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1 Introduction

1.1. In this paper we consider problems of stopping a Brownian motion or a geometric Brownian motion optimally on a finite time interval when it has a *disorder*, i. e. its drift coefficient changes at some unknown moment of time from a positive value to a negative one. We look for the stopping time that maximizes the expected value of the stopped process.

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Suppose we sequentially observe the process $X = (X_t)_{t\geq 0}$,

$$X_t = \mu_1 t + (\mu_2 - \mu_1)(t-\theta)^+ + \sigma B_t,$$

or, equivalently in stochastic differentials,

$$dX_t = \big[\mu_1 \mathbf{I}(t < \theta) + \mu_2 \mathbf{I}(t \geq \theta)\big] dt + \sigma dB_t, \qquad X_0 = 0,$$

where $\mu_1 > 0 > \mu_2$, $\sigma > 0$ are known numbers and θ is an unknown *time of a disorder* – a moment when the drift coefficient of X changes from value μ_1 to value μ_2. The process X is called a (linear) *Brownian motion with a disorder.*

Adopting the Bayesian approach, we let θ be a random variable defined on $(\Omega, \mathcal{F}, \mathsf{P})$ and independent of B. In this paper, in view of applications (see below), we assume that θ is *uniformly distributed* on a finite interval $[0, T]$. Without loss of generality, we take $T = 1$, and suppose the distribution function $G(t) = \mathsf{P}(\theta \leq t)$ of θ is given by

$$G(t) = G(0) + \rho t \text{ for } 0 \leq t < 1, \qquad G(1) = 1,$$

where $G(0) \in [0, 1)$ is the probability that the disorder presents from the beginning, and $0 < \rho < 1 - G(0)$ is the density of θ. The probability $\mathsf{P}(\theta = 1)$ may be strictly positive.

Let $\mathfrak{M}_1$ denote the class of all stopping times $\tau \leq 1$ of the process X. We consider the following two optimal stopping problems for X and the exponent of X (a *geometric Brownian motion with a disorder*):

$$V^{(l)} = \sup_{\tau \in \mathfrak{M}_1} \mathsf{E} X_\tau, \qquad V^{(g)} = \sup_{\tau \in \mathfrak{M}_1} \mathsf{E} \exp(X_\tau - \sigma^2 \tau/2). \tag{1}$$

The problems consist in finding the values $V^{(l)}$, $V^{(g)}$ and finding the stopping times $\tau_*^{(l)}$ and $\tau_*^{(g)}$ at which the infima are attained (if such stopping times exist). The superscript (l) stands for the problem for a *linear* Brownian motions, while (g) stands for a *geometric* Brownian motion.

Observe that, roughly speaking, the processes X_t and $\exp(X_t - \sigma^2 t/2)$ increase on average "up to time θ" and decrease on average "after time θ". But since θ is not a stopping time, we cannot simply take $\tau_* = \theta$, and need to stop by detecting the disorder based on sequential observation of X.

We provide a solution to problems (1) using the results obtained in the recent paper [10]. The central idea is based on a reduction to a Markovian optimal stopping problem using a change of measure. This approach has already been applied in the literature, but we were able to generalize it to the case of a finite time interval.

1.2. For economic applications, the problems considered are related to the question when to quit a financial "bubble".

Suppose that the price of an asset is modelled by a geometric Brownian motion with a disorder $S = (S_t)_{t \ge 0}$:

$$S_t = \exp(X_t - \sigma^2 t/2),$$

or, equivalently,

$$dS_t = \big[\mu_1 \mathbf{I}(t < \theta) + \mu_2 \mathbf{I}(t \ge \theta)\big] S_t dt + \sigma S_t dB_t, \qquad S_0 = 1.$$

In other words, the price initially has a positive trend, but then the bubble bursts (at an unknown time θ) and the trend becomes negative.

Let $t = 0$ correspond to the "current" moment of time, when one holds an asset with positive trend. Usually, it is possible to predict that the bubble will burst until some (maybe, distant) time T in the future. Then one is interested in the question when it is optimal to sell the asset maximizing the gain.

If nothing is known about the actual distribution of θ, it is natural to assume that θ is uniformly distributed on $[0, T]$ (since the uniform distribution has the maximum entropy on a finite interval). Interpreting the quantity $\mathsf{E} S_\tau$ as the average gain achieved by selling the asset at time τ, problem (1) for a geometric Brownian motion seeks for the optimal time to sell the asset. The problem for a linear Brownian motion can be thought of as a problem of finding the optimal time to sell the asset provided that a trader has a logarithmic utility function, i. e. maximizes $\mathsf{E}\log(S_\tau)$, which is equivalent to maximizing $\mathsf{E}X_\tau$ with $\mu_1' = \mu_1 - \sigma^2/2$, $\mu_2' = \mu_2 - \sigma^2/2$.

An interesting result that follows from the solution of problems (1) is the qualitative difference between *risk-neutral* traders who maximize $\mathsf{E}S_\tau$ and *risk-averse* traders who maximize $\mathsf{E}\log(S_\tau)$: if $\mathsf{P}(\theta < 1) = 1$ (i. e. the distribution of θ has no mass at $t = 1$), then a risk-neutral trader will sell the asset strictly before time $T = 1$ with probability one ($\mathsf{P}(\tau_*^{(g)} < 1) = 1$), while a risk-averse trader will wait until the end of the time interval with positive probability ($\mathsf{P}(\tau_*^{(l)} = 1) > 0$) (see the Theorem and Remark 1 in Section 2).

1.3. Disorder detection problems have a long history starting with work of Page, Roberts, Shewart, Shiryaev and others in the 1930–60s. Problems (1), for example, was considered in the papers [1, 2, 7], assuming that the moment of disorder θ is exponentially distributed. In [1], the problem for a geometric Brownian motion was solved. It was shown that if μ_1, μ_2, σ satisfy some relation, the optimal stopping time is the first hitting time of the posterior probability process $\pi_t = \mathsf{P}(\theta \le t \mid \mathcal{F}_t^X)$, where $\mathcal{F}_t^X = \sigma(X_s; s \le t)$, to some level. In [2] this result was extended to all values of μ_1, μ_2, σ and the optimal stopping level was found. In [7], the problem for a linear Brownian motion was considered on a finite interval, i. e. assuming that one should choose τ not exceeding some time horizon T (but the disorder may happen after T). It turned out that the problem is equivalent to the original Bayesian setting of the disorder problem when one seeks for a stopping time minimizing the average detection delay and the probability of a false alarm (see e. g. [4, 8, 9]). The paper [7] also briefly discusses the optimal stopping problem for a geometric Brownian motion on a finite interval, but does not provide an explicit solution.

2 The main results

2.1. Let $\mu = (\mu_1 - \mu_2)/\sigma$ denote the *signal-to-noise ratio*. For convenience of notation, introduce the process $\widetilde{X} = (\widetilde{X}_t)_{t\ge0}$, $\widetilde{X}_t = (X_t - \mu_1 t)/\sigma$, which is a Brownian motion with the unit diffusion coefficient and the drift coefficient changing at time θ from value 0 to value $(-\mu)$.

Introduce the process $\psi = (\psi_t)_{t\ge0}$, the so-called *Shiryaev–Roberts statis-*

tic[1]:

$$\psi_t = \exp\bigl(-\mu \widetilde{X}_t - \mu^2 t/2\bigr) \left(\psi_0 + \rho \int_0^t \exp\bigl(\mu \widetilde{X}_s + \mu^2 s/2\bigr) ds\right) \tag{2}$$

with $\psi_0 = G(0)$. Applying the Itô formula it is easy to see that ψ satisfies the stochastic differential equation

$$d\psi_t = \rho dt - \mu \psi_t d\widetilde{X}_t. \tag{3}$$

On the measurable space $(\Omega, \mathcal{F}_1^X)$, $\mathcal{F}_1^X = \sigma(X_t; t \le 1)$, define the probability measures $\mathsf{P}^{(l)}$ and $\mathsf{P}^{(g)}$ such that $\widetilde{X}_t$ is a standard Brownian motion under $\mathsf{P}^{(l)}$ and $(\widetilde{X}_t - \sigma t)$ is a standard Brownian motion under $\mathsf{P}^{(g)}$. These measures will be used to solve the problems $V^{(l)}$ and $V^{(g)}$ respectively. It is well-known (see, e. g., [5, Ch. 7]) that $\mathsf{P}^{(l)}$ and $\mathsf{P}^{(g)}$ are equivalent on the space $(\Omega, \mathcal{F}_1^X)$.

For any $x \ge 0$, by $\mathsf{E}_x^{(l)}[\,\cdot\,]$ and $\mathsf{E}_x^{(g)}[\,\cdot\,]$ we denote the mathematical expectations of functionals of the process $(\psi_t)_{t\ge 0}$ defined by (2)–(3) with the initial condition $\psi_0 = x$, when $\widetilde{X}$ is respectively a standard Brownian motion or a Brownian motion with drift σ. For brevity, instead of $\mathsf{E}_{G(0)}^{(l)}[\,\cdot\,]$ and $\mathsf{E}_{G(0)}^{(g)}[\,\cdot\,]$ we simply write $\mathsf{E}^{(l)}[\,\cdot\,]$ and $\mathsf{E}^{(g)}[\,\cdot\,]$.

The main result of the paper is the following theorem.

Theorem. *The optimal stopping times in the problems $V^{(l)}$ and $V^{(g)}$ are given respectively by*

$$\tau_*^{(l)} = \inf\{t \ge 0 : \psi_t \ge a^{(l)}(t)\} \wedge 1,$$
$$\tau_*^{(g)} = \inf\{t \ge 0 : \psi_t \ge a^{(g)}(t)\} \wedge 1,$$

where $a^{(l)}(t)$ and $a^{(g)}(t)$ are non-increasing non-negative functions on $[0,1]$ being the unique continuous solutions of the integral equations ($t \in [0,1]$)

$$\int_0^{1-t} \mathsf{E}_{a^{(l)}(t)}^{(l)}\bigl[\bigl(\mu_1 - (\mu_1-\mu_2)\psi_s\bigr)\mathbf{I}\{\psi_s < a^{(l)}(t+s)\} ds\bigr] ds = 0, \tag{4}$$

$$\int_0^{1-t} \mathsf{E}_{a^{(g)}(t)}^{(g)}\bigl[e^{\mu_1 s}\bigl(\mu_1(1-G(t+s)) - |\mu_2|\psi_s\bigr)\mathbf{I}\{\psi_s < a^{(g)}(t+s)\}\bigr] ds = 0, \tag{5}$$

[1] In a general case, the Shiryaev–Roberts statistic is $\psi_t = \int_0^t (d\mathsf{P}_t^s/d\mathsf{P}_t) dG(s)$, where $\mathsf{P}_t = \mathsf{P} \mid \mathcal{F}_t^X$, $\mathsf{P}_t^s = \mathsf{P}^s \mid \mathcal{F}_t^X$ for the initial probability measure P and the measure P^s, corresponding to the disorder occurring at a fixed time s (for details, see the proof of the Lemma in Section 3).

satisfying the conditions

$$a^{(l)}(t) \geq \frac{\mu_1}{\mu_1 - \mu_2} \text{ for } t \in [0,1), \qquad a^{(l)}(1) = \frac{\mu_1}{\mu_1 - \mu_2}, \tag{6}$$

$$a^{(g)}(t) \geq \frac{\mu_1}{|\mu_2|}(1 - G(t)) \text{ for } t \in [0,1), \quad a^{(g)}(1) = \frac{\mu_1}{|\mu_2|}(1 - G(1-)). \tag{7}$$

The values $V^{(l)}$ and $V^{(g)}$ can be found by the formulas

$$V^{(l)} = \int_0^1 \mathsf{E}^{(l)}\big[\big(\mu_1 - (\mu_1 - \mu_2)\psi_s\big)\mathbf{I}\{\psi_s < a^{(l)}(s)\}ds\big]ds, \tag{8}$$

$$V^{(g)} = 1 + \int_0^1 \mathsf{E}^{(g)}\big[e^{\mu_1 s}\big(\mu_1(1 - G(s)) - |\mu_2|\psi_s\big)\mathbf{I}\{\psi_s < a^{(g)}(s)\}\big]ds. \tag{9}$$

Remarks. 1. Regarding the question of the difference between a risk-averse trader and a risk-neutral trader mentioned in Section 1, observe that if no mass of θ is concentrated at the point $T = 1$ (i. e. $\mathsf{P}(\theta < 1) = 1$), then $\mathsf{P}(\tau_*^{(g)} < 1) = 1$ because $a^{(g)}(1) = 0$, while $\mathsf{P}(\tau_*^{(l)} < 1) < 1$ because $a^{(l)}(1) > 0$ (in the latter case one can check that the process ψ stays below $a^{(l)}(t)$ on the whole interval $[0,1]$ with positive probability).

2. In the above mentioned papers [1, 2, 7], solutions to problems (1) when θ is exponentially distributed were given in terms of the posterior probability process $\pi_t = \mathsf{P}(\theta \leq t \mid \mathcal{F}_t^X)$. Using the Bayes formula, one can check that the processes π and ψ are connected by the formula $\psi_t = \pi_t(1 - G(t))/(1 - \pi_t)$ (see [10]). Consequently, it is easy to reformulate the Theorem in a such way that $\tau_*^{(l)}$ and $\tau_*^{(g)}$ are the first moments of time when the process π crosses time-dependent levels. We prefer to work with the process ψ because it has a somewhat simpler form.

2.2. Equations (4), (5) can be solved numerically by *backward induction*: we fix a partition $0 \leq t_0 < t_1 < \ldots < t_n = 1$ of the segment $[0,1]$ and sequentially find the values $a(t_n),\ a(t_{n-1}),\ \ldots, a(t_0)$. The value $a(1)$ can be found from condition (6) or (7) respectively. Having found the values $a(t_k), a(t_{k+1}), \ldots, a(t_n)$ and numerically computing integral (4) or (5) for $t = t_{k-1}$ through the values of the integrand at points $t_{k-1}, t_k, \ldots, t_n$, we obtain the equation, from which the value $a(t_{k-1})$ can be found. Repeating this procedure, we find the value of $a(t)$ at every point of the partition.

To compute the mathematical expectations in (4), (5), (8), (9), one can use the Monte–Carlo method or use the explicit formula for the transitional density of ψ (see e. g. [10], where it was obtained from the joint law of an exponent of a Brownian motion and its integral that was found in [6]).

As an example of a numerical solution of equations (4), (5), on Fig. 1 we present the optimal stopping boundaries for the case $\mu_1 = 1$, $\mu_2 = -1$, $\sigma = 1$, $\mathsf{P}(\theta \le t) = t$ for $t \le 1$.

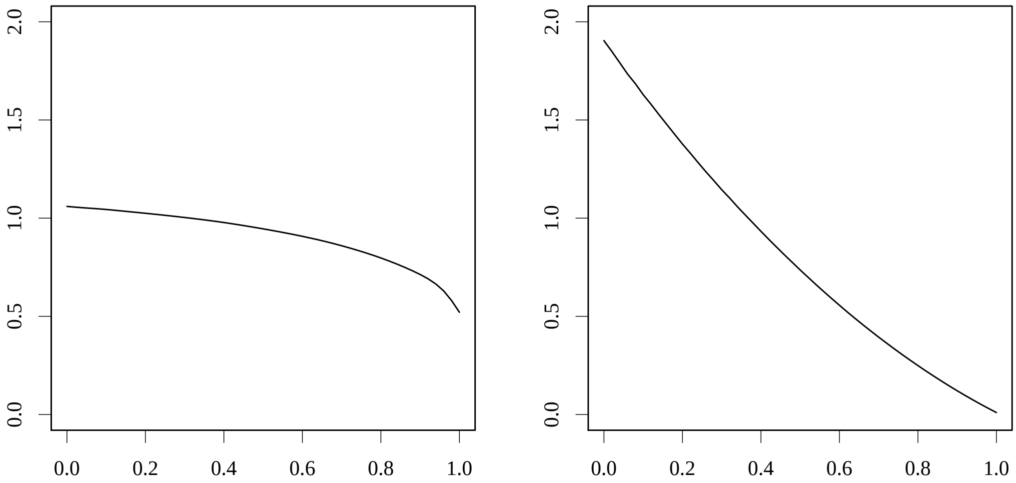

Figure 1: The optimal stopping boundaries in the problems $V^{(l)}$ (left) and $V^{(g)}$ (right) in the case $\mu_1 = 1$, $\mu_2 = -1$, $\sigma = 1$, $\mathsf{P}(\theta \le t) = t$ for $t \le 1$.

3 Proof of the theorem

3.1. To prove the theorem, we first reduce problems $V^{(l)}$ and $V^{(g)}$ to optimal stopping problems for the Shiryaev–Roberts statistic ψ, and then apply the Proposition from the Appendix, which was proved in [10]. The method we use is based on the ideas of [1, 3].

Lemma. *The following formulas hold:*

$$V^{(l)} = \sup_{\tau \in \mathfrak{M}_1} \mathsf{E}^{(l)} \int_0^\tau \big(\mu_1 - (\mu_1 - \mu_2)\psi_s\big) ds,$$

$$V^{(g)} = 1 + \sup_{\tau \in \mathfrak{M}_1} \mathsf{E}^{(g)} \int_0^\tau e^{\mu_1 s} \big[\mu_1(1 - G(s)) - |\mu_2|\psi_s\big] ds.$$

The supremum in each formula is attained at the stopping time which is optimal in the corresponding problem (1).

Proof. It is sufficient to show that for any stopping time $\tau \le 1$

$$\mathsf{E}X_\tau = \mathsf{E}^{(l)} \int_0^\tau \big(\mu_1 - (\mu_1 - \mu_2)\psi_s\big) ds, \tag{10}$$

$$\mathsf{E}\exp(X_\tau - \sigma^2\tau/2) = 1 + \mathsf{E}^{(g)} \int_0^\tau e^{\mu_1 s}\big[\mu_1(1 - G(s)) - |\mu_2|\psi_s\big] ds. \tag{11}$$

On the measurable space $(\Omega, \mathcal{F}_1^X)$ define the family of probability measures $(\mathsf{P}^u)_{0\le u\le 1}$ such that under P^u the disorder occurs at the fixed time u, i. e. for each $0 \le u \le 1$ the process X can be represented as $X_t = \mu_1 t + (\mu_2 - \mu_1)(t - u)^+ + \sigma B_t^{(u)}$, where $B^{(u)}$ is a standard Brownian motion under P^u. By $\mathsf{E}^u[\,\cdot\,]$ we denote the mathematical expectation with respect to P^u and by P_t, P_t^u, $\mathsf{P}_t^{(l)}$, $\mathsf{P}_t^{(g)}$ we denote the restrictions of the corresponding measures to the σ-algebra $\mathcal{F}_t^X = \sigma(X_s; s \le t)$, $0 \le t \le 1$.

Let us prove (10). Since the Brownian motion B is a martingale, we have

$$\mathsf{E}X_\tau = \mathsf{E}\big[\mu_1\tau + (\mu_2 - \mu_1)(\tau - \theta)^+\big]. \tag{12}$$

Consider the second term in the sum:

$$\mathsf{E}(\tau - \theta)^+ = \int_0^1 \mathsf{E}^u[(\tau - u)\mathbf{I}(\tau > u)]dG(u). \tag{13}$$

Observe that for any $0 \le u \le 1$ the following relation is valid:

$$\begin{aligned}\mathsf{E}^u[(\tau - u)\mathbf{I}(\tau > u)] &= \int_u^1 \mathsf{E}^u\mathbf{I}(s \le \tau) ds \\ &= \int_u^1 \mathsf{E}^{(l)}\big[\exp\big(-\mu(\widetilde{X}_s - \widetilde{X}_u) - \mu^2(s-u)/2\big)\mathbf{I}(s \le \tau)\big] ds,\end{aligned} \tag{14}$$

where we use that $\mathbf{I}(s \le \tau)$ is an $\mathcal{F}_s^X$-measurable random variable and

$$\frac{d\mathsf{P}_s^u}{d\mathsf{P}_s^{(l)}} = \exp\big(-\mu(\widetilde{X}_s - \widetilde{X}_u) - \mu^2(s-u)/2\big)$$

(the explicit formula for the density of the measure generated by one Itô process with respect to the measure generated by another Itô process can be found in e. g. [5]). From (13)–(14), changing the order of integration we find

$$\mathsf{E}(\tau - \theta)^+ = \int_0^\tau \int_0^s \exp\big(-\mu(\widetilde{X}_s - \widetilde{X}_u) - \mu^2(s-u)/2\big) dG(u) ds = \int_0^s \psi_s ds.$$

Then using (12), we obtain representation (10).

Let us prove (11). We have

$$\frac{d\mathsf{P}_t^s}{d\mathsf{P}_t^{(g)}} = \begin{cases} \exp\bigl(-\sigma\widetilde{X}_t + \sigma^2 t/2\bigr)\cdot\exp\bigl(-\mu(\widetilde{X}_t-\widetilde{X}_s)-\mu^2(t-s)/2\bigr), & s\le t,\\ \exp\bigl(-\sigma\widetilde{X}_t + \sigma^2 t/2\bigr), & s>t,\end{cases}$$

which implies

$$\begin{aligned}\frac{d\mathsf{P}_t}{d\mathsf{P}_t^{(g)}} &= \int_0^1 \frac{d\mathsf{P}_t^s}{d\mathsf{P}_t^{(g)}}dG(s)\\ &= \exp\bigl(-\sigma\widetilde{X}_t+\sigma^2t/2\bigr)\\ &\qquad\times\Bigl(\int_0^t \exp\bigl(-\mu(\widetilde{X}_t-\widetilde{X}_s)-\mu^2(t-s)/2\bigr)\,dG(s)+1-G(t)\Bigr)\\ &= \exp\bigl(-\sigma\widetilde{X}_t+\sigma^2t/2\bigr)\bigl(\psi_t+1-G(t)\bigr).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathsf{E}\exp(X_\tau-\sigma^2\tau/2) &= \mathsf{E}^{(g)}\Bigl(\frac{d\mathsf{P}_\tau}{d\mathsf{P}_\tau^{(g)}}\exp(X_\tau-\sigma^2\tau/2)\Bigr)\\ &= \mathsf{E}^{(g)}\bigl[e^{\mu_1\tau}(\psi_\tau+1-G(\tau))\bigr].\end{aligned}$$

Applying the Itô formula, we get

$$\begin{aligned}&e^{\mu_1\tau}(\psi_\tau+1-G(\tau))\\ &\qquad=1+\int_0^\tau e^{\mu_1 s}\left[\mu_2\psi_s+\mu_1(1-G(s))\right]ds+\mu\int_0^\tau \psi_s d(\widetilde{X}_s-\sigma s),\end{aligned}$$

Taking the mathematical expectation $\mathsf{E}^{(g)}[\,\cdot\,]$ of the both sides and using that $(\widetilde{X}_s-\sigma s)$ is a standard Brownian motion under $\mathsf{P}^{(g)}$ and, hence, the expectation of the stochastic integral equals zero, we obtain (11). $\square$

3.2. Now the proof of the Theorem follows from the Proposition in the Appendix – for the problem $V^{(l)}$ we use that the process ψ satisfies equation (3) with $\widetilde{X}$ being the Brownian motion under the measure $\mathsf{P}^{(l)}$, and for the problem $V^{(g)}$ we use that ψ satisfies the equation

$$d\psi_t = \bigl(\rho+(\mu_2-\mu_1)\psi_t\bigr)dt+\mu\psi_t d(\widetilde{X}_t-\sigma t), \tag{15}$$

where $(\widetilde{X}_t-\sigma t)$ is a Brownian motion under $\mathsf{P}^{(g)}$.

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Appendix. An auxiliary optimal stopping problem

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ and $\psi = (\psi_s)_{s\geq 0}$ be a stochastic process satisfying the stochastic differential equation (cf. (3), (15))

$$d\psi_s = (\rho + b\psi_s)ds - \mu\psi_s dB_s, \tag{16}$$

where $b, \mu \in \mathbb{R}$ and $\rho > 0$.

Consider the optimal stopping problem consisting in finding the quantity

$$V = \sup_{\tau\leq 1} \mathsf{E} \int_0^\tau e^{\lambda s}(f(s) - \psi_s)ds, \tag{17}$$

where $\lambda \in \mathbb{R}$, and $f(s)$ is a non-increasing bounded function which is continuous and strictly positive on $[0, 1)$. The supremum in (17) is taken over all stopping times τ of the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the condition $\tau \leq 1$ a.s.

For arbitrary $x \geq 0$, let $\mathsf{E}_x[\,\cdot\,]$ denote the mathematical expectation of functionals of the process $(\psi_t)_{t\geq 0}$, satisfying (16) with the initial condition $\psi_0 = x$.

The following result was proved in [10].

Proposition. *The optimal stopping time in problem* (17) *is given by*

$$\tau_* = \inf\{t \geq 0 : \psi_t \geq a(t)\} \wedge 1,$$

where $a(t)$ is a non-increasing non-negative function on $[0, 1]$ being the unique continuous solution of the equation ($t \in [0, 1]$)

$$\int_0^{1-t} \mathsf{E}_{a(t)}\big[e^{\lambda s}(f(t+s) - \psi_s)\mathbf{I}\{\psi_s < a(t+s)\}\big]ds = 0$$

satisfying the conditions

$$a(t) \geq f(t) \text{ for } t \in [0, 1), \qquad a(1) = f(1-).$$

The quantity V can be found by the formula

$$V = \int_0^1 \mathsf{E}_{\psi_0}\big[e^{\lambda s}(\psi_s - f(s))\mathbf{I}\{\psi_s < a(s)\}\big]ds.$$

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