mathbb{E}_{P_0}[\rho(\lambda(\theta)' m(z, \theta))] &\leq \rho(\lambda(\theta)' m(\theta)) \\ &< 0; \end{aligned}$$

a contradiction. The first inequality holds by Jensen's inequality from $\rho(\cdot) < 0$ and the strict concavity of $\rho(\cdot)$ on $\mathcal{V}$ by Assumption A.2-GEL(b). The second inequality follows from $\lambda(\theta)' m(\theta) > 0$ since $m^j(\theta) = 0$, $j \in \{1, ..., b\}$, and $\lambda^j(\theta) \geq 0$ with at least one

$\lambda^j(\theta) > 0$, $j \in \{b+1, ..., d_m\}$, from above. Hence, the associated auxiliary parameter $\lambda_j(\theta) = 0$, $j \in \{b+1, ..., d_m\}$, and $\mathbb{E}_{P_0}[\rho(\lambda(\theta)' m(z,\theta))]$ is maximised at $\rho(0)$ by setting $\lambda^j(\theta) = 0$, $j \in \{1, ..., d_m\}$. Therefore, $\hat{P}^\rho(\theta) = 0$ if $\theta \in \Theta_{P_0}$, i.e., $\Theta_{P_0} \subseteq \hat{\Theta}^\rho_{P_0}$.

To conclude, suppose $a \neq 0$, i.e., $\theta \in \Theta/\Theta_{P_0}$, and so there exists $j \in \{1, ..., a\}$ such that $m^j(\theta) < 0$. Now, as above, $\mathbb{E}_{P_0}[\rho(\lambda' m(z,\theta))] = 0$ and $\partial \mathbb{E}_{P_0}[\rho(\lambda' m(z,\theta)) - \rho(0)]/\partial \lambda^j = \mathbb{E}_{P_0}[\rho_1(\lambda' m(z,\theta)) m^j(z,\theta)] > 0$ at $\lambda = 0$. Define λ such that $\lambda^j = \epsilon$ for some small $\epsilon > 0$ and $\lambda^k = 0$ for $k \neq j$. Then, by continuity, $\hat{P}^\rho(\theta) \geq \mathbb{E}_{P_0}[\rho(\lambda^j m^j(z,\theta))] > 0$. Cf. Canay (2010) Proof of Lemma B.3, p.423. Hence, $\theta \in \Theta \backslash \hat{\Theta}^\rho_{P_0}$, i.e., $\hat{\Theta}^\rho_{P_0} \subseteq \Theta_{P_0}$.■

Appendix E: Alternative GEL Criteria

A number of alternative but equivalent GEL criteria may also be defined.

Mirroring the GMM criterion (3.1), the introduction of the d_m-vector of complementary slackness parameters $\tau \geq 0$, cf. (2.3), directly into (3.4) defines the alternative GEL criterion

$$\tilde{P}^{\rho,a}_n(\theta, \lambda, \tau) = \sum_{i=1}^n \rho(\lambda'(m_i(\theta) - \tau))/n. \tag{E.1}$$

The GEL criterion $\tilde{P}^{\rho,a}_n(\theta, \lambda, \tau)$ (E.1) is then optimised over $\lambda \in \tilde{\Lambda}^a_n(\theta, \tau)$, where $\tilde{\Lambda}^a_n(\theta, \tau) = \{\lambda : \lambda'(m_i(\theta) - \tau) \in \mathcal{V}, i = 1, ..., n\}$, and $\tau \in \mathcal{T}$ for given $\theta \in \Theta$ with the slackness parameter space $\mathcal{T} = \{\tau \in \mathcal{R}^{d_m} : \tau \geq 0, \|\tau\| \leq C\}$ and $C > 0$ defined by the boundedness condition in Assumption A.1(b). The slackness parameter estimator $\tilde{\tau}^a_n(\theta)$ solves the corresponding f.o.c. with respect to τ, i.e., $\partial \tilde{P}^{\rho,a}_n(\theta, \tilde{\lambda}^a_n(\theta), \tilde{\tau}^a_n(\theta))/\partial \tau \geq 0$, $\tau \geq 0$. Now $\tilde{\lambda}^a_n(\theta) \geq 0$ since $\partial \tilde{P}^{\rho,a}_n(\theta, \tilde{\lambda}^a_n(\theta), \tilde{\tau}^a_n(\theta))/\partial \tau = -\sum_{i=1}^n \rho_1(\tilde{\lambda}^a_n(\theta)'(m_i(\theta) - \tilde{\tau}^a_n(\theta)))\tilde{\lambda}^a_n(\theta)/n$ and $\sum_{i=1}^n \rho_1(\tilde{\lambda}^a_n(\theta)'(m_i(\theta) - \tilde{\tau}^a_n(\theta))) < 0$ from Assumption A.2-GEL(b). In particular, either $\tilde{\lambda}^{a,j}_n(\theta) = 0$ and $\tilde{\tau}^{a,j}_n(\theta) > 0$ or $\tilde{\lambda}^{a,j}_n(\theta) > 0$ and $\tilde{\tau}^{a,j}_n(\theta) = 0$, $(j = 1, ..., d_m)$, and, thus, $\tilde{\lambda}^a_n(\theta)' \tilde{\tau}^a_n(\theta) = 0$. Hence, the auxiliary parameter constraint space $\tilde{\Lambda}^a_n(\theta, \tau)$ simplifies to $\hat{\Lambda}_n(\theta)$. The auxiliary parameter estimator $\tilde{\lambda}^a_n(\theta)$ solves the corresponding f.o.c. with respect to λ, i.e., $\sum_{i=1}^n \rho_1(\tilde{\lambda}^a_n(\theta)'(m_i(\theta) - \tilde{\tau}^a_n(\theta)))(m_i(\theta) - \tilde{\tau}^a_n(\theta))/n = 0$. Consequently, the slackness parameter estimator $\tilde{\tau}^a_n(\theta)$ satisfies

$$\begin{aligned}
\tilde{\tau}^a_n(\theta) &= \frac{\sum_{i=1}^n \rho_1(\tilde{\lambda}^a_n(\theta)'(m_i(\theta) - \tilde{\tau}^a_n(\theta))) m_i(\theta)}{\sum_{k=1}^n \rho_1(\tilde{\lambda}^a_n(\theta)'(m_k(\theta) - \tilde{\tau}^a_n(\theta)))} \\
&= \sum_{i=1}^n \tilde{\pi}^\rho_i(\theta, \tilde{\lambda}^a_n(\theta)) m_i(\theta),
\end{aligned}$$

since $\tilde{\lambda}_n^a(\theta)'\tilde{\tau}_n^a(\theta)=0$; cf. (3.6). Therefore, $\tilde{\lambda}_n^a(\theta)=\hat{\lambda}_n(\theta)$ and, thus, $\tilde{P}_n^{\rho,a}(\theta,\tilde{\lambda}_n(\theta),\tilde{\tau}_n(\theta))=\hat{P}_n^{\rho}(\theta)$.

Remark E.1. Note that $\lim_{n\to\infty}\mathcal{P}\{\tilde{\tau}_n^a(\theta)\in\mathcal{T}\}=1$ since $\sup_{\theta\in\Theta}\|\hat{m}_n(\theta)-m(\theta)\|=o_p(1)$ by UWL from Assumption A.1(b). Thus, the upper bound C is not binding in $\mathcal{T}$ w.p.a.1.

Remark E.2. The GEL implied probabilities defined from (E.2),

$$\tilde{\pi}_i^{\rho,a}(\theta,\lambda,\tau)=\frac{\rho_1(\lambda'(m_i(\theta)-\tau))}{\sum_{k=1}^n\rho_1(\lambda'(m_k(\theta)-\tau))},(i=1,...,n), \tag{E.2}$$

are non-negative and sum to unity. Moreover, since $\tilde{\lambda}_n^a(\theta)'\tilde{\tau}_n^a(\theta)=0$, $\tilde{\pi}_i^{\rho,a}(\theta,\tilde{\lambda}_n^a(\theta),\tilde{\tau}_n^a(\theta))=\hat{\pi}_i^{\rho}(\theta,\tilde{\lambda}_n^a(\theta))$, $(i=1,...,n)$.

The GEL criterion (E.1) may be re-centred by separating out the slackness parameter $\tau\geq 0$ to form

$$\tilde{P}_n^{\rho,b}(\theta,\lambda,\tau)=\sum_{i=1}^n[\rho(\lambda'm_i(\theta))-\rho(\lambda'\tau)]/n, \tag{E.3}$$

which is then optimised over $\lambda\in\tilde{\Lambda}_n^b(\theta,\tau)$, where $\tilde{\Lambda}_n^b(\theta,\tau)=\{\lambda:\lambda'm_i(\theta)\in\mathcal{V},i=1,...,n,\lambda'\tau\in\mathcal{V}\}$ and $\tau\in\mathcal{T}$ for given $\theta\in\Theta$. As above $\tilde{\lambda}_n^a(\theta)\geq 0$ since it follows that $\partial\tilde{P}_n^{\rho,b}(\theta,\tilde{\lambda}_n^b(\theta),\tilde{\tau}_n^b(\theta))/\partial\tau=-\rho_1(\tilde{\lambda}_n(\theta)'\tilde{\tau}_n(\theta))\tilde{\lambda}_n(\theta)\geq 0$ noting $\rho_1(\tilde{\lambda}_n(\theta)'\tilde{\tau}_n(\theta))<0$ from Assumption A.2-GEL(b). Similarly, either $\tilde{\lambda}_n^{b,j}(\theta)=0$ and $\tilde{\tau}_n^{b,j}(\theta)>0$ or $\tilde{\lambda}_n^{b,j}(\theta)>0$ and $\tilde{\tau}_n^{b,j}(\theta)=0$, $(j=1,...,d_m)$, and, thus, $\tilde{\lambda}_n^b(\theta)'\tilde{\tau}_n^b(\theta)=0$. Examining the f.o.c. with respect to λ, i.e., $\sum_{i=1}^n\rho_1(\tilde{\lambda}_n^b(\theta)'m_i(\theta))m_i(\theta)/n-\rho_1(\tilde{\lambda}_n^b(\theta)'\tilde{\tau}_n^b(\theta))\tilde{\tau}_n^b(\theta))=0$, the slackness parameter estimator $\tilde{\tau}_n^b(\theta)$ satisfies

$$\tilde{\tau}_n^b(\theta)=\frac{\sum_{i=1}^n\rho_1(\tilde{\lambda}_n^b(\theta)'m_i(\theta))m_i(\theta)/n}{\rho_1(\tilde{\lambda}_n^b(\theta)'\tilde{\tau}_n^b(\theta))}. \tag{E.4}$$

Hence, the auxiliary parameter constraint space $\tilde{\Lambda}_n^b(\theta,\tau)$ is not fully binding and reduces to $\hat{\Lambda}_n(\theta)$ as previously. Consequently, $\tilde{P}_n^{\rho,b}(\theta,\tilde{\lambda}_n^b(\theta),\tilde{\tau}_n^b(\theta))=\hat{P}_n^{\rho}(\theta)$.

Remark E.3. Noting $\rho_1(0)=-1$, since $\tilde{\lambda}_n^b(\theta)'\tilde{\tau}_n^b(\theta)$, the slackness parameter estimator (E.4) $\tilde{\tau}_n^b(\theta)=-\sum_{i=1}^n\rho_1(\tilde{\lambda}_n^b(\theta)'m_i(\theta))m_i(\theta)/n$; cf. (3.6). The GEL implied probabilities implicitly defined from (E.4) as $\rho_1(\lambda'm_i(\theta))/n\rho_1(\theta'\tau)$, $(i=1,...,n)$, although non-negative by Assumption A.2-GEL(b), do not sum to unity. Even if evaluated at

$\tilde{\lambda}^b_n(\theta)$ and $\tilde{\tau}^b_n(\theta)$, the GEL implied probabilities $-\rho_1(\tilde{\lambda}^b_n(\theta)'m_i(\theta))/n$, $(i = 1, ..., n)$, do not sum to unity. Exploiting (3.7) guarantees non-negativity and unit summability, i.e.,

$$\tilde{\pi}^{\rho,b}_i(\theta, \lambda, \tau) = \frac{\rho_1(\lambda' m_i(\theta))}{\sum_{k=1}^n \rho_1(\lambda' m_k(\theta))}, (i = 1, ..., n). \tag{E.5}$$

Moreover, $\tilde{\pi}^{\rho,b}_i(\theta, \tilde{\lambda}^b_n(\theta), \tilde{\tau}^b_n(\theta)) = \hat{\pi}^{\rho}_i(\theta, \tilde{\lambda}^b_n(\theta))$, $(i = 1, ..., n)$.

Consider the Lagrangean

$$\tilde{P}^{\rho}_n(\theta, \lambda, \tau) = \sum_{i=1}^n \rho(\lambda' m_i(\theta))/n + \lambda'\tau \tag{E.6}$$

in which the slackness parameter vector τ now denotes a d_m-vector of Lagrange multipliers associated with the inequality constraint $\lambda \geq 0$; cf. $G^*(\theta, v, \lambda)$ defined in Moon and Schorfheide (2009) eq. (16), p.140. Here the GEL criterion $\tilde{P}^{\rho}_n(\theta, \lambda, \tau)$ (E.6) is optimised over $\lambda \in \hat{\Lambda}_n(\theta)$ and $\tau \in \mathcal{T}$ for given $\theta \in \Theta$. The Lagrange multiplier parameter estimator $\tilde{\tau}_n(\theta)$ satisfies $\partial \tilde{P}^{\rho}_n(\theta, \tilde{\lambda}_n(\theta), \tilde{\tau}_n(\theta))/\partial\tau \geq 0$, $\tau \geq 0$. Thus, the auxiliary parameter estimator $\tilde{\lambda}_n(\theta) \geq 0$ as $\partial \tilde{P}^{\rho}_n(\theta, \tilde{\lambda}_n(\theta), \tilde{\tau}_n(\theta))/\partial\tau = \tilde{\lambda}_n(\theta)$. Moreover, $\tilde{\lambda}_n(\theta)'\tilde{\tau}_n(\theta) = 0$ with, in particular, $\tilde{\lambda}^j_n(\theta) = 0$ and $\tilde{\tau}^j_n(\theta) > 0$ or $\tilde{\lambda}^j_n(\theta) > 0$ and $\tilde{\tau}^j_n(\theta) = 0$, $(j = 1, ..., d_m)$. From the f.o.c. with respect to λ, i.e., $\sum_{i=1}^n \rho_1(\tilde{\lambda}_n(\theta)'m_i(\theta))m_i(\theta)/n + \tilde{\tau}_n(\theta) = 0$, the Lagrange multiplier estimator $\hat{\tau}_n(\theta) \geq 0$ satisfies

$$\tilde{\tau}_n(\theta) = -\sum_{i=1}^n \rho_1(\tilde{\lambda}_n(\theta)'m_i(\theta))m_i(\theta)/n, \tag{E.7}$$

cf. (3.6). Substituting $\tilde{\lambda}_n(\theta)$ and $\tilde{\tau}_n(\theta)$, $\tilde{P}^{\rho}_n(\theta, \tilde{\lambda}_n(\theta), \tilde{\tau}_n(\theta)) = \hat{P}^{\rho}_n(\theta, \tilde{\lambda}_n(\theta))$. Therefore, from the strict concavity of $\rho(\cdot)$ on $\mathcal{V}$ by Assumption A.2-GEL(b), $\tilde{\lambda}_n(\theta) = \hat{\lambda}_n(\theta)$ and, likewise, $\tilde{P}^{\rho}_n(\theta, \tilde{\lambda}_n(\theta), \tilde{\tau}_n(\theta)) = \hat{P}^{\rho}_n(\theta)$.

Remark E.4. The GEL implied probabilities defined from (E.7) as $-\rho_1(\tilde{\lambda}_n(\theta)'m_i(\theta))/n$, $(i = 1, ..., n)$, are non-negative by Assumption A.2-GEL(b) but do not sum to unity. The redefinition $\tilde{\pi}^{\rho}_i(\theta, \tilde{\lambda}_n(\theta), \tilde{\tau}_n(\theta)) = \rho_1(\tilde{\lambda}_n(\theta)'m_i(\theta))/\sum_{k=1}^n \rho_1(\tilde{\lambda}_n(\theta)'m_k(\theta))$ guarantees non-negativity and unit summability; cf. Remark D.3. Moreover, $\tilde{\pi}^{\rho}_i(\theta, \tilde{\lambda}_n(\theta), \tilde{\tau}_n(\theta)) = \hat{\pi}^{\rho}_i(\theta, \tilde{\lambda}_n(\theta))$, $(i = 1, ..., n)$.

E.1 GEL Estimator Equivalence

Lemma E.1. *The solutions to the saddle point problems (3.4) and (E.1) are identical, i.e.,* **(a)** *if $(\tilde{\lambda}(\theta), \tilde{\tau}(\theta))$, where $\tilde{\tau}(\theta) \in int(\mathcal{T})$, is a saddlepoint of $\tilde{P}^{\rho,a}_n(\theta, \lambda, \tau)$ then $\tilde{\lambda}(\theta)$ is* *also a maximiser of $\hat{P}_n^{\rho}(\theta,\lambda)$;* **(b)** *if $\hat{\lambda}(\theta)$ is a maximiser of $\hat{P}_n^{\rho}(\theta,\lambda)$ and $\hat{\tau}(\theta)\in int(\mathcal{T})$, where $\hat{\tau}^j(\theta)=\sum_{i=1}^n\hat{\pi}_i^{\rho}(\theta,\hat{\lambda}(\theta))m_i^j(\theta)$ if $\hat{\lambda}^j(\theta)=0$ and 0 if $\hat{\lambda}^j(\theta)>0$, $(j=1,...,d_m)$, then $(\hat{\lambda}(\theta),\hat{\tau}(\theta))$ is a saddlepoint of $\tilde{P}_n^{\rho,c}(\theta,\lambda,\tau)$.*

Proof. To prove (a), note that $\tilde{\lambda}\geq 0$, since the solution $\tilde{\tau}$ satisfies $\partial\tilde{P}_n^{\rho,a}(\theta,\tilde{\lambda},\tilde{\tau})/\partial\tau\geq 0$ and $\partial\tilde{P}_n^{\rho,a}(\theta,\lambda,\tau)/\partial\tau=-\sum_{i=1}^n\rho_1(\lambda'(m_i(\theta)-\tau))\lambda/n$ with $\sum_{i=1}^n\rho_1(\lambda'(m_i(\theta)-\tau))/n<0$ by Assumption A.2-GEL(b). In particular, $\tilde{\lambda}^j=0$ and $\tilde{\tau}^j>0$ or $\tilde{\lambda}^j>0$ and $\tilde{\tau}^j=0$, $(j=1,...,d_m)$, and $\tilde{\lambda}'\tilde{\tau}=0$. The solution $\tilde{\lambda}$ satisfies $\partial\tilde{P}_n^{\rho,a}(\theta,\tilde{\lambda},\tilde{\tau})/\partial\lambda=0$, i.e., $\sum_{i=1}^n\rho_1(\tilde{\lambda}'(m_i(\theta)-\tilde{\tau}))(m_i(\theta)-\tilde{\tau})/n=0$, and, thus, $\tilde{\tau}^j=0$ if $\tilde{\lambda}^j>0$ or $\sum_{i=1}^n\rho_1(\tilde{\lambda}'m_i(\theta))m_i^j(\theta)/\sum_{k=1}^n\rho_1(\tilde{\lambda}'m_k(\theta))>0$ if $\tilde{\lambda}^j=0$, $(j=1,...,d_m)$. Now $\tilde{P}_n^{\rho,a}(\theta,\lambda,\tilde{\tau})=\hat{P}_n^{\rho}(\theta,\lambda)-\sum_{i=1}^n\rho_1(\lambda'(m_i(\theta)-\tau_*))\lambda'\tilde{\tau}/n\geq\hat{P}_n^{\rho}(\theta,\lambda)$ for $\tau_*\in(0,\tilde{\tau})$ since $\lambda\geq 0$ and $\sum_{i=1}^n\rho_1(\lambda'(m_i(\theta)-\tau_*))<0$ by Assumption A.2-GEL(b). Therefore, from the saddlepoint property with respect to λ, $\hat{P}_n^{\rho}(\theta,\tilde{\lambda})=\tilde{P}_n^{\rho,a}(\theta,\tilde{\lambda},\tilde{\tau})\geq\tilde{P}_n^{\rho,a}(\theta,\lambda,\tilde{\tau})\geq\hat{P}_n^{\rho}(\theta,\lambda)$.

For (b), $\hat{\lambda}'\hat{\tau}=0$ from the definition of $\hat{\tau}$ with $\hat{\tau}^j=0$ and $\hat{\lambda}^j>0$ or $\hat{\tau}^j>0$ and $\hat{\lambda}^j=0$ $(j=1,...,d_m)$, from the first order condition $\partial\hat{P}_n^{\rho}(\theta,\lambda)/\partial\lambda\leq 0$, $\lambda\geq 0$, cf. (3.6). For the saddle point property with respect to $\tau\geq 0$, $\tilde{P}_n^{\rho,a}(\theta,\hat{\lambda},\hat{\tau})=\hat{P}_n^{\rho}(\theta,\hat{\lambda})\leq\tilde{P}_n^{\rho,a}(\theta,\hat{\lambda},\tau)$ since $\hat{\lambda}\geq 0$ and $\tilde{P}_n^{\rho,a}(\theta,\hat{\lambda},\tau)=\hat{P}_n^{\rho}(\theta,\hat{\lambda})-\sum_{i=1}^n\rho_1(\hat{\lambda}'(m_i(\theta)-\tau_*))\hat{\lambda}'\tau/n$ for $\tau_*\in(0,\tau)$ with $\sum_{i=1}^n\rho_1(\hat{\lambda}'(m_i(\theta)-\tau_*))/n<0$ from Assumption A.2-GEL(b). For λ, $\tilde{P}_n^{\rho,a}(\theta,\hat{\lambda},\hat{\tau})=\hat{P}_n^{\rho}(\theta,\hat{\lambda})\geq\tilde{P}_n^{\rho,a}(\theta,\lambda,\hat{\tau})$ since, noting $\partial\tilde{P}_n^{\rho,a}(\theta,\hat{\lambda},\hat{\tau})/\partial\lambda=0$, $\tilde{P}_n^{\rho,a}(\theta,\lambda,\hat{\tau})=\tilde{P}_n^{\rho,a}(\theta,\hat{\lambda},\hat{\tau})+\sum_{i=1}^n\rho_2(\lambda_*'(m_i(\theta)-\hat{\tau}))[(m_i(\theta)-\hat{\tau})'(\lambda-\hat{\lambda})]^2/2n\leq\tilde{P}_n^{\rho,a}(\theta,\hat{\lambda},\hat{\tau})$ for $\lambda_*\in(\lambda,\hat{\lambda})$ and $\rho_2(\cdot)<0$ by the concavity of $\rho(\cdot)$ from Assumption A.2-GEL(b).■

Lemma E.2. *The solutions to the saddle point problems (3.4) and (E.3) are identical, i.e.,* **(a)** *if $(\tilde{\lambda}(\theta),\tilde{\tau}(\theta))$, where $\tilde{\tau}(\theta)\in int(\mathcal{T})$, is a saddlepoint of $\tilde{P}_n^{\rho,b}(\theta,\lambda,\tau)$ then $\tilde{\lambda}(\theta)$ is also a maximiser of $\hat{P}_n^{\rho}(\theta,\lambda)$;* **(b)** *if $\hat{\lambda}(\theta)$ is a maximiser of $\hat{P}_n^{\rho}(\theta,\lambda)$ and $\hat{\tau}(\theta)\in int(\mathcal{T})$, where $\hat{\tau}^j(\theta)=-\sum_{i=1}^n\rho_1(\hat{\lambda}(\theta)'m(\theta))m_i^j(\theta)/n$ if $\hat{\lambda}^j(\theta)=0$ and 0 if $\hat{\lambda}^j(\theta)>0$, $(j=1,...,d_m)$, then $(\hat{\lambda}(\theta),\hat{\tau}(\theta))$ is a saddlepoint of $\tilde{P}_n^{\rho,b}(\theta,\lambda,\tau)$.*

Proof. The proof follows on similar lines to that for Lemma E.1.

For (a), $\tilde{\lambda}\geq 0$ since $\tilde{\tau}$ satisfies $\partial\tilde{P}_n^{\rho,b}(\theta,\tilde{\lambda},\tilde{\tau})/\partial\tau\geq 0$ and $\partial\tilde{P}_n^{\rho,b}(\theta,\lambda,\tau)/\partial\tau=-\rho_1(\lambda'\tau)\lambda$ with $\rho_1(\lambda'\tau)<0$ by Assumption A.2-GEL(b). Likewise, $\tilde{\lambda}'\tilde{\tau}=0$ with $\tilde{\lambda}^j=0$ and $\tilde{\tau}^j>0$ or $\tilde{\lambda}^j>0$ and $\tilde{\tau}^j=0$, $(j=1,...,d_m)$. In this case $\tilde{\lambda}$ satisfies $\partial\tilde{P}_n^{\rho,b}(\theta,\tilde{\lambda},\tilde{\tau})/\partial\lambda=0$, i.e., $\sum_{i=1}^n\rho_1(\tilde{\lambda}'m_i(\theta))m_i(\theta)/n-\rho_1(\tilde{\lambda}'\tilde{\tau})\tilde{\tau}=0$, and, thus, $\tilde{\tau}^j=-\sum_{i=1}^n\rho_1(\tilde{\lambda}'m_i(\theta))m_i^j(\theta)/n$ if $\tilde{\lambda}^j=0$ or 0 if $\tilde{\lambda}^j>0$, $(j=1,...,d_m)$, from the normalisation $\rho_1(0)=-1$ of Remark 3.3. Now $\tilde{P}_n^{\rho,b}(\theta,\lambda,\tilde{\tau})=\hat{P}_n^{\rho}(\theta,\lambda)-\rho_1((\lambda'\tilde{\tau})_*)\lambda'\tilde{\tau}\geq\hat{P}_n^{\rho}(\theta,\lambda)$ for $(\lambda'\tilde{\tau})_*\in(0,(\lambda'\tilde{\tau}))$ since $\lambda\geq 0$

and $\rho_1((\lambda'\tilde{\tau})_*) < 0$ by Assumption A.2-GEL(b). Therefore, $\hat{P}^{\rho}_n(\theta,\tilde{\lambda}) = \tilde{P}^{\rho,b}_n(\theta,\tilde{\lambda},\tilde{\tau}) \geq \tilde{P}^{\rho,b}_n(\theta,\lambda,\tilde{\tau}) \geq \hat{P}^{\rho}_n(\theta,\lambda)$.

For the proof of (b), as in the Proof of Lemma E.1(b), $\hat{\lambda}'\hat{\tau} = 0$ with $\tilde{\tau}^j = 0$ and $\hat{\lambda}^j > 0$ or $\tilde{\tau}^j > 0$ and $\hat{\lambda}^j = 0$, $(j = 1, ..., d_m)$. For the saddle point property with respect to $\tau \geq 0$, $\tilde{P}^{\rho,b}_n(\theta,\hat{\lambda},\hat{\tau}) = \hat{P}^{\rho}_n(\theta,\hat{\lambda}) \leq \tilde{P}^{\rho,b}_n(\theta,\hat{\lambda},\tau)$ since $\hat{\lambda} \geq 0$ and $\tilde{P}^{\rho,b}_n(\theta,\hat{\lambda},\tau) = \hat{P}^{\rho}_n(\theta,\hat{\lambda}) - \rho_1(\hat{\lambda}'\tau_*)\hat{\lambda}'\tau$ for $\tau_* \in (0,\tau)$ with $\rho_1(\hat{\lambda}'\tau_*) < 0$. For λ, $\tilde{P}^{\rho,b}_n(\theta,\hat{\lambda},\hat{\tau}) = \hat{P}^{\rho}_n(\theta,\hat{\lambda}) \geq \tilde{P}^{\rho,b}_n(\theta,\lambda,\hat{\tau})$ since, noting $\partial\tilde{P}^{\rho,b}_n(\theta,\hat{\lambda},\hat{\tau})/\partial\lambda = 0$ and $\rho_2(\cdot) < 0$, it follows that $\hat{P}^{\rho,b}_n(\theta,\lambda,\hat{\tau}) = \tilde{P}^{\rho,b}_n(\theta,\hat{\lambda},\hat{\tau}) + \sum_{i=1}^n \rho_2(\lambda_*'m_i(\theta))[m_i(\theta)'(\lambda-\hat{\lambda})]^2/2n + \sum_{i=1}^n \rho_2(\lambda_*'\hat{\tau})[\hat{\tau}'(\lambda-\hat{\lambda})]^2/2n \leq \tilde{P}^{\rho,a}_n(\theta,\hat{\lambda},\hat{\tau})$ for $\lambda_* \in (\lambda,\hat{\lambda})$.■

Lemma E.3. *The solutions to the saddle point problems (3.4) and (E.6) are identical, i.e.,* **(a)** *if $(\tilde{\lambda}(\theta),\tilde{\tau}(\theta))$, where $\tilde{\tau}(\theta) \in int(\mathcal{T})$, is a saddlepoint of $\tilde{P}^{\rho}_n(\theta,\lambda,\tau)$ then $\tilde{\lambda}(\theta)$ is also a maximiser of $\hat{P}^{\rho}_n(\theta,\lambda)$;* **(b)** *if $\hat{\lambda}(\theta)$ is a maximiser of $\hat{P}^{\rho}_n(\theta,\lambda)$ and $\hat{\tau}(\theta) \in int(\mathcal{T})$, where $\hat{\tau}^j(\theta) = -\sum_{i=1}^n \rho_1(\hat{\lambda}(\theta)'m_i(\theta))m_i^j(\theta)/n$ if $\hat{\lambda}^j(\theta) = 0$ and 0 if $\hat{\lambda}^j(\theta) > 0$, $(j = 1, ..., d_m)$, then $(\hat{\lambda}(\theta),\hat{\tau}(\theta))$ is a saddlepoint of $\tilde{P}^{\rho}_n(\theta,\lambda,\tau)$.*

Proof. The proof again follows along similar lines to the Proof of Lemma E.1. Cf. Moon and Schorfheide (2009) Lemma A.1, p.150.

For (a), $\tilde{\lambda} \geq 0$ since $\tilde{\tau}$ satisfies $\partial\tilde{P}^{\rho}_n(\theta,\tilde{\lambda},\tilde{\tau})/\partial\tau \geq 0$ and $\partial\tilde{P}^{\rho}_n(\theta,\lambda,\tau)/\partial\tau = \lambda$. Likewise, $\tilde{\lambda}'\tilde{\tau} = 0$ with $\tilde{\lambda}^j = 0$ and $\tilde{\tau}^j > 0$ or $\tilde{\lambda}^j > 0$ and $\tilde{\tau}^j = 0$, $(j = 1, ..., d_m)$. In this case $\tilde{\lambda}$ satisfies $\partial\tilde{P}^{\rho,b}_n(\theta,\tilde{\lambda},\tilde{\tau})/\partial\lambda = 0$, i.e., $\sum_{i=1}^n \rho_1(\tilde{\lambda}'m_i(\theta))m_i(\theta)/n + \tilde{\tau} = 0$, and, thus, $\tilde{\tau}^j = -\sum_{i=1}^n \rho_1(\tilde{\lambda}'m_i(\theta))m_i^j(\theta)/n$ if $\tilde{\lambda}^j = 0$ or 0 if $\tilde{\lambda}^j > 0$, $(j = 1, ..., d_m)$. Now $\tilde{P}^{\rho}_n(\theta,\lambda,\tilde{\tau}) = \hat{P}^{\rho}_n(\theta,\lambda) + \lambda'\tilde{\tau} \geq \hat{P}^{\rho}_n(\theta,\lambda)$ since $\lambda \geq 0$. Therefore, $\hat{P}^{\rho}_n(\theta,\tilde{\lambda}) = \tilde{P}^{\rho}_n(\theta,\tilde{\lambda},\tilde{\tau}) \geq \tilde{P}^{\rho}_n(\theta,\lambda,\tilde{\tau}) \geq \hat{P}^{\rho}_n(\theta,\lambda)$.

For (b), $\hat{\lambda}'\hat{\tau} = 0$ with $\tilde{\tau}^j = 0$ and $\hat{\lambda}^j > 0$ or $\tilde{\tau}^j > 0$ and $\hat{\lambda}^j = 0$, $(j = 1, ..., d_m)$, from the first order condition $\partial\hat{P}^{\rho}_n(\theta,\lambda)/\partial\lambda \leq 0$, $\lambda \geq 0$. For the saddle point property with respect to $\tau \geq 0$, $\tilde{P}^{\rho}_n(\theta,\hat{\lambda},\hat{\tau}) = \hat{P}^{\rho}_n(\theta,\hat{\lambda}) \leq \tilde{P}^{\rho}_n(\theta,\hat{\lambda},\tau)$ since $\hat{\lambda} \geq 0$. For λ, $\tilde{P}^{\rho}_n(\theta,\hat{\lambda},\hat{\tau}) = \hat{P}^{\rho}_n(\theta,\hat{\lambda}) \geq \tilde{P}^{\rho}_n(\theta,\lambda,\hat{\tau})$ since, noting $\partial\tilde{P}^{\rho}_n(\theta,\hat{\lambda},\hat{\tau})/\partial\lambda = 0$ and $\rho_2(\cdot) < 0$, $\hat{P}^{\rho}_n(\theta,\lambda,\hat{\tau}) = \tilde{P}^{\rho}_n(\theta,\hat{\lambda},\hat{\tau}) + \sum_{i=1}^n \rho_2(\lambda_*'m_i(\theta))[m_i(\theta)'(\lambda-\hat{\lambda})]^2/2n \leq \tilde{P}^{\rho,a}_n(\theta,\hat{\lambda},\hat{\tau})$ for $\lambda_* \in (\lambda,\hat{\lambda})$.■

E.2 Identified Set

Alternative but equivalent population versions of the GEL identified set $\hat{\Theta}^{\rho}_{P_0}$ (3.9), cf. Canay (2010) for EL, may be defined corresponding to the alternative GEL criteria $\tilde{P}^{\rho}_n(\theta,\lambda,\tau)$ (E.6) and $\tilde{P}^{\rho,k}_n(\theta,\lambda,\tau)$, $(k = a, b)$, (E.1), (E.3), described in Appendix E.1. The respective population criteria are defined by $\tilde{P}^{\rho}(\theta) = \inf_{\tau\in\mathcal{T}} \sup_{\lambda\in\mathcal{R}^{d_m}} \tilde{P}^{\rho}(\theta,\lambda,\tau)$ with

$\tilde{P}^\rho(\theta,\lambda,\tau) = \mathbb{E}_{P_0}[\rho(\lambda' m(z,\theta))] + \lambda'\tau$ and $\tilde{P}^{\rho,k}(\theta) = \inf_{\tau\in\mathcal{T}}\sup_{\lambda\in\mathcal{R}^{d_m}} \tilde{P}^{\rho,k}(\theta,\lambda,\tau)$, $(k = a,b)$, with $\tilde{P}^{\rho,a}(\theta,\lambda,\tau) = \mathbb{E}_{P_0}[\rho(\lambda'(m(z,\theta)-\tau))]$ and $\tilde{P}^{\rho,b}(\theta,\lambda,\tau) = \mathbb{E}_{P_0}[\rho(\lambda' m(z,\theta)) - \rho(\lambda'\tau)]$. The respective GEL population counterparts to identified set Θ_{P_0} are

$$\tilde{\Theta}^\rho_{P_0} = \{\theta \in \Theta : \theta = \arg\min_{\theta\in\Theta} \tilde{P}^\rho(\theta)\}, \tag{E.8}$$

and

$$\tilde{\Theta}^{\rho,k}_{P_0} = \{\theta \in \Theta : \theta = \arg\min_{\theta\in\Theta} \tilde{P}^{\rho,k}(\theta)\}, (k = a,b). \tag{E.9}$$

Recall the notation $m(\theta) = \mathbb{E}_{P_0}[m(z,\theta)]$ and $\hat{\Theta}^\rho_{P_0} = \{\theta \in \Theta : \theta = \arg\min_{\theta\in\Theta} \hat{P}^\rho(\theta)\}$ (3.9).

Lemma E.4. *Suppose that Assumptions A.1 and A.2-GEL are satisfied. Then* $\tilde{\Theta}^{\rho,k}_{P_0} = \Theta_{P_0}$, $(k = a,b)$.

Proof. Let $\hat{P}^\rho(\theta,\lambda) = \mathbb{E}_{P_0}[\rho(\lambda' m(z,\theta))]$.

First, consider the alternative GEL population criterion $\tilde{P}^{\rho,a}(\theta,\lambda,\tau) = \mathbb{E}_{P_0}[\rho(\lambda'(m(z,\theta)-\tau))]$ corresponding to (E.1). Now $\tilde{P}^{\rho,a}(\theta) = \inf_{\tau\in\mathcal{T}}\sup_{\lambda\in\mathcal{R}^{d_m}} \tilde{P}^{\rho,a}(\theta,\lambda,\tau)$. The solution $\tau(\theta) \geq 0$ satisfies $\partial\tilde{P}^{\rho,a}(\theta,\lambda,\tau)/\partial\tau \geq 0$. Thus, since $\partial\tilde{P}^{\rho,a}(\theta,\lambda,\tau)/\partial\tau = -\mathbb{E}_{P_0}[\rho_1(\lambda'(m(z,\theta)-\tau))]\lambda$ and $\mathbb{E}_{P_0}[\rho_1(\lambda'(m(z,\theta)-\tau))] < 0$ by Assumption A.2-GEL(b), $\lambda^j \geq 0$, $(j = 1,...,d_m)$, and $\lambda'\tau(\theta) = 0$. The solution $\lambda(\theta)$ satisfies $\partial\tilde{P}^{\rho,a}(\theta,\lambda,\tau)/\partial\lambda = 0$, i.e., $\mathbb{E}_{P_0}[\rho_1(\lambda(\theta)'(m(z,\theta)-\tau(\theta)))(m(z,\theta)-\tau(\theta))] = 0$ and, thus,

$$\begin{aligned} \tau(\theta) &= \frac{\mathbb{E}_{P_0}[\rho_1(\lambda(\theta)'(m(z,\theta)-\tau(\theta)))m(z,\theta)]}{\mathbb{E}_{P_0}[\rho_1(\lambda(\theta)'(m(z,\theta)-\tau(\theta)))]} \\ &= \frac{\mathbb{E}_{P_0}[\rho_1(\lambda(\theta)' m(z,\theta))m(z,\theta)]}{\mathbb{E}_{P_0}[\rho_1(\lambda(\theta)' m(z,\theta))]} \geq 0. \end{aligned}$$

Now $\tilde{P}^{\rho,a}(\theta,\lambda,\tau(\theta)) = \hat{P}^\rho(\theta,\lambda) - \mathbb{E}_{P_0}[\rho_1(\lambda'(m(z,\theta)-\tau_*))]\lambda'\tau(\theta) \geq \hat{P}^\rho(\theta,\lambda)$ for $\tau_* \in (0,\tau(\theta))$ since $\lambda \geq 0$ and $\mathbb{E}_{P_0}[\rho_1(\lambda'(m(z,\theta)-\tau_*))] < 0$ by Assumption A.2-GEL(b). Hence, as $\hat{P}^\rho(\theta,\lambda(\theta)) = \tilde{P}^{\rho,a}(\theta,\lambda(\theta),\tau(\theta)) \geq \tilde{P}^{\rho,a}(\theta,\lambda,\tau(\theta))$, $\hat{P}^\rho(\theta,\lambda(\theta)) \geq \hat{P}^\rho(\theta,\lambda)$. That is, $\lambda(\theta)$ also optimises the GEL criterion $\hat{P}^\rho(\theta,\lambda)$ (3.4) and therefore $\tilde{\Theta}^{\rho,a}_{P_0} = \hat{\Theta}^\rho_{P_0}$.

Secondly, the population criterion for the GEL criterion (E.3) is $\tilde{P}^{\rho,b}(\theta,\lambda,\tau) = \mathbb{E}_{P_0}[\rho(\lambda' m(z,\theta)) - \rho(\lambda'\tau)]$ with $\tilde{P}^{\rho,b}(\theta) = \inf_{\tau\in\mathcal{T}}\sup_{\lambda\in\mathcal{R}^{d_m}} \tilde{P}^\rho(\theta,\lambda,\tau)$. The solution $\tau(\theta) \geq 0$ satisfies $\partial\tilde{P}^{\rho,b}(\theta,\lambda,\tau)/\partial\tau \geq 0$. Likewise, since $\partial\tilde{P}^{\rho,b}(\theta,\lambda,\tau)/\partial\tau = -\rho_1(\lambda'\tau)\lambda$, by Assumption A.2-GEL(b), $\lambda^j(\theta) \geq 0$, $(j = 1,...,d_m)$, and $\lambda'\tau(\theta) = 0$ as above. The solution

$\lambda(\theta)$ satisfies $\partial\tilde{P}^{\rho,b}(\theta,\lambda,\tau)/\partial\lambda = 0$, i.e., $\mathbb{E}_{P_0}[\rho_1(\lambda(\theta)'m(z,\theta))m(z,\theta)] - \rho_1(\lambda(\theta)'\tau(\theta))\tau(\theta) = 0$ or

$$\tau(\theta) = \frac{\mathbb{E}_{P_0}[\rho_1(\lambda(\theta)'m(z,\theta))m(z,\theta)]}{\rho_1(\lambda(\theta)'\tau(\theta))} \geq 0.$$

By similar reasoning $\tilde{P}^{\rho,b}(\theta,\lambda,\tau(\theta)) = \hat{P}^{\rho}(\theta,\lambda) - \rho_1(\lambda'\tau_*)\lambda'\tau(\theta) \geq \hat{P}^{\rho}(\theta,\lambda)$ for $\tau_* \in (0,\tau(\theta))$ since $\lambda \geq 0$ and $\rho_1(\cdot) < 0$ by Assumption A.2-GEL(b). Hence, as $\hat{P}^{\rho}(\theta,\lambda(\theta)) = \tilde{P}^{\rho,b}(\theta,\lambda(\theta),\tau(\theta)) \geq \tilde{P}^{\rho,b}(\theta,\lambda,\tau(\theta))$, $\hat{P}^{\rho}(\theta,\lambda(\theta)) \geq \hat{P}^{\rho}(\theta,\lambda)$, i.e., $\lambda(\theta)$ also optimises the GEL criterion $\hat{P}^{\rho}(\theta,\lambda)$ (3.4). Therefore, $\tilde{\Theta}^{\rho,b}_{P_0} = \hat{\Theta}^{\rho}_{P_0}$.■

Lemma E.5. *Suppose that Assumptions A.1 and A.2-GEL are satisfied. Then* $\tilde{\Theta}^{\rho}_{P_0} = \Theta_{P_0}$.

Proof. The population criterion corresponding to the alternative sample GEL criterion (E.6) is given by $\tilde{P}^{\rho}(\theta,\lambda,\tau) = \mathbb{E}_{P_0}[\rho(\lambda'm(z,\theta)) - \rho(0)] + \lambda'\tau$ with $\tilde{P}^{\rho}(\theta) = \inf_{\tau\in\mathcal{T}}\sup_{\lambda\in\mathcal{R}^{d_m}} \tilde{P}^{\rho}(\theta,\lambda,\tau)$. The solution $\tau(\theta)$ satisfies $\partial\tilde{P}^{\rho}(\theta,\lambda,\tau)/\partial\tau \geq 0$. Thus, since $\partial\tilde{P}^{\rho}(\theta,\lambda,\tau)/\partial\tau = \lambda$, $\lambda^j \geq 0$, $(j = 1,...,d_m)$, and $\lambda'\tau(\theta) = 0$. The solution $\lambda(\theta)$ satisfies $\partial\tilde{P}^{\rho}(\theta,\lambda,\tau)/\partial\lambda = 0$, i.e., $\mathbb{E}_{P_0}[\rho_1(\lambda'm(z,\theta))m(z,\theta)] + \tau(\theta) = 0$ or

$$\tau(\theta) = -\mathbb{E}_{P_0}[\rho_1(\lambda'm(z,\theta))m(z,\theta)] \geq 0.$$

Now $\tilde{P}^{\rho}(\theta,\lambda,\tau(\theta)) = \hat{P}^{\rho}(\theta,\lambda) + \lambda'\tau(\theta) \geq \hat{P}^{\rho}(\theta,\lambda)$ since $\lambda \geq 0$. Hence, as $\hat{P}^{\rho}(\theta,\lambda(\theta)) = \tilde{P}^{\rho}(\theta,\lambda(\theta),\tau(\theta)) \geq \tilde{P}^{\rho}(\theta,\lambda,\tau(\theta))$, $\hat{P}^{\rho}(\theta,\lambda(\theta)) \geq \hat{P}^{\rho}(\theta,\lambda)$, i.e., $\lambda(\theta)$ also optimises the GEL criterion $\hat{P}^{\rho}(\theta,\lambda)$ (3.4). Therefore, $\tilde{\Theta}^{\rho}_{P_0} = \hat{\Theta}^{\rho}_{P_0}$.■

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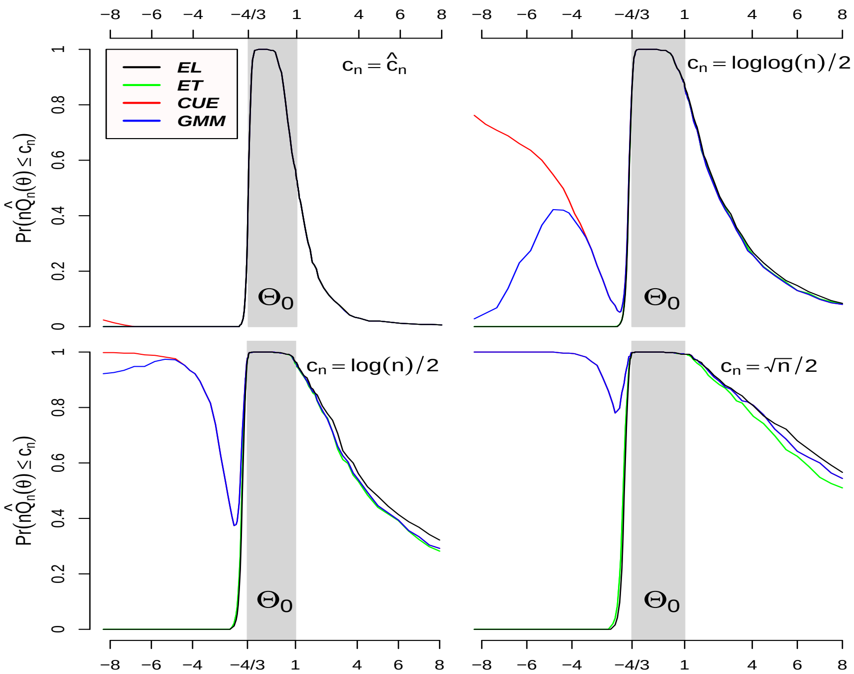
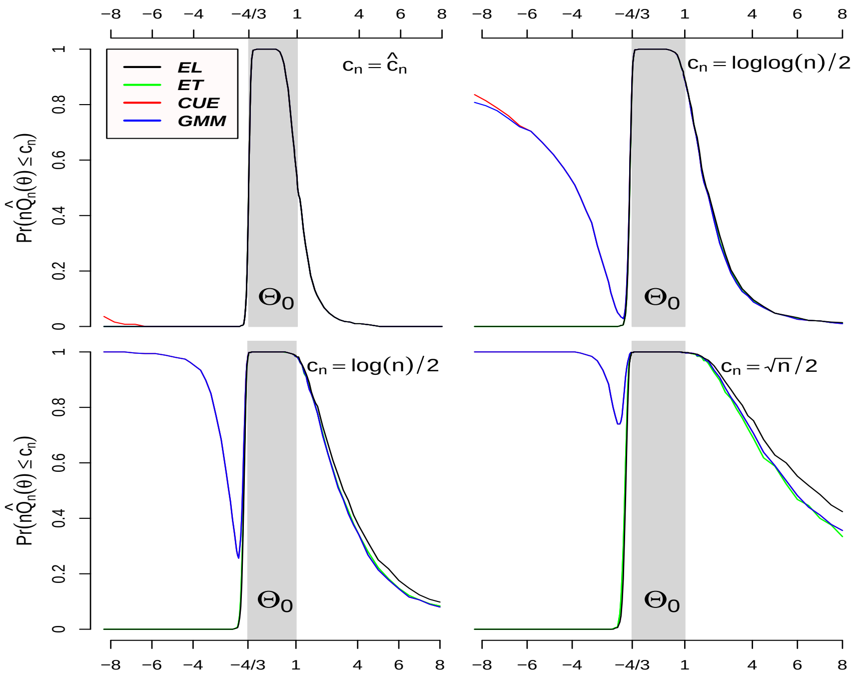


(a) $n = 50$



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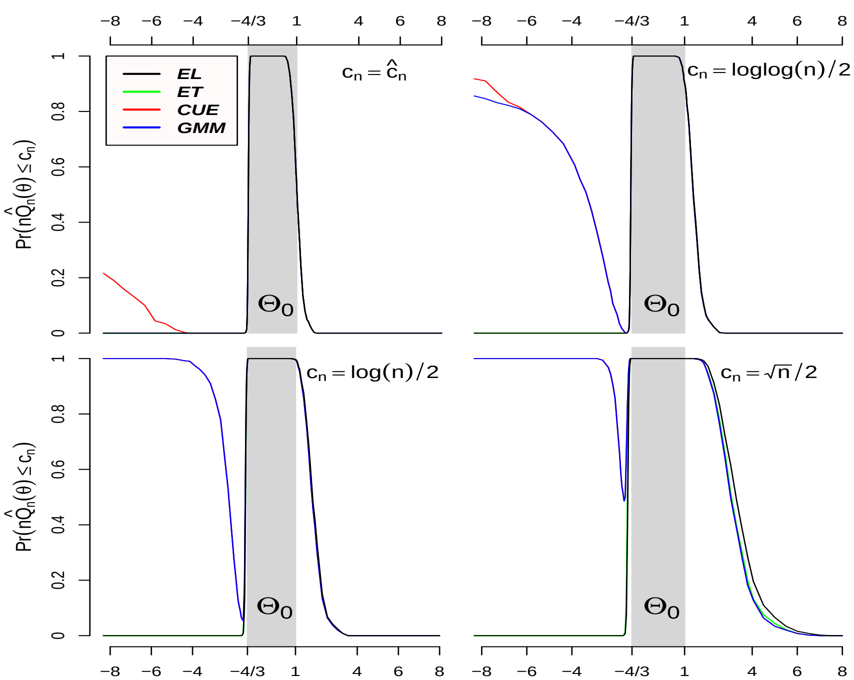

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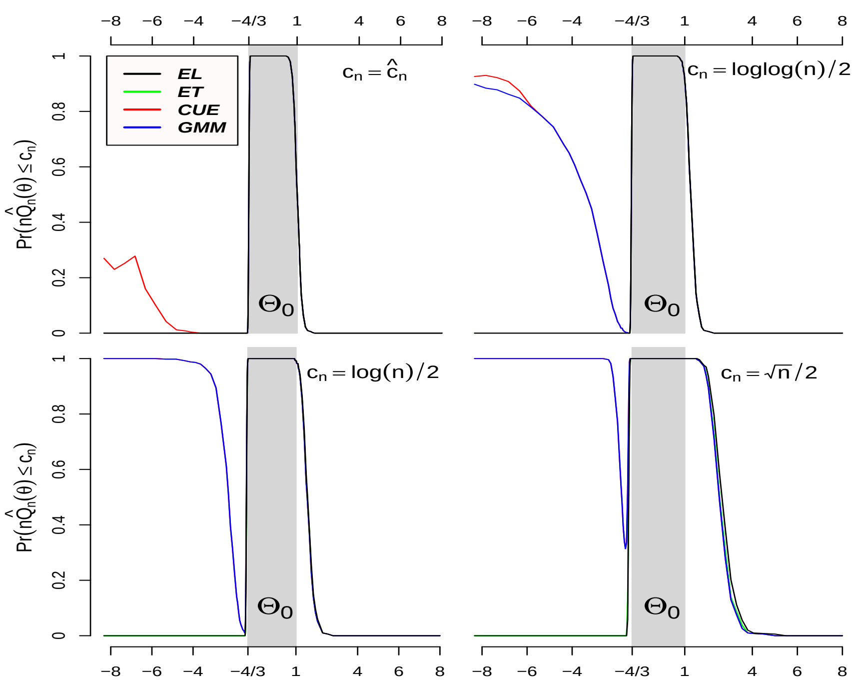
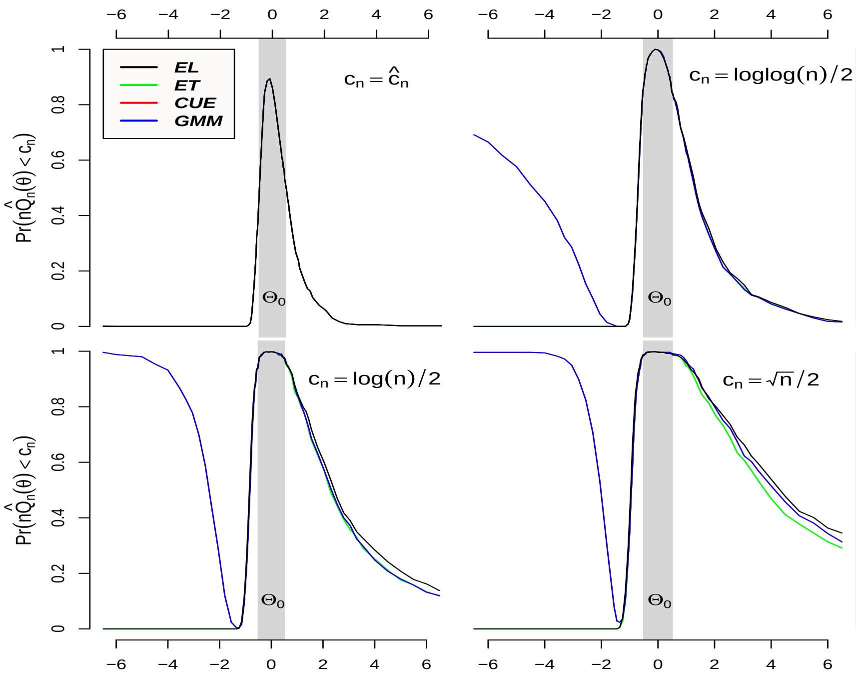
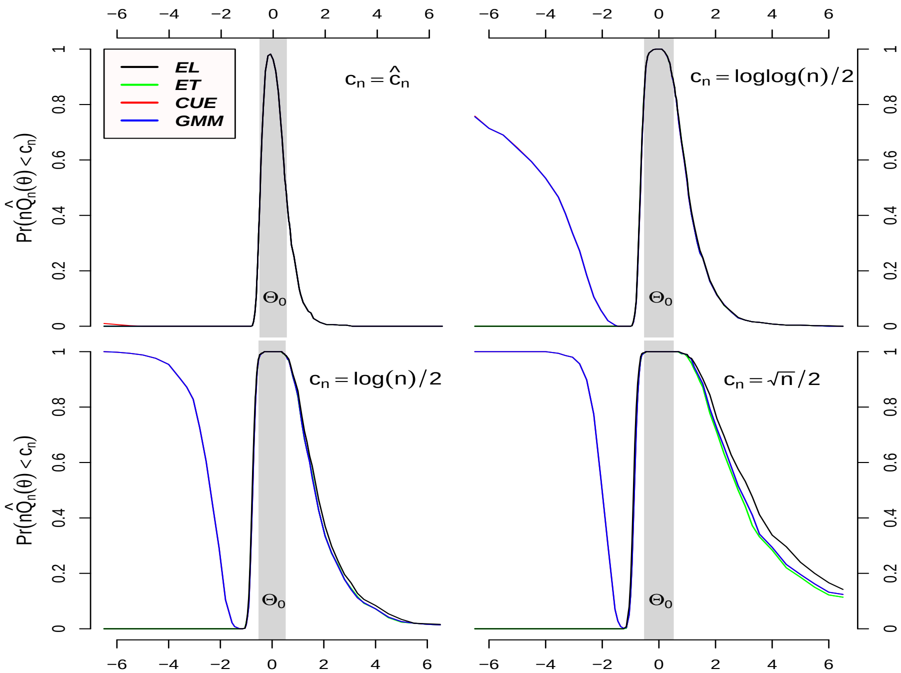


(d) $n = 1000$

Figure 1: **Coverage Probabilities for** $\{n\hat{Q}_n^j(\theta) \le c_n^j\}$ **Design 1**.



(a) $n = 50$



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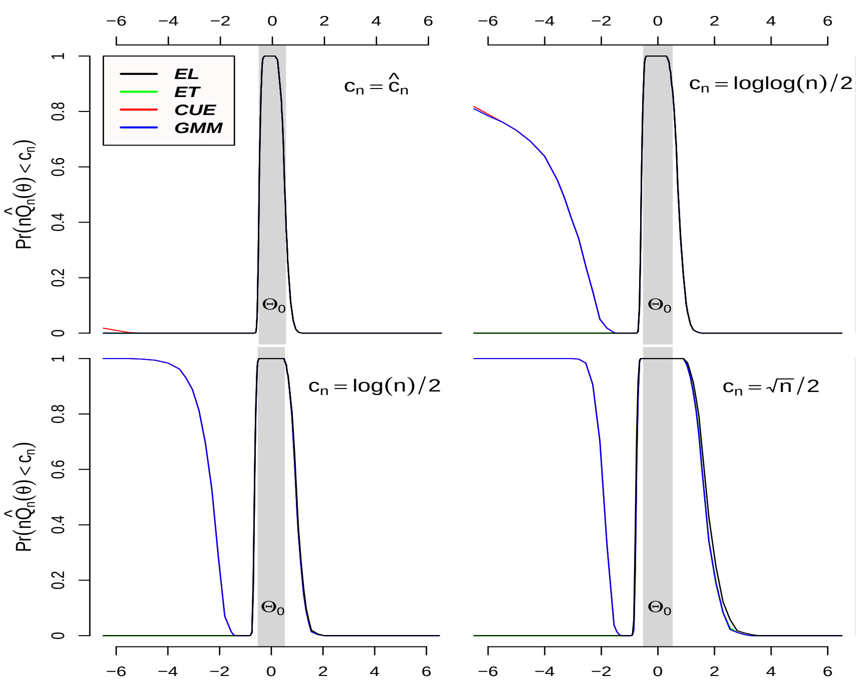
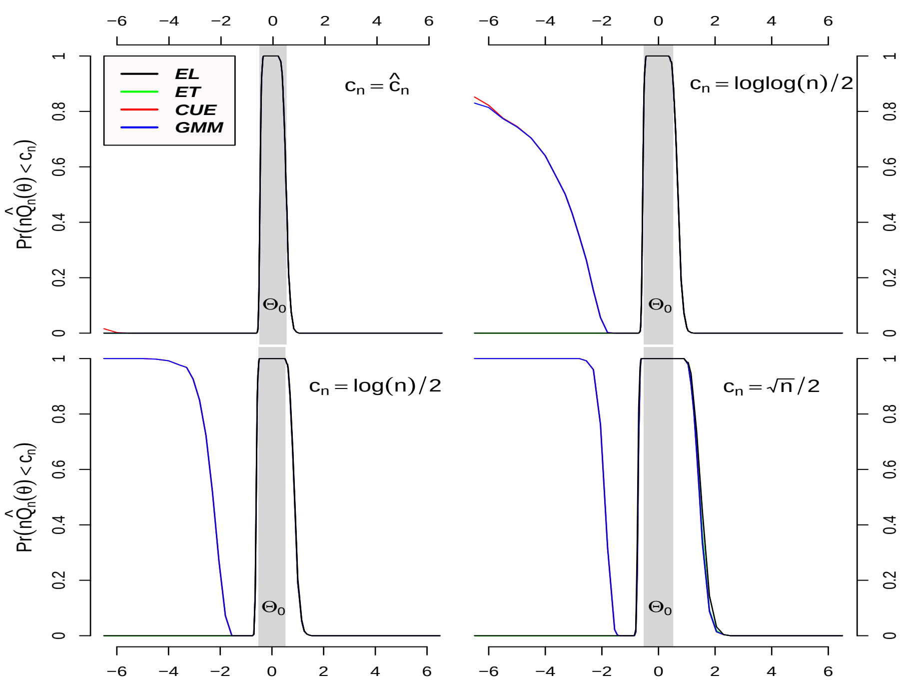
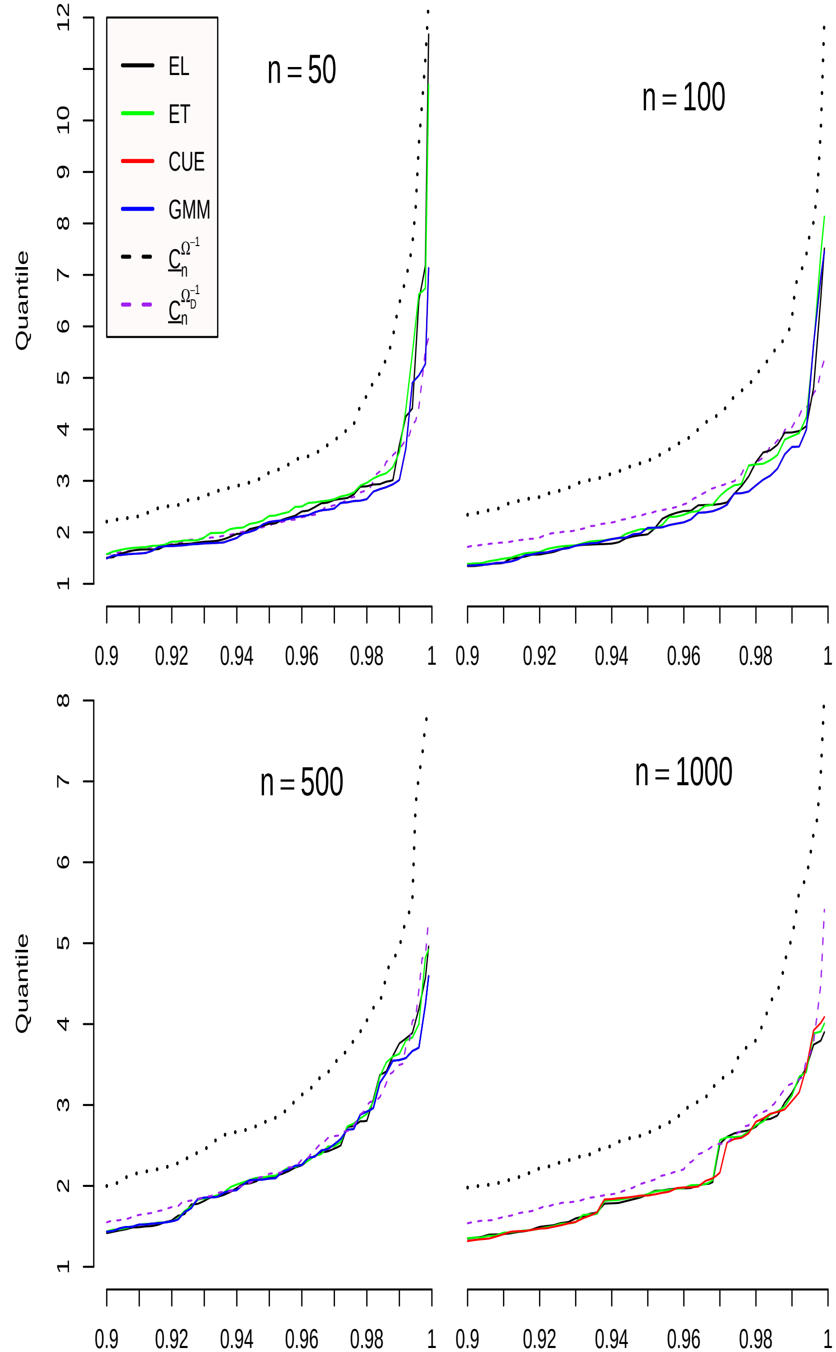


(b) $n = 500$

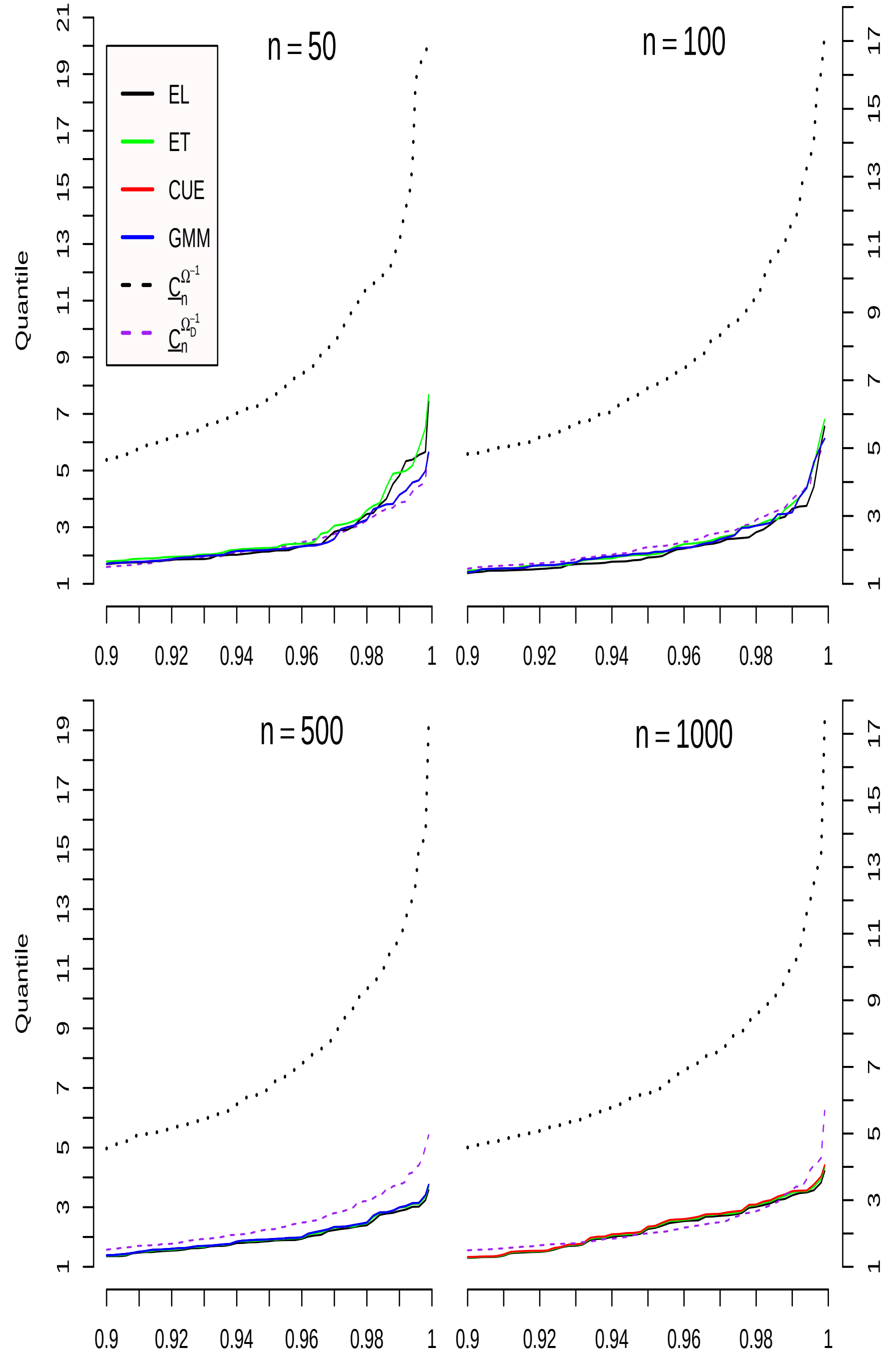


(d) $n = 1000$

Figure 2: **Coverage Probabilities for $\{n\hat{Q}_n^j(\theta) \leq c_n^j\}$ Design 2.**



(a) Design 1



(b) Design 2

Figure 3: **Quantiles of** $\hat{\underline{C}}_n^{\Omega^{-1}*}$, $\hat{C}_n^{\Omega_D^{-1}*}$ **and** $\sup_{\theta\in\Theta_{P_0}} n\hat{Q}_n^j(\theta)$.

Table 1: **Small Sample Properties of Identified Set Estimators. Design 1.**

		$c_n = \hat{c}_n$			$c_n = \log\log(n)/2$			$c_n = \log(n)/2$		
		Bounds	MSE	$p_{\Theta_{P_0}}$	Bounds	MSE	$p_{\Theta_{P_0}}$	Bounds	MSE	$p_{\Theta_{P_0}}$
$n = 50$	GMM	(-1.327, 1.280)	1.198	87.24	(-4.333, 2.946)	4.719	97.65	(-8.178, 4.735)	8.236	99.55
	CUE	(-1.482, 1.281)	1.586	87.30	(-6.707, 2.946)	6.767	97.74	(-8.321, 4.735)	8.337	99.55
	ET	(-1.327, 1.281)	1.199	87.24	(-1.467, 2.965)	2.936	97.38	(-1.587, 4.694)	4.515	99.42
	EL	(-1.327, 1.281)	1.199	87.24	(-1.463, 3.030)	3.004	97.44	(-1.564, 4.912)	4.701	99.51
$n = 100$	GMM	(-1.340, 1.143)	0.714	90.91	(-7.008, 2.180)	6.536	98.75	(-8.333, 3.516)	7.691	99.84
	CUE	(-1.523, 1.142)	1.336	90.97	(-7.202, 2.180)	6.640	98.8	(-8.333, 3.516)	7.691	99.84
	ET	(-1.340, 1.143)	0.714	90.91	(-1.442, 2.197)	1.796	98.58	(-1.528, 3.543)	3.224	99.79
	EL	(-1.340, 1.143)	0.714	90.91	(-1.440, 2.229)	1.831	98.63	(-1.517, 3.721)	3.407	99.82
$n = 500$	GMM	(-1.330, 1.011)	0.261	95.22	(-7.333, 1.380)	6.496	99.46	(-8.333, 1.788)	7.057	99.98
	CUE	(-2.379, 1.011)	2.714	95.37	(-8.021, 1.380)	6.845	99.48	(-8.333, 1.788)	7.057	99.98
	ET	(-1.330, 1.011)	0.261	95.22	(-1.380, 1.383)	0.505	99.38	(-1.426, 1.798)	0.908	99.97
	EL	(-1.330, 1.011)	0.261	95.22	(-1.380, 1.387)	0.508	99.40	(-1.423, 1.816)	0.924	99.98
$n = 1000$	GMM	(-1.329, 0.996)	0.188	96.19	(-7.622, 1.248)	6.641	99.61	(-8.333, 1.522)	7.024	99.98
	CUE	(-2.886, 0.996)	3.304	96.36	(-8.188, 1.248)	6.903	99.61	(-8.333, 1.522)	7.024	99.98
	ET	(-1.329, 0.996)	0.188	96.19	(-1.365, 1.249)	0.329	99.55	(-1.400, 1.526)	0.593	99.98
	EL	(-1.329, 0.996)	0.188	96.19	(-1.365, 1.250)	0.329	99.55	(-1.398, 1.537)	0.603	99.98

Table 2: **Small Sample Properties of Identified Set Estimators. Design 2.**

		$c_n = \hat{c}_n$			$c_n = \log\log(n)/2$			$c_n = \log(n)/2$		
		Bounds	MSE	$p_{\Theta_{P_0}}$	Bounds	MSE	$p_{\Theta_{P_0}}$	Bounds	MSE	$p_{\Theta_{P_0}}$
$n = 50$	GMM	(-0.468, 0.633)	0.759	81.44	(-4.667, 1.591)	5.260	95.64	(-6.478, 2.763)	6.672	99.51
	CUE	(-0.480, 0.633)	0.805	81.44	(-4.667, 1.591)	5.260	95.64	(-6.478, 2.763)	6.672	99.51
	ET	(-0.469, 0.633)	0.759	81.44	(-0.672, 1.597)	1.745	94.29	(-0.799, 2.745)	2.955	98.85
	EL	(-0.468, 0.633)	0.759	81.44	(-0.677, 1.636)	1.793	94.45	(-0.815, 2.904)	3.116	98.92
$n = 100$	GMM	(-0.476, 0.540)	0.448	86.31	(-5.066, 1.133)	5.301	97.99	(-6.500, 1.857)	6.253	99.52
	CUE	(-0.537, 0.540)	0.749	86.31	(-5.078, 1.133)	5.308	97.99	(-6.500, 1.857)	6.253	99.52
	ET	(-0.476, 0.540)	0.448	86.31	(-0.630, 1.136)	0.956	97.24	(-0.732, 1.858)	1.778	99.73
	EL	(-0.476, 0.540)	0.448	86.31	(-0.634, 1.145)	0.969	97.33	(-0.742, 1.941)	1.873	98.77
$n = 500$	GMM	(-0.500, 0.482)	0.175	92.61	(-5.372, 0.710)	5.407	99.07	(-6.500, 0.949)	6.021	100
	CUE	(-0.609, 0.482)	0.824	92.61	(-0.543, 0.710)	5.440	99.07	(-6.500, 0.949)	6.021	100
	ET	(-0.500, 0.482)	0.175	92.61	(-0.557, 0.711)	0.297	98.89	(-0.634, 0.954)	0.531	99.99
	EL	(-0.500, 0.482)	0.175	92.61	(-0.578, 0.713)	0.298	98.91	(-0.637,0.967)	0.526	99.99
$n = 1000$	GMM	(-0.493, 0.509)	0.125	95.77	(-5.489, 0.665)	5.470	99.64	(-6.500, 0.834)	6.011	100
	CUE	(-0.590, 0.509)	0.769	95.77	(-0.565, 0.665)	5.557	99.65	(-6.500, 0.834)	6.011	100
	ET	(-0.493, 0.509)	0.125	95.77	(-0.551, 0.666)	0.215	99.42	(-0.600, 0.836)	0.385	99.99
	EL	(-0.493, 0.509)	0.125	95.77	(-0.551, 0.666)	0.215	99.42	(-0.602, 0.841)	0.375	99.99

Table 3: **Confidence Region Volume and Coverage. Design 1.**

		$\underline{\hat{C}}_n^{\Omega^{-1}*}$						$\hat{C}_n^{\Omega_D^{-1}*}$					
		$1-\alpha=0.90$		$1-\alpha=0.95$		$1-\alpha=0.99$		$1-\alpha=0.90$		$1-\alpha=0.95$		$1-\alpha=0.99$	
		vol	cov	vol	cov	vol	cov	vol	cov	vol	cov	vol	cov
$n=50$	GMM	13.62	0.984	14.52	0.992	15.89	0.998	12.41	0.946	13.35	0.976	14.72	0.994
	CUE	13.65	0.984	14.52	0.992	15.89	0.998	12.68	0.946	13.40	0.976	14.72	0.994
	ET	6.87	0.956	7.71	0.988	9.23	0.994	5.92	0.902	6.62	0.948	7.92	0.990
	EL	7.08	0.964	7.95	0.988	9.25	0.994	6.07	0.908	6.81	0.950	8.14	0.990
$n=100$	GMM	11.93	0.984	12.70	0.994	14.55	1.000	11.30	0.970	11.77	0.982	13.13	0.996
	CUE	11.93	0.984	12.70	0.994	14.55	1.000	11.30	0.970	11.77	0.982	13.13	0.996
	ET	5.16	0.968	5.96	0.986	7.78	0.996	4.54	0.938	4.99	0.964	6.39	0.994
	EL	5.32	0.972	6.22	0.986	8.10	0.998	4.65	0.942	5.15	0.962	6.70	0.994
$n=500$	GMM	9.85	0.970	10.07	0.992	10.31	0.998	9.69	0.946	9.89	0.972	10.13	0.996
	CUE	9.90	0.970	10.07	0.992	10.31	0.998	9.74	0.948	9.93	0.972	10.13	0.996
	ET	2.99	0.936	3.17	0.976	3.44	0.996	2.89	0.900	3.01	0.938	3.23	0.982
	EL	3.00	0.936	3.19	0.980	3.47	0.996	2.90	0.904	3.02	0.940	3.25	0.982
$n=1000$	GMM	9.68	0.970	9.79	0.982	9.95	0.998	9.57	0.956	9.69	0.972	9.83	0.988
	CUE	9.71	0.970	9.79	0.982	9.95	0.998	9.66	0.956	9.71	0.972	9.83	0.988
	ET	2.76	0.946	2.85	0.968	3.04	0.998	2.70	0.920	2.77	0.948	2.90	0.984
	EL	2.77	0.948	2.86	0.970	3.05	1.000	2.70	0.918	2.77	0.950	2.91	0.986

Table 4: **Confidence Region Volume and Coverage. Design 2.**

		$\underline{\hat{C}}_n^{\Omega^{-1}*}$						$\hat{C}_n^{\Omega_D^{-1}*}$					
		$1-\alpha=0.90$		$1-\alpha=0.95$		$1-\alpha=0.99$		$1-\alpha=0.90$		$1-\alpha=0.95$		$1-\alpha=0.99$	
		vol	cov	vol	cov	vol	cov	vol	cov	vol	cov	vol	cov
$n=50$	GMM	11.38	0.998	12.35	1.000	12.99	1.000	8.56	0.938	9.25	0.978	10.83	0.998
	CUE	11.40	0.998	12.35	1.000	13.00	1.000	8.56	0.938	9.25	0.978	10.83	0.980
	ET	5.52	0.986	6.49	0.998	7.54	1.000	3.02	0.870	3.55	0.922	5.04	0.984
	EL	5.86	0.988	6.71	0.998	7.57	1.000	3.13	0.876	3.73	0.932	5.34	0.984
$n=100$	GMM	9.70	1.000	10.69	1.000	12.74	1.000	7.82	0.970	8.28	0.988	8.98	1.000
	CUE	9.69	1.000	10.70	1.000	12.70	1.000	7.82	0.970	8.28	0.988	8.98	1.000
	ET	3.95	0.996	4.88	1.000	6.89	1.000	2.20	0.924	2.51	0.958	3.25	1.000
	EL	4.29	0.996	5.37	1.000	7.20	1.000	2.26	0.926	2.59	0.964	3.45	0.994
$n=500$	GMM	7.64	1.000	7.81	1.000	8.28	1.000	7.17	0.946	7.36	0.980	7.55	1.000
	CUE	7.64	1.000	7.81	1.000	8.28	1.000	7.17	0.946	7.36	0.980	7.55	1.000
	ET	1.82	1.000	2.01	1.000	2.55	1.000	1.40	0.924	1.49	0.970	1.70	1.000
	EL	1.85	1.000	2.05	1.000	2.69	1.000	1.40	0.924	1.50	0.974	1.72	1.000
$n=1000$	GMM	7.40	1.000	7.47	1.000	7.60	1.000	7.08	0.978	7.20	0.980	7.30	0.996
	CUE	7.40	1.000	7.47	1.000	7.60	1.000	7.09	0.978	7.20	0.980	7.30	0.996
	ET	1.52	1.000	1.61	1.000	1.75	1.000	1.28	0.922	1.32	0.940	1.40	0.982
	EL	1.53	1.000	1.62	1.000	1.77	1.000	1.28	0.924	1.32	0.946	1.40	0.984