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Zhiwei Liu
Nicholas C. Yannelis

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School of Social Sciences
The University of Manchester
Manchester M13 9PL

Implementation under ambiguity: the maximin core*

Zhiwei Liu† and Nicholas C. Yannelis‡

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Abstract

We introduce the idea of implementation under ambiguity. In particular, we study an ex ante maximin core notion of an ambiguous asymmetric information economy where agents' preferences are maximin *à la* Gilboa-Schmeidler [7]. The interest on the maximin core notion lies on the fact that it is always incentive compatible (de Castro-Yannelis [3]) and of course efficient, a result which is false with Bayesian preferences. A noncooperative notion called maximin equilibrium is introduced which provides a noncooperative foundation for the maximin core. Specifically, we show that given any arbitrary maximin core allocation, there is a direct revelation mechanism that yields the core allocation as its unique maximin equilibrium outcome. Thus, an incentive compatible and efficient outcome can be reached by means of noncooperative behavior under ambiguity.

1 Introduction

We go beyond the Bayesian (standard) asymmetric information economy. In particular, we study an *ambiguous asymmetric information economy*, i.e., an economy consisting of a finite set of states of nature, a finite set of agents, each of whom is characterized by an *information partition*, a (possibly incomplete[1]) *private prior* over the states of nature, a *random initial endowment* and a privately known *ex post utility function.* An ambiguous asymmetric information economy differs from the Bayesian one, in that we

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†Department of Economics, the University of Illinois, Urbana-Champaign, USA. liu108@illinois.edu

‡Department of Economics, the University of Iowa, Iowa City, USA; the University of Manchester, Manchester, UK; and the University of Illinois, Urbana-Champaign, USA. nicholasyannelis@gmail.com

[1]It will be made clear in the definition of an ambiguous asymmetric information economy, that an agent's private prior is defined on the sigma-algebra generated by his information partition. Therefore, had he observed a non singleton event in his information partition, he does not know the probability of each of the states within the event, that is, he faces ambiguity. As for where the ambiguity comes from, we refer interested readers to Ellsberg [6] and Riedel and Sass [16].

do not require agents to be able to form a prior probability on every state of nature, nor do we require the agents' priors and their ex post utility functions to be common knowledge. To accommodate the agents' information constraints, we postulate that they evaluate (random) *allocations*, hereafter allocations, with the help of the *maximin expected utility* (see Gilboa and Schmeidler [7]).

For such an ambiguous asymmetric information economy, we adopt an ex ante version of the *maximin core*[2] first defined in de Castro-Yannelis [3]. The interest on the maximin core arises from the fact that with maximin preferences, any Pareto optimal allocation is incentive compatible (de Castro-Yannelis [3]), in other words, maximin preferences solve the conflict between incentive compatibility and efficiency (recall that in the standard expected utility/ Bayesian framework, an efficient allocation may not be incentive compatible as it was shown in Holmström-Myerson [10]). We will show by means of an example that the maximin core notion is different than the rational expectations equilibrium (Radner [13]) or Walrasian expectations equilibrium (Radner [12], [14]) or private core (Yannelis [18]). In particular, we show that the maximin core not only exists in a situation that the Walrasian expectations equilibrium fails to exist, but also achieves higher efficiency than all the above concepts.

It turns out that the maximin core, a cooperative solution concept, exists under the standard continuity and concavity assumptions (de Castro-Yannelis [3]), also it is incentive compatible (de Castro-Yannelis [3]) and obviously efficient. But, could one provide a noncooperative foundation for the maximin core in terms of some game theoretic solution concept? In other words, can the maximin core allocations be reached by means of noncooperative behavior? What would be the appropriate game theoretic solution concept?

In view of the ambiguous asymmetric information economy, one should not expect to employ any Bayesian Nash type equilibrium notion. Indeed, we will show by means of an example that the Bayesian Nash equilibrium notion fails. To this end, we introduce the idea of a *maximin equilibrium.* Roughly speaking, in a maximin equilibrium, each agent maximizes his payoff lowest bound, that is, each agent simply maximizes the payoff that takes into account the worst actions of all the other agents against him and also the worst state that can occur.

The main result of the paper is that given any arbitrary maximin core allocation, there is a *direct revelation mechanism* that yields the core allocation as its unique maximin equilibrium outcome, i.e., each maximin core allocation is implementable as a maximin equilibrium. Therefore efficient and incentive compatible outcomes can be reached by means of noncooperative behavior under ambiguity.

The paper is organized as follows. Section 2 defines an ambiguous asymmetric information economy. Section 3 discusses the cooperative concept of the paper – the maximin core. In Section 4, we introduce the direct revelation mechanisms, the maximin equilibrium, and present the main result of the paper. Finally, we conclude in Section 5. Appendix contains proofs.

[2]Loosely speaking, the maximin core is the set of feasible allocations, that cannot be "improved upon" by any coalition of agents. The "improved upon" idea is now based on the maximin preferences.

2 Ambiguous asymmetric information economy

An ambiguous asymmetric information economy is an asymmetric information economy, in which agents have incomplete private priors and privately known utility functions.

Formally, let Ω denote a finite set of states of nature, $\omega \in \Omega$ a state of nature, $\mathbb{R}_+^l$ the l good commodity space, and I the set of N agents, i.e., $I = \{1, \cdots, N\}$. *An ambiguous asymmetric information economy* $\mathcal{E}$ is a set

$$\mathcal{E} = \{\Omega; (\mathcal{F}_i, \mu_i, e_i, u_i) : i \in I\}$$

where for each $i \in I$,

1. the partition $\mathcal{F}_i$ of Ω denotes the set of all possible interim information of agent i. More precisely, let $E_i^{\mathcal{F}_i} \in \mathcal{F}_i$ denote an event, and $\omega \in E_i^{\mathcal{F}_i}$ a state in the event. Then, in the interim, if the state ω occurs, agent i only knows that the event $E_i^{\mathcal{F}_i}$ has occurred. Also, we impose the standard assumption, that when a state occurs, and all agents truthfully report their information, they will know the realized state[3]. That is,

   Assumption 1. *For each ω, $\bigcap_{j \in I} E_j^{\mathcal{F}_j}(\omega) = \{\omega\}$, where $E_j^{\mathcal{F}_j}(\omega)$ denotes the element in $\mathcal{F}_j$ that contains the state ω.*

2. $\mu_i : \sigma(\mathcal{F}_i) \to [0, 1]$ is agent i's private prior, where $\sigma(\mathcal{F}_i)$ denotes the σ-algebra generated by the partition $\mathcal{F}_i$. Note, if $E_i^{\mathcal{F}_i} = \left\{\omega, \omega'\right\}$ with $\omega \neq \omega'$, then the probability of the event $E_i^{\mathcal{F}_i}$ is well defined, but not the probability of the event $\{\omega\}$ or the event $\left\{\omega'\right\}$, i.e., an agent's prior maybe incomplete. Here, we assume

   Assumption 2. *For each i and for each event $E_i^{\mathcal{F}_i} \in \mathcal{F}_i$, $\mu_i\left(E_i^{\mathcal{F}_i}\right) > 0$.*

3. $e_i : \Omega \to \mathbb{R}_+^l$ is agent i's random initial endowment. We require e_i to be $\mathcal{F}_i$-measurable[4]. So that the information partition $\mathcal{F}_i$ indeed contains all possible interim information of agent i.

4. $u_i : \mathbb{R}_+^l \times \Omega \to \mathbb{R}$ is agent i's ex post utility function, taking the form of $u_i(c_i; \omega)$, where c_i denotes agent i's consumption. The ex post utility function u_i is strictly monotone in consumption[5], and only agent i knows the form of u_i.

[3]This assumption is without loss of generality, since if there exist two different states ω and ω', such that no agent is able to distinguish them, then the two states may as well be treated as one state.

[4]That is, e_i is constant on each element in $\mathcal{F}_i$. More precisely, let agent i's partition be $\mathcal{F}_i$ and fix any $\omega_k \in \Omega$. If $e_i : \Omega \to \mathbb{R}_+^l$ is $\mathcal{F}_i$ - measurable, then $e_i(\omega) = e_i(\omega_k)$ for any $\omega \in E_i^{\mathcal{F}_i}(\omega_k)$. Clearly, if each e_i is state independent, then it is automatically $\mathcal{F}_i$-measurable.

[5]For each fixed ω, we have $u_i(c_i; \omega) < u_i(c_i + \epsilon; \omega)$, whenever ϵ is a none zero vector in $\mathbb{R}_+^l$.

Let $x_i : \Omega \to \mathbb{R}^l_+$ denote *agent i's allocation* (or in short, *i-allocation*). Denote by L_i the set of all possible allocations of agent i, and by $x = (x_1, \cdots, x_N)$ *an allocation* of the above economy $\mathcal{E}$. An allocation x is said to be *feasible*, if for each $\omega \in \Omega$, $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$.

To accommodate the agents' information constraints, we postulate that the agents have the maximin preferences axiomatized by Gilboa and Schmeidler [7]. Unlike the standard ex ante expected utility, the maximin ex ante expected utility of each allocation $x_i \in L_i$ is well defined.

Let Δ_i be the set of all probability measures over 2^Ω, that agree with the agent i's prior μ_i, formally,

$$\Delta_i = \left\{ \text{probability measure } \ \pi_i : 2^\Omega \to [0,1] \mid \pi_i(A) = \mu_i(A), \forall A \in \sigma(\mathcal{F}_i) \right\}.$$

Take any two allocations of agent i, f_i and h_i, from the set L_i. Agent i prefers f_i to h_i, $f_i \succeq_i^{MP} h_i$, if

$$\min_{\pi_i \in \Delta_i} \sum_{\omega \in \Omega} u_i(f_i(\omega); \omega) \pi_i(\omega) \geq \min_{\pi_i \in \Delta_i} \sum_{\omega \in \Omega} u_i(h_i(\omega); \omega) \pi_i(\omega). \tag{1}$$

de Castro-Yannelis [3] adopt the following equivalent formulation to (1),

$$\sum_{E_i^{\mathcal{F}_i} \in \mathcal{F}_i} \left(\min_{\omega \in E_i^{\mathcal{F}_i}} u_i(f_i(\omega); \omega) \right) \mu_i\left(E_i^{\mathcal{F}_i}\right) \geq \sum_{E_i^{\mathcal{F}_i} \in \mathcal{F}_i} \left(\min_{\omega \in E_i^{\mathcal{F}_i}} u_i(h_i(\omega); \omega) \right) \mu_i\left(E_i^{\mathcal{F}_i}\right). \tag{2}$$

We employ the utility formulation by de Castro-Yannelis.

Furthermore, we say agent i strictly prefers f_i to h_i, $f_i \succ_i^{MP} h_i$, if he prefers f_i to h_i but not the reverse, i.e. $f_i \succeq_i^{MP} h_i$ but $h_i \not\succeq_i^{MP} f_i$.

The last assumption we impose is that

Assumption 3. *For each i and for each fixed $c_i \in \mathbb{R}^l_+$, $u_i(c_i; \cdot)$ is $\mathcal{F}_i$-measurable. That is, given any $c_i \in \mathbb{R}^l_+$, and any two states ω, $\hat{\omega} \in \Omega$, with $\omega \neq \hat{\omega}$, we have $u_i(c_i; \omega) = u_i(c_i; \hat{\omega})$, whenever $\omega \in E_i^{\mathcal{F}_i}(\hat{\omega})$.*

The $\mathcal{F}_i$-measurability of the ex post utility functions is, in fact, often assumed in games with incomplete information. Indeed, one may regard, $\mathcal{F}_i$ as agent i's type space, and $E_i^{\mathcal{F}_i} \in \mathcal{F}_i$ as a possible type of agent i. Then clearly, assuming the function $u_i(c_i; \cdot)$ to be $\mathcal{F}_i$-measurable, is the same as assuming u_i to depend on agent i's type.

3 Maximin core – a cooperative solution concept

Loosely speaking, an ex ante core of an economy is a collection of desirable allocations, in the sense that, each allocation provides the best possible insurance for the agents.

The maximin core (an ex ante core concept), first defined in de Castro-Yannelis [3], is the set of feasible allocations, that cannot be "improved upon" by any coalition of agents. The "improved upon" idea is now based on the ex ante maximin expected utility. Formally,

Definition 1. *A maximin core allocation is a feasible allocation $x = (x_i)_{i \in I}$ for which there is no coalition $C \subseteq I$, $C \neq \emptyset$, and an allocation of the coalition $(y_i)_{i \in C}$ satisfying $\sum_{i \in C} y_i(\omega) = \sum_{i \in C} e_i(\omega)$, for all $\omega \in \Omega$, $y_i \succeq_i^{MP} x_i$ for all $i \in C$, and $y_i \succ_i^{MP} x_i$ for at least one $i \in C$.*

de Castro-Yannelis [3] point out that the existence of the maximin core allocations follows from the standard balancedness condition (e.g. Scarf [17]), provided that the ex post utility functions are concave and continuous in consumption. Furthermore, they [3] show that each maximin core allocation is incentive compatible.

We indicate by means of an example that the maximin core notion is different than the rational expectations equilibrium (Allen [1], Radner [13]), or the Walrasian expectations equilibrium (Radner [12], [14]) or the private core notion (Yannelis [18]). We show that the maximin core not only exists in a situation that the Walrasian expectations equilibrium (Radner [12]) fails to exist, but also achieves higher efficiency than all the above concepts.

The rational expectations equilibrium, Walrasian expectations equilibrium and the private core were first defined in a Bayesian asymmetric information economy[6].

In a Bayesian (standard) asymmetric information economy, for each i-allocation x_i, the *standard ex ante expected utility* of player i is given by

$$\nu_i(x_i) = \sum_{\omega \in \Omega} u_i(x_i(\omega);\omega)\,\pi_i(\omega),$$

where $\pi_i(\omega)$ denotes the likelihood that agent i assigns to the state ω.

Denote by $\succeq_i$, the preference relation of agent i. Agent i weakly prefers the i-allocation, f_i to h_i (written as $f_i \succeq_i h_i$), if the standard ex ante expected utility of f_i is greater than or equal to the standard ex ante expected utility of h_i, i.e.,

$$\sum_{\omega \in \Omega} u_i(f_i(\omega);\omega)\,\pi_i(\omega) \geq \sum_{\omega \in \Omega} u_i(h_i(\omega);\omega)\,\pi_i(\omega).$$

We write $f_i \succ_i h_i$, whenever we have $f_i \succeq_i h_i$ but $h_i \not\succeq_i f_i$.

Furthermore, for each agent i, let $\mathcal{G}_i$ be a partition of Ω. For $\omega \in \Omega$, denote by $E_i^{\mathcal{G}_i}(\omega)$ the element of $\mathcal{G}_i$ containing ω. Agent i's *Bayesian conditional probability* is defined as

$$\pi_i\left(\omega' \mid E_i^{\mathcal{G}_i}(\omega)\right) = \begin{cases} 0 & \text{if } \omega' \notin E_i^{\mathcal{G}_i}(\omega) \\ \frac{\pi_i\left(\omega'\right)}{\pi_i\left(E_i^{\mathcal{G}_i}(\omega)\right)} & \text{if } \omega' \in E_i^{\mathcal{G}_i}(\omega), \end{cases}$$

where $\pi_i\left(E_i^{\mathcal{G}_i}(\omega)\right) := \sum_{\omega' \in E_i^{\mathcal{G}_i}(\omega)} \pi_i\left(\omega'\right)$.

[6] A Bayesian asymmetric information economy is similar to an ambiguous asymmetric information economy, except that there is no ambiguity – agents' priors are defined at every element of the state space; furthermore, the partitions, the priors, the random initial endowments, and the ex ante utility functions are all common knowledge.

The *Bayesian interim expected utility* function of agent i, $\nu_i\left(x_i \mid E_i^{\mathcal{G}_i}(\omega)\right)$, is given by

$$\nu_i\left(x_i \mid E_i^{\mathcal{G}_i}(\omega)\right) = \sum_{\omega' \in \Omega} u_i\left(x_i\left(\omega'\right); \omega'\right) \pi_i\left(\omega' \mid E_i^{\mathcal{G}_i}(\omega)\right).$$

Definition 2. *Let $C \subset I$ denote a coalition. An allocation $x = (x_1, \cdots, x_N)$ is said to be $(\mathcal{F}_i)_{i \in C}$ -measurable, if for each $i \in C$, x_i is $\mathcal{F}_i$-measurable*[7] *.*

Note, if an i-allocation x_i is not $\mathcal{F}_i$-measurable, then the standard ex ante expected utility of x_i is not well defined. Take a very simple example. Suppose the state space is $\Omega = \{a, b\}$ and agent i's partition is $\mathcal{F}_i = \{\{a, b\}\}$, that is, he cannot distinguish the states a and b even in the interim. By definition, his private prior is given by $\mu_i(\{a, b\}) = 1$, $\mu_i(\emptyset) = 0$. Now, it is clear that he cannot evaluate the allocation

$$x_i = (x_i(a), x_i(b)) = (1, 0),$$

which fails to be $\mathcal{F}_i$-measurable, based on the standard ex ante expected utility. Indeed, the standard ex ante expected utility of agent i is $u_i(x_i(a); a)\mu_i(a) + u_i(x_i(b); b)\mu_i(b)$, and it is not well defined, since he does not know $\mu_i(a)$ or $\mu_i(b)$.

Now, let the non-zero function $p : \Omega \to \mathbb{R}_+^l$ denote a *price system.* Let $\sigma(p)$ be the smallest[8] σ-algebra of Ω for which the price system p is measurable, and let $\sigma(\mathcal{G}_i) = \sigma(p) \vee \sigma(\mathcal{F}_i)$ denote the smallest σ-algebra of Ω containing both $\sigma(p)$ and $\sigma(\mathcal{F}_i)$ [9]. Then, $\mathcal{G}_i$ is the partition of Ω that generates the σ-algebra $\sigma(\mathcal{G}_i)$.

The two definitions below are taken from Allen [1] and Radner [13], [14], [12].

Definition 3. *A rational expectations equilibrium (REE), (p^*, x^*), consists of a price system p^* and a $(\mathcal{G}_i)_{i \in I}$ -measurable*[10] *allocation $x^* = (x_1^*, \cdots, x_N^*)$, such that*

1. *for each i and for each ω, $x_i^*(\omega)$ maximizes $\nu_i\left(x_i \mid E_i^{\mathcal{G}_i}(\omega)\right)$ subject to the budget constraint $p^*(\omega) x_i(\omega) \leq p^*(\omega) e_i(\omega)$;*
2. *$\sum_{i=1}^N x_i^*(\omega) = \sum_{i=1}^N e_i(\omega)$, for each $\omega \in \Omega$.*

Definition 4. *A Walrasian expectations equilibrium (WEE), (p^*, x^*), consists of a price system p^* and an $(\mathcal{F}_i)_{i \in I}$ -measurable allocation $x^* = (x_1^*, \cdots, x_N^*)$, such that*

1. *for each i, the i-allocation x_i^* maximizes $\nu_i(x_i)$, subject to the budget set*

$$B_i(p^*) := \left\{ x_i : \Omega \to \mathbb{R}_+^l \mid x_i \textit{ is } \mathcal{F}_i\textit{-measurable, and } \sum_{\omega \in \Omega} p^*(\omega) \cdot x_i(\omega) \leq \sum_{\omega \in \Omega} p^*(\omega) \cdot e_i(\omega) \right\}.$$

2. *$\sum_{i=1}^N x_i^*(\omega) = \sum_{i=1}^N e_i(\omega)$, for each $\omega \in \Omega$.*

[7] That is, x_i is constant on each element in $\mathcal{F}_i$.

[8] Let σ_1, σ_2 be two σ-algebras of Ω. σ_1 is smaller than σ_2, if $\sigma_1 \subset \sigma_2$.

[9] Recall, $\sigma(\mathcal{F}_i)$ denotes the σ-algebra generated by the partition $\mathcal{F}_i$.

[10] For each i, the i-allocation x_i^* is $\mathcal{G}_i$-measurable.

If we allow the total consumption at each state to be less than the total endowment at that state, $\sum_{i=1}^{N} x_i^* (\omega) \leq \sum_{i=1}^{N} e_i (\omega)$ for each $\omega \in \Omega$, and have the total spending to be the same as the total income, $\sum_{\omega \in \Omega} p(\omega) \sum_{i=1}^{N} x_i^* (\omega) = \sum_{\omega \in \Omega} p(\omega) \sum_{i=1}^{N} e_i (\omega)$, then we have a *free disposal* WEE as defined by Radner [14].

The definition below is taken from Yannelis [18].

Definition 5. *A private core allocation is a feasible and $(\mathcal{F}_i)_{i \in I}$ -measurable allocation $x = (x_i)_{i \in I}$ for which there is no coalition $C \subseteq I$, $C \neq \emptyset$, and an $(\mathcal{F}_i)_{i \in C}$ -measurable allocation for the coalition $(y_i)_{i \in C}$ satisfying $\sum_{i \in C} y_i (\omega) = \sum_{i \in C} e_i (\omega)$, for all $\omega \in \Omega$, $y_i \succeq_i x_i$ for all $i \in C$, and $y_i \succ_i x_i$ for at least one $i \in C$.*

But these three notions, rational expectations equilibrium, Walrasian expectations equilibrium and the private core, can be readily applied to our ambiguous asymmetric information economy[11].

The difference of these three notions and the maximin core can be seen most clearly by means of the following example.

Example 1. There are two agents, one commodity, and three possible states of nature $\Omega = \{a, b, c\}$. The ex post utility function of each agent i is $u_i (c_i; \omega) = \sqrt{c_i}$. The agents' random initial endowments, information partitions and private priors are:

$$(e_1 (a), e_1 (b), e_1 (c)) = (5, 5, 0); \quad \mathcal{F}_1 = \{\{a, b\}, \{c\}\}$$

$$(e_2 (a), e_2 (b), e_2 (c)) = (5, 0, 5); \quad \mathcal{F}_2 = \{\{a, c\}, \{b\}\}$$

$$\mu_1 (\{a, b\}) = \tfrac{2}{3}; \quad \mu_1 (\{c\}) = \tfrac{1}{3}$$

$$\mu_2 (\{a, c\}) = \tfrac{2}{3}; \quad \mu_2 (\{b\}) = \tfrac{1}{3}$$

When calculating the private core, the REE and the WEE, we assume that each agent ignores his information constraint, and completes his prior μ_i by assigning a none zero probability to each state of nature[12]. In our setting, regardless of the ways the agents complete their priors, the following results[13] hold.

[11] In an ambiguous asymmetric information economy with $\mathcal{F}_i$-measurable utility functions, the standard ex ante expected utility of any $\mathcal{F}_i$-measurable i-allocation is well defined. Indeed, let x_i be an $\mathcal{F}_i$-measurable i-allocation, then

$$\nu_i (x_i) = \sum_{E_i^{\mathcal{F}_i} \in \mathcal{F}_i} u_i \left(x_i \left(E_i^{\mathcal{F}_i} \right); E_i^{\mathcal{F}_i} \right) \mu_i \left(E_i^{\mathcal{F}_i} \right),$$

where $u_i \left(x_i \left(E_i^{\mathcal{F}_i} \right); E_i^{\mathcal{F}_i} \right) := u_i (x_i (\omega); \omega)$ for some $\omega \in E_i^{\mathcal{F}_i}$. Similarly, the Bayesian interim expected utility of a $\mathcal{G}_i$-measurable allocation, $\nu_i \left(x_i \mid E_i^{\mathcal{G}_i} (\omega) \right)$, is also well defined.

[12] $\mu_i (\omega) > 0$, for all ω and for all i, is a standard assumption imposed on a Bayesian asymmetric information economy (for example, Yannelis [18]).

[13] See Glycopantis-Yannelis [8] for detailed calculations.

No trade (i.e., the initial endowment) is the unique private core allocation and the unique REE allocation. Furthermore, it can be easily checked that a WEE with positive prices[14] does not exist. If we allow for free disposal, then

$$z = \begin{pmatrix} z_1(a) & z_1(b) & z_1(c) \\ z_2(a) & z_2(b) & z_2(c) \end{pmatrix} = \begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

is a (free disposal) WEE allocation, which requires each agent to throw away a unit of the good at the state a.

Notice the allocation z is not incentive compatible. Indeed, suppose that the realized state of nature is a, agent 1 is in the event $\{a, b\}$ and he reports $\{c\}$. Observe that agent 2 cannot distinguish between a and c, and may believe that state c has occurred. In this case, agent 1 gets one unit from agent 2. His Bayesian interim expected utility from lying is[15]

$$\begin{aligned} u_1(e_1(a) + z_1(c) - e_1(c); a) \times \pi_1(a \mid \{a, b\}) + u_1(z_1(b); b) \times \pi_1(b \mid \{a, b\}) \\ = \sqrt{6} \times \pi_1(a \mid \{a, b\}) + \sqrt{4} \times \pi_1(b \mid \{a, b\}), \end{aligned}$$

which is higher than the Bayesian interim expected utility of telling the truth,

$$\begin{aligned} u_1(z_1(a); a) \times \pi_1(a \mid \{a, b\}) + u_1(z_1(b); b) \times \pi_1(b \mid \{a, b\}) \\ = \sqrt{4} \times \pi_1(a \mid \{a, b\}) + \sqrt{4} \times \pi_1(b \mid \{a, b\}), \end{aligned}$$

where $\pi_1(a \mid \{a, b\}) > 0$ and $\pi_1(b \mid \{a, b\}) > 0$ are the conditional probabilities.

The initial endowment fails to be a maximin core allocation, since there exists an alternative feasible allocation x that Pareto improves it[16], where

$$x = \begin{pmatrix} x_1(a) & x_1(b) & x_1(c) \\ x_2(a) & x_2(b) & x_2(c) \end{pmatrix} = \begin{pmatrix} 5 & 4 & 1 \\ 5 & 1 & 4 \end{pmatrix}.$$

The feasible, individually rational, and Pareto optimal[17] allocation x is in fact a maximin core allocation.

Clearly, in this example, the maximin core, the private core, the REE and the WEE are different. Most importantly, the maximin core allows the agents to trade and reach

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[14]That is, a WEE with $(p(a), p(b), p(c)) \in \mathbb{R}^3_+$.

[15]Agent 1 can only successfully lie in the states that agent 2 cannot distinguish.

[16]For each agent i, we have $x_i \succ_i^{MP} e_i$, since

$$\frac{2}{3}\sqrt{\min\{5,4\}} + \frac{1}{3}\sqrt{1} = 1.67 > \frac{2}{3}\sqrt{\min\{5,5\}} + \frac{1}{3}\sqrt{0} = 1.49.$$

[17]The allocation x solves

$$\max_x \left(\frac{2}{3}\sqrt{\min\{x_1(a), x_1(b)\}} + \frac{1}{3}\sqrt{x_1(c)}\right) + \left(\frac{2}{3}\sqrt{\min\{x_2(a), x_2(c)\}} + \frac{1}{3}\sqrt{x_2(b)}\right)$$

subject to $x_1(a) + x_2(a) = 10$, $x_1(b) + x_2(b) = 5$ and $x_1(c) + x_2(c) = 5$.

a Pareto superior outcome, which is also incentive compatible (see de Castro-Yannelis [3] for a rigorous definition).

de Castro-Yannelis [3] show that with the allocation x, no agent has an incentive to misreport his privately observed event under the maximin preferences.

Indeed, if state a is realized, then agent 1 sees the event $\{a, b\}$. He can report the true event $\{a, b\}$ or he can lie and report the event $\{c\}$. Suppose agent 1 lies (i.e., reports the event $\{c\}$). Notice that agent 2 cannot distinguish the states a from state c, and may believe that state c has occurred. In this case, agent 1 gets $e_1(a) + x_1(c) - e_1(c) = 6$.

His maximin expected utility from lying is

$$\min\{u_1(e_1(a) + x_1(c) - e_1(c); a), u_1(x_1(b); b)\} = \min\left\{\sqrt{6}, \sqrt{4}\right\} = \sqrt{4}.$$

When agent 1 does not misreport, he gets

$$\min\{u_1(x_1(a); a), u_1(x_1(b); b)\} = \min\left\{\sqrt{5}, \sqrt{4}\right\} = \sqrt{4}.$$

Consequently, agent 1 does not gain by misreporting.

Clearly, the maximin core allocation x outperforms the free disposal WEE allocation z. More precisely, the allocation x is both efficient and incentive compatible in the maximin framework, whereas the allocation z is neither first best efficient nor incentive compatible in the Bayesian framework.

Hence, the maximin core seems to be a desirable cooperative solution concept in the sense that it exists under the continuity and concavity assumptions, it is incentive compatible, and obviously efficient. But, could one provide a noncooperative foundation for the maximin core? That is, can the maximin core allocations be reached by means of noncooperation? We address this question in the next sections.

4 Implementation of a maximin core allocation

4.1 The direct revelation mechanism

A direct revelation mechanism, associated with a maximin core allocation and its underlying ambiguous asymmetric information economy, is a noncooperative game, in which agents (players) need to decide what to report after a state of nature is realized. To ease the understanding, we describe the game first, and then define it formally.

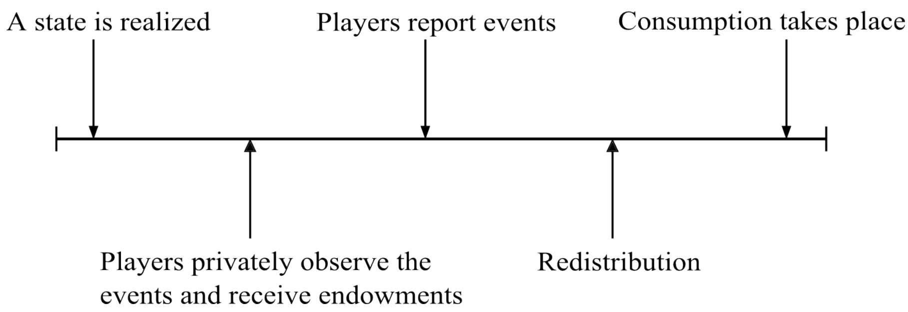
In the interim, a state of nature ω is realized, each player i privately observes the event $E_i^{\mathcal{F}_i}(\omega)$ and receives the initial endowment $e_i(\omega)$. Then, each player i strategically writes down his report $E_i^{\mathcal{F}_i} \in \mathcal{F}_i$ on a piece of paper and puts it in a sealed envelope.

The report $E_i^{\mathcal{F}_i}$, however, may or may not be truthful. That is, for player i, the reported event $E_i^{\mathcal{F}_i}$ may be different from the observed true event $E_i^{\mathcal{F}_i}(\omega)$.

Definition 6. *Suppose the realized state (the true state) is ω. Then, a report of player i, $E_i^{\mathcal{F}_i} \in \mathcal{F}_i$, is a lie, if it differs from the event $E_i^{\mathcal{F}_i}(\omega)$.*

The players' envelopes are opened at the same time. Based on the players' reports, redistribution takes place. Figure 1 shows the time line.

Figure 1: Time line



A planned redistribution is the adjustments needed to go from the initial endowment e to a planned allocation x.

Definition 7. *If x is a maximin core allocation of an economy $\mathcal{E}$, then the maximin core redistribution (a planned redistribution) is given by $x - e$.*

For example, given the maximin core allocation of *Example 1*, the maximin core redistribution $x - e$ is:

$$(x_1(a) - e_1(a), x_1(b) - e_1(b), x_1(c) - e_1(c)) = (0, -1, 1);$$

$$(x_2(a) - e_2(a), x_2(b) - e_2(b), x_2(c) - e_2(c)) = (0, 1, -1).$$

It says, if the players agree that state a has occurred, then everyone keeps what they have; but if the players agree that state b has occurred, then player 1 is to give one unit of the good to player 2; etc.

The actual redistribution, on the other hand, depends on the planned redistribution, the players' reports $E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}$, and the realized state of nature ω. From Assumption 1, we know for any collection of reports $E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}$, the $\bigcap_{i \in I} E_i^{\mathcal{F}_i}$ is either singleton or empty. So clearly, for the reports to be compatible, they must not contradict with each other. Yet, this requirement is not sufficient. We also need every player to have enough endowment to carry out the planned redistribution. More precisely,

Definition 8. *We say the reports $E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}$ are compatible at the state ω, if*

1. *$\bigcap_{i \in I} E_i^{\mathcal{F}_i} = \{\tilde{\omega}\}$, and*

2. $e_i(\omega) + (x_i(\tilde{\omega}) - e_i(\tilde{\omega})) \in \mathbb{R}^l_+$ *for all* $i \in I$.

Furthermore, we refer the state $\tilde{\omega}$ *as the implied state (the agreed state).*

When the reports $E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}$ are compatible at the state ω, the players will end up with $e(\omega)+x(\tilde{\omega})-e(\tilde{\omega})$, where $\tilde{\omega}$ is the implied state, and $x(\tilde{\omega})-e(\tilde{\omega})$ is the planned redistribution specified for the state $\tilde{\omega}$. Clearly, if all the players tell the truth, then $\tilde{\omega} = \omega$ and the players get what they planned to get, $e(\omega)+x(\omega)-e(\omega) = x(\omega)$. But, since some player may successfully lie, $\tilde{\omega}$ may not be the true state. As a consequence, $e(\omega) + x(\tilde{\omega}) - e(\tilde{\omega})$ may differ from $x(\omega)$, i.e., the players may not end up with the planned allocation.

If the reports are not compatible at the realized state ω, then the players redistribute their initial endowments according to the planned redistribution specified for the realized state[18]. Consequently, the players get, $e(\omega) + x(\omega) - e(\omega) = x(\omega)$.

Definition 9. *Let* $x - e$ *denote a planned redistribution,* $\left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right)$ *a list of reports, and* ω *a realized state of nature. Then the actual redistribution is given by*

$$D\left(x - e, \left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right), \omega\right) = \begin{cases} x(\tilde{\omega}) - e(\tilde{\omega}) & \textit{if the reports are compatible at } \omega \\ x(\omega) - e(\omega) & \textit{otherwise;} \end{cases}$$

where $\tilde{\omega}$ *and* ω *denote the implied state and the realized state respectively, and* $D_i\left(x - e, \left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right), \omega\right)$ *denotes the amount player* i *gives to or takes from the others.*

Now, we gradually define the game. A *decision node* of player i is a circumstance that he might be called upon to act (in Figure 2, it is denoted by a dot.). An *information set* of player i, $\mathcal{I}_i$, is the set of all of his decision nodes that look the same to him (the players' information sets are illustrated by the dotted lines in Figure 2). Let $\mathbb{I}_i$ denote the set of player i's information sets. That is, $\mathbb{I}_i$ contains all of the distinct circumstances that player i might be called upon to act.

A *strategy* of player i is a function, $s_i : \mathbb{I}_i \to \mathcal{F}_i$. In words, player i's strategy is a complete plan of reports, that specifies a report for the player conditional on each distinct circumstance that he might be called upon to act. But each information set corresponds to a unique event. Indeed, if state ω is realized and the player i ends up at the information set $\mathcal{I}_i \mid_\omega$, then he only knows that the event $E_i^{\mathcal{F}_i}(\omega)$ has occurred. So for simplicity, we slightly abuse the notation, defining player i's strategy as a function that goes from the set $\mathcal{F}_i$ to the set itself.

Definition 10. *A strategy of player* i *is a function* $s_i : \mathcal{F}_i \to \mathcal{F}_i$.

In words, it says a strategy of player i is a complete plan of reports, that specifies a report for him conditional on each possible event that he might observe from nature.

[18]The players believe that whenever their reports fail to be compatible, they will go back to the planned allocation.

Let S_i denote player i's strategy set – the collection of all possible strategies of player i; $S := \times_{i\in I} S_i$ the strategy set, and $s \in S$ a strategy profile.

Furthermore, with a slightly abused notation, we use $s(\omega)$ to denote the players' reports, when they adopt the strategy profile s, and the realized state is ω. That is, $s(\omega) := \left(s_1\left(E_1^{\Pi_1}(\omega)\right), \cdots, s_N\left(E_N^{\Pi_N}(\omega)\right)\right)$. Clearly, for any $\omega \in \Omega$, $s(\omega) \in \times_{i\in I}\mathcal{F}_i$.

Definition 11. *A direct revelation mechanism, associated with a maximin core allocation x and its underlying ambiguous asymmetric information economy $\mathcal{E} = \{\Omega; (\mathcal{F}_i, \mu_i, e_i, u_i)_{i\in I}\}$, denoted by $\Gamma = \left\langle I, S, x-e, \{g_i\}_{i\in I}, \{u_i\}_{i\in I}\right\rangle$, is a set, where*

1. *$I = \{1, \cdots, N\}$ is the set of N players;*
2. *S is the players' strategies set; for each $s \in S$, we have $s_i : \mathcal{F}_i \to \mathcal{F}_i$ for all i;*
3. *$x - e$ denotes the planned redistribution;*
4. *$g_i : \mathcal{F}_1 \times \cdots \times \mathcal{F}_N \times \Omega \to \mathbb{R}_+^l$ is the outcome function for player i. It depends on the reports of all the players $\left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right) \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_N$ and the realized state of nature $\omega \in \Omega$. It takes the form of*
$$g_i\left(\left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right), \omega\right) = e_i(\omega) + D_i\left(x - e, \left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right), \omega\right), \tag{3}$$
where $e_i(\omega) + D_i\left(x - e, \left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right), \omega\right)$ is the quantity of the goods, that player i ends up consuming. In particular, if the players adopt the strategy profile s and the state ω is realized, then we have
$$g_i(s(\omega), \omega) = e_i(\omega) + D_i(x - e, s(\omega), \omega);$$
5. *and finally, $u_i : \mathbb{R}_+^l \times \Omega \to \mathbb{R}$ is player i's ex post utility function, taking the form of $u_i(c_i; \omega)$, where c_i denotes agent i's consumption (as defined in the economy $\mathcal{E}$).*

For convenience, we define, for each player i, a final payoff function. It tells us the final payoff that the player i ends up, given a list of reports and a realized state of nature. Formally,

Definition 12. *Denote by $v_i : \mathcal{F}_1 \times \cdots \times \mathcal{F}_N \times \Omega \to \mathbb{R}$, $v_i := u_i \circ g_i$, the final payoff function of player i. It depends on the reports of all the players $\left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right) \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_N$ and the realized state of nature $\omega \in \Omega$, taking the form of*
$$\begin{aligned} v_i\left(\left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right); \omega\right) &= u_i\left(g_i\left(\left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right), \omega\right); \omega\right) \\ &= u_i\left(e_i(\omega) + D_i\left(x - e, \left(E_1^{\mathcal{F}_1}, \cdots, E_N^{\mathcal{F}_N}\right), \omega\right); \omega\right). \end{aligned}$$

To ease the understanding, we illustrate the mechanism with an example.

4.2 An example

Example 2. Consider the ambiguous asymmetric information economy of *Example 1*. That is, there are two agents, one commodity, and three possible states of nature $\Omega = \{a, b, c\}$. The ex post utility function of each agent i is $u_i(c_i; \omega) = \sqrt{c_i}$. The agents' random initial endowments, information partitions and private priors are:

$$(e_1(a), e_1(b), e_1(c)) = (5, 5, 0); \quad \mathcal{F}_1 = \{\{a, b\}, \{c\}\}$$

$$(e_2(a), e_2(b), e_2(c)) = (5, 0, 5); \quad \mathcal{F}_2 = \{\{a, c\}, \{b\}\}$$

$$\mu_1(\{a, b\}) = \tfrac{2}{3}; \quad \mu_1(\{c\}) = \tfrac{1}{3}$$

$$\mu_2(\{a, c\}) = \tfrac{2}{3}; \quad \mu_2(\{b\}) = \tfrac{1}{3}$$

Suppose the planned allocation is the maximin core allocation of *Example 1*. Then, the planned redistribution $x - e$ is

$$(x_1(a) - e_1(a), x_1(b) - e_1(b), x_1(c) - e_1(c)) = (0, -1, 1);$$

$$(x_2(a) - e_2(a), x_2(b) - e_2(b), x_2(c) - e_2(c)) = (0, 1, -1).$$

When a state of nature is realized, each player privately observes an event and receives the initial endowment. Then, each player writes down his or her report on a piece of paper and puts it in a sealed envelope. All the envelopes are opened at the same time, and redistribution takes place.

For simplicity, let $\mathcal{F}_1 = \{A_1, c_1\}$, where $A_1 := E_1^{\mathcal{F}_1}(a) = E_1^{\mathcal{F}_1}(b) = \{a, b\}$, $c_1 := E_1^{\mathcal{F}_1}(c) = \{c\}$, and similarly let $\mathcal{F}_2 = \{A_2, b_2\}$, where $A_2 = \{a, c\}$, $b_2 = \{b\}$.

For purposes of clarity, we present the game in an informal game tree[19] (Figure 2).

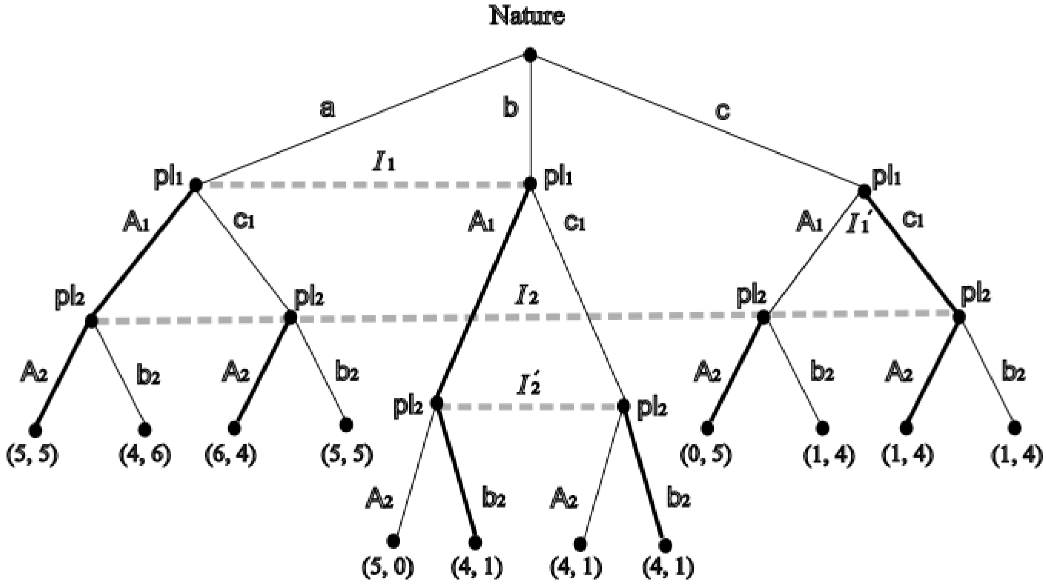
Nature chooses a state, a, b or c. Player 1 (pl_1) cannot distinguish between a and b, and player 2 (pl_2) between a and c; furthermore, when a player acts, he does not know what the other does. This accounts for the none singleton information sets $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_2^{'}$ in Figure 2. At such an information set, the player of the move cannot distinguish between the decision nodes within, and therefore his (or her) decisions are common to all of them.

For example, if the state a is realized, then player 1 (she) observes the event A_1 (she finds herself at the information set $\mathcal{I}_1$) and player 2 (he) observes the event A_2 (he finds himself at the information set $\mathcal{I}_2$). Player 1 can report 'I have seen the event A_1' or 'I have seen the event c_1'; player 2 can report 'I have seen the event A_2' or 'I have seen the event b_2'.

The quantity of the good that each player ends up consuming, is determined by the realized state and the reports of all the players, according to the outcome function (equation (3)).

[19] It is an informal game tree, since the pair assigned to each terminal node of the tree does not denote the players' final payoffs, but rather the quantity of the good that player 1 and player 2 end up consuming respectively.

Figure 2: An informal game tree
(Simultaneous moves)



To illustrate, suppose the state a is realized, player 1 reports the event c_1 (she lies), and player 2 truthfully reports the event A_2 (we are looking at the path ac_1A_2 on the tree). Clearly, the reports are compatible at the state a, and the implied state is c. The actual redistribution is then $D(x-e,(c_1,A_2),a)=x(c)-e(c)$, and the outcome functions tell us that player 1 ends up with 6 units and player 2 ends up with 4 units, i.e.,

$$\begin{aligned} g_1((c_1,A_2),a) &= e_1(a)+D_1(x-e,(c_1,A_2),a) \\ &= e_1(a)+x_1(c)-e_1(c)=5+1-0=6; \end{aligned}$$

and

$$\begin{aligned} g_2((c_1,A_2),a) &= e_2(a)+D_2(x-e,(c_1,A_2),a) \\ &= e_2(a)+x_2(c)-e_2(c)=5+4-5=4. \end{aligned}$$

We record this outcome $(g_1((c_1,A_2),a),g_2((c_1,A_2),a))=(6,4)$ at the end of the path ac_1A_2 in Figure 2.

As a consequence, the final payoff each player enjoys is given by

$$v_1((c_1,A_2);a)=u_1(g_1((c_1,A_2),a);a)=u_1(6;a)=\sqrt{6}$$

and

$$v_2((c_1,A_2);a)=u_2(g_2((c_1,A_2),a);a)=u_2(4;a)=\sqrt{4}.$$

Furthermore, in this game, a strategy profile of the game can be (as indicated by the bold lines on the tree)

$$s=\left(s_1(\mathcal{I}_1)=A_1,s_1\left(\mathcal{I}_1'\right)=c_1;s_2(\mathcal{I}_2)=A_2,s_2\left(\mathcal{I}_2'\right)=b_2\right),$$

or in our simplified notation,

$$s = (s_1(A_1) = A_1, s_1(c_1) = c_1; s_2(A_2) = A_2, s_2(b_2) = b_2).$$

In words, this strategy profile says every player reports exactly what he or she sees.

Given this strategy profile s and a state of nature ω, the list of reports $s(\omega)$ is uniquely determined. For example, $s(a) = (s_1(A_1), s_2(A_2)) = (A_1, A_2)$.

So if the players act according to the strategy profile s and state a is realized (we are looking at the path aA_1A_2 on the tree), then both players will end up with 5 units of the good, i.e.,

$$g_1(s(a), a) = g_1((A_1, A_2), a) = e_1(a) + x_1(a) - e_1(a) = 5,$$

and

$$g_2(s(a), a) = g_2((A_1, A_2), a) = e_2(a) + x_2(a) - e_2(a) = 5.$$

Hence, the players will end up at the pair $(g_1((A_1, A_2), a), g_2((A_1, A_2), a)) = (5, 5)$, which is recorded at the end of the path aA_1A_2 in Figure 2.

4.3 The problem of implementation

We study the implementation of each maximin core allocation with the help of its corresponding direct revelation mechanism. In particular, we say a maximin core allocation x is implementable, if x can be realized through an equilibrium of the direct revelation mechanism $\Gamma = \left\langle I, S, x - e, \{g_i\}_{i \in I}, \{u_i\}_{i \in I} \right\rangle$.

Let $\mathbb{S}$ denote a game theoretic solution concept, and $\mathbb{S}(\Gamma)$ the set of $\mathbb{S}$ equilibria of the mechanism $\Gamma = \left\langle I, S, x - e, \{g_i\}_{i \in I}, \{u_i\}_{i \in I} \right\rangle$.

Definition 13. *Let x denote a maximin core allocation of an ambiguous asymmetric information economy $\mathcal{E}$, and $\mathbb{S}(\Gamma)$ the set of $\mathbb{S}$ equilibria of the mechanism $\Gamma = \left\langle I, S, x - e, \{g_i\}_{i \in I}, \{u_i\}_{i \in I} \right\rangle$. We say the allocation x is implementable as an $\mathbb{S}$ equilibrium of the mechanism Γ if,*

$$\exists s^* \in \mathbb{S}(\Gamma), \text{ such that } g_i(s^*(\omega), \omega) = x_i(\omega),$$

for each $\omega \in \Omega$ and for each $i \in I$.

Definition 14. *We say a strategy profile s is truth telling, if according to it, every player reports the truth whenever it is his turn to report. That is, for each player i and for each state ω, if he observes the event $E_i^{\mathcal{F}_i}(\omega)$ from nature, then his action is to report the true event, i.e., $s_i\left(E_i^{\mathcal{F}_i}(\omega)\right) = E_i^{\mathcal{F}_i}(\omega)$. We denote such a strategy profile by s^T.*

Remark 1. Clearly, if the truth telling strategy profile s^T constitutes an $\mathbb{S}$ equilibrium of the mechanism Γ, i.e., $s^T \in \mathbb{S}(\Gamma)$, then the allocation x is implementable as an $\mathbb{S}$ equilibrium of the mechanism $\Gamma = \left\langle I, S, x - e, \{g_i\}_{i \in I}, \{u_i\}_{i \in I} \right\rangle$.

Indeed, under the truth telling strategy profile s^T, the list of reports associated with each state ω is

$$s^T(\omega) = \left(E_1^{\mathcal{F}_1}(\omega), \cdots, E_N^{\mathcal{F}_N}(\omega)\right).$$

That is, the players always tell the truth. As a consequence, we have

$$\begin{aligned} g_i\left(s^T(\omega), \omega\right) &= g_i\left(\left(E_1^{\mathcal{F}_1}(\omega), \cdots, E_N^{\mathcal{F}_N}(\omega)\right), \omega\right) \\ &= e_i(\omega) + D_i\left(x - e, \left(E_1^{\mathcal{F}_1}(\omega), \cdots, E_N^{\mathcal{F}_N}(\omega)\right), \omega\right) \\ &= e_i(\omega) + x_i(\omega) - e_i(\omega) = x_i(\omega), \end{aligned}$$

for each $\omega \in \Omega$ and for each $i \in I$ – the requirement of definition 13.

Furthermore, when $s^T \in \mathbb{S}(\Gamma)$, we say Γ has a *truth telling* $\mathbb{S}$ *equilibrium*.

An immediate question is that, what would be a reasonable solution concept for Γ? It is helpful, to clarify first, what does a player know and not know before his turn to act. The players, in an ambiguous asymmetric information economy, have incomplete private priors and privately known utility functions. These information constraints give rise to interesting implications.

If a player has an incomplete prior, then upon observing a non-singleton event, not only he does not know which state in the event is the true state, but also he is unable to form a probabilistic assessment over the states in the event.

The lack of mutual knowledge of utility functions implies that a player cannot form a probabilistic assessment over the possible actions of his opponents.

Clearly, the standard Bayesian Nash solution concept is not suitable here, in the sense that it cannot accommodate the players' information constraints. Nevertheless, we can still use the Bayesian Nash solution concept, if we were to ignore these information constraints, and assume that the players are able to assign a probability to everything they do not know. However, the predicted outcomes may fail to be convincing. Indeed, it is well known by now, from the Ellsberg's paradox (recall the Ellsberg example), that if we assume the players know more than what they actually know, we may fail to explain their actions.

In Section 4.4, we introduce the maximin solution concept, which seems to be a suitable equilibrium notion in our framework.

To illustrate the differences between the Bayesian Nash solution concept and the maximin solution concept, we compare their predictions. In the remaining of this section, we show, in Example 3, that the Bayesian Nash solution concept suggests no truth telling. That is, the maximin core allocation x fails to be Bayesian incentive compatible[20].

Example 3. Recall, in *Example 2*, the players' information partitions are $\mathcal{F}_1 = \{A_1, c_1\}$ and $\mathcal{F}_2 = \{A_2, b_2\}$, where $A_1 = \{a, b\}$, $c_1 = \{c\}$, $A_2 = \{a, c\}$, and $b_2 = \{b\}$. Their private priors are

[20] As defined in Holmström and Myerson [10].

$$\mu_1(\{a,b\}) = \tfrac{2}{3}, \quad \mu_1(\{c\}) = \tfrac{1}{3}; \quad \mu_2(\{a,c\}) = \tfrac{2}{3}, \quad \mu_2(\{b\}) = \tfrac{1}{3}.$$

In order to apply the Bayesian Nash equilibrium solution concept to *Example 2*, we need to ignore the players' information constraints on the state of nature, and assume player 1's private priors are $\mu_1(\{a\}) = p$, $\mu_1(\{b\}) = \frac{2}{3} - p$ and $\mu_1(\{c\}) = \frac{1}{3}$, where $0 < p < \frac{2}{3}$; player 2's are $\mu_2(\{a\}) = q$, $\mu_2(\{b\}) = \frac{1}{3}$ and $\mu_2(\{c\}) = \frac{2}{3} - q$, where $0 < q < \frac{2}{3}$.

The Bayesian Nash solution concept predicts no truth telling equilibrium. Indeed, suppose that the realized state of nature is a, agent 1 is in the event $A_1 = \{a,b\}$, and she reports $c_1 = \{c\}$. With the updated beliefs $\mu_1(\{a\} \mid A_1) = \mu_1(\{a\}) / \mu_1(A_1) = \frac{3p}{2}$ and $\mu_1(\{b\} \mid A_1) = \mu_1(\{b\}) / \mu_1(A_1) = 1 - \frac{3p}{2}$, agent 1's Bayesian interim expected utility from lying is

$$u_1(g_1(c_1, A_2, a); a) \cdot \frac{3p}{2} + u_1(g_1(c_1, b_2, b); b) \cdot \left(1 - \frac{3p}{2}\right)$$

$$= u_1(e_1(a) + x_1(c) - e_1(c); a) \cdot \frac{3p}{2} + u_1(x_1(b); b) \cdot \left(1 - \frac{3p}{2}\right) = \sqrt{6} \cdot \frac{3p}{2} + \sqrt{4} \cdot \left(1 - \frac{3p}{2}\right),$$

which is higher than the Bayesian interim expected utility of telling the truth,

$$u_1(g_1(A_1, A_2, a); a) \cdot \frac{3p}{2} + u_1(g_1(A_1, b_2, b); b) \cdot \left(1 - \frac{3p}{2}\right)$$

$$= u_1(x_1(a); a) \cdot \frac{3p}{2} + u_1(x_1(b); b) \cdot \left(1 - \frac{3p}{2}\right) = \sqrt{5} \cdot \frac{3p}{2} + \sqrt{4} \cdot \left(1 - \frac{3p}{2}\right).$$

Here, the Bayesian Nash equilibrium solution concept suggests no truth telling. We will show that the maximin solution concept predicts truth telling.

4.4 Maximin equilibrium

We postulate that the players do not define their objectives based on something that they do not know; and in an equilibrium, every player's action choice is the best with respect to his objective.

Clearly, with a limited ability to form probabilities, maximizing one's standard expected payoff may not be well defined. Here, we postulate that the players maximize their payoff lower bound. This objective is well defined – every player knows the worst possible payoff associated with each of his actions; and every player clearly knows what to do to achieve his objective.

With this objective, an equilibrium becomes a situation in which, no matter what event a player observes from nature, his action insures him the best worst payoff, hence the name, a maximin equilibrium. In particular, each agent maximizes his payoff lowest bound, i.e., each agent maximizes the payoff that takes into account the worst actions of all the other agents against him and also the worst state that can occur.

Definition 15. *In a direct revelation mechanism* $\Gamma = \left\langle I, S, x-e, \{g_i\}_{i\in I}, \{u_i\}_{i\in I} \right\rangle$, *a strategy profile* $s^* = (s_1^*, \cdots, s_N^*)$ *constitutes a maximin equilibrium* ($\mathbb{ME}$), *if for each player* i, *his strategy* s_i^* *maximizes his interim payoff lower bound, that is, the function* $s_i^* : \mathcal{F}_i \to \mathcal{F}_i$ *satisfies, for each* $E_i^{\mathcal{F}_i} \in \mathcal{F}_i$,

$$\min_{E_{-i}^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\ \omega' \in E_i^{\mathcal{F}_i}} v_i \left(s_i^* \left(E_i^{\mathcal{F}_i} \right), E_{-i}^{\mathcal{F}_{-i}}; \omega' \right) \geq \min_{E_{-i}^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\ \omega' \in E_i^{\mathcal{F}_i}} v_i \left(\hat{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}; \omega' \right), \quad (4)$$

for all $\hat{E}_i^{\mathcal{F}_i} \in \mathcal{F}_i$; *where* $E_{-i}^{\mathcal{F}_{-i}}$ *denotes the reports from all the other players, so* $E_{-i}^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i} := \times_{j \neq i} \mathcal{F}_j$.

Definition 16. *If the truth telling strategy profile* s^T [21] *constitutes a maximin equilibrium of the mechanism* $\Gamma = \left\langle I, S, x-e, \{g_i\}_{i\in I}, \{u_i\}_{i\in I} \right\rangle$, *then we say that the mechanism* Γ *has a truth telling maximin equilibrium.*

4.5 Implementation

This section presents our main result, i.e., in the mechanism $\Gamma = \left\langle I, S, x-e, \{g_i\}_{i\in I}, \{u_i\}_{i\in I} \right\rangle$, no player has an incentive to lie. It implies that every maximin core allocation x of an ambiguous asymmetric information economy $\mathcal{E}$ is implementable through its corresponding direct revelation mechanism $\Gamma = \left\langle I, S, x-e, \{g_i\}_{i\in I}, \{u_i\}_{i\in I} \right\rangle$. Formally,

> **Main Theorem:** *Denote by* x *a maximin core allocation of an ambiguous asymmetric information economy* $\mathcal{E}$, *and* $\mathbb{ME}(\Gamma)$ *the set of maximin equilibria of the direct revelation mechanism* $\Gamma = \left\langle I, S, x-e, \{g_i\}_{i\in I}, \{u_i\}_{i\in I} \right\rangle$. *Then, there exists a truth telling maximin equilibrium* s^T, *which is the unique maximin equilibrium of the mechanism* Γ (*i.e.,* $\left\{ s^T \right\} = \mathbb{ME}(\Gamma)$), *for which we have* $g_i\left(s^T(\omega), \omega\right) = x_i(\omega)$, *for each* $\omega \in \Omega$ *and for each* $i \in I$, *i.e., the maximin core allocation* x *is implementable as a maximin equilibrium of its corresponding mechanism* Γ.

Remark 2. The implementation shares some similarities with the truthful implementation of Dasgupta, Hammond and Maskin [3, *p*.189] – an allocation can be truthfully implemented, if there exists a direct revelation mechanism (a game in which players report their private information) for which truth telling is its equilibrium (based on some game theoretic solution concept), and the truth telling equilibrium yields the allocation as its outcome.

Remark 3. We differ from the full implementation of Jackson [11], Palfrey and Srivastava's [15], and Hahn and Yannelis [9], in that, we do not implement the set of core allocations. Instead, we show, given any arbitrary maximin core allocation, its corresponding mechanism $\Gamma = \left\langle I, S, x-e, \{g_i\}_{i\in I}, \{u_i\}_{i\in I} \right\rangle$ yields the core allocation x as its *unique* maximin equilibrium outcome.

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[21] The truth telling strategy profile s^T takes the form of $s_i^T\left(E_i^{\mathcal{F}_i}\right) = E_i^{\mathcal{F}_i}$, for all $E_i^{\mathcal{F}_i} \in \mathcal{F}_i$, and for all i.

Here, we illustrate the main theorem by means of an example.

Example 4. We reconsider *Example 2*, i.e., there are two agents, one commodity, and three possible states of nature $\Omega = \{a, b, c\}$. The ex post utility function of each agent i is $u_i(c_i; \omega) = \sqrt{c_i}$. The agents' random initial endowments, information partitions and private priors are:

$$(e_1(a), e_1(b), e_1(c)) = (5, 5, 0); \quad \mathcal{F}_1 = \{\{a, b\}, \{c\}\}$$

$$(e_2(a), e_2(b), e_2(c)) = (5, 0, 5); \quad \mathcal{F}_2 = \{\{a, c\}, \{b\}\}$$

$$\mu_1(\{a, b\}) = \tfrac{2}{3}; \quad \mu_1(\{c\}) = \tfrac{1}{3}$$

$$\mu_2(\{a, c\}) = \tfrac{2}{3}; \quad \mu_2(\{b\}) = \tfrac{1}{3}$$

The planned allocation is the maximin core allocation.

$$x = \begin{pmatrix} x_1(a) & x_1(b) & x_1(c) \\ x_2(a) & x_2(b) & x_2(c) \end{pmatrix} = \begin{pmatrix} 5 & 4 & 1 \\ 5 & 1 & 4 \end{pmatrix}.$$

The planned redistribution $x - e$ is

$$(x_1(a) - e_1(a), x_1(b) - e_1(b), x_1(c) - e_1(c)) = (0, -1, 1);$$

$$(x_2(a) - e_2(a), x_2(b) - e_2(b), x_2(c) - e_2(c)) = (0, 1, -1).$$

The game tree is presented again in Figure 3, in which for simplicity, we let $A_1 = \{a, b\}$, $c_1 = \{c\}$, $A_2 = \{a, c\}$, and $b_2 = \{b\}$.

We will show that the truth telling strategy profile constitutes the only maximin equilibrium of the game, and the immediate consequence is that the allocation x is implemented.

Formally, we will show that the strategy profile

$$s = (s_1(A_1) = A_1, s_1(c_1) = c_1; s_2(A_2) = A_2, s_2(b_2) = b_2),$$

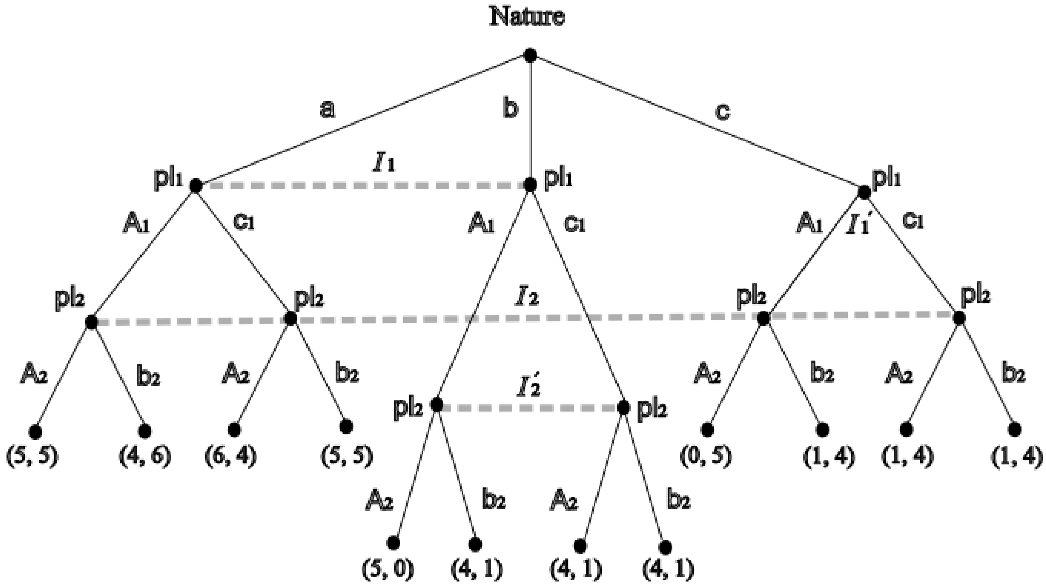
constitutes the only maximin equilibrium of the game.

We look at player 1 first, she has two information sets $\mathcal{I}_1$ and $\mathcal{I}_1'$. (Figure 4)

If she is at $\mathcal{I}_1$, then she must have seen the event A_1 from nature. She can either tell the truth A_1 or the lie c_1. Recall, player 1 cannot distinguish the two decision nodes within the set $\mathcal{I}_1$, so her action is common at the two nodes. Figure 4a shows that, being truthful (reports A_1), she may end up with, from the left to the right, 5, 4, 5 or 4 units of the good[22]; and by lying (reports c_1), she may end up with, from the left to the right, 6, 5, 4 or 4 units of the good.

[22] That is, at the information set $\mathcal{I}_1$, if player 1 tells the truth, then she may go down one of the four paths 'aA_1A_2', 'aA_1b_2', 'bA_1A_2' and 'bA_1b_2', for which she ends up with $g_1((A_1, A_2), a) = 5$, $g_1((A_1, b_2), a) = 4$, $g_1((A_1, A_2), b) = 5$, and $g_1((A_1, b_2), b) = 4$ units of the good respectively.

Figure 3: An informal game tree
(Simultaneous moves)



Clearly, when player 1 observes the event A_1, telling the truth (reports A_1) gives her a lower bound payoff of

$$\begin{aligned} &\min \quad \{v_1(A_1, A_2; a), v_1(A_1, b_2; a), v_1(A_1, A_2; b), v_1(A_1, b_2; b)\} \\ &= \quad \min\left\{\sqrt{5}, \sqrt{4}, \sqrt{5}, \sqrt{4}\right\} = \sqrt{4}; \end{aligned}$$

lying (reports c_1) gives her a lower bound payoff of

$$\begin{aligned} &\min \quad \{v_1(c_1, A_2; a), v_1(c_1, b_2; a), v_1(c_1, A_2; b), v_1(c_1, b_2; b)\} \\ &= \quad \min\left\{\sqrt{6}, \sqrt{5}, \sqrt{4}, \sqrt{4}\right\} = \sqrt{4}. \end{aligned}$$

So when she observes the event A_1, she has no incentive to lie, i.e., $s_1(A_1) = A_1$ constitutes part of a maximin equilibrium of the game.

If player 1 is at $\mathcal{I}_1'$, then she must have seen the event c_1 from nature. Figure 4b shows that, if she tells the truth (reports c_1), then she will get 1 unit of the good, no matter what player 2 reports; but if she lies (reports A_1), then she may end up with, 0 or 1 unit of the good. Here, telling the truth (reports c_1) gives her a lower bound payoff of

$$\min\{v_1(c_1, A_2; c), v_1(c_1, b_2; c)\} = \min\left\{\sqrt{1}, \sqrt{1}\right\} = 1;$$

lying (reports A_1) gives her a lower bound payoff of

$$\min\{v_1(A_1, A_2; c), v_1(A_1, b_2; c)\} = \min\left\{\sqrt{0}, \sqrt{1}\right\} = 0.$$

Figure 4: At player 1's information sets
(simultaneous moves)

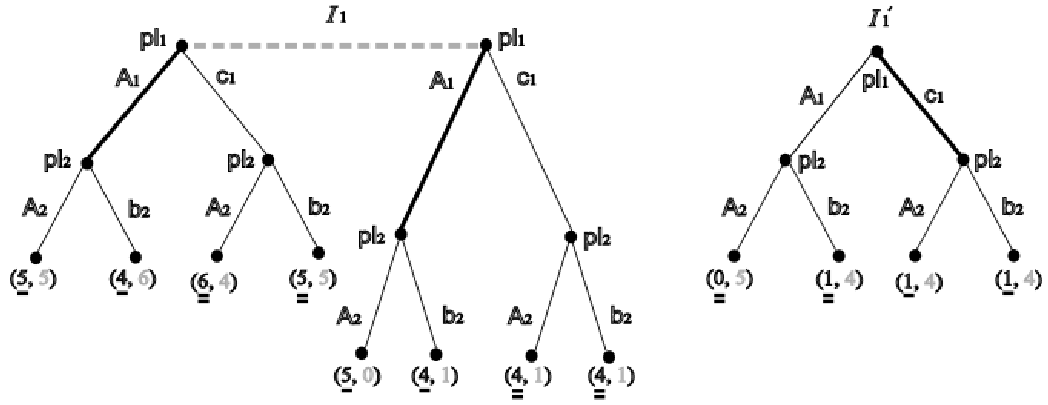


Figure 4a Figure 4b

So when she observes the event c_1, she should report the event c_1, i.e., $s_1(c_1) = c_1$ constitutes part of a maximin equilibrium of the game.

Now, turn to player 2. He has two information sets also, $\mathcal{I}_2$ and $\mathcal{I}_2^{'}$. (Figure 5)

If player 2 is at $\mathcal{I}_2$, then he must have seen the event A_2 from nature. Figure 5a shows that, being truthful (reports A_2), he may end up with, from the left to the right, 5, 4, 5 or 4 units of the good; and by lying (reports b_2), he may end up with, from the left to the right, 6, 5, 4 or 4 units of the good.

Clearly, telling the truth (reports A_2) gives him a lower bound payoff of

$$\begin{aligned} \min & \quad \{v_2(A_1, A_2; a), v_2(c_1, A_2; a), v_2(A_1, A_2; c), v_2(c_1, A_2; c)\} \\ = & \quad \min\left\{\sqrt{5}, \sqrt{4}, \sqrt{5}, \sqrt{4}\right\} = \sqrt{4}; \end{aligned}$$

lying (reports b_2) gives her a lower bound payoff of

$$\begin{aligned} \min & \quad \{v_2(A_1, b_2; a), v_2(c_1, b_2; a), v_2(A_1, b_2; c), v_2(c_1, b_2; c)\} \\ = & \quad \min\left\{\sqrt{6}, \sqrt{5}, \sqrt{4}, \sqrt{4}\right\} = \sqrt{4}. \end{aligned}$$

So when he observes the event A_2, he has no incentive to lie, i.e., $s_2(A_2) = A_2$ constitutes part of a maximin equilibrium of the game.

If player 2 is at $\mathcal{I}_2^{'}$, then he must have seen the event b_2 from nature. Figure 5b shows that if he tells the truth, then he will get 1 unit of the good, no matter what player 1 reports; but if he lies, then he may end up with, 0 or 1 unit of the good. Here, telling the truth (reports b_2) gives him a lower bound payoff of

$$\min\{v_2(A_1, b_2; b), v_2(c_1, b_2; b)\} = \min\left\{\sqrt{1}, \sqrt{1}\right\} = 1;$$

Figure 5: At player 2's information sets
(simultaneous moves)

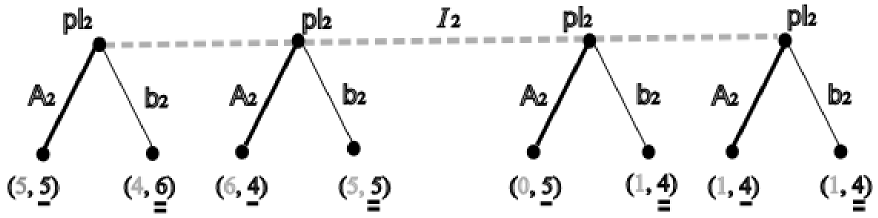


Figure 5a

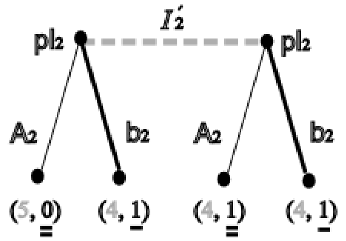


Figure 5b

lying (reports A_2) gives him a lower bound payoff of

$$\min\left\{v_2\left(A_1, A_2; b\right), v_2\left(c_1, A_2; b\right)\right\} = \min\left\{\sqrt{0}, \sqrt{1}\right\} = 0.$$

So when he observes the event b_2, he should report the event b_2, i.e., $s_2\left(b_2\right) = b_2$ constitutes part of a maximin equilibrium of the game.

Now, put together, the strategy profile

$$s = \left(s_1\left(A_1\right) = A_1, s_1\left(c_1\right) = c_1; s_2\left(A_2\right) = A_2, s_2\left(b_2\right) = b_2\right),$$

is a maximin equilibrium of the game. It is, in fact, the only maximin equilibrium of the game[23].

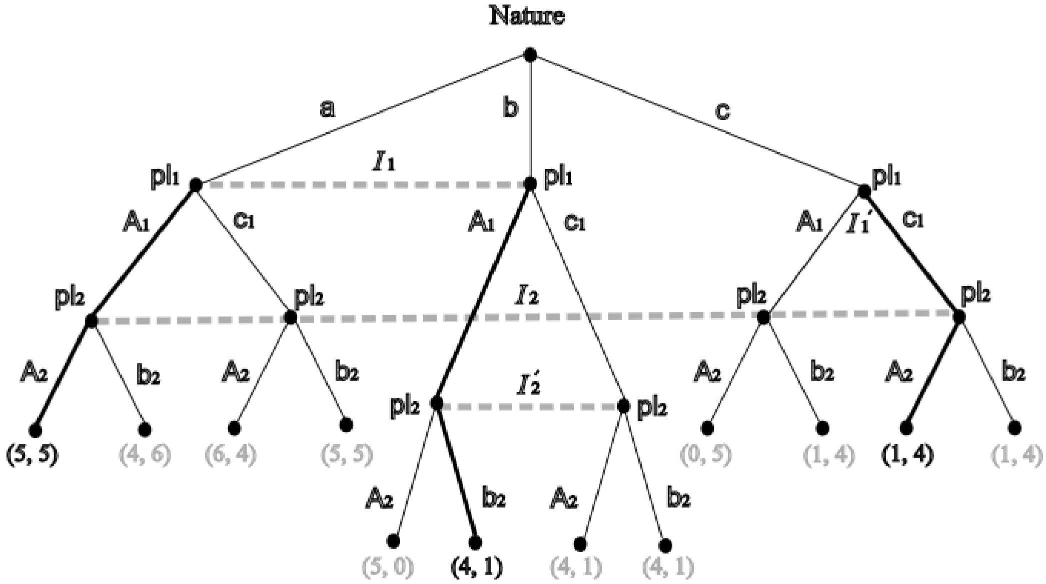
The equilibrium report paths are $s\left(a\right) = \left(s_1\left(A_1\right), s_2\left(A_2\right)\right) = \left(A_1, A_2\right)$, $s\left(b\right) = \left(s_1\left(A_1\right), s_2\left(b_2\right)\right) = \left(A_1, b_2\right)$ and $s\left(c\right) = \left(s_1\left(c_1\right), s_2\left(A_2\right)\right) = \left(c_1, A_2\right)$. (as marked in Figure 6)

Now, it can be easily checked that the maximin core allocation

$$x = \begin{pmatrix} x_1\left(a\right) & x_1\left(b\right) & x_1\left(c\right) \\ x_2\left(a\right) & x_2\left(b\right) & x_2\left(c\right) \end{pmatrix} = \begin{pmatrix} 5 & 4 & 1 \\ 5 & 1 & 4 \end{pmatrix}$$

[23]We assume that a player lies, only if he can benefit from doing so.

Figure 6: The equilibrium report paths
(Simultaneous moves)



is implemented, since we have

$$g_1\left(s\left(a\right),a\right) = g_1\left(\left(A_1, A_2\right),a\right) = 5 + 5 - 5 = 5 = x_1\left(a\right),$$

and similarly, we have $g_2\left(s\left(a\right),a\right) = 5 = x_2\left(a\right)$, $g_1\left(s\left(b\right),b\right) = 4 = x_1\left(b\right)$, $g_2\left(s\left(b\right),b\right) = 1 = x_2\left(b\right)$, $g_1\left(s\left(c\right),c\right) = 1 = x_1\left(c\right)$, $g_2\left(s\left(c\right),c\right) = 4 = x_2\left(c\right)$. These outcomes are illustrated in Figure 6, as pairs following the equilibrium paths.

4.6 Proof of the main theorem

To ease the explanation, we introduce a few notations. We use $\mathcal{F}_{-i} := \times_{j\neq i}\mathcal{F}_j$ to denote the action set of all the players except player i, and $E_{-i}^{\mathcal{F}_{-i}} := \left(E_1^{\mathcal{F}_1}, \cdots, E_{i-1}^{\mathcal{F}_{i-1}}, E_{i+1}^{\mathcal{F}_{i+1}}, \cdots, E_N^{\mathcal{F}_N}\right) \in \mathcal{F}_{-i}$ reports of all the players expect player i.

Furthermore, we write $\omega \in E_{-i}^{\mathcal{F}_{-i}}$ or $E_{-i}^{\mathcal{F}_{-i}}\left(\omega\right)$, if the state ω belongs to each element in the list $\left(E_1^{\mathcal{F}_1}, \cdots, E_{i-1}^{\mathcal{F}_{i-1}}, E_{i+1}^{\mathcal{F}_{i+1}}, \cdots, E_N^{\mathcal{F}_N}\right)$; and we use $E_i^{\mathcal{F}_i} \bigcap E_{-i}^{\mathcal{F}_{-i}} := \bigcap_{j\in I} E_j^{\mathcal{F}_j}$ to denote the information revealed by the reports of all the players.

Let x be a maximin core allocation and suppose that the mechanism Γ does not have a truth telling maximin equilibrium. Then, there must exist a player i, an event $E_i^{\mathcal{F}_i}$, and a lie $\tilde{E}_i^{\mathcal{F}_i} \in \mathcal{F}_i$ (clearly, $\tilde{E}_i^{\mathcal{F}_i} \neq E_i^{\mathcal{F}_i}$), such that when the player i observes the event $E_i^{\mathcal{F}_i}$, he can insure a better lower bound payoff by lying, i.e.,

$$\min_{\substack{E_{-i}^{\mathcal{F}_{-i}}\in\mathcal{F}_{-i};\\ \omega'\in E_i^{\mathcal{F}_i}}} \left\{v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}; \omega'\right)\right\} < \min_{\substack{E_{-i}^{\mathcal{F}_{-i}}\in\mathcal{F}_{-i};\\ \omega'\in E_i^{\mathcal{F}_i}}} \left\{v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}; \omega'\right)\right\}. \tag{5}$$

We will show, in Step 1 and 2, that (5) cannot hold, and therefore every game Γ has a truth telling maximin equilibrium.

To ease the explanation, denote the LHS of (5) by

$$v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) := \min_{E_{-i}^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\, \omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}; \omega'\right)\right\},$$

where $E_{-i}^* \in \mathcal{F}_{-i}$ and $\omega^* \in E_i^{\mathcal{F}_i}$ solve the minimization problem above.

Step 1 We will show that if $v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) = u_i\left(x_i\left(\omega^*\right); \omega^*\right)$ and (5) holds, then x fails to be a maximin core allocation.

Notice, $v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) = u_i\left(x_i\left(\omega^*\right); \omega^*\right)$ implies[24]

$$v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) = \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\}. \tag{6}$$

Also, (5) implies[25]

$$v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) < \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\}. \tag{7}$$

Now, (6) and (7) together imply

$$\min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\} < \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\}. \tag{8}$$

Finally, Lemma 1 (see the Appendix) shows that if (8) holds, then x fails to be a maximin core allocation, a contradiction.

Step 2 We will show that if $v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) \neq u_i\left(x_i\left(\omega^*\right); \omega^*\right)$ and (5) holds, then x fails to be a maximin core allocation.

[24] Note $\omega^* \in E_i^{\mathcal{F}_i}$, and

$$\begin{aligned} v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) &:= \min_{\substack{E_{-i}^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\\ \omega' \in E_i^{\mathcal{F}_i}}} \left\{ v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}; \omega'\right)\right\} \le \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\} \\ &\le v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega^*\right); \omega^*\right) = u_i\left(x_i\left(\omega^*\right); \omega^*\right), \end{aligned}$$

imply that we must have equality throughout.

[25] Since by the definition of a minimum, we have that

$$\min_{\substack{E_{-i}^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\\ \omega' \in E_i^{\mathcal{F}_i}}} \left\{ v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}; \omega'\right)\right\} \le \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\}.$$

Notice, $v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) \neq u_i\left(x_i\left(\omega^*\right); \omega^*\right)$ holds, only if $E_i^{\mathcal{F}_i}$, E_{-i}^* are compatible at the state ω^*, and $E_i^{\mathcal{F}_i} \cap E_{-i}^* = \{\tilde{\omega}\}$ where $\tilde{\omega}$ is some state different from ω^*. This implies that we must have

$$v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) = u_i\left(e_i\left(\omega^*\right) + x_i\left(\tilde{\omega}\right) - e_i\left(\tilde{\omega}\right); \omega^*\right) = u_i\left(x_i\left(\tilde{\omega}\right); \omega^*\right) = u_i\left(x_i\left(\tilde{\omega}\right); \tilde{\omega}\right) \tag{9}$$

(recall that both the endowment e_i and the utility function u_i are $\mathcal{F}_i$-measurable).

Now $\tilde{\omega} \in E_i^{\mathcal{F}_i}$ and the equation (9) together imply that

$$v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) = \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\}. \tag{10}$$

It follows from (5) that

$$v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^*; \omega^*\right) < \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\}. \tag{11}$$

Combining (10) and (11), it follows that

$$\min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\} < \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\}. \tag{12}$$

By Lemma 1 in the Appendix, if (12) holds, then x fails to be a maximin core allocation, a contradiction.

Therefore, we conclude that the mechanism Γ has a truth telling maximin equilibrium, i.e., $s^T \in \mathbb{ME}(\Gamma)$.

We now show that the truth telling maximin equilibrium is the only maximin equilibrium of the mechanism Γ, i.e., $\left\{s^T\right\} = \mathbb{ME}(\Gamma)$. So suppose otherwise, that is, suppose both s^T and s^* are maximin equilibria of the mechanism Γ, and $s^T \neq s^*$.

The truth telling strategy profile s^T is different from the strategy profile s^*, implies that there must exist a player i and an event $E_i^{\mathcal{F}_i}$, such that

$$s_i^T\left(E_i^{\mathcal{F}_i}\right) = E_i^{\mathcal{F}_i} \neq \tilde{E}_i^{\mathcal{F}_i} = s_i^*\left(E_i^{\mathcal{F}_i}\right). \tag{13}$$

But $s_i^*\left(E_i^{\mathcal{F}_i}\right) = \tilde{E}_i^{\mathcal{F}_i} \neq E_i^{\mathcal{F}_i}$ holds, only if lying makes player i strictly better off upon observing the event $E_i^{\mathcal{F}_i}$, i.e.,

$$\min_{\substack{E_{-i}^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\\ \omega' \in E_i^{\mathcal{F}_i}}} \left\{v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}; \omega'\right)\right\} < \min_{\substack{E_{-i}^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\\ \omega' \in E_i^{\mathcal{F}_i}}} \left\{v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}; \omega'\right)\right\},$$

a contradiction to the fact that the truth telling strategy profile constitutes a maximin equilibrium of the mechanism.

Clearly, the maximin core allocation x is implemented. Indeed, under the truth telling strategy profile s^T, the list of reports associated to each state ω is

$$s^T(\omega) = \left(E_1^{\mathcal{F}_1}(\omega), \cdots, E_N^{\mathcal{F}_N}(\omega)\right).$$

That is, the players always tell the truth. As a consequence, we have

$$\begin{aligned} g_i\left(s^T(\omega), \omega\right) &= g_i\left(\left(E_1^{\mathcal{F}_1}(\omega), \cdots, E_N^{\mathcal{F}_N}(\omega)\right), \omega\right) \\ &= e_i(\omega) + D_i\left(x - e, \left(E_1^{\mathcal{F}_1}(\omega), \cdots, E_N^{\mathcal{F}_N}(\omega)\right), \omega\right) \\ &= e_i(\omega) + x_i(\omega) - e_i(\omega) = x_i(\omega), \end{aligned}$$

for each $\omega \in \Omega$ and for each $i \in I$ – the requirement of definition 13.

5 Concluding remarks

The maximin core notion (a cooperative concept) introduces a collection of desirable allocations, which are both incentive compatible and efficient, a property that the rational expectations equilibrium and the Walrasian expectations equilibrium in the sense of Radner [12] [14] fail to have.

We showed that any maximin core allocation is implementable by means of noncooperative behavior under ambiguity. That is, given any arbitrary maximin core allocation, its corresponding direct revelation mechanism yields the core allocation as its unique maximin equilibrium outcome.

The new equilibrium notion (maximin equilibrium) takes into account the agents' information constraints – the inability to assign a probability to every state of nature, and to each possible action of his opponents. In a maximin equilibrium, each agent maximizes his payoff lowest bound, i.e., each agent maximizes the payoff that takes into account the worst actions of all the other agents against him and also the worst state that can occur. It turns out that, such a noncooperative behavior (i.e., the maximin equilibrium) enables agents to reach a desirable outcome, i.e., a maximin core allocation, which is both incentive compatible and efficient.

Interestingly, our counter example (Example 3) in Section 4.3 shows that, the Bayesian Nash solution concept fails. In particular, the Bayesian Nash solution concept does not predict truth telling, contrary to the maximin equilibrium solution concept. This further highlights the disagreements in predictions, created by assuming the agents know more than what they actually know, as originally demonstrated in the Ellsberg's experiment [6].

It is an open question whether or not the result of this paper holds in the presence of infinitely many states. The difficulty arises from the fact that the minimum of the utility over even countably many states may not exist.

6 Appendix

Lemma 1. *Given a direct revelation mechanism* $\Gamma = \left\langle I, S, x-e, \{g_i\}_{i\in I}, \{u_i\}_{i\in I}\right\rangle$*, if the allocation* x *is a maximin core allocation, then there does not exist a player* i*, an event* $E_i^{\mathcal{F}_i}$*, and a lie* $\tilde{E}_i^{\mathcal{F}_i} \in \mathcal{F}_i$ *(clearly,* $\tilde{E}_i^{\mathcal{F}_i} \neq E_i^{\mathcal{F}_i}$*), such that*[26]

$$\min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\} < \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\}. \tag{14}$$

Proof. Suppose that there exist a player i, an event $E_i^{\mathcal{F}_i}$, and a lie $\tilde{E}_i^{\mathcal{F}_i} \neq E_i^{\mathcal{F}_i}$, such that (14) holds. We will show that the feasible allocation x fails to be Pareto optimal under the maximin preferences, and therefore fails to be a maximin core allocation – an idea similar to the one in theorem 4.1 of de Castro-Yannelis [3].

Note, for each $\omega' \in E_i^{\mathcal{F}_i}$, we have

$$v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right) = u_i\left(x_i\left(\omega'\right); \omega'\right),$$

and therefore, the LHS of (14) can be rewritten as

$$\min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(E_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\} = \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ u_i\left(x_i\left(\omega'\right); \omega'\right)\right\}.$$

Define an i-allocation of player i, $z_i(\cdot)$, such that for each $\omega' \in E_i^{\mathcal{F}_i}$,

$$v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right) = u_i\left(z_i\left(\omega'\right); \omega'\right),$$

and therefore, the RHS of (14) can be rewritten as

$$\min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ v_i\left(\tilde{E}_i^{\mathcal{F}_i}, E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right); \omega'\right)\right\} = \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ u_i\left(z_i\left(\omega'\right); \omega'\right)\right\}.$$

It follows from (14) that

$$\min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ u_i\left(x_i\left(\omega'\right); \omega'\right)\right\} < \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{ u_i\left(z_i\left(\omega'\right); \omega'\right)\right\}. \tag{15}$$

Since u_i is strictly monotone in consumption, and $\mathcal{F}_i$-measurable, (15) then implies,

$$\text{for each } \omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{ u_i\left(x_i\left(\omega''\right); \omega''\right)\right\}, \text{ we have } z_i\left(\omega'\right) = x_i\left(\omega'\right) + \epsilon\left(\omega'\right), \tag{16}$$

where $\epsilon\left(\omega'\right)$ is a none zero vector in $\mathbb{R}_+^l$. For (16) to hold, it must be the case that for *each* $\omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{ u_i\left(x_i\left(\omega''\right); \omega''\right)\right\}$, *there exists a state* $\tilde{\omega}$, such that

[26]In words, (14) says that if all the other players are truthful, then player i can insure a higher lower bound payoff by lying under the event $E_i^{\mathcal{F}_i}$.

1. $\tilde{E}_i^{\mathcal{F}_i} \cap E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right) = \{\tilde{\omega}\}$,

2. $\tilde{E}_i^{\mathcal{F}_i}$, $E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right)$ are compatible at ω', and

3. $z_i\left(\omega'\right) = e_i\left(\omega'\right) + x_i\left(\tilde{\omega}\right) - e_i\left(\tilde{\omega}\right) = x_i\left(\omega'\right) + \epsilon\left(\omega'\right)$, where $\epsilon\left(\omega'\right)$ is a none zero vector in $\mathbb{R}_+^l$.

Let $\left\{\omega', \tilde{\omega}\right\}$ denote a set, containing a state $\omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{u_i\left(x_i\left(\omega''\right); \omega''\right)\right\}$ and *its* corresponding[27] $\tilde{\omega}$. It follows by 1 above that, for each $\omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{u_i\left(x_i\left(\omega''\right); \omega''\right)\right\}$, the set $\left\{\omega', \tilde{\omega}\right\}$ is a subset of $E_{-i}^{\mathcal{F}_{-i}}\left(\omega'\right)$.

Now, we are ready to define an allocation y that Pareto improves x under the maximin preferences. Define for each $j \in I$, the j-allocation $y_j(\cdot)$ by

$$y_j(\omega') = \begin{cases} z_j\left(\omega'\right) = e_j\left(\omega'\right) + x_j\left(\tilde{\omega}\right) - e_j\left(\tilde{\omega}\right) & \text{if } \omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{u_i\left(x_i\left(\omega''\right); \omega''\right)\right\} \\ x_j\left(\omega'\right) & \text{otherwise.} \end{cases}$$

Notice that the allocation y is feasible.

Indeed, for a state $\omega' \notin \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{u_i\left(x_i\left(\omega''\right); \omega''\right)\right\}$, we have

$$\sum_{j \in I} y_j\left(\omega'\right) = \sum_{j \in I} x_j\left(\omega'\right) = \sum_{j \in I} e_j\left(\omega'\right);$$

and for a state $\omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{u_i\left(x_i\left(\omega''\right); \omega''\right)\right\}$, we have

$$\sum_{j \in I} y_j\left(\omega'\right) = \sum_{j \in I} z_j\left(\omega'\right) = \sum_{j \in I} e_j\left(\omega'\right) + \sum_{j \in I} x_j\left(\tilde{\omega}\right) - \sum_{j \in I} e_j\left(\tilde{\omega}\right) = \sum_{j \in I} e_j\left(\omega'\right)$$

(recall that x is a feasible allocation at the state $\tilde{\omega}$).

From (16) and the definition of y_i, we have

$$\min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{u_i\left(y_i\left(\omega'\right); \omega'\right)\right\} > \min_{\omega' \in E_i^{\mathcal{F}_i}} \left\{u_i\left(x_i\left(\omega'\right); \omega'\right)\right\} \tag{17}$$

under the event $E_i^{\mathcal{F}_i}$; and for any other event $\hat{E}_i^{\mathcal{F}_i} \in \mathcal{F}_i$, we have

$$\min_{\omega' \in \hat{E}_i^{\mathcal{F}_i}} \left\{u_i\left(y_i\left(\omega'\right); \omega'\right)\right\} = \min_{\omega' \in \hat{E}_i^{\mathcal{F}_i}} \left\{u_i\left(x_i\left(\omega'\right); \omega'\right)\right\}.$$

[27] To avoid confusion, it is worthwhile to re-emphasize that different $\omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{u_i\left(x_i\left(\omega''\right); \omega''\right)\right\}$ may be matched with a different $\tilde{\omega}$.

Therefore, combined with the assumption on $\mu_i(\cdot)$ (Assumption 2), we conclude that, for the player i,

$$\sum_{E_i \in \mathcal{F}_i} \left(\min_{\omega' \in E_i} u_i \left(y_i \left(\omega' \right); \omega' \right) \right) \mu_i(E_i) > \sum_{E_i \in \mathcal{F}_i} \left(\min_{\omega' \in E_i} u_i \left(x_i \left(\omega' \right); \omega' \right) \right) \mu_i(E_i). \quad (18)$$

Here we abuse the notations in (18) slightly, in particular, E_i denotes an arbitrary event in $\mathcal{F}_i$. Using the compact notation as in (2), we can rewrite (18) as $y_i \succ_i^{MP} x_i$, i.e., player i strictly prefers the i-allocation y_i to the i-allocation x_i. Now, it remains to show that for any other player $k \neq i$, we have $y_k \succeq_k^{MP} x_k$.

Fix an arbitrary player $k \neq i$, and an arbitrary event that player k may observe, $E_k^{\mathcal{F}_k} \in \mathcal{F}_k$. Notice, if the event $E_k^{\mathcal{F}_k}$ contains a state $\omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{ u_i \left(x_i \left(\omega'' \right); \omega'' \right) \right\}$, then it contains the set $\left\{ \omega', \tilde{\omega} \right\}$. So, by the $\mathcal{F}_k$-measurability of e_k, we have $z_k \left(\omega' \right) = e_k \left(\omega' \right) + x_k(\tilde{\omega}) - e_k(\tilde{\omega}) = x_k(\tilde{\omega})$. Now, for the event $E_k^{\mathcal{F}_k}$, define $X_k = \left\{ x_k \left(\omega' \right) : \omega' \in E_k^{\mathcal{F}_k} \right\}$ and $Y_k = \left\{ y_k \left(\omega' \right) : \omega' \in E_k^{\mathcal{F}_k} \right\}$. We have $Y_k \subset X_k$. Indeed, if $\omega' \in E_k^{\mathcal{F}_k}$ and $\omega' \in \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{ u_i \left(x_i \left(\omega'' \right); \omega'' \right) \right\}$, then

$$y_k \left(\omega' \right) = z_k \left(\omega' \right) = x_k(\tilde{\omega}) \in X_k;$$

and if $\omega' \in E_k^{\mathcal{F}_k}$ and $\omega' \notin \arg\min_{\omega'' \in E_i^{\mathcal{F}_i}} \left\{ u_i \left(x_i \left(\omega'' \right); \omega'' \right) \right\}$, then

$$y_k \left(\omega' \right) = x_k \left(\omega' \right) \in X_k.$$

Therefore, we have that

$$\min_{\omega' \in E_k^{\mathcal{F}_k}} \left\{ u_k \left(y_k \left(\omega' \right); \omega' \right) \right\} \geq \min_{\omega' \in E_k^{\mathcal{F}_k}} \left\{ u_k \left(x_k \left(\omega' \right); \omega' \right) \right\}.$$

Since the event $E_k^{\mathcal{F}_k} \in \mathcal{F}_k$ is arbitrary, we conclude that

$$\sum_{E_k^{\mathcal{F}_k} \in \mathcal{F}_k} \left(\min_{\omega' \in E_k^{\mathcal{F}_k}} u_k \left(y_k \left(\omega' \right); \omega' \right) \right) \mu_k \left(E_k^{\mathcal{F}_k} \right) \geq \sum_{E_k^{\mathcal{F}_k} \in \mathcal{F}_k} \left(\min_{\omega' \in E_k^{\mathcal{F}_k}} u_k \left(x_k \left(\omega' \right); \omega' \right) \right) \mu_k \left(E_k^{\mathcal{F}_k} \right).$$

Also, since player $k \neq i$ is arbitrary, we have for every player $k \neq i$, $y_k \succeq_k^{MP} x_k$.

Thus, the allocation y Pareto improves the allocation x under the maximin preferences, i.e., x fails to be a maximin core allocation. This contradiction completes the proof of Lemma 1. □

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