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Abstract

A model of choice under purely subjective uncertainty, *Piecewise-Additive Choquet Expected* (PACE) utility, is introduced. PACE utility allows for optimism and pessimism simultaneously, but represents a minimal departure from expected utility. It can be seen as a continuous version of NEO-expected utility (Chateauneuf et al, 2007) and, as such, is especially suited for applications with rich state spaces. The main theorem provides a preference foundation for PACE utility in the Savage framework of purely subjective uncertainty with an arbitrary outcome set.

Keywords: Optimism, Pessimism, Inverse-S, Choquet Expected Utility, NEO-additive capacities, Probabilistic Sophistication.

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1 Introduction

Economic life is rich with uncertainties, most of which look nothing like probabilistic risk. In the subjective expected utility theory of Savage (1954), the decision maker's

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beliefs about uncertainty can be quantified probabilistically. All uncertainty reduces to risk. This theory also excludes probabilistic risk attitudes towards this subjective risk. Attitudes such as *optimism* and *pessimism*, for which there is considerable empirical support. This paper studies a simple modification of expected utility to allow for optimism and pessimism.

Cohen (1992) argued that the most important departures from expected utility can be explained by the *security* and *potential factors*. These are the worst and best possible outcomes of a decision (Lopes, 1987). Axiomatic models have been developed by Gilboa (1988), Jaffray (1988), Cohen (1992), Essid (1997), Chateauneuf et al. (2007), and Schmidt and Zimper (2007). The *NEO-expected utility* (NEO-EU) model of Chateauneuf et al. (2007) evaluates choices using the formula:

$$\gamma(\text{worst possible utility}) + (1 - \gamma - \delta)(\text{expected utility}) + \delta(\text{best possible utility}).$$

NEO-EU is a special case of *Choquet expected utility* (Gilboa, 1987; Schmeidler, 1989; Wakker, 1989) with a Non-Extremal-Outcome additive (NEO-additive) capacity. A NEO-additive capacity is a transformation of a probability measure that is linear for non-extreme probabilities, and departs from expected utility if and only if it is discontinuous at zero or one. As such, it retains subjective expected utility's *probabilistic sophistication* property (Machina and Schmeidler, 1992) whilst allowing for nonexpected utility attitudes. Wakker (2001, 2005) gave model-free, behavioural definitions of pessimism and optimism using an uncertainty analogue of the *common consequence effect* (Allais, 1953).[1] According to these definitions, a NEO-EU maximiser, with γ and δ strictly positive, responds to her probabilistic beliefs by exhibiting both pessimism and optimism simultaneously; a condition called *ambivalence* or *inverse-S* behaviour. Ambivalence is the most prevalent uncertainty attitude observed in experiments (Wakker, 2010: 203-243, 290-292).

NEO-EU has been applied extensively (Abdellaoui et al. 2010; Dominiak et al. 2012;

[1] *Event optimism* and *event pessimism* (Wakker, 2005:114) are relevant for the framework of this paper.

Dominiak and Lefort, 2013; Ford et al. 2013; Eichberger et al. 2012; Eichberger and Kelsey, 2014; Ludwig and Zimper, 2014; Romm, 2014; Teitelbaum, 2007; Zimper, 2012). Its tractability stems from both: the probabilistic sophistication property, and the simple way such probabilities are used in the evaluation. Naively applied, it is not compatible with the ambiguity aversion examples of Ellsberg (1961). It has been argued, however, that *source dependence* of uncertainty attitudes is appropriate for explaining Ellsberg's examples (Heath and Tversky, 1991; Fox and Tversky, 1995; Chow and Sarin, 2001; Wakker, 2001; Abdellaoui et al. 2011). For example, a British investor may respond more optimistically to his beliefs about the UK market than his beliefs about foreign markets, even if beliefs are probabilistic in both cases. By allowing for source dependence of uncertainty attitudes, probabilistic sophistication can be derived *within* each source of uncertainty, without requiring it to hold *across* each source (Chew and Sagi, 2008). Allowing parameters of decision models to be source-dependent adds further layers of difficulty for empirical applications. NEO-expected utility's simple form, however, makes such an approach tractable, as demonstrated by Abdellaoui et al. (2011).

Chateauneuf et al. (2007) gave a preference foundation for NEO-EU in a purely subjective, Savage-style framework. In particular, using simple acts that map an arbitrary state space to a connected and separable outcome set. By deriving a continuous, cardinal utility on this rich outcome space, the subjective mixture techniques of Ghirardato et al. (2003) could be employed to give elegant preference axioms. In certain economic applications, most notably when outcomes are monetary, assuming rich topological structure on the outcome set is appropriate. In other applications, it is not natural to assume such a structure. For example, when the outcomes are: health outcomes, environmental outcomes, durable goods, and so on.

In this paper, the original Savage framework with an arbitrary outcome set is considered. The state space is at least countably infinite. There are several problems with NEO-EU in this framework. Because the probability transformation can be discontinuous, the axiomatic foundations are complicated. For *risk*, which is a special case of the Savage framework, Webb and Zank (2011) provided a preference foundation for NEO-expected utility. Complicated trade-off axioms, consistent optimism and consistent pessimism, were used to measure and ensure consistency of the discontinuities, and additional structural assumptions were necessary to derive cardinal utility and unique parameters. A similar approach in the purely subjective setting would be no less complex. For arbitrary outcome sets, the NEO-EU parameters need not be uniquely determined (Webb and Zank, 2011, 710, Example 8). For applications based on such a framework, it may be difficult to use NEO-EU to generate behavioural predictions related to comparative optimism or pessimism. Futhermore, applications of NEO-EU to frameworks with infinite states are problematic. Discontinuities in the evaluation formula mean even simple applications are vulnerable to problems such as empty best responses, leading to nonexistence of equilibria.[2]

To resolve both of the problems above, a continuous version of NEO-EU is developed in this paper. We start with NEO-EU on a finite state space, where the above mentioned problems do not arise, and consider the simplest extension to infinite states that has a continuous probability transformation function. The resulting theory is called *Piecewise Additive Choquet Expected* (PACE) utility. It will be shown that PACE utility admits a simple axiomatisation in the Savage framework and can be applied in cases where NEO-EU fails. If NEO-EU is the smallest departure from expected utility to allow inverse-S behaviour, then PACE utility is the smallest *continuous* departure from expected utility to allow inverse-S behaviour.

The remainder of the paper is structured as follows: Section 2 presents the theoretical background of this paper; Section 3 considers the problem of extending NEO-additive capacities to infinite state spaces and introduces PACE utility; and Section 4 studies the preference foundations in the Savage framework, with the paper's main theorem delivering an axiomatic characterisation of those preferences that admit PACE utility representations.

[2]Finite games with mixed strategies, for example, are affected by both of these problems. Allowing players to mix strategies using any probability in the $[0, 1]$ interval means considering preferences over a finite outcome set with a continuum of states.

2 Preliminaries

This section outlines the framework for choice under uncertainty and models for ambiguity. Let $\mathscr{S}$ a set of *states* and $\mathscr{E}$ be a σ-algebra of *events*. We allow for the case where $\mathscr{E}$ is the power set of $\mathscr{S}$. Some richness will later be imposed on $\mathscr{S}$ via a solvability condition, which will imply that $\mathscr{S}$ contains infinitely many states.[3] Let $\mathscr{X}$ be a set of *outcomes*. The set of outcomes can be finite or infinite. States and outcomes are the only primitives; from these all other definitions are derived.

An *act* is a function $f : \mathscr{S} \to \mathscr{X}$ that is measurable with respect to $\mathscr{E}$. It is assumed that acts are *simple*, that is, they take only finitely many values. Acts will also be written $f = [A_1, f_2; \ldots; A_n, f_n]$, denoting the act with outcome f_i if the state belongs to event A_i. Acts are the objects of choice. By choosing act f, the decision maker receives outcome $f(s)$ if state s obtains. The act results in outcome x if the state belongs to $f^{-1}(x) \in \mathscr{E}$. The set of acts is $\mathscr{A}$. An act f is *constant* if $f(s) = x$ for all $s \in \mathscr{S}$ (we will write $f = x$). An act may be defined by its *subacts*. For $f, g \in \mathscr{A}$ and $A \in \mathscr{E}$, $f_A g := \{h \in \mathscr{A} : s \in A \Rightarrow h(s) = f(s), s \notin A \Rightarrow h(s) = g(s)\}$. An event $A \in \mathscr{E}$ is *null* if $f_A h \sim g_A h$ for all $f, g, h \in \mathscr{A}$, otherwise it is *non-null*.

A set function $\nu : \mathscr{E} \to \mathbb{R}$ is *normalised* if $\nu(\emptyset) = 0$ and $\nu(\mathscr{S}) = 1$. It is *monotonic* if, for all $A, B \in \mathscr{E}$, $A \subseteq B$ implies $\nu(A) \leqslant \nu(B)$. It is *additive* if, for all disjoint $A, B \in \mathscr{E}$, $\nu(A \cup B) = \nu(A) + \nu(B)$. A *capacity* is a real-valued set function $\nu : \mathscr{E} \to \mathbb{R}$ that is normalised and monotonic. A capacity $\nu : \mathscr{E} \to [0, 1]$ is *convex-valued* if for all $\alpha \in [0, 1]$ there exists $A \in \mathscr{E}$ such that $\nu(A) = \alpha$. A *probability measure* is an additive capacity. A capacity $\nu : \mathscr{E} \to [0, 1]$ is a *probability transformation* if there is a strictly increasing function $\phi : [0, 1] \to [0, 1]$ and a probability measure $p : \mathscr{E} \to [0, 1]$ such that $\nu = \phi \circ p$.

The decision maker exists only to maximise a preference relation $\succcurlyeq$ defined over $\mathscr{A}$. A utility function U over acts $\mathscr{A}$ is a real-valued function that represents preferences

[3] Our assumptions allow for an uncountably infinite $\mathscr{S}$, but do not imply uncountability of $\mathscr{S}$. See observations 3 and 4 of Wakker (1993).

such that $f \succcurlyeq g$ if and only if $U(f) \geqslant U(g)$. Preferences $\succcurlyeq$ over acts $\mathscr{A}$ conform to *subjective expected utility* if they are represented by:

$$E(p,u)(f) = \sum_{x \in \mathscr{X}} p(f^{-1}(x))u(x)$$

where $p : \mathscr{E} \to [0,1]$ is a probability measure and $u : \mathscr{X} \to \mathbb{R}$ is a utility function for outcomes. Additivity of the probability measure ensures that, for all $x \in \mathscr{X}$, the following holds:

$$p(f^{-1}(x)) = p\Big(\bigcup_{y \succcurlyeq x} f^{-1}(y)\Big) - p\Big(\bigcup_{y \succ x} f^{-1}(y)\Big).$$

The subjective expected utility formula can therefore be written as:

$$E(p,u)(f) = \sum_{x \in \mathscr{X}} \Big[p\Big(\bigcup_{y \succcurlyeq x} f^{-1}(y)\Big) - p\Big(\bigcup_{y \succ x} f^{-1}(y)\Big)\Big]u(x).$$

This exercise helps one clearly distinguish between subjective expected utility and the following model, *Choquet expected utility* (Schmeidler, 1989; Wakker, 1989), in which the probability measure p in the above expression is replaced with ν, a (possibly) nonadditive capacity. Preferences $\succcurlyeq$ over acts $\mathscr{A}$ conform to Choquet expected utility if they are represented by:

$$E(\nu,u)(f) = \sum_{x \in \mathscr{X}} \Big[\nu\Big(\bigcup_{y \succcurlyeq x} f^{-1}(y)\Big) - \nu\Big(\bigcup_{y \succ x} f^{-1}(y)\Big)\Big]u(x)$$

where $\nu : \mathscr{E} \to [0,1]$ is a capacity and $u : \mathscr{X} \to \mathbb{R}$ is a utility function for outcomes. Notice that we use the shorthand $E(\nu,u)(f)$ for the Choquet expected utility of act $f \in \mathscr{A}$ using capacity ν and utility for outcomes u.

Choquet expected utility allows for non-neutral attitudes to ambiguity. A capacity $\nu : \mathscr{E} \to [0,1]$ is *convex* if $\nu(A \cup B) - \nu(B)$ is non-decreasing as B $\supseteq$-increases. Similarly, a capacity is *concave* if $\nu(A \cup B) - \nu(B)$ is non-increasing as B $\supseteq$-increases.

Wakker (2001, 2005) gave behavioural definitions of pessimism and optimism and characterised such behaviour in context of Choquet expected utility. Under Wakker's definitions, a Choquet expected utility maximiser is pessimistic if and only if the capacity is convex, and optimistic if and only if the capacity is concave. The most prevalent attitude found in experiments is *ambivalence*; a composition of pessimistic and optimistic responses to uncertainty. Because such capacities are initially concave and then convex, ambivalence is often called *inverse-S behaviour*.

A special case of Choquet expected utility, presented and axiomatised by Chateauneuf, Eichberger and Grant (2007), is the *NEO-additive capacities*. These are discussed further in the next section. The Choquet expected utility of an act f with respect to a NEO-additive capacity can be shown to be a convex combination of subjective expected utility, the utility of the act's best outcome, $u(f^*)$, and utility of the act's worst outcome, $u(f_*)$:

$$NEO(f) = \gamma u(f_*) + (1 - \gamma - \delta)E(p, u)(f) + \delta u(f^*)$$

with $\gamma, \delta \geqslant 0$ and $\gamma + \delta < 1$. Here, $\gamma + \delta$ dictates the extent of the departure from expected utility; the *degree of ambiguity*. It is apparent from the representation that two acts with identical (or indifferent) best and worst outcomes will be ranked according to their expected utilities. Choquet expected utility using NEO-additive capacities is sometimes called *NEO-expected utility* (NEO-EU).

3 PACE Utility

In this section, the problem of extending NEO-additive capacities to Savage's infinite state space is addressed. Consider a NEO-additive capacity ω defined over a finite set of states $S \subset \mathscr{S}$, $S = \{s_1, \ldots, s_n\}$. Let $\mathscr{E}_S = \{\emptyset, A_1, \ldots, A_i, \ldots, \mathscr{S}\}$ denote the set of 2^n events, formed by all subsets of S. Suppose, for the sake of presentational simplicity, that all non-empty events in $\mathscr{E}_S$ occur with positive probability. It is known that such NEO-additive capacities have the following form:

$$\begin{aligned}
\omega \quad : \quad & \mathscr{E}_S \to [0,1], \\
& \emptyset \mapsto 0, \\
& A_1 \mapsto (1-\gamma-\delta)p(A_1)+\delta, \\
& \vdots \\
& A_i \mapsto (1-\gamma-\delta)p(A_i)+\delta, \\
& \vdots \\
& S \mapsto 1,
\end{aligned}$$

where $\gamma, \delta \geqslant 0$ and $\gamma + \delta < 1$.

The NEO-expected utility model is tractable for economic applications assuming finitely many states. Consider the following example, which will be used throughout this section:

Example 1 (A simple insurance model with finite states): An agent with monetary wealth w faces a loss of l with probability $(1-p)$. Full insurance is available at premium z, which is actuarially fair, $w - z = pw + (1-p)(w-l)$. The agent chooses the probability q that she receives full insurance.[4] Suppose there are finitely many options: $0 = q_1 < \cdots < q_{n-1} < q_n = 1$. The problem is, choose q to make the lottery $(q, w-z; (1-q)p, w; (1-q)(1-p), w-l)$ as preferable as possible. Suppose the agent is a CEU maximiser with a strictly increasing and strictly concave utility for money u and probability transformation $\omega = \phi \circ p$. If ϕ is the identity (expected utility), then $u(w-z) > pu(w) + (1-p)u(w-l) := e$, and full insurance is strictly preferred to no insurance. Maximising $qu(w-z) + (1-q)e$ yields $q = q_n = 1$ as the unique solution. If ϕ is NEO-additive then, for $q \in (0,1)$, her utility is given by $\gamma u(w-l) + (1-\gamma-\delta)[qu(w-z) + (1-q)e] + \delta u(w)$, which is strictly increasing in

[4] This could be done by rearranging her portfolio in some unmodelled way, or could be a deliberate randomisation / mixed strategy. This approach is not standard, but serves to highlight the differences between the models under consideration.

q. It is possible, however, that $q = 1$ is not optimal. Utility for $q \in (0, 1)$ is bounded below by $\gamma u(w - l) + (1 - \gamma - \delta)e + \delta u(w)$ which is greater than $u(w - z)$ for δ sufficiently close to one. In that case, when the agent has a high degree of optimism, optimality occurs where $q = q_{n-1}$. $\square$

An *extension* of ω to the infinite state space $\mathscr{S}$, is a capacity $\nu : \mathscr{E} \to [0, 1]$ that coincides with ω wherever ω is defined, $\nu|_{\mathscr{E}_S} = \omega$. One extension of a NEO-additive capacity to consider is the probability transformation such that, for all $A \in \mathscr{E}$:

$$\nu(A) = \begin{cases} 0 & \text{if } p(A) = 0, \\ (1 - \gamma - \delta)p(A) + \delta & \text{if } 0 < p(A) < 1, \\ 1 & \text{if } p(A) = 1, \end{cases}$$

with $\gamma, \delta \geqslant 0$ and $\gamma + \delta < 1$. That is, $\nu = \phi \circ p$ with the transformation ϕ that is strictly increasing everywhere, linear for all probabilities between zero and one, but possibly discontinuous at zero and/or at one. The probability transformation above is certainly the most obvious extension of a NEO-additive capacity to infinite states. Indeed, we call such ϕ a *NEO-additive transformation function.* There are, however, some problems with the NEO-expected utility model that results. In particular, the discontinuity present in this transformation function presents difficulties for even simple applications. In the following example there is a *continuum* (a compact and connected set) of states, which is typical of many economic applications:

Example 2 (A simple insurance model with a continuum of states): Consider the model of example above, except the problem now involves choosing $q \in [0, 1]$. If ϕ is the identity, maximising $qu(w - z) + (1 - q)e$ yields $q = 1$ as the unique solution. If ϕ is continuous, the problem has at least one solution. If ϕ is NEO-additive then, utility is strictly increasing for $q \in (0, 1)$. If δ sufficiently close to one, $q = 1$ is not optimal. In that case there is no well-defined solution to the agent's problem. $\square$

These issues do not arise for probability transformations that are continuous. Therefore, we now seek the simplest extension of a NEO-additive capacity that has a *continuous* transformation function. Recall that the NEO-additive transformation function above linearly transforms all non-extreme probabilities:

$$A \mapsto (1-\gamma-\delta)p(A)+\delta \quad \text{if } 0<p(A)<1.$$

Because of this, a NEO-additive transformation function ϕ is continuous if and only if it is the identity. Consider the following, minor weakenening of the above requirement. Let $\kappa \in [\frac{1}{2}, 1]$ and consider a capacity $\nu = \phi \circ p$ that satisfies:

$$A \mapsto (1-\gamma-\delta)p(A)+\delta \quad \text{if } 1-\kappa<p(A)<\kappa.$$

By taking κ close to one, this capacity is empirically indistinguishable from a NEO-additive capacity. Under this weaker requirement, however, continuity of ϕ can be retained. To do so, we must specify ϕ on $[0, 1-\kappa]$ and $[\kappa, 1]$. The simplest assumption, and therefore most in keeping with the NEO-EU spirit, is that ϕ is also strictly increasing and linear on these intervals. The only capacity to achieve all of this is the capacity ν such that, for all $A \in \mathscr{E}$:

$$\nu(A) = \begin{cases} (1-\gamma-\frac{\delta\kappa}{1-\kappa})p(A) & \text{if } p(A) \leqslant 1-\kappa \\ (1-\gamma-\delta)p(A)+\delta & \text{if } 1-\kappa \leqslant p(A) \leqslant \kappa \\ \frac{1-[(1-\gamma-\delta)\kappa+\delta]}{1-\kappa}p(A)+\frac{(1-\gamma-\delta)\kappa+\delta-\kappa}{1-\kappa} & \text{if } \kappa \leqslant p(A) \end{cases}$$

with $p: \mathscr{E} \to [0,1]$ a probability measure, $\kappa \in [\frac{1}{2}, 1]$, $\kappa \leqslant \frac{1-\gamma}{1-\gamma-\delta}$ and $\kappa \leqslant \frac{1-\delta}{1-\gamma-\delta}$ and $\gamma+\delta<1$. That is, $\nu = \phi \circ p$ with the transformation ϕ being continuous and strictly increasing everywhere, and linear on $[0, 1-\kappa]$, $[1-\kappa, \kappa]$ and $[\kappa, 1]$. We call such capacities *piecewise-additive.* Notice that $\gamma, \delta \geqslant 0$ is not required, hence departures from additivity can, but need not, be of the inverse-S variety.

Definition 3 (Piecewise Additive Choquet Expected (PACE) Utility): PACE utility holds if preferences are represented by $PACE(\cdot) = E(\nu, u)(\cdot)$; Choquet expected utility with a piecewise-additive capacity.

The PACE utility model can be thought of as "expected utility with *kinks*", hence the κ. In applications, it can be easier to deal with "kinks" than to deal with "jumps". In the insurance example developed above, the decision maker's best response will *always* be non-empty if ϕ is continuous. The problem encountered with NEO-EU in Example 2 cannot arise under PACE utility. Also, PACE utility remains tractable enough to obtain a closed-form solution:

Example 4 (A simple insurance model with a continuum of states): Suppose, in the example above, that the agent chooses $q \in [0, 1]$ and her capacity is piecewise-additive. Utility in this case varies continuously with $q \in [0, 1]$, hence the problem has a well-defined solution. It is possible for the solution to differ from the expected utility case. To see this, let $\kappa > p$, and let δ be close to one. Then, utility increases with q on $[0, \frac{\kappa-p}{1-p}]$ and decreases with q on $[\frac{\kappa-p}{1-p}, 1]$. Optimality occurs where $q = \frac{\kappa-p}{1-p}$. At this q, the probability of getting at least $w - z$ is κ, and the agent is more sensitive to the unlikely, best outcome w. This agent prefers full insurance to no insurance, but most prefers to gamble on being insured. $\square$

A NEO-EU maximiser stratifies events into "impossible", "uncertain" and "certain", being probability zero, probability in $(0, 1)$, and probability one respectively, and behaves as an expected utility maximiser *within* each class, but not *across* the classes. PACE utility is based on a similar trichotomy, buts allows the classes to be *subjective*. Events are now stratified into "unlikely", "moderate" and "likely", corresponding to probability "low enough", probability "not too low or too high", and probability "high enough" respectively. To operationalise this idea, we used a personal parameter, $\kappa \in [\frac{1}{2}, 1]$, such that "unlikely", "moderate" and "likely" correspond to probability not greater than $1 - \kappa$, in $[1 - \kappa, \kappa]$, and not less than κ, respectively. For example, if $\kappa = 2/3$, then events occurring with probability less than $1/3$ are dubbed "unlikely" and events occurring with probability $2/3$ or greater are dubbed "likely".[5]

[5]Parry et al (2007: 27), for example, used such a stratification (with the same numbers) to quantify claims about climate change. "It is likely that we will see increases in hurricane intensity

PACE utility reduces to expected utility within the unlikely, moderate, and likely classes of events. It permits departures from expected utility when comparing events from different classes.

The main theorem of this paper, Theorem 4.1, presents a preference foundation for PACE utility under purely subjective uncertainty. The axiomatic foundations of Choquet expected utility with NEO-additive capacities are complicated in the rich state space, arbitrary outcome set framework (Webb and Zank, 2011). PACE utility will be derived here, however, from a simple weakening of expected utility's sure-thing principle.

4 A Preference Foundation

Here we recall the axioms for subjective expected utility. We assume there are at least three outcomes $x, y, z \in \mathscr{X}$ such that $x \succ y \succ z$.

Axiom 1 (Ordering): Preferences $\succcurlyeq$ over acts $\mathscr{A}$ are a weak order.

Axiom 2 (Monotonicity): For acts $f, g \in \mathscr{A}$, $f(s) \succcurlyeq g(s)$ for all $s \in \mathscr{S}$ implies $f \succcurlyeq g$.

An Archimedean axiom is required. Here, a *rank-dependent* axiom is used. A simpler axiom could be used at this point. But rank-dependence is required later. The benefit of using a slightly more complicated Archimedean axiom is that only one such axiom will be used throughout the paper. For an act f and event A, the event B *dominates* A *under* f if, for all $\tilde{s} \in B$ and $s \in A$, $f(\tilde{s}) \succcurlyeq f(s)$. The *rank* of an event A under f, denoted $\mathcal{R}(A, f)$, is the largest event that dominates A under f. Hence, $f(\tilde{s}) \succcurlyeq f(s)$ holds for all $\tilde{s} \in \mathcal{R}(A, f)$ and $s \in A$ holds, and also $f(\tilde{s}) \preccurlyeq f(s)$ holds for all $\tilde{s} \notin \mathcal{R}(A, f)$ and $s \in A$ because $\mathcal{R}(A, f)$ is the $\supseteq$-maximal dominating event. For a simple act, $f = [A_1, f_1; \ldots; A_n, f_n]$ we may label the outcomes of f so that $f_1 \prec \cdots \prec f_n$. Then, the rank of an event A_i under f is given by $\mathcal{R}(A_i, f) = \cup_{j=i}^{n} A_j$.

during the 21st century," and so on.

Axiom 3 (Archimedeanity): If $S = A^1, A^2, \ldots,$ is a sequence of non-null events such that:

$$x_{A^1} f \succ x_{A^1} g \text{ and } x_{A^i} f \sim x_{A^{i+1}} g$$

with $\mathcal{R}(A^i, x_{A^i} f) = \mathcal{R}(A^i, x_{A^i} g)$, for all $i = 1, 2, \ldots,$ then S is finite.

The following condition, *solvability*, relates to the structure of $\mathscr{E}$, so is separated from the preference conditions above. Solvability holds if, for all acts $f, g, h \in \mathscr{A}$ with $f \succ g \succ h$ there is an event A such that $g \sim f_A h$.

Let $\mathscr{B}$ denote the set of *binary acts*, acts taking at most two values. Given $x \in \mathscr{X}$, $A \in \mathscr{E}$ and $f \in \mathscr{B}$, the act $x_A f$ takes at most three values. The following axiom concerns consistency of a likelihood order revealed using such acts:

Axiom 4 (Comparative Likelihood Consistency): For all $f, g, \tilde{f}, \tilde{g} \in \mathscr{B}$, all $A, B \in \mathscr{E}$, $x, y, \tilde{x}, \tilde{y} \in \mathscr{X}$ with $x \prec y$ and $\tilde{x} \prec \tilde{y}$, the implication:

$$x_A f \sim x_B g, \;\; \tilde{x}_A \tilde{f} \sim \tilde{x}_B \tilde{g}, \;\; \& \;\; y_A f \succcurlyeq y_B g \;\; \Rightarrow \;\; \tilde{y}_A \tilde{f} \succcurlyeq \tilde{y}_B \tilde{g},$$

holds if $\mathcal{R}(A, j)$ and $\mathcal{R}(B, k)$ are constant, where $j = x_A f, \tilde{x}_A \tilde{f}, y_A f, \tilde{y}_A \tilde{f}$ and $k = x_B g, \tilde{x}_B \tilde{g}, y_B g, \tilde{y}_B \tilde{g}$.

Axiom 4 implies Savage's axiom P4.[6] Axiom 4 is the well-known P2* axiom of Gilboa (1987), restricted to ranked, three-outcome acts.[7] Abdellaoui and Wakker (2005) provides an extensive treatment of such axioms. Define an order $\succ$ over events, written $A \succ^L B$ and read, "A is, subjectively, more likely than B," whenever $x_A y \succ x_B y$ for some (for *all*, by axiom 4) $x, y \in \mathscr{X}$ with $x \succ y$. Definition 2.1 of Abdellaoui and Wakker (2005) referred to the same condition as *revealed more likely in a basic sense*. Define $\prec^L$, $\succcurlyeq^L$, $\preccurlyeq^L$, and $\sim^L$ in the usual way. It can be shown that $\succcurlyeq^L$ is a well-defined weak-order over $\mathscr{E}$.

Subjective expected utility preferences and PACE utility preferences both satisfy

[6] Observation 2.4.1 of Gilboa (1987). Proof: Take $f = g = x$ and $\tilde{f} = \tilde{g} = \tilde{x}$.

[7] Gilboa considered P2* a replacement for P2, and so, having noted that P4 was implied, dropped P4 without replacement from the axiom set.

axioms 1-4. The following axiom, the *sure-thing principle*, when combined with axioms 1-4, characterises subjective expected utility:

Axiom 5 (The Sure-Thing Principle): For all events $A \in \mathscr{E}$ and acts $f, \tilde{f}, g, h \in \mathscr{A}$, the following implication holds: $f_A g \succcurlyeq f_A h \;\Rightarrow\; \tilde{f}_A g \succcurlyeq \tilde{f}_A h$.

The sure-thing principle is will be suitably modified to account for the type of uncertainty attitudes permitted under PACE utility. It is useful to first consider the how PACE utility compares with NEO-EU. Under NEO-EU, a decision maker will conform to expected utility when comparing acts with common best and worst outcomes. Indeed, even if acts do not have common best and worst outcomes, a sure-thing principle holds whenever common outcomes are changed in a way that leaves best and worst outcomes unaffected. For an act $f \in \mathscr{A}$, let $b(f)$ and $w(f)$ denote the ranks of the best and worst outcomes of f, respectively. Of course, $w(f) \sim^L \mathscr{S}$. Then, NEO-EU necessarily satisfies the following condition:

Definition 5 (The NEO-Sure-Thing Principle): For all events $A \in \mathscr{E}$ and acts $f, \tilde{f}, g, h \in \mathscr{A}$, the implication:

$$f_A g \succcurlyeq f_A h \;\Rightarrow\; \tilde{f}_A g \succcurlyeq \tilde{f}_A h$$

holds if $w(j) \succ^L \mathcal{R}(A, j) \succ^L b(j)$, for all $j = f_A g, f_A h, \tilde{f}_A g, \tilde{f}_A h$. The NEO-sure-thing principle is necessary but, when combined with axioms 1-4, it is not sufficient for NEO-EU. See example 10 of Webb and Zank (2011:711). To pin down NEO-EU, Webb and Zank (2011) employed further axioms, *consistent optimism* and *consistent pessimism*. The derivation of PACE utility, however, will require only a weakening of the sure-thing principle.

Consider a PACE utility representation, and let K be an event with $p(K) = \kappa$. The key properties of PACE utility will be sure-thing principles, the above implication, that hold whenever:

1. Events are ranked as likely: $\mathcal{R}(A, j) \succcurlyeq^L K$ for all j.

2. Events are ranked as unlikely: $K^{\complement} \succcurlyeq^{L} \mathcal{R}(A, j)$ for all j.

3. Events are ranked as moderate: $K \succcurlyeq^{L} \mathcal{R}(A, j) \succcurlyeq^{L} K^{\complement}$ for all j.

In each case above, $j = f_A g, f_A h, \tilde{f}_A g, \tilde{f}_A h$. Sure-thing principles based on the above conditions are necessary for PACE utility. They are not, however, falsifiable axioms because they assume *a priori* knowledge of an event K with the required properties. A preference axiom, based on falsifiable conditions, is now developed that will imply both the existence of such a K and the corresponding behaviour within in each class of events. This axiom will be called the *piecewise-sure-thing principle.* Before stating the piecewise-sure-thing principle, it is necessary to formulate various local versions of the sure-thing principle. For a *given* event, $A \in \mathscr{E}$, we define *upper*, *lower*, *outer* and *inner* sure-thing principles that hold "at A".

Definition 6 (Upper Sure-Thing Principle at A**)**: For $A \in \mathscr{E}$, the implication:

$$f_B g \succcurlyeq f_B h \quad \Rightarrow \quad \tilde{f}_B g \succcurlyeq \tilde{f}_B h$$

holds if $\mathcal{R}(B, j) \succcurlyeq^{L} A$, for all $j = f_B g, f_B h, \tilde{f}_B g, \tilde{f}_B h$.

The upper sure-thing principle at A implies the sure-thing principle holds for outcomes ranked likelier than A. This is a simple and testable condition. Conforming to the standard sure-thing principle is often seen as normatively desirable. One might appeal to a weaker criterion such as this, when violations of the sure-thing principle are permitted. The upper sure-thing principle at K necessarily holds under PACE utility.

Definition 7 (Lower Sure-Thing Principle at A**)**: For $A \in \mathscr{E}$, the implication:

$$f_B g \succcurlyeq f_B h \quad \Rightarrow \quad \tilde{f}_B g \succcurlyeq \tilde{f}_B h$$

holds if $A \succcurlyeq^{L} \mathcal{R}(B, j)$, for all $j = f_B g, f_B h, \tilde{f}_B g, \tilde{f}_B h$.

The lower sure-thing principle at A implies the sure-thing principle holds for outcomes ranked less likely than A. This seems to carry the same normative content as the upper-sure thing principle. Under PACE utility, events are considered unlikely only if they are no more likely than $K^{\complement}$. Then, the lower sure-thing principle at $K^{\complement}$ necessarily holds under PACE utility. Consider an event A, with A likelier than its complement $A^{\complement}$. Given their apparently equivalent normative status, if one conforms to the upper sure-thing principle at A, then conforming to the lower sure-thing principle at $A^{\complement}$ is reasonable. We call this the *outer sure-thing principle*:

Definition 8 (Outer Sure-Thing Principle at A): The upper sure-thing principle at A and the lower sure-thing principle at $A^{\complement}$ both hold, or the upper sure-thing principle at $A^{\complement}$ and the lower sure-thing principle at A both hold.

Under PACE utility, preferences must satisfy the outer sure-thing principle at K. That is, the sure-thing principle holds for likely outcomes and unlikely outcomes. The third class, moderate likelihood, is covered by the following condition:

Definition 9 (Inner Sure-Thing Principle at A): For $A \in \mathscr{E}$, with $A \succcurlyeq^{L} A^{\complement}$, the implication:

$$f_B g \succcurlyeq f_B h \quad \Rightarrow \quad \tilde{f}_B g \succcurlyeq \tilde{f}_B h$$

holds if $A \succcurlyeq^{L} \mathcal{R}(B, j) \succcurlyeq^{L} A^{\complement}$, for all $j = f_B g, f_B h, \tilde{f}_B g, \tilde{f}_B h$.

The inner sure-thing principle applies to events with ranks of moderate likelihood. Under PACE utility, the inner sure-thing principle at K necessarily holds, where K and $K^{\complement}$ are labelled so that $K \succcurlyeq^{L} K^{\complement}$. In probability terms this amounts to expected utility when the outcomes are ranked with probability in the "inner" interval $[1-\kappa, \kappa]$. The outer sure-thing principle refers to the "outer" intervals $[0, 1-\kappa]$ and $[\kappa, 1]$.

The key axiom for PACE utility can now be stated:

Axiom 5κ (The Piecewise-Sure-Thing Principle): For all $A \in \mathscr{E}$, at least one of the inner sure-thing principle at A or the outer sure-thing principle at A holds.

The piecewise-sure-thing principle takes the content of the above conditions, then adds a simplifying assumption. It forces *every* event to fall into at least one of three categories: a set of events where the upper sure-thing principle holds, a set of events where the lower sure-thing principle holds, or a set of events where the inner sure-thing principle holds. The following theorem characterises PACE utility:

Theorem 4.1. *Let solvability hold. Then, the following statements are equivalent:*

1. *The preference relation $\succcurlyeq$ satisfies axioms 1, 2, 3, 4 and 5κ (weak order, monotonicity, Archimedeanity, comparative likelihood consistency, and the piecewise-sure-thing principle).*

2. *There exists a convex-valued, piecewise-additive capacity ν_κ over $\mathscr{E}$ and a real-valued, strictly $\succcurlyeq$-increasing utility function u over outcomes $\mathscr{X}$ such that $\succcurlyeq$ is represented by $E(\nu_\kappa, u)$. That is, PACE utility holds.*

In statement 2, the capacity is unique and utility is cardinal.

5 Closing Comments

This paper has presented a simple way of integrating optimism and pessimism into subjective expected utility. To get PACE utility, 'kinks' were incorporated into expected utility. In rough terms, approximating 'inverse-S' with a 'Z'. An intuitive weakening of the sure-thing principle called the piecewise-sure-thing principle was introduced. Theorem 4.1 provided a behavioural foundation for PACE utility.

Chateauneuf et al. (2007) have shown how NEO-expected utility can be applied to resolve well-known phenomena that are difficult to reconcile with expected utility, such as the coexistence of gambling and insurance and the equity premium puzzle. Since, for finite state spaces, NEO-expected utility is obtained as a special case, PACE utility can generate the same results. In applications with rich state spaces, PACE utility is a more tractable alternative.

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A Appendix

A.1 General Results on Additive Separability

The proof of Theorem 4.1 uses the general results of Wakker (1991) so the main result of that paper is repeated here. We restrict attention to the case of three or more coordinates here, although Wakker (1991) allows for two coordinates.

Let $\mathscr{Y}$ be a non-empty set of *outcomes*, and $\succcurlyeq'$ a weak order on $\mathscr{Y}$. Any n-tuple $\mathbf{y} = (y_1, \ldots, y_n) \in \mathscr{Y}^n$ is called a *rank-ordered alternative* if $y_1 \succcurlyeq' \cdots \succcurlyeq' y_n$. The set of rank-ordered alternatives is $\mathscr{Y}^n_{\succcurlyeq'}$. An outcome y is *minimal* if $y \succ' x$ for no $x \in \mathscr{Y}$ and is *maximal* if $x \succ' y$ for no $x \in \mathscr{Y}$. An *extreme alternative* $\mathbf{y}$ has its first (so every) coordinate a minimal outcome, or its last (so every) coordinate a maximal outcome.

A weak order $\succcurlyeq$ is defined on $Y \subseteq \mathscr{Y}^n_{\succcurlyeq'}$, a subset of the rank-ordered alternatives. A constant alternative $(\alpha,\dots,\alpha)$ is identified with the outcome $\alpha \in \mathscr{Y}$. It is therefore ensured that $\succcurlyeq$ and $\succcurlyeq'$ are *in agreement* in the following sense: $(\alpha,\dots,\alpha) \succ (\beta,\dots,\beta) \Rightarrow \alpha \succ' \beta$, and $(\alpha,\dots,\alpha) \sim (\beta,\dots,\beta) \Rightarrow \alpha \sim' \beta$.

We write $\alpha_i\mathbf{y}$ for the alternative $\mathbf{y} \in Y$ with coordinate y_i replaced by α. Replacing more than one coordinate of $\mathbf{y}$ the notation $\alpha_i\beta_j\mathbf{y}$ is clear. Replacing a set of coordinates $I \subseteq \{1,\dots,n\}$ write $\alpha_I\mathbf{y}$. *Solvability* holds if, for all alternatives $\alpha_i\mathbf{z}, \mathbf{y}, \gamma_i\mathbf{z} \in \mathscr{Y}$ with $\alpha_i\mathbf{z} \succ \mathbf{y} \succ \gamma_i\mathbf{z}$ there is an outcome β such that $\beta_i\mathbf{z} \sim \mathbf{y}$. An order is *monotonic* if the following equivalence holds: $\alpha \succcurlyeq \beta \Leftrightarrow \alpha_i\mathbf{y} \succcurlyeq \beta_i\mathbf{y}$ for all $\alpha_i\mathbf{y}, \beta_i\mathbf{y} \in Y$. The order is *coordinate independent* if the following holds: $\alpha_i\mathbf{y} \succcurlyeq \alpha_i\mathbf{z} \Leftrightarrow \beta_i\mathbf{y} \succcurlyeq \beta_i\mathbf{z}$ when all the alternatives are in Y.

Call a sequence $\alpha^1, \alpha^2, \dots$ a *standard sequence* if, for some $j \in \{1,\dots,n\}$, $\alpha^1_j\mathbf{y} \succ \alpha^1_j\mathbf{z}$ and $\alpha^k_j\mathbf{y} \sim \alpha^{k+1}_j\mathbf{z}$ for $k = 1, 2, \dots$. Standard sequences could be finite or infinite. A standard sequence $\alpha^1, \alpha^2, \dots$ is *bounded* if for all k there exist outcomes β, γ such that $(\beta,\dots,\beta) \succcurlyeq (\alpha^k,\dots,\alpha^k) \succcurlyeq (\gamma,\dots,\gamma)$ and there are alternatives $\mathbf{y}, \mathbf{z} \in Y$ with $y_i = \beta$ and $z_i = \gamma$. An order is *Archimedean* if every bounded standard sequence is finite.

The extended reals is $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. A preference relation $\succcurlyeq$ on Y has an *extended additive representation* if: there are functions $V_1,\dots,V_n$ so that $\sum_{j=1}^n V_j : Y \to \overline{\mathbb{R}}$ represents $\succcurlyeq$ on Y. To characterise those orders for which an extended additive representation exists, Wakker (1991) offered the following theorem:

Theorem A.1.1 (Wakker, 1991)**.** *Let $\mathscr{Y}$ be a non-empty set, and $\succcurlyeq$ a preference relation on a set $Y \subseteq \mathscr{Y}^n_r$ ($n \geqslant 3$) of rank-ordered alternatives, where rank-ordering is with respect to the weak order $\succcurlyeq'$ on $\mathscr{Y}$ agreeing with the binary relation $\succcurlyeq$ on Y restricted to constant alternatives. Let $\succcurlyeq$ satisfy solvability on Y. The following two statements are equivalent:*

1. *The preference relation $\succcurlyeq$ on Y is an Archimedean, monotonic, weak order satisfying coordinate independence.*

2. *The preference relation $\succcurlyeq$ on Y is represented by an extended additive function that is real-valued for all non-extreme alternatives.*

$$\mathbf{y} \succcurlyeq \mathbf{z} \quad \Leftrightarrow \quad \sum_{i=1}^{n} V_i(y_i) \geqslant \sum_{i=1}^{n} V_i(z_i)$$

Further, the additive representation is an interval scale on $Y \setminus \{extremes\}$ unless $\succcurlyeq$ on Y has exactly two equivalence classes and $n > 3$.

A.2 Proof of Theorem 4.1

If preferences are represented by PACE utility, with a convex-valued capacity, one can verify weak order, the piecewise-sure-thing principle, monotonicity, comparative likelihood consistency and Archimedeanity by substitution of the preference functional. Hence, we assume statement 1 of the theorem and derive statement 2. For finite $X \subseteq \mathscr{X}$, the set of acts with outcomes only in X is written $\mathscr{A}_X$. Enumerate the outcomes in X to be increasing in terms of preference: $X = \{x_0, \ldots, x_n\}$ with $x_0 \prec \cdots \prec x_n$. For $f \in \mathscr{A}_X$ the *decumulative representation*, written F as follows:

$$F = (F_1, \ldots, F_n) \quad \text{ where } F_i = \bigcup_{x_j \succcurlyeq x_i} f^{-1}(x_j)$$

so each F_i is the union of all events for which the act f yields x_i or any other preferred outcome. Clearly, $F_n = f^{-1}(x_n)$ for all $f \in \mathscr{A}$ as no outcome in X is preferred to x_n. Also, one would find $F_0 = \mathscr{S}$ for all acts. Therefore, we drop the F_0 from the notation and write $F = (F_1, \ldots, F_n)$. No relevant information is lost when transforming an act to its decumulative representation, that is, $F = G$ iff $f = g$. The set of all decumulative acts is $\mathscr{A}^*$. The preference relation $\succcurlyeq$ over $\mathscr{A}$ naturally induces a preference relation over $\mathscr{A}^*$, that we also write $\succcurlyeq$. As we proceed we will refer to elements of $\mathscr{A}^*_X$ as F, F', G, G' and so on without referring to the underlying acts f, f', g, g' unless necessary.

Comparative likelihood consistency, axiom 4, affects the decision maker's preferences for acts with, at most, three possible outcomes. For X with $|X| = 3$, preferences over $\mathscr{A}_X$ satisfy the axioms of Theorem 4.1.4 (and Corollary 4.1.4) of Gilboa (1987). Hence, preferences admit a Choquet expected utility representation on $\mathscr{A}_X$, with utility u_X and convex-valued capacity ν_X. The utility is cardinal and the capacity is unique.

Let $\mathscr{L}$ denote the set of events A for which $\succcurlyeq^L$ satisfies the lower sure-thing principle at A, let $\mathscr{U}$ denote the set of events A for which $\succcurlyeq^L$ satisfies the upper sure-thing principle at A, and let $\mathscr{M}$ denote the set of events A for which $\succcurlyeq^L$ satisfies the inner sure-thing principle at A. By the piecewise-sure-thing principle, every event belongs to at least one of $\mathscr{U}, \mathscr{M}$ or $\mathscr{L}$.

An order satisfies $\mathscr{Z}$*-cancellation* if the following holds:

$$[A, B, A\cup C, B\cup C \in \mathscr{Z}, A\cap C = B\cap C = \emptyset, A_iF \succcurlyeq B_iG] \Rightarrow (A\cup C)_iF \succcurlyeq (B\cup C)_iG$$

An order satisfies $\mathscr{K}$-cancellation if it satisfies $\mathscr{Z}$-cancellation for $\mathscr{Z} \in \{\mathscr{U}, \mathscr{M}, \mathscr{L}\}$.

Lemma A.2.1. *Preferences $\succcurlyeq$ over $\mathscr{A}^*$ satisfy $\mathscr{K}$-cancellation.*

Proof. The details for $\mathscr{U}$-cancellation are given, the other conditions follow similar reasoning. Let $i \geqslant 1$. Let h and $\tilde{h}$ be the acts associated with decumulative acts A_iF and B_iG. Define $E := h^{-1}(x_{i-1})$ and $\tilde{E} := \tilde{h}^{-1}(x_{i-1})$. Suppose that, for $C \in \mathscr{E}$ with $A\cap C = B\cap C = \emptyset$, the decumulative acts $(A\cup C)_iF$ and $(B\cup C)_iG$ are well defined. Then it must be[8] that $E \supseteq C$ and $\tilde{E} \supseteq C$. So, both acts coincide on the event C (with outcome x_{i-1}). By assumption, $\mathcal{R}(h^{-1}(x_i), h) \succcurlyeq^L K$ and $\mathcal{R}(\tilde{h}^{-1}(x_i), \tilde{h}) \succcurlyeq^L K$, so $\mathcal{R}(E, h) = F_{i-1} \succcurlyeq^L K$ and $\mathcal{R}(\tilde{E}, \tilde{h}) = G_{i-1} \succcurlyeq^L K$. By the upper sure-thing

[8] Because such acts are obtained from h and $\tilde{h}$ by replacing x_{i-1} with x_i on C.

principle at K, we may replace x_{i-1} with x_i on an event $C \subset E$, without reversing the expressed preference:

$$x_{iC}x_{i-1E\setminus C}f \succcurlyeq x_{iC}x_{i-1E\setminus C}g.$$

The map that transforms these acts to their decumulative representation is as follows: $x_{iC}x_{i-1E\setminus C}f \mapsto (F_1, \ldots, F_{i-1}, A \cup C, F_{i+1}, \ldots F_n)$ and $x_{iC}x_{i-1E\setminus C}g \mapsto (G_1, \ldots, G_{i-1}, B \cup C, G_{i+1}, \ldots, G_n)$, as required. $\square$

A capacity ν is *$\mathscr{Z}$-modular* if the following holds:

$$[A, B, A \cup C, B \cup C \in \mathscr{Z}, A \cap C = B \cap C = \emptyset] \Rightarrow \nu(A \cup C) - \nu(A) = \nu(B \cup C) - \nu(B).$$

The capacity is $\mathscr{K}$-modular if it is $\mathscr{Z}$-modular for $\mathscr{Z} \in \{\mathscr{U}, \mathscr{M}, \mathscr{L}\}$. The following lemma establishes the $\mathscr{K}$-modularity of the ν_X obtained above.

Lemma A.2.2. *The capacity ν_X is $\mathscr{K}$-modular.*

Proof. Let $X = \{x_0, x_1, x_2\}$ be enumerated in increasing order of preference: $x_0 \prec x_1 \prec x_2$. We prove that ν_X is $\mathscr{U}$-modular. The remaining conditions are entirely similar. In the decumulative framework, an act f translates to (A, E), where $E = f^{-1}(x_2)$ and $A = E \cup f^{-1}(x_1)$. Take any A, B, C such that $A \cap C = B \cap C = \emptyset$ and $A, B, A \cup C, B \cup C \in \mathscr{U}$. Let $\nu_X(A) \geqslant \nu_X(B)$. We require: $\nu_X(A \cup C) - \nu_X(A) = \nu_X(B \cup C) - \nu_X(B)$. Without loss of generality, we fix $u_X(x_0) = 0$. Define $d := [\nu_X(A) - \nu_X(B)]u_X(x_1) > 0$ and $e := \nu_X(B)[u_X(x_2) - u_X(x_1)]$. We separate the proof of step 1 into two cases.

Case 1: $d \leqslant e$. We know $\nu_X(A)u_X(x_1) \geqslant \nu_X(B)u_X(x_1)$. This holds if and only if, $(A, \emptyset) \succcurlyeq (B, \emptyset)$. Also, $d \leqslant e$ if and only if $\nu_X(A)u_X(x_1) \leqslant \nu_X(B)u_X(x_1) + \nu_X(B)[u_X(x_2) - u_X(x_1)]$, equivalent to $(A, \emptyset) \preccurlyeq (B, B)$. Appealing to convex-valuedness of the ν_X functions, there exists an E (which could be $\emptyset$) such that $(A, \emptyset) \sim (B, E)$. Furthermore, by $\mathscr{K}$-cancellation (Lemma A.2.1) applied twice, the above indifference holds only if $(A \cup C, \emptyset) \sim (B \cup C, E)$. Substituting the additive representation for these two indifferences yields:

$$\nu_X(A)u_X(x_1) = \nu_X(B)u_X(x_1) + \nu_X(E)[u_X(x_2) - u_X(x_1)],$$

and

$$\nu_X(A \cup C)u_X(x_1) = \nu_X(B \cup C)u_X(x_1) + \nu_X(E)[u_X(x_2) - u_X(x_1)],$$

which jointly imply: $\nu_X(A) - \nu_X(B) = \nu_X(A \cup C) - \nu_X(B \cup C)$. Hence, ν_X is $\mathscr{U}$-modular.

Case 2: $d > e$. This is the more difficult case, in that the proof is indirect. We reduce the proof to case 1 holding for some other, appropriately constructed, $\tilde{A}, \tilde{B}$.

We now have, $d > e$, or: $\nu_X(A)u_X(x_1) > \nu_X(B)u_X(x_1) + \nu_X(B)[u_X(x_2) - u_X(x_1)]$. Equivalently, $(A, \emptyset) \succ (B, B) \succcurlyeq (B, \emptyset)$, the second preference being implied by monotonicity. By convex-valuedness, there exists $Z \in \mathscr{E}$ with $A \succcurlyeq^L Z \succcurlyeq^L B$ such that $(Z, \emptyset) \sim (B, B)$. Let Y be defined so that $A \setminus Y = Z$. So, $(A \setminus Y, \emptyset) \sim (B, B)$. One can verify that $B \cup Y$ and $B \cup Y \cup C$ are in $\mathscr{U}$. By $\mathscr{K}$-cancellation, $(A \setminus Y \cup C, \emptyset) \sim (B \cup C, B)$. Again, by $\mathscr{K}$-cancellation, $(A, \emptyset) \sim (B \cup Y, B)$. And, again, by $\mathscr{K}$-cancellation, $(A \cup C, \emptyset) \sim (B \cup Y \cup C, B)$. Substituting the Choquet expected utility representation for each of these four indifferences, subtracting the obtained equations from each other and cancelling the common utility terms, the following equations can be established:

$$\nu_X(A) - \nu_X(B) = \nu_X(B \cup Y) - \nu_X(A \setminus Y),$$

and,

$$\nu_X(A \cup C) - \nu_X(B \cup C) = \nu_X(B \cup Y \cup C) - \nu_X(A \setminus Y \cup C).$$

Hence, given the last two equations, we can establish the $\mathscr{U}$-modularity property for the chosen A, B, C only if the $\mathscr{U}$-modularity property can be established using $\tilde{A}, \tilde{B}, C$ with $\tilde{A} = A \setminus Y$ and $\tilde{B} = B \cup Y$. Define $\tilde{d}$ and $\tilde{e}$ in the same manner as

d and e, but using $\tilde{A}$ and $\tilde{B}$ where A and B were used. Then, it is immediate that $\tilde{d} = e$ and $\tilde{e} > e$, hence case 1 holds ($\tilde{d} \leqslant \tilde{e}$) and $\mathscr{U}$-modularity is proved. □

The following lemma contains an implication of the piecewise-sure-thing principle. This lemma is crucial in deriving the "kinks" endogenously. By the piecewise-sure-thing principle, every event belongs to at least one of $\mathscr{U}, \mathscr{M}$ or $\mathscr{L}$. It is apparent from the definitions that $A \in \mathscr{U}$ and $B \succcurlyeq^L A$ only if $B \in \mathscr{U}$. Also, by the piecewise-sure-thing principle, $A^{\complement} \in \mathscr{L}$ and, then, $B^{\complement} \in \mathscr{L}$ and also $C \preccurlyeq^L A$ only if $C \in \mathscr{L}$. Suppose there exist $A \in \mathscr{L}$ and $B \in \mathscr{U}$ with $A \succ^L B$ ($\mathscr{U}$ and $\mathscr{L}$ "overlap"). Then, ν_X is modular within $\mathscr{U}$ and $\mathscr{L}$, hence also in their intersection, so ν_X must be modular globally. Then ν_X is additive, hence the following holds:

Lemma A.2.3. *There exist $A \in \mathscr{L}$ and $B \in \mathscr{U}$ with $A \succ^L B$ ($\mathscr{U}$ and $\mathscr{L}$ "overlap") if and only if the sure-thing principle holds.*

To avoid repeatedly adding "if expected utility does not hold" qualifiers to everything that follows, we assume for the remainder of the paper that preferences are non-expected utility unless otherwise stated. Equivalently, we assume, $\mathscr{U}$, $\mathscr{M}$ and $\mathscr{L}$ do not overlap. Let K be the $\succcurlyeq^L$-infimum of $\mathscr{U}$. It must be that $K \succcurlyeq^L K^{\complement}$, or else $\mathscr{U}$ and $\mathscr{L}$ would overlap. K is the $\succcurlyeq^L$-supremum of $\mathscr{M}$, $K^{\complement}$ is the $\succcurlyeq^L$-supremum of $\mathscr{L}$, and $K^{\complement}$ is the $\succcurlyeq^L$-infimum of $\mathscr{M}$. Clearly, the inner and outer sure-thing principles at K hold. The same holds for $J \in \mathscr{E}$ if and only if $J \sim^L K$.

We now let X be finite and have at least four outcomes. The following lemma establishes coordinate independence for decumulative acts:

Lemma A.2.4. *Preferences $\succcurlyeq$ over $\mathscr{A}^*$ are coordinate independent.*

Proof. We show that, for all $F_iG, F_iH, \tilde{F}_iG, \tilde{F}_iH \in \mathscr{A}^*_X$, $F_iG \succcurlyeq F_iH$ only if $\tilde{F}_iG \succcurlyeq \tilde{F}_iH$. Let $\tilde{F}_i = F_i \cup A$ for some A with $A \cap F_i = \emptyset$, so that $\tilde{F}_i \succcurlyeq^L F_i$. Then, there are six cases to consider:

i. $\{\tilde{F}_i \succcurlyeq^L F_i \succcurlyeq^L K\}$. $F_iG \succcurlyeq F_iH$ only if $\tilde{F}_iG \succcurlyeq \tilde{F}_iH$ by $\mathscr{U}$-cancellation. *ii.* $\{\tilde{F}_i \succcurlyeq^L K \succcurlyeq^L F_i \succcurlyeq^L K^{\complement}\}$. $F_iG \succcurlyeq F_iH$ only if $K_iG \succcurlyeq K_iH$ by $\mathscr{M}$-cancellation. Then, $K_iG \succcurlyeq K_iH$ only if $\tilde{F}_iG \succcurlyeq \tilde{F}_iH$ by $\mathscr{U}$-cancellation. *iii.* $\{\tilde{F}_i \succcurlyeq^L K \succcurlyeq^L K^{\complement} \succcurlyeq^L F_i\}$. $F_iG \succcurlyeq F_iH$ only if $K_i^{\complement}G \succcurlyeq K_i^{\complement}H$ by $\mathscr{L}$-cancellation. Then, $K_i^{\complement}G \succcurlyeq K_i^{\complement}H$ only if $K_iG \succcurlyeq K_iH$ by $\mathscr{M}$-cancellation. Finally, $K_iG \succcurlyeq K_iH$ only if $\tilde{F}_iG \succcurlyeq \tilde{F}_iH$ by $\mathscr{U}$-cancellation. *iv.* $\{K \succcurlyeq^L \tilde{F}_i \succcurlyeq^L F_i \succcurlyeq^L K^{\complement}\}$. $F_iG \succcurlyeq F_iH$ only if $\tilde{F}_iG \succcurlyeq \tilde{F}_iH$ by $\mathscr{M}$-cancellation. *v.* $\{K \succcurlyeq^L \tilde{F}_i \succcurlyeq^L K^{\complement} \succcurlyeq^L F_i\}$. $F_iG \succcurlyeq F_iH$ only if $K_i^{\complement}G \succcurlyeq K_i^{\complement}H$ by $\mathscr{L}$-cancellation. Then, $K_i^{\complement}G \succcurlyeq K_i^{\complement}H$ only if $\tilde{F}_iG \succcurlyeq \tilde{F}_iH$ by $\mathscr{M}$-cancellation. *vi.* $\{K^{\complement} \succcurlyeq^L \tilde{F}_i \succcurlyeq^L F_i\}$. $F_iG \succcurlyeq F_iH$ only if $\tilde{F}_iG \succcurlyeq \tilde{F}_iH$ by $\mathscr{L}$-cancellation. □

By construction, $\mathscr{A}^*$ is the rank-ordered set $\mathscr{S}^n_{\supseteq}$ with $n \geqslant 3$. The preference relation $\succcurlyeq$ restricted to constant acts agrees with $\supseteq$. Weak ordering of $\succcurlyeq$ on $\mathscr{A}^*$ is inherited from the same property on $\mathscr{A}$. Lemmas B.1 and B.3 of Abdellaoui and Wakker (2005: 44-45) ensure that monotonicity and solvability in the decumulative framework hold. Lemma A.2.4 established the coordinate independence of $\succcurlyeq$ on $\mathscr{A}^*$. Our Archimedean axiom (A5) translates to the following: bounded standard sequences on coordinate i, that is $A_i^1F \succ A_i^1G$ and $A_i^kF \sim A_i^{k+1}G$ for $k = 1, 2, \ldots$, are finite for $i = 1, \ldots, n$. That is, preferences over $\mathscr{A}^*$ are Archimedean in the sense of Wakker (1991). We now invoke Theorem A.1.1, so there exist functions $V_1, \ldots, V_n$ such that $\sum_{j=1}^n V_j$ represents $\succcurlyeq$ on $\mathscr{A}^*$. Each V_j is real-valued, for $j = 2, \ldots, n-1$. It is possible that $V_1(\emptyset) = -\infty$ and/or $V_n(\mathscr{S}) = +\infty$. But, $\succcurlyeq$ has a real-valued, additive representation on $\mathscr{A}^* \setminus \{extremes\}$.

Choquet expected utility holds over $\mathscr{A}_X$ with $|X| = 3$. We now extend this to arbitrary, finite X by a routine argument appealing to uniqueness properties on overlapping outcome subsets. To see this, consider the four-outcome case, $X = \{x_0, x_1, x_2, x_3\}$. We know preferences over decumulative acts $\mathscr{A}^*_X$ are represented by $V_1 + V_2 + V_3$. Let $Y = \{x_0, x_1, x_3\}$ and $Z = \{x_0, x_2, x_3\}$. We then obtain Choquet expected utility representations with utilities u_Y, u_Z and capacities ν_Y, ν_Z. Rescale these functions so that $u_Y(x_3) = u_Z(x_3) = 1$ and $u_Y(x_0) = u_Z(x_0) = 0$. Rescale the additive representation so that $V_j(\emptyset) = 0$ for $j = 1, 2, 3$ and $\sum_{j=1}^3 V_j(\mathscr{S}) = 1$.

Since $\sum_j V_j$ represents preferences when restricted to $\mathscr{A}_Y^*$ and $\mathscr{A}_Z^*$, we have $V_j(\cdot) = \nu_Y(\cdot)[u_Y(x_j) - u_Y(x_{j-1})] = \nu_Z(\cdot)[u_Z(x_j) - u_Z(x_{j-1})]$ for $j = 1, 2, 3$. So, $\nu_Y = \nu_Z$ and then $u_Y = u_Z$. Let $\nu = \nu_Y$ and define u so that $u|_Y = u_Y$ and $u|_Z = u_Z$. Then, Choquet expected utility with utility u and capacity ν represents preferences over $\mathscr{A}_X$. In the same manner, Choquet expected utility on $\mathscr{A}_X$ can be obtained for all finite X. We now extend this to preferences $\succcurlyeq$ over $\mathscr{A}$, the set of simple acts over $\mathscr{X}$. If $\mathscr{X}$ has finitely many equivalence classes under $\sim$ then, passing to the quotient, the proof above applies. If $\mathscr{X}$ has infinitely many equivalence classes under $\sim$, the representation can be obtained as in Abdellaoui and Wakker (2005, p60-61).

The capacity ν is *ordinally additive* if the following holds:

$$[A, B, A \cup C, B \cup C \in \mathscr{E}, A \cap C = B \cap C = \emptyset, \nu(A) < \nu(B)] \Rightarrow \nu(A \cup C) < \nu(B \cup C).$$

Notice that ordinal additivity concerns events within and across the $\mathscr{K}$ sets, over all of $\mathscr{E}$. The following lemma establishes the ordinal additivity of ν.

Lemma A.2.5. *The capacity ν is ordinally additive.*

Proof. Let A, B, C satisfy the prerequisite conditions of the ordinally additive implication. If $A, B, A \cup C, B \cup C \in \mathscr{Z}$ for some $\mathscr{Z} \in \{\mathscr{U}, \mathscr{M}, \mathscr{L}\}$ then the implication follows from $\mathscr{K}$-cancellation established in Lemma A.2.1. There are various cases to consider, although the proofs are sufficiently similar to fully explain the proof when $A \in \mathscr{M}$ and $A \cup C, B, B \cup C \in \mathscr{U}$. Then, $\nu(A) \leqslant \nu(K) \leqslant \nu(B)$, with at least one inequality strict, so we take $\nu(A) < \nu(K)$. There exists, by convex-valuedness, E such that $\nu(A \cup (C \setminus E)) = \nu(K)$. Then, by $\mathscr{U}$-modularity and the capacity's monotonicity, $\nu(A \cup (C \setminus E) \cup E) = \nu(A \cup C) \leqslant \nu(B \cup E) \leqslant \nu(B \cup E \cup (C \setminus E)) = \nu(B \cup C)$, as required. $\square$

Lemma A.2.1 can be amended now, $\succcurlyeq^L$ satisfies *cancellation*, so it is a *qualitative probability* order (Savage, 1954: 32). We now show that $\succcurlyeq^L$ is a *quantitative probability* order. That is, for all $A, B \in \mathscr{E}$, with $A \succ^L B$, there exists a finite partition

$\{C_1, \ldots, C_n\}$ of $\mathscr{S}$ such that $A \succ^L B \cup C_i$ for all $i = 1, \ldots, n$. Savage (1954) showed that any quantitative probability order may be represented by a convex-valued, finitely additive probability measure, with such a probability uniquely determined (see also Krantz, Luce, Suppes and Tversky, 1971: 202-208). In showing that $\succcurlyeq^L$ is a quantitative probability, we prove:[9]

Lemma A.2.6. *There exists a convex-valued and finitely-additive probability measure p on $\mathscr{E}$ and a real-valued, strictly increasing function $\phi : [0,1] \to [0,1]$ such that $\nu = \phi \circ p$. Such ϕ and p are unique.*

Proof. Let $A \succ^L B$. We show there exists a finite partition $\{C_1, \ldots, C_n\}$ of $\mathscr{S}$ such that $A \succ^L B \cup C_i$ for all $i = 1, \ldots, n$. The order $\succcurlyeq^L$ is *tight* if $A \cup C \succcurlyeq^L B$ for all disjoint $C \succ^L \emptyset$ and $B \cup D \succcurlyeq^L A$ for all disjoint $D \succ^L \emptyset$ jointly imply $A \sim^L B$. The order is *fine* if, for all $A \succ^L \emptyset$, there exists a finite partition $\{C_1, \ldots, C_n\}$ of $\mathscr{S}$ such that $A \succ^L C_i$ for all $i = 1, \ldots, n$. An equivalent formulation, due to Theorem 4 of Savage (1954: 38), of the finite partition condition is that $\succcurlyeq^L$ is fine and tight. If $A \succ^L B$, then the existence of disjoint C such that $A \succ B \cup C$ can be established using convex-valuedness of ν. Then, $A \cup C \succcurlyeq^L B$ for all disjoint $C \succ^L \emptyset$ with $A \succ^L B$ is excluded and $\succcurlyeq^L$ is tight.

Lemma C.3 of Abdellaoui and Wakker (2005) guarantees, for all $\epsilon > 0$, the existence of a partition $\{C_1, \ldots, C_n\}$ of $\mathscr{S}$ such that $\nu(\bigcup_{j=i}^n C_j) - \nu(\bigcup_{j=i-1}^n C_j) < \epsilon$ for all $i = 1, \ldots, n$.[10] To show $\succcurlyeq^L$ is fine, let $A \succ^L \emptyset$, choose $\epsilon < \min\{\nu(A), \frac{\nu(K^\complement)}{2}\}$ and take such a partition. Clearly $\nu(C_n) < \epsilon$. Given $\nu(C_{n-1} \cup C_n) - \nu(C_n) < \epsilon$, exists $E \in \mathscr{E}$ disjoint from C_n such that $\nu(C_n \cup E) = \nu(C_n) + \epsilon < 2\epsilon < \nu(K^\complement)$. By choosing ϵ as stated, $C_n \cup E \in \mathscr{L}$, which includes $\emptyset$, and by $\mathscr{L}$-modularity (equivalent to additivity), $\nu(E) = \epsilon$. By ordinal additivity, $\nu(C_{n-1} \cup C_n) < \nu(C_n \cup E)$ holds only if $\nu(C_{n-1}) < \nu(E) = \epsilon$ as required. Continuing in this way, $\nu(C_i) < \epsilon$ holds for all $i = 1, \ldots, n$, and $\succcurlyeq^L$ is fine.

[9] The same idea is used in Gilboa (1985) under more general conditions.

[10] Kopylov (2007) called probability measures with this property *finely ranged*.

The capacity induces an order which satisfies the Savage axioms for a probability representation, hence the capacity is a strictly increasing transformation of such a probability measure. Uniqueness of p is given by Savage's result. Uniqueness of ϕ follows from that of ν. $\square$

Lemma A.2.7. *The capacity ν is piecewise-additive.*

Proof. According to lemma A.2.6, the capacity ν and the probability measure p obtained above both represent $\succcurlyeq^l$, the likelihood order over $\mathscr{E}$. By additivity, $p(A \cup B) - p(B) = p(A)$ for all A, B with $A \cap B = \emptyset$, hence p is $\mathscr{K}$-modular.

Let $\mathscr{Z} \in \{\mathscr{U}, \mathscr{M}, \mathscr{L}\}$ and take $A, B, C, D, A \cup D, B \cup C \in \mathscr{Z}$ with $A \cap D = B \cap C = \emptyset$. We show $p(A) - p(B) = p(C) - p(D)$ if and only if $\nu(A) - \nu(B) = \nu(C) - \nu(D)$. Assume the first equation holds and, without loss of generality, that $p(B) < p(A) < p(C)$. Then, there exists E such that $p(A \cup E) = p(C)$, so $p(D) = (B \cup E)$ by $\mathscr{K}$-modularity, so $\nu(D) = \nu(B \cup E)$. There also exists $\tilde{E}$ such that $p(A) = p(B \cup \tilde{E})$ and $B \cap \tilde{E} = E \cap \tilde{E} = \emptyset$. By $\mathscr{K}$-modularity of ν: $\nu(B \cup \tilde{E}) - \nu(B) = \nu(B \cup \tilde{E} \cup E) - \nu(B \cup E)$, which is equivalent to $\nu(A) - \nu(B) = \nu(C) - \nu(D)$. As solvability holds, the $\mathscr{Z}$ sets are sufficiently rich to ensure that ν and p are affinely related. It is readily verified, for example, that the conditions of Theorem 4.2 of Krantz, Luce, Suppes and Tversky (1971) hold.[11] Then, there are $a_0, a_1, a_3, b_1, b_2 \in \mathbb{R}$ such that:

$$\phi(A) = \begin{cases} a_0 p(A) & \text{if } A \in \mathscr{L} \\ a_1 p(A) + b_1 & \text{if } A \in \mathscr{M} \\ a_2 p(A) + b_2 & \text{if } A \in \mathscr{U} \end{cases}$$

with $a_1, a_2, a_3 > 0$, and $b_1, b_2 \in \mathbb{R}$ such that monotonicity and normalisation hold. We have already shown the following:

$$\mathscr{U} := \{A \in \mathscr{E} : A \succcurlyeq^L K\}, \;\; \mathscr{M} := \{A \in \mathscr{E} : K \succcurlyeq^L A \succcurlyeq^L K^{\complement}\}, \;\; \mathscr{L} := \{A \in \mathscr{E} : K^{\complement} \succcurlyeq^L A\}.$$

[11]To see this, define an order $\succcurlyeq^*$ over $\mathscr{Z} \times \mathscr{Z}$ such that $(A, B) \succcurlyeq^* (C, D)$ if and only if $p(A) - p(B) \geqslant p(C) - p(D)$. Conditions 1-5 of definition 4.1 of KLST are routinely confirmed.

Define $\kappa := p(K)$. Because $\mathscr{L}$ and $\mathscr{M}$ share a common element $K^{\complement}$, for ν to be well-defined we must have $a_0(1-\kappa) = a_1(1-\kappa) + b_1$. Similarly, because $\mathscr{U}$ and $\mathscr{M}$ share a common element K, for ν to be well-defined we must have $a_1\kappa + b_1 = a_2\kappa + b_2$. Now uniquely define γ and δ such that $\gamma = 1 + a_1 + b_1$ and $\delta = b_1$, and we are done.

$\square$

It is now apparent that preferences $\succcurlyeq$ over $\mathscr{A}$ are represented by Choquet expected utility with capacity ν, with piecewise-additive. That is, PACE utility holds on $\mathscr{A}$. $\blacksquare$

A.3 Further Comments on Theorem 4.1

Theorem 4.1 characterises a special case of Choquet expected utility, with a convex-valued (solvable) capacity. Applying Theorem 6 of Wakker (1996) gives the following observation:

Observation A.3.1. *If, in statement 1 of Theorem 4.1, the piecewise-sure-thing principle (axiom 5κ) is replaced with the sure-thing principle (axiom 5), then the capacity obtained in statement 2 is additive (subjective expected utility holds).*

Consider the following weakening of the sure-thing principle:

Axiom 6 (The Rank-Dependent Sure-Thing Principle): For all outcomes $x, y \in \mathscr{X}$ and acts g, $h \in \mathscr{A}$, the implication:

$$x_A g \succcurlyeq x_A h \quad \Rightarrow \quad y_A g \succcurlyeq y_A h$$

holds whenever $\mathcal{R}(A, x_A g) = \mathcal{R}(A, x_A h) = \mathcal{R}(A, y_A g) = \mathcal{R}(A, y_A h)$.

The rank-dependent sure-thing principle asserts that the sure-thing principle holds whenever the common outcomes have common ranks and are changed in a way that preserves the ranks. By a direct translation of the above definition, it can be shown that the rank-dependent sure-thing principle for preferences over acts is equivalent to the coordinate independence condition for preferences over acts in their decumulative representation. Therefore, Lemma A.2.4 in Appendix A.2, has shown that the rank-dependent sure-thing principle can be derived from axioms 1-4 and 5κ. Following that Lemma, as part of the proof of Theorem 4.1, a Choquet expected utility representation was proved for preferences satisfying axioms 1-4 over acts, and satisfying coordinate independence for acts in their decumulative representation. Hence the following observation is proved:

Observation A.3.2. *If, in statement 1 of Theorem 4.1, the piecewise-sure-thing principle (axiom 5κ) is replaced with the rank-dependent sure-thing principle (axiom 6), then Choquet expected utility with a convex-valued capacity holds in statement 2.*

In the representation obtained in Theorem 4.1, the capacity is a probability transformation: $\nu_\kappa = \phi \circ p$ with ϕ strictly increasing and continuous. As in Savage (1954), however, the probability measure p is not, in general, countably additive.[12] Under solvability, the power set of $\mathscr{S}$ is too large a set of events to derive a convex-valued, countably additive probability measure. One must restrict $\mathscr{E}$ to be some other, sufficiently small, σ-algebra. Having done so, one may apply the *monotone continuity* axiom:

Axiom 7 (Monotone Continuity): Given acts $f, g \in \mathscr{A}$ with $f \succ g$, outcome $x \in \mathscr{X}$, and countable collection of events $E_1, E_2, \ldots$, with $E_1 \supseteq E_2 \supseteq \cdots$, and $\cap_{i=1}^{\infty} E_i = \emptyset$, there exists $j \geqslant 1$ such that $x_{E_j} f \succ g$ and $f \succ x_{E_j} g$.

Following Villegas (1964), Arrow (1970) introduced monotone continuity in order derive subjective expected utility with a countably additive probability measure. The same axiom was used by Chateauneuf, Maccheroni, Marinacci and Taillon (2005) to characterise countable additivity of all probability measures in a set of priors associated with a multiple priors representation. Machina and Schmeidler (1995:771)

[12] A probability measure p is countably additive if, for all countable collections of disjoint events $A_1, A_2, \ldots$, we have $p(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} p(A_i)$.

also noted that this axiom would lead to countable additivity of their probabilistic sophistication model. PACE utility is a special case of probabilistic sophistication, and collapses to expected utility in the two-outcome case, hence we note the following:

Observation A.3.3. *If, in statement 1 of Theorem 4.1, the set of events* $\mathscr{E}$ *is restricted to be a sufficiently small* σ*-algebra, the set of acts* $\mathscr{A}$ *is restricted to those acts measurable with respect to the restricted set of events, the axioms in statement 1 hold on this restricted set of acts, and the monotone continuity axiom is further assumed, then the capacity obtained in statement 2 is a strictly increasing and continuous transformation of a convex-valued, countably additive probability measure.*