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Loss Aversion in Contests

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Abstract

We study pure strategy Nash equilibria of rent-seeking contests in which contestants value gains less than losses of similar magnitude. We demonstrate that, if the degree of loss aversion is sufficiently great, there may be multiple equilibria, even for the simplest contest success functions and discuss condition which ensure uniqueness. We investigate comparative statics when these conditions are satisfied. For symmetric contests, we establish that there is a unique symmetric equilibrium, derive an explicit expression for this equilibrium and show that, in the presence of other equilibria, the symmetric equilibrium may display perverse comparative statics. We use these results in a comparison of contests with divisible and indivisible prizes and conclude by drawing lessons for the design of experimental contests.

Key words: rent-seeking, contests, loss aversion.

JEL classification: C72, D72, D80

1 Introduction[1]

In this article, we examine the impact of loss aversion on equilibria in contests. The risky prospect faced by an individual in a rent-seeking contest with indivisible rent has two outcomes: a gain (rent net of outlay on rent seeking) for the winner and a loss equal to the outlay for a losers. It is well established experimentally that individuals' responses to such a prospect may exhibit loss aversion: a loss from some reference level figures larger than a gain of similar magnitude. Furthermore, this effect is maintained even when such gains and losses are small. This is in contrast to risk aversion, where Rabin [18] and Thaler [23] have argued that concave utility functions have difficulty in reconciling individual's responses to lotteries entailing small changes in wealth with those where the change is large. Further discussions of loss aversion can be found in [10], [20], [25] and [26]. If subjects import such behavior into the laboratory, comparison of the results of experiments in which these subjects take part in experiments, with the predictions of models that assume loss-neutrality may be misleading.

In this article, we examine the impact of loss aversion on equilibria in contests, focussing on incompletely discriminating contests as introduced by Tullock [24] to model the strategic aspects of rent seeking. This model and many of the theoretical and experimental studies that followed assumed that contestants seek to maximize their expected wealth. (Useful surveys are provided by Nitzan [17] and in Konrad [12].) One feature of many of these analyses is that, when the probability of winning is proportional to expenditure, cost functions are linear and the number of contestants is large, close to 100% of the value of the rent will be dissipated in the form of resources used up in the competition for that rent. Many authors felt this "explained too much" (c.f. Riley [19] and Lockhard and Tullock [14]) and some limited empirical support for this view was provided by Sobel and Garrett [22] and by Hazlett and Michaels [8].

Motivated in part by such concerns authors turned to modifications to the basic model that reduced the extent of rent seeking and a number of them (for example, [9], [11] and [3]) have focussed on risk aversion as such a factor. However, whilst important for large rents, this effect may be weak for small rents, at least if utility functions are smooth. For example, in [5], we derived a formula for the proportion of rent dissipated in a contest with many risk-averse contestants. Provided utility functions are continuously differentiable at current wealth, this proportion approaches unity as the size of the rent approaches zero. This raises the possibility that the effects of risk aversion may not be significant when the prize is small as is typical in experiments.

In contrast to the empirical results, a consistent theme in the experimental literature [15], [2], [1] is that aggregate expenditures exceed those in the Nash equilibrium and can even exceed the size of the rent in the long run, although Shogren and Baik [21] offer evidence that such results may be sensitive

[1] RH would like to thank participants at the Young Researchers' Workshop in Contests and Tournaments, Magdeburg, 2009 and particularly the discussant Magnus Hoffmann for many helpful comments which led to significant improvements in the paper.

to the design of the experiment. The equilibria studied in these experiments are typically derived under the assumption that contestants choose expenditures to maximize expected wealth. A notable early exception is Millner and Pratt [16] who found that expenditure in contests played by more risk averse subjects was significantly lower than those where the contestants were less risk averse. However, since risk attitudes were assessed using binary lotteries, these results might also be explained by loss aversion. Indeed one of the main results of our analysis is that loss aversion reduces aggregate lobbying and, with the type of contest success function widely used in experiments (including Millner and Pratt), reduces aggregate expenditure. We note, however, a less palatable result: loss aversion may lead to multiple equilibria and the possibility of coordination failure. Aggregate expenditure in such a miscoordinated strategy profile may even exceed the rent.

Loss averse individuals display *status quo* bias and to apply such a behavioral assumption requires specification of the reference level. Initially, we make the simplest assumption: *ex post* wealth is referenced to current wealth and a change in wealth above the reference level is worth $\theta < 1$ times an equal change below. Otherwise, our model is a conventional incompletely discriminating contest. We suppose that the contest success function takes a standard Tullock form, although we do not always insist on its being symmetric and that the rent is indivisible (although we also discuss a divisible rent in the final two sections). One way to interpret such a function is to suppose that each contestant is endowed with a constant elasticity production function transforming expenditure into lobbying effort and that the probability of winning the indivisible prize is proportional to that lobbying effort. Thus contestant i is characterized by two parameters: a production elasticity r_i and coefficient of loss aversion θ_i. Provided both are members of the interval $(0, 1]$ for all i, the contest has an equilibrium in pure strategies. This result is non-trivial; discontinuity at the origin prevents direct application of general existence results. Instead, we use the method of share correspondences, building on the methods described in [4] and [5]. One benefit of this approach is that it also allows us to study uniqueness and comparative statics. We find that loss aversion has two main effects on contest equilibria. The first is that loss averse contestants compete less aggressively than loss neutral contestants. Secondly, in contrast to a contest with loss neutral contestants, equilibria may not be unique.

Multiple equilibria can occur if r_i is large enough and θ_i small enough. Indeed, we show that there is a critical value for the loss aversion parameter, dependent on r_i such that, if θ_i exceeds this value for all i the contest has a unique equilibrium and does not display perverse comparative statics. Notably, if $r_i = 1$, the critical value is $1/2$, a rather typical observed value of θ_i and we show that, if contestants are more loss averse than this, there will be multiple equilibria if the contest is large enough. Conversely, if $r_i \leq 1/2$, the equilibrium is unique no matter how averse contestants are to loss, showing that it is possible to design the contest to avoid multiple equilibria.

These results also allow us to conclude that aggregate lobbying (and aggregate expenditure if all $r_i = 1$) in any equilibrium is lower than in the unique equilibrium of a contest with the same contest success function but loss-neutral contestants. We can draw a similar conclusion for the unique symmetric equilibrium of a symmetric contest. Indeed, we derive an explicit formula for aggregate expenditure in such an equilibrium and show that, if there are many contestants, the reduction in aggregate expenditure is approximately equal to the common coefficient of loss aversion. These results allow us to draw comparisons between a conventional contest with an indivisible prize in which the contest success function determines the probability of winning the prize and a second contest with a divisible prize which is divided according to shares determined by the contest success function and conclude that equilibrium lobbying will be greater when the prize is divisible.

In Section 2, we define share correspondences and derive some of their key properties. In particular, we show that this correspondence is always the inverse of a function and discuss when this function is monotonic, implying that the share correspondence is actually a function. We apply these results in Section 3 to establish existence of an equilibrium and to derive conditions under which this equilibrium is unique. The results of Section 2 also permit us to study comparative statics with respect to entry as well as changes in the degree of loss aversion. In Section 4, we obtain an expression for the symmetric equilibrium strategy of a symmetric contest and discuss several implications of this formula, including the possibility that, if the contest has alternative, asymmetric equilibria, aggregate expenditure may fall as the number of contestants increases. Section 5 compares divisible and indivisible prizes and suggests a simple experimental test of the effect of loss aversion on equilibria. In the concluding section, we discuss some implications of these results for the design and analysis of experiments.

2 Share correspondences and their properties

For any strategy profile, each contestant in a rent-seeking contest faces an uncertain prospect with two outcomes. A simple and common way to capture loss aversion in such a context is to suppose that a gain in wealth is evaluated at θ times the same loss in wealth, where $0 < \theta < 1$. This can be applied to a standard Tullock rentseeking model with n contestants in which contestant i chooses a level of expenditure x_i to devote to contesting an exogenously fixed and indivisible rent R. We study the classic Tullock form of contest success function in which the probability that contestant i wins is given by $x_i^{r_i} / \sum_{j=1}^{n} x_j^{r_j}$, where $r_i \leq 1$ captures the returns to scale of the rent-seeking technology available to that contestant. The contestants are therefore engaged in a simultaneous-move game in which the payoff of contestant i for strategy profile $\mathbf{x} = (x_1, \ldots, x_n) \neq \mathbf{0}$ is

$$\pi_i(\mathbf{x}) = \frac{x_i^{r_i}}{\sum_{j=1}^{n} x_j^{r_j}} \min\{\theta_i(R - x_i), R - x_i\} - \left[1 - \frac{x_i^{r_i}}{\sum_{j=1}^{n} x_j^{r_j}}\right] x_i,$$

recalling that the reference level of wealth is zero for each contestant. Note that we have allowed the parameter θ to vary amongst contestants, but we still impose the conditions $0 < \theta_i \leq 1$. If $\mathbf{x} = \mathbf{0}$, we shall assume[2] that $\pi_i = 0$. Note that any x_i exceeding R is strictly dominated (by $x_i = 0$) so $\min \{\theta_i (R - x_i), R - x_i\}$ can simply be replaced by $\theta_i (R - x_i)$ without changing the set of Nash equilibria.

In this and the next two sections, we investigate pure strategy Nash equilibria of this game, focussing on existence and uniqueness. As in [4] and [5], our approach circumvents the difficulties of handling multi-dimensional self-mappings by using share correspondences and, in this section, we start by introducing these correspondences and deriving their key properties.

It proves convenient to work with the transformed variables: $y_i = x_i^r$ in terms of which the payoff function (**??**) can be re-written:

$$\widetilde{\pi}_i (y_i, Y) = \frac{\theta_i R y_i}{Y} - y_i^{u_i} + \frac{(1 - \theta_i) y_i^{u_i + 1}}{Y}, \tag{1}$$

where $u_i = 1/r_i$ and $Y = \sum_{j=1}^n y_j$. We can view x_i^r as a production function transforming expenditure into lobbying effort and so we will refer to $X = \sum_{j=1}^n x_j$ as *aggregate expenditure* and Y as *aggregate lobbying*.

For any contestant i and any $Y > 0$, our approach first determines necessary and sufficient conditions on $\widehat{y}_i \in [0, Y]$ such that there is an equilibrium in which contestant i spends $\widehat{y}_i$ and $\sum_{j=1}^n \widehat{y}_j = Y$. We then let $S_i (Y)$ denote the corresponding set of $\widehat{y}_i / Y$:

$$S_i (Y) = \left\{ \frac{\widehat{y}_i}{Y} : \exists \text{ equilibrium profile } (\widehat{y}_1, \dots, \widehat{y}_n) \text{ such that } Y = \sum_{j=1}^n \widehat{y}_j \right\}.$$

We refer to S_i as the *share correspondence of contestant i*.

The importance of share correspondences arises from the (easily verified) fact that $\widehat{\mathbf{y}}$ is an equilibrium of the contest if and only if

$$\frac{\widehat{y}_j}{\widehat{Y}} \in S_j \left(\widehat{Y} \right) \text{ for } j = 1, \dots, n. \tag{2}$$

This means that $\widehat{Y}$ is an equilibrium value of Y if and only if $1 \in \sum_{j=1}^n S_j \left(\widehat{Y} \right)$ (the *aggregate share correspondence*), using conventional set addition. Note that contestants have no best response if all rivals are inactive, so at least two contestants must be active in equilibrium. Consequently there cannot be an equilibrium with $Y = 0$ and, if $Y > 0$, then $1 \notin S_i (Y)$.

The next proposition, proved in the appendix, uses the first-order conditions to characterize share correspondences.

[2] Other assumptions such as $\pi_i = 1/n$ are possible, but a discontinuity at the origin is unavoidable and our results are unchanged.

Proposition 2.1 *For $i = 1, \ldots, n$, we have $0 \in S_i(Y)$ if and only if $r_i = 1$ and $Y \geq \theta_i R$. Furthermore, $\sigma \in S_i(Y) \cap (0,1)$ if and only if $Y = \varphi_i(\sigma)$, where $\varphi_i : (0,1) \longrightarrow \mathbb{R}_{++}$ satisfies $\varphi_i(\sigma) = [\theta_i R / D_i(\sigma)]^{r_i}$ and*

$$D_i(\sigma) = u_i \sigma^{u_i - 1} - (1 - \theta_i)\sigma^{u_i} + \frac{\theta_i u_i \sigma^{u_i}}{1 - \sigma}. \quad (3)$$

The preceding proposition says that, excluding zero shares, the share correspondence S_i of contestant i is the inverse of a continuous, positive real-valued function φ_i defined on $(0,1)$. We refer to φ_i as the *inverse share function* and list several, largely qualitative, properties of this function in the next proposition. The proof may be found in the appendix.

Proposition 2.2 *For $i = 1, \ldots, n$, the inverse share function $\varphi_i : (0,1) \longrightarrow \mathbb{R}_{++}$ is continuous and has the following properties.*

1. *If $r_i = 1$, then $\varphi_i(\sigma) \longrightarrow \theta_i R$ as $\sigma \longrightarrow 0$.*
2. *If $r_i < 1$, then $\varphi_i(\sigma) \longrightarrow \infty$ as $\sigma \longrightarrow 0$.*
3. *For all $r_i \leq 1$, $\varphi_i(\sigma) \longrightarrow 0$ as $\sigma \longrightarrow 1$.*

Furthermore, there exists a function $\overline{\theta} : (0,1] \longrightarrow \mathbb{R}_+$ with the following properties: (a) $\overline{\theta}(1) = 1/2$, (b) $\overline{\theta}(r)$ is non decreasing in r for $1/2 \leq r \leq 1$, (c) $\overline{\theta}(r) = 0$ if $r \leq 1/2$. This determines the shape of φ_i as follows.

4. *If $\theta_i \geq \overline{\theta}(r_i)$, then $\varphi_i'(\sigma) < 0$ for all $\sigma \in (0,1)$.*
5. *If $r_i = 1$ and $\theta_i < 1/2$, then φ_i has a unique local and global maximum in $(0,1)$.*
6. *If $r_i < 1$ and $\theta_i < \overline{\theta}(r_i)$, then as σ increases from 0 to 1, $\varphi_i(\sigma)$ decreases to a local minimum and then increases to a local maximum in $(0,1)$.*

If $S_i(Y)$ is a singleton $\{s_i(Y)\}$ for all $Y > 0$, we refer to s_i as the *share function of contestant i* and note that $\widehat{Y}$ is an equilibrium value of Y if and only if $\sum_{j=1}^{n} s_j\left(\widehat{Y}\right) = 1$. The following corollary notes a key property of share functions which follows directly from Part 4 of the proposition.

Corollary 2.3 *If $\theta_i \geq \overline{\theta}(r_i)$, contestant i has a share function s_i which is strictly decreasing where positive.*

This corollary will prove useful for establishing uniqueness of equilibrium. However, there are values of θ_i and r_i for which the share correspondence is not a function.

Proposition 2.2 is illustrated in Figures 1–4. In each figure, we first graph D in panel (a), and then display φ in panel (b), using the formula $(\theta R/D)^r$. The share correspondence is obtained by reflecting φ in the 45^0 line (and, if $r = 1$, adding that portion of the Y-axis to the right of $Y = \theta R$). This is shown in panel (c). Figures 1 and 3 illustrate cases in which $\theta < \overline{\theta}(r)$. In Figure 1, we have $r = 1$ and, in Figure 3, $r < 1$. Figures 2 and 4 illustrate $\theta > \overline{\theta}(r)$, with $r = 1$ in Figure 2 and $r < 1$ in Figure 4. In these two figures, there is a share function and it is easy to check that the figures exhibit the property stated in Corollary 2.3.

3 Existence, uniqueness and comparative statics of equilibria

3.1 Existence

The discontinuity in payoffs at the origin prevents direct application of standard theorems for existence of pure strategy equilibria, such as those of Glicksberg [7] or Dasgupta and Maskin [6] (in the latter case because payoffs are not semi-continuous). However, the propositions of the previous section can be used to establish not only existence but, with additional restrictions on the parameters, uniqueness of an equilibrium. Examination of the forms of share correspondence in Figures 1–4 shows that all members the aggregate share correspondence exceed 1 for small enough Y and are less than 1 for large enough Y. When this correspondence is a function, the existence of a value of Y for which the aggregate share function equals 1 (and therefore an equilibrium exists) follows from the intermediate value theorem, since the function is easily seen to be continuous. Furthermore, the fact that it is decreasing means the equilibrium is unique.

When the correspondence is not singleton-valued, we can use the fact that demonstrating existence of a Nash equilibrium is equivalent to showing that there is a positive $\widehat{Y}$ and shares in each $S_i\left(\widehat{Y}\right)$ summing to 1. When all $r_i < 1$ (so the share correspondences does not contain 0), this requirement can be re-expressed in terms of the inverse share functions as showing the existence of $\widehat{Y} > 0$ and $\widehat{\sigma}_i \geq 0$ for all i satisfying

$$\varphi_i\left(\widehat{\sigma}_i\right) = \widehat{Y} \text{ for } i = 1, \ldots, n \text{ and } \sum_{j=1}^{n} \widehat{\sigma}_j = 1. \tag{4}$$

In the appendix, we prove that (4) always has a solution and then extend this result to allow for $r_i = 1$.

Theorem 3.1 *If $0 < r_i, \theta_i \leq 1$ for $i = 1, \ldots, n$, the contest has a Nash equilibrium.*

3.2 Uniqueness

This equilibrium need not be unique. This is particularly clear in the case of a symmetric contest in which all $r_i = 1$ and all $\theta_i < 1/2$. For, Proposition 2.2 implies that, for all large enough integers m, there is a $Y^m(\geq \theta R)$ such that both 0 and $1/m$ are members of $S_i(Y^m)$. This is clearly seen[3] in Figure 1. It follows that, if the number of contestants is $n > m$, there are asymmetric equilibria in which m contestants choose $y_i = Y^m/m$ and the remainder choose $y_i = 0$ in addition to the symmetrical equilibrium in which all contestants choose $y_i = Y^n/n$. For example, if $n = 3$ and $\theta = 0.2$, the symmetric equilibrium is $(0.08R, 0.08R, 0.08R)$. An alternative Nash equilibrium is $(0.125R, 0.125R, 0)$ since we can check that the requirement $\varphi(1/2) = 0.25R > \theta R$ is satisfied. This is illustrated in Figure 5.

Corollary 3.2 *A symmetric contest in which $r_i = 1$ and $\theta_i < 1/2$ for $i = 1, \ldots, n$ exhibits multiple equilibria for all large enough n.*

This result sheds an interesting light on equilibria of contests with risk-averse contestants. Suppose contestant i is loss-neutral, but risk averse with von Neumann-Morganstern utility function u_i which has positive, decreasing marginal utility u_i' and let

$$\delta_i = \inf_{x \in (0,R)} \frac{u_i'(R-x)}{u_i'(-x)}.$$

In [5], we showed that, if all $\delta_i \geq 1/2$, the contest has a unique equilibrium. It follows from Corollary 3.2 that this bound is best possible. For, if $\delta < 1/2$, the contest in which $u_i(x) = x$ for $x < 0$ and $u_i(x) = \delta x$ for $x > 0$ for all i and all $r_i = 1$ is identical to a contest with risk-neutral, but loss-averse contestants in which $\theta = \delta$. The corollary then implies that the contest does not have a unique equilibrium. Of course, the proposed utility function is not differentiable, but this can be fixed by smoothing the utility function close enough to the origin to have no effect on the equilibrium.

The fact that the contest in Corollary 3.2 has multiple equilibria for all large enough n is, in part, a peculiarity of assuming all $r_i = 1$. Indeed, if all $r_i < 1$, the equilibrium is unique whether or not $\theta_i < \overline{\theta}(r_i)$ provided there are enough contestants. (This does not preclude multiple equilibria for smaller n.) We can see this by observing from the graphs of share correspondences in Figures 3 and 4, or by using Proposition 2.2 that that there will be a $\overline{Y}$ such that (i) $S(\overline{Y}) = \{\overline{\sigma}\}$, (ii) $S(Y)$ is singleton-valued and strictly decreasing for $Y \geq \overline{Y}$ and (iii) $S(Y) > \overline{\sigma}$ for $Y < \overline{Y}$, where we have dropped the subscript. Indeed, we can take $\overline{Y}$ to be any value of Y exceeding the value of φ at its local maximum if φ is non-monotonic (see Figure 3) or any positive value of Y if φ is decreasing (see Figure 4). We may conclude that the equilibrium with $\widetilde{Y} = \varphi(1/n)$ must be unique if $n > 1/\overline{\sigma}$. For it follows from (i), (ii) and (iii) that $\widetilde{Y} \geq \overline{Y}$ and therefore any member of $\sum_{j=1}^{n} S_j(Y)$ exceeds 1 for $Y < \widetilde{Y}$ and is less than 1 for $Y < \widetilde{Y}$.

Corollary 3.3 *A contest in which $r_1 = \cdots = r_n = r < 1$ has a unique equilibrium for all large enough n.*

[3] In addition, $S_i(Y)$ also contains a value of $\sigma > 1/m$.

Even if it is not symmetric, a contest will have a unique equilibrium if the share correspondence of every contestant is singleton-valued, since it follows from Corollary 2.3 that the corresponding share functions are strictly decreasing and this property is inherited by the aggegate share function $\sum_{j=1}^{n} s_j$. It follows that there can only be one value of Y for which $\sum_{j=1}^{n} s_j(Y) = 1$ and this precludes multiple equilibria. Using the condition in Corollary 2.3 gives the following theorem.

Theorem 3.4 *If $\theta_i \geq \overline{\theta}(r_i)$ for $i = 1, \ldots, n$, the contest has a unique equilibrium.*

We state explicitly as corollaries two special cases which follow from properties of $\overline{\theta}$ set out in Proposition 2.2. The first such property is that $\overline{\theta}(1) = 1/2$.

Corollary 3.5 *If $r_i = 1$ and $\theta_i \geq 1/2$ for $i = 1, \ldots, n$, the contest has a unique equilibrium.*

Corollaries 3.2 and 3.5 show that the critical value of θ distinguishing between unique and multiple equilibria for the simple lottery contest success function is $1/2$. It is an interesting coincidence that Tversky and Kahneman [25] suggest that a value of θ of about $1/2$ is consistent with the much of the experimental and empirical evidence, at least for small or moderate changes in wealth (though much smaller values may also be observed, for example when health risks are involved).

The second corollary follows from the fact that $\overline{\theta}(r) = 0$ if $r \leq 1/2$.

Corollary 3.6 *If $r_i \leq 1/2$ for $i = 1, \ldots, n$, the contest has a unique equilibrium.*

3.3 Comparative statics

When every contestant has a share function, we can also deduce results in comparative statics. Since we have discussed this extensively in [4] and [5], and the methods and results for the present model are very similar, we omit most of the details. We also confine attention to the effects of adding contestants and changing loss-aversion parameters.

The key observation is that, since share functions are strictly decreasing where positive, the same is true of the aggregate share function $\mathcal{S}$. Since the equilibrium value of Y satisfies $\mathcal{S}(Y) = 1$, it follows that any change to the contest which decreases the aggregate share function also decreases equilibrium Y. For example, if an active contestant leaves the contest equilibrium Y decreases. Note that aggregate expenditure $X = \sum_{j=1}^{n} x_j$ is not directly related to Y, so we cannot sign changes in X (except in the case $r_i = 1$ for all i when $X = Y$). However, the facts that a contestant's share function is just their probability of winning and this function is decreasing, means that the remaining contestants are more likely to win in the smaller contest. It can also be shown [5] that $\widetilde{\pi}_i(s_i(Y), Y)$ decreases with Y, which means that the remaining contestants are better off.

Corollary 3.7 *If an active contestant leaves a contest which satisfies the hypotheses of Theorem 3.4 and has at least three players, aggregate lobbying decreases in equilibrium. The probability of winning and the payoff of the remaining active contestants rise.*

Another interesting change is a reduction in loss aversion. Note that (3) implies

$$\frac{D_i(\sigma)}{\theta_i} = \frac{(u_i - \sigma)}{\theta_i}\sigma^{u_i - 1} + \text{ terms independent of } \theta_i.$$

Since $u_i \geq 1$, we see that D_i/θ_i is decreasing with θ_i. From Proposition 2.1, we deduce an increase in θ_i increases φ_i and moves the graph of the share correspondence S_i to the right.

Now consider two contests $\mathcal{C}$ and $\mathcal{C}'$ with the number of contestants and the same r_i in both contests and suppose that $\theta_i' \geq \theta_i$ for all i with at least one inequality strict. Let S_i' denote the share correspondence of contestant i in $\mathcal{C}'$. Suppose further that all contestants in $\mathcal{C}'$ have a share function, which we write s_i' for contestant i. Since this is strictly decreasing where positive and lies to the right of S_i, it must also lie above it. It follows that, if $(\widehat{y}_1, \ldots, \widehat{y}_n)$ is an equilibrium profile of $\mathcal{C}$ and $\widehat{Y} = \sum_{j=1}^n \widehat{y}_j$, then $\widehat{y}_i/\widehat{Y} \in S_i\left(\widehat{Y}\right)$ and therefore $s_i'\left(\widehat{Y}\right) \geq \widehat{y}_i/\widehat{Y}$ for all i. Since at least one of these inequalities must be strict, we can sum over i to deduce that $\sum_{j=1}^n s_j'\left(\widehat{Y}\right) > 1$, which implies that $\widehat{Y}'$, the unique equilibrium value of aggregate lobbying in $\mathcal{C}'$ satisfies $\widehat{Y}' > \widehat{Y}$. If, in addition, there is a contestant i such that $\theta_j' = \theta_j$ for all $j \neq i$, the fact that share functions are decreasing where positive implies that the probability of contestant $j \neq i$ winning (and its payoff) falls. Since the sum of the probabilities of winning remains constant, this means that contestant i is more likely to win in $\mathcal{C}'$.

Corollary 3.8 *If $\theta_i' \geq \overline{\theta}(r_i)$ for $i = 1, \ldots, n$, equilibrium aggregate lobbying in $\mathcal{C}'$ exceeds aggregate lobbying at any equilibrium of $\mathcal{C}$. If the degree of aversion to loss is unchanged for all but one contestant, the probability of that contestant winning increases and the winning probabilities and payoffs of the remaining contestants fall (strictly if they were initially active).*

If $r_j = 1$ for all j, then $Y = \sum_{j=1}^n x_j$. For this widely studied contest success function, we may deduce that aggregate expenditure increases. In particular, aggregate expenditure for loss averse contestants is less than that for otherwise identical but loss neutral contestants.

It is important to note that Corollaries 3.7 and 3.8 require all contestants to have share functions. Although this condition also implies uniqueness of equilibrium, uniqueness is not itself sufficient to ensure the 'intuitive' comparative statics properties stated in the corollaries. A counterexample can be constructed by supposing that all but one contestant (say $i = 1$) has a share function and that S_1 is multi-valued (as in Figures 1 and 3). This means that the aggregate share function is also multivalued. In particular, it is the inverse of a function (say Φ) and therefore contains unity at a unique value of Y.

This means that the contest has a unique equilibrium at $\widetilde{Y} = \Phi(1)$. However, it is possible that this equilibrium exhibits perverse comparative statics. For example, Φ may be decreasing in a neighbourhood of unity, which means that a sufficiently small decrease in the aggregate share correspondence will lead to an *increase* in the value of Y at which the new aggregate share correspondence includes the value 1. For example, a small increase in loss aversion of contestant $i \neq 1$, or equivalently a small decrease in θ_i, can lead to an *increase* in the equilibrium value of Y. This means that contestants other than 1 and i are less likely to win the contest and their payoffs fall. Note that the probability that contestant 1 wins will rise, so we can no longer determine whether contestant i is more or less likely to win as a result of the change. Similarly, if an active contestant leaves the contest, equilibrium Y increases and the remaining contestants other than 1 are *less* likely to win the contest and are worse off, provided the level of expenditure of the entrant is sufficiently small.

4 Symmetric contests

In this section, we focus on symmetric contests in which $r_i = r$ and $\theta_i = \theta$ for $i = 1, \ldots, n$ and we will drop the subscript i throughout. Symmetric contests always have a symmetric equilibrium; furthermore, this will be unique. To see this note that that $y_i = \widehat{y}$ for all i is a symmetric equilibrium if and only if

$$\frac{1}{n} \in S(n\widehat{y}).$$

This can be re-written as $\widehat{y} = \varphi(1/n)/n$, where φ is the inverse share function and shows that there is a unique symmetric equilibrium.

Transforming $\widehat{y} = \varphi(1/n)/n$ back to the original strategic variables, writing $\widehat{x}^n(\theta)$ for the expenditure of each contestant and using Proposition 2.2 gives

$$\widehat{x}^n(\theta) = \left[\frac{\varphi(1/n)}{n}\right]^u = \frac{\theta R}{nD(1/n)}.$$

Using the formula (3) for D gives an expression for the symmetric equilibrium.

Theorem 4.1 *If $0 < r \leq 1$, there is a unique symmetric Nash equilibrium in which the expenditure of each contestant is*

$$\widehat{x}^n(\theta) = \frac{\theta r R(n-1)}{n^2 - (1-\theta)(n + rn - r)}.$$

We can make a number of observations on this theorem.

1. Comparative statics of the symmetric equilibrium with respect to changes in loss aversion are as expected: increasing aversion to loss reduces equilibrium expenditure. This is readily seen by re-writing $\widehat{x}^n(\theta)$ in the form

$$\widehat{x}^n(\theta) = \frac{rR(n-1)}{n + rn - r}\left\{1 - \frac{(n-1)(n-r)}{n^2 - (1-\theta)(n + rn - r)}\right\}.$$

which shows that, as θ falls, so does $\widehat{x}^n(\theta)$. In particular, loss averse contestants devote less effort to rent-seeking than loss neutral ($\theta = 1$) contestants facing the same contest success function. It is illuminating to compare this with the results on comparative statics of loss aversion in Section 3. There we showed that aggregate lobbying in asymmetric contests increases as contestants become less averse to loss but needed to impose the additional restriction $\theta_i \geq \overline{\theta}(r_i)$ for all i. To derive a similar conclusion for aggregate expenditure required the additional restriction that all $r_i = 1$.

2. As $n \longrightarrow \infty$, so $n\widehat{x}^n(\theta) \longrightarrow \theta rR$. Equilibrium expenditure devoted to rent seeking in a large contest is reduced by a factor equal to the loss aversion ratio. In the frequently studied case where $r = 1$, loss neutral contestants would exhaust the whole value of the rent whereas, if loss averse, only a proportion θ would be spent. For a "typical" value [25] of θ, this would halve aggregate expenditure.

3. The symmetric equilibrium may display counter-intuitive comparative statics with respect to the number of contestants. For example, if $r = 1$ and $\theta < \frac{1}{2}$, aggregate expenditure on rent-seeking: $n\widehat{x}^n(\theta)$ is decreasing in n for all large enough n. This can be seen by direct study of the expression for $\widehat{x}^n(\theta)$ in the theorem or, perhaps more insightfully, from the fact that $n\widehat{x}^n(\theta) = \varphi(1/n)$ and that $\varphi(\sigma)$ is strictly increasing for all small enough positive σ, as shown in Figure 1. Note that this also implies that the limit in the previous observation is approached from above. Consequently, the maximum (over n) rent dissipation will occur for a finite value of n.

   The possibility that φ is strictly increasing over part of its domain is not restricted to $r = 1$. This is illustrated in Figure 3 and it follows that aggregate expenditure may decrease with n even if $r < 1$, though, unlike the case $r = 1$, this can only happen for a restricted range of n. Indeed, it follows from Corollary 3.3 that if n is large enough, there is a unique equilibrium.

4. Whether entry leads to a rise or fall in aggregate expenditure it unambiguously makes incumbents worse off in the symmetric equilibrium. When $\theta \geq \overline{\theta}(r)$, this is a special case of Corollary 3.7, but remains valid even if this inequality does not hold. Indeed, if we let $\widehat{\pi}^n$ denote the equilibrium payoff with $n\,(\geq 1)$ contestants, we find after some manipulation that

$$\widehat{\pi}^{n+1} - \widehat{\pi}^n = -\frac{(1+r)\,\theta^2 + (2n-1)\,\theta + (1-r)\,n\,(n-1)}{E(n)\,E(n+1)}R,$$

   where

$$E(n) = (n-1)(n-r) + \theta n + \theta r\,(n-1)\,.$$

   Since $E(n)$ is positive for $n \geq 1$, we conclude that $\widehat{\pi}^n$ is strictly decreasing in n.

We note that $\widehat{\pi}^m$ is also the payoff of an active contestant in an asymmetric equilibrium in which $n-m$ inactive contestants choose $x_i = 0$ and the rest choose the same strategy. Since $\widehat{\pi}^m > \widehat{\pi}^n$, contestants prefer to be active players in such an asymmetric equilibrium to playing the symmetric equilibrium, when the latter is not unique.

5. In some instances — for example, if a loss averse manager is given a positive revenue target — it may be appropriate to analyze a reference level, a, different from zero. If we maintain the same loss aversion ratio, θ, the payoff can be written

$$\pi_i = \frac{x_i^r}{\sum_{j=1}^n x_j^r}\theta\left(R^* - x_i\right) - \left[1 - \frac{x_i^r}{\sum_{j=1}^n x_j^r}\right]x_i - a,$$

where $R^* = R - a + a/\theta$, provided winnings net of expenditure exceed a and expenditures exceed $-a$. Hence, if $0 < a + x^* < R^*$, where x^* is given by the formula for $\widehat{x}^n(\theta)$ in the theorem but with R replaced by R^*, the profile $x_i = x^*$ is a Nash equilibrium. It follows that, if $0 \leq a < R^*$, aggregate equilibrium expenditure approaches $r\left[\theta R + (1-\theta)a\right]$ as $n \longrightarrow \infty$: a positive reference level may offset some of the reduction arising from loss aversion.

The intuition behind the theorem and subsequent observations rests on the fact that winning the rent increases wealth and therefore loss aversion decreases its value by a factor of $\theta < 1$. However, loss aversion also effectively decreases the cost of expenditure on rent-seeking in the case (and only in the case) of a win. Hence, the aggregate expenditure is decreased relative to the case of risk neutrality but by a factor less than θ. Furthermore, if θ increases, the increase in effective value of the rent leads to an increase in expenditure. Observation 1 shows that the partial offset due to the effective reduction in the cost of successful rent-seeking does not reverse this conclusion. When there are many contestants, the expenditure of each contestant is sufficiently small relative to the rent that it can be ignored, in which case the reduction in aggregate expenditure is equal to θ (cf. Observation 2).

These observations allow us to discuss several extensions of these results, for example to the case where the technological coefficient r (but not the degree of loss aversion) differs between the contestants, although we do not undertake a formal analysis. Even in this asymmetric case, the effective reduction in the value of winning means the aggregate lobbying will be reduced (by a factor between θ and 1).

Returning to the symmetric case, another generalization is to contest success functions of the form $f(x_i)/\sum_{j=1}^n f(x_j)$, where f is a concave function satisfying $f(0) = 0$. In [**?**], we showed that in the absence of loss aversion, if the elasticity $xf'(x)/f(x)$ has a limit η as $x \longrightarrow 0$, aggregate expenditure approaches ηR as $n \longrightarrow \infty$. The intuition is that Tullock's contest success function with $r = \eta$ is a good approximation for the small individual expenditures found in symmetric equilibria when n is large. For the reasons set out above, loss aversion will again reduce limiting aggregate expenditure by the factor θ if the contest is large and less than this if the offsetting effect of reduced winner's expenditure is significant.

When production of lobbying is subject to increasing returns ($r > 1$), there is no symmetric equilibrium in pure strategies; mixed strategies are necessary. Even in this case, the effective reduction of the value of the prize will reduce the expected aggregate expenditure by a factor of θ or less.

5 Divisible and indivisible rents

The standard contest in which all players are risk and loss neutral is open to an alternative interpretation in which the rent is regarded as divisible and the contest success function determines shares of the rent. Specifically, if the strategy profile is $(x_1, \ldots, x_n)$, the share received by contestant i is $p_i = x_i^r / \sum_{j=1}^n x_j^r$. In the absence of loss and risk aversion, the two games - the one with an indivisible rent, the other with a divisible rent - are strategically equivalent. In particular, both yield the same relationship between the proportion of rent dissipated in equilibrium and the number of contestants. Under loss aversion, however, the two games have distinct equilibria.

The analysis of a contest in which a divisible rent of R is shared in proportion to x_i^r is straightforward. With no entry fees, contestant i's payoff function is

$$\pi_i^D = \min \left\{ \theta \left(p_i R - x_i \right), \left(p_i R - x_i \right) \right\}.$$

In equilibrium, payoffs can never be negative, so $\pi_i^D = \theta \left(p_i R - x_i \right)$ and the Nash equilibrium will be the same as in a loss-neutral contest with the same contest success function. Equilibria of such a game can be found by applying the results in preceding sections with $\theta = 1$ and this makes it easy to compare equilibria in the divisible and indivisible cases.

Consider, for example, a contest with an indivisible rent in which $\theta_i \geq \overline{\theta} \left(r_i \right)$ for all i, so that the equilibrium is unique. Then application of Corollary 3.8 shows that increasing θ_i to 1 also increases the equilibrium value of Y. We note that $1 \geq \overline{\theta} \left(r_i \right)$ for all i and all r_i, which means that the hypothesis of Corollary 3.8 are satisfied.

Corollary 5.1 *Aggregate lobbying is higher in the unique equilibrium when the rent is divisible than in any equilibrium when the rent is divisible. If $r_i = 1$ for all i, the same inequality holds for aggregate expenditure.*

The final assertion is a consequence of the fact that, if $r_i = 1$ for all i, aggregate lobbying is equal to aggregate expenditure. We can also compare aggregate expenditure if we confine attention to symmetric equilibria of symmetric contests, for then we can use the expression for the equilibrium strategy $\widehat{x}^n \left(\theta \right)$ in Theorem 4.1. In Observation 1, we showed that $\widehat{x}^n \left(\theta \right)$ is increasing in θ and therefore $\widehat{x}^n \left(\theta \right) < \widehat{x}^n \left(1 \right)$. Since $\widehat{x}^n \left(1 \right)$ is the unique equilibrium strategy when the rent is divisible, we have the following result.

Corollary 5.2 *The ratio of the aggregate expenditures in the symmetric equilibrium when the rent is indivisible to that when the rent is divisible exceeds unity. This ratio approaches $1/\theta$ as $n \longrightarrow \infty$.*

The final assertion is a restatement of the limit $\widehat{x}^n(1)/\widehat{x}^n(\theta) \longrightarrow \theta^{-1}$ as $n \longrightarrow \infty$, which follows directly from the formula in Theorem 4.1.

For larger rents, nonlinear value functions for positive arguments may be appropriate to reflect risk aversion. For divisible rents and increasing value functions, this makes no difference to pure strategy equilibrium. However, if the rent is indivisible and there are many contestants, individual expenditures will be small so their value will be an approximately linear function of their magnitudes (assuming that the only kink in the value function is at the origin) whereas the value of winning will be reduced below θR. Hence, even in this case, equilibrium expenditure will be smaller for an indivisible rent as compared to a divisible rent. Indeed, even for loss-neutral but risk averse contestants equilibrium expenditure is reduced. However, if contestants are risk averse and have a smooth utility function over wealth, for small rents, to first order, contestants' behavior will be risk neutral [18] and the difference between divisible and indivisible rents may be too small to be detectable in an experimental setting. By contrast, loss aversion implies that the utility function cannot be accurately approximated by a linear function. Experiments should therefore be capable of detecting a difference between the contests even for small rents.

6 Conclusion

We have examined the possibility that loss aversion can lead to significant modifications of the set of equilibria of a Tullock contest. Since we expect them to persist even when the prize is small, these modifications may have implications for both the design of experimental contests and the interpretation of results.

When either contestants are not too loss averse or production elasticities are sufficiently small, the equilibrium is unique and exhibits "non-perverse" comparative statics (Theorem 3.4 and Corollaries 3.7 and 3.8), although even in this case, loss aversion will reduce aggregate lobbying (Corollary 3.8). When the contest is symmetric aggregate expenditure will fall and, if there are many players, this reduction will be proportional to the common loss aversion parameter, θ (Theorem 4.1).

However, if production elasticities are large and contestants are more averse to loss, the contest may have multiple equilibria. When the contest is symmetric, this may be accompanied by perverse comparative statics (Theorem 4.1 and subsequent observations). When the aversion to loss of contestants is not known, the possibility that $\theta_i < \overline{\theta}(r_i)$ for some i needs to be considered. For example, if $r_i = 1$ for all i and contestants' risk aversion parameters are drawn randomly from a distribution with mean value close to $1/2$ and wide enough support, multiple equilibria are likely. Indeed, as the number of contestants becomes large, the probability that the equilibrium is unique approaches zero.

Furthermore, the existence of alternative equilibria raises the possibility of coordination failure. For example, if all contestants play the active strategy in an asymmetric equilibrium in which some contestants are inactive, aggregate expenditure will exceed the equilibrium value and may even exceed the value of the rent. What is more, as the number of contestants grows, total expenditure in such a miscoordinated strategy profile is unbounded.

In the laboratory, a typical finding is that "... subjects spend significantly more than the Nash equilibrium ..." [1]. This equilibrium is calculated assuming loss neutrality. Taking account of loss aversion in experiments is complicated by uncertainty in how subjects set their reference levels, particularly when they are asked to participate in a sequence of contests. However, if we take it to be current wealth when each contest is played, our results point in two directions: loss aversion reduces expenditure in equilibrium, but can also make coordination on an equilibrium harder to achieve. We note that, if loss aversion parameters are continuously distributed amongst subjects with a median value around $1/2$ and $r_i = 1$ for all i, as in most reported experiments, the probability that at least one subject has a multi-valued share correspondence, with the attendant possibility of coordination failure, is positive and approaches one as the number of contestants becomes large.

These observations suggest that some thought may usefully be given to the choice of contest success function in experimental design, at least when contestants' attitudes to loss are uncontrolled. It follows from Theorem 3.4 and Proposition 2.2 that, if the degree of contestants' loss aversion is unknown, lower production elasticities reduce the risk of multiple equilibria. Indeed, if $r_i \leq 1/2$, for all i the equilibrium is unique no matter how averse to loss contestants are (Corollary 3.6). For large contests, Corollary 3.3 implies that simply choosing production elasticities less than one is enough to ensure uniqueness. An alternative and perhaps more readily implementable way of ensuring uniqueness for any profile of production elasticities is simply to make the prize divisible and interpret the contest success function as determining shares of that prize. Indeed, in the spirit of the comparison discussed in Section 5, it would be instructive to try both designs and compare the outcomes.

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7 Appendix

Proof of Proposition 2.1. For ease of exposition, we drop the subscript i throughout the proof. The definition of the share correspondence means that $\widehat{y}/Y \in S(Y)$ if and only if $\widehat{y}$ maximizes $\psi(y)$ over $y \geq 0$, where

$$\psi(y;\widehat{y}) = \widetilde{\pi}(y, Y - \widehat{y} + y).$$

We see from the formula for $\widetilde{\pi}$ in equation 1 that ψ is continuously differentiable and satisfies $\psi(0;\widehat{y}) = 0 > \psi(y;\widehat{y})$ if $y > R^r$ (in which case, expenditure exceeds the rent). Since ψ_i is continuous in y, it must have at least one maximizer in the interval $[0, R^r]$. Furthermore, $\psi(y)$ is quasiconcave. This follows from the expression for the first and second derivatives:

$$\psi'(y;\widehat{y}) = \theta \frac{(Y-\widehat{y})R}{(Y-\widehat{y}+y)^2} - uy^{u-1} + (1-\theta)\Delta(y;\widehat{y}),$$

where primes indicate derivatives with respect to y and

$$\Delta(y;\widehat{y}) = \frac{uy^u}{Y-\widehat{y}+y} + \frac{(Y-\widehat{y})y^u}{(Y-\widehat{y}+y)^2},$$

and

$$\psi''(y;\widehat{y}) = -2\theta \frac{(Y-\widehat{y})R}{(Y-\widehat{y}+y)^3} - u(u-1)y^{u-2} + (1-\theta)\Delta'.$$

If $\psi'(y) = 0$, we find, after some manipulation that

$$\psi''(y;\widehat{y}_i) = -2\frac{uy^{u-1}}{Y-\widehat{y}+y} - u(u-1)y^{u-2} + (1-\theta)\frac{u(u+1)y^{u-1}}{Y-\widehat{y}+y}.$$

Using $\theta > 0$, we obtain

$$\psi''(y;\widehat{y}_i) < u(u-1)\left[\frac{y^{u-1}}{Y-\widehat{y}+y} - y^{u-2}\right] < 0$$

since $\widehat{y} < Y$, which confirms (strict) quasiconcavity.

It follows that $0 \in S(Y)$ if and only if $\psi'(0;0) \leq 0$. When $r < 1$, we have $u > 1$ and $\psi'(0;0) = \theta R/Y > 0$, so we cannot have $0 \in S(Y)$ in this case. When $r = 1$, we have $\psi'(0;0) = \theta R/Y - 1$, which implies $0 \in S(Y)$ if and only if $Y \geq \theta R$.

Furthermore, $\sigma \in (0,1)$ satisfies $\sigma \in S(Y)$ if and only if $\psi'(\sigma Y;\sigma Y) = 0$. Since

$$\Delta(\sigma Y;\sigma Y) = u\sigma^u Y^{u-1} + (1-\sigma)\sigma^u Y^{u-1},$$

we have

$$\begin{aligned}
\psi'(\sigma Y;\sigma Y) &= \theta\frac{(1-\sigma)R}{Y} - u\sigma^{u-1}Y^{u-1} + (1-\theta)\Delta(\sigma Y;\sigma Y) \\
&= \frac{\theta R(1-\sigma)}{Y} - (1-\sigma)D(\sigma)Y^{u-1}.
\end{aligned}$$

Hence, $\psi'(\sigma Y;\sigma Y)=0$ if and only if $\theta R = D(\sigma)Y^u$, or $Y=[\theta R/D(\sigma)]^r$. ■

Proof. Proposition 2.1 asserts that the inverse share function is given by $\varphi_i(\sigma)=[\theta_i R/D_i(\sigma)]^{r_i}$, where D_i is given in (3). Hence, $\varphi_i'(\sigma)=-r_i D_i'(\sigma)[\theta_i R]^{r_i}[D_i(\sigma)]^{-r_i-1}$, which implies that the sign of the slope of φ_i is the opposite of that of D_i and that turning points of φ_i occur at the same values of σ as those of D_i. Consequently, the numbered assertions in the proposition follow from the following properties of D_i: (i) if $u_i=1$, then $D_i(\sigma)\longrightarrow 1$ as $\sigma\longrightarrow 0$, (ii) if $u_i>1$, then $D_i(\sigma)\longrightarrow 0$ as $\sigma\longrightarrow 0$, (iii) $D_i(\sigma)\longrightarrow\infty$ as $\sigma\longrightarrow 0$, (iv) there is a $\overline{\theta}:(0,1]\longrightarrow(0,1]$ such that $D_i'(\sigma)>0$ for all $\sigma\in(0,1)$ if and only if $\theta\geq\overline{\theta}(r_i)$, (v) if $u_i=1$ and $\theta<\overline{\theta}(r_i)$, D_i has a unique minimum in $(0,1)$, (vi) if $u_i>1$ and $\theta<\overline{\theta}(r_i)$, D_i increases to a local maximum and then decreases to a local minimum in $(0,1)$. Properties (i), (ii) and (iii) are obvious from (3). For ease of exposition in proving the remaining parts, we drop the subscript i.

When $u=r=1$, (3) simplifies to

$$D(\sigma)=(1-\theta)(1-\sigma)+\frac{\theta}{1-\sigma},$$

which has a unique minimum in $\sigma<1$ at

$$\sigma=1-\sqrt{\frac{\theta}{1-\theta}}.$$

This minimum is positive if and only if $\theta<1/2$ and $D'(\sigma)>0$ for $0<\sigma<1$ if $\theta\geq 1/2$. This establishes Property (iv) in the special case $r=1$ with $\overline{\theta}_i(1)=1/2$ as well as Property (v).

If $u>1$, a little manipulation shows that

$$D'(\sigma)=\frac{u\sigma^{u-2}}{(1-\sigma)^2}Q(\sigma;\theta),$$

where

$$\begin{aligned} Q(\sigma;\theta,u) &= \sigma^2(1-\theta)(u+1-\sigma) \\ &\quad +(u-1)(1-\sigma)+(-u+\theta+\theta u)\sigma. \end{aligned}$$

Since $Q(0;\theta,u)=u-1>0$ and $Q(1;\theta,u)=\theta>0$, the cubic (in σ) equation $Q(\sigma;\theta,u)=0$ has at most two roots in $(0,1)$. If $-u+\theta+\theta u>0$, then $Q(\sigma;\theta,u)>0$ for all $\sigma\in(0,1)$, which shows that Q has no roots for all θ close enough to 1. Furthermore, if $\overline{\sigma}\in(0,1)$ satisfies $Q(\overline{\sigma};\theta,u)=0$ and $\theta'<\theta$, then

$$Q\left(\overline{\sigma};\theta',u\right)-Q(\overline{\sigma};\theta,u)=\left[\sigma^3+(u+1)\sigma(1-\sigma)\right]\left(\theta'-\theta\right)<0.$$

Since $Q\left(\sigma;\theta',u\right)$ is a continuous function of σ, it must have (two) roots in $(0,1)$. A straightforward continuity argument allows us to conclude that the set of θ for which $Q(\sigma;\theta,u)$ has no roots is closed and therefore takes the form $\left[\overline{\theta}(r),1\right)$. For any $\theta\in\left[\overline{\theta}(r),1\right)$, D is strictly increasing for all $\sigma\in(0,1)$, which establishes

Property (iv) when $r < 1$. If $\theta < 0$ satisfies $\theta < \overline{\theta}(r)$, D has two turning points; taken together with Properties (ii) and (iii), this proves Property (vi).

It remains to justify the properties of $\overline{\theta}$ in the last sentence of the Proposition. We have shown above that D has positive slope if and only if $\theta \geq 1/2$, which establishes assertion (a) in the proposition. To complete the proof, it is helpful to rewrite the expression for Q as follows:

$$Q(\sigma;\theta,u) = uq(\sigma;\theta,u) + \sigma^2(1-\sigma)(1-\theta) - (1-\sigma) + \theta\sigma, \tag{5}$$

where

$$q(\sigma;\theta,u) = (1-\sigma)^2 + \theta\sigma(1-\sigma) > 0. \tag{6}$$

For any θ satisfying $0 < \theta < \overline{\theta}(r)$, there must be $\overline{\sigma} \in (0,1)$ such that $Q(\overline{\sigma};\theta,u) = 0$, where $u = 1/r$. If $r' > r$, then $u' < u$ and (5) and (5) imply that so $Q(\overline{\sigma};\theta,u') < 0$, which means that Q has two roots in $(0,1)$ and therefore $\overline{\theta}(r') > \theta$. It follows that $\overline{\theta}(r') \geq \overline{\theta}(r)$, proving (b). When $r \leq 1/2$ and therefore $u \geq 2$, we have $uq(\sigma;\theta) \geq 2q(\sigma;\theta)$, which shows, after some rearrangement, that

$$Q(\sigma;\theta,u) \geq (1-\sigma)^3 + 3\theta\sigma(1-\sigma) + \theta\sigma^3 > 0.$$

Hence, D is strictly increasing in $(0,1)$ for any $\theta \geq 0$, which sets $\overline{\theta}(r)$ at 0, proving (c). ∎

Proof of Theorem 3.1. We start by showing that (4) has a solution when $r_i < 1$ for all i and commence with an extension of the domain of φ_i to $\sigma \in (0,1]$ by setting $\varphi_i(1) = 0$. Proposition 2.2 implies that φ_i is continuous on the extended domain. Since $r_i < 1$ implies $u_i > 1$, we have $\varphi_i(\sigma) \longrightarrow \infty$ as $\sigma \longrightarrow 0$ (Proposition 2.2). It follows that there is a $Y_L > 0$ such that, if $\sigma \in (0,1)$ satisfies $\varphi_i(\sigma) \geq Y_L$, then $\sigma < 1/(n+1)$.

For any $\sigma \in (0,1]$, define $\zeta_i(\sigma) = \min\{\varphi_i(\sigma), Y_L\}$. If we further define $\zeta_i(0;\theta_i) = Y_L$, the conclusion of the preceding paragraph and the definition of φ_i imply that $\zeta_i(\sigma)$ is continuous for $\sigma \in [0,1]$. Let Δ denote the simplex $\left\{(\sigma_1,\ldots,\sigma_{n-1}) : \sigma_1,\ldots,\sigma_{n-1} \geq 0, \sum_{j=1}^{n-1}\sigma_j = 1\right\}$ and, for $i = 1,\ldots,n-1$, define $\xi_i : \Delta \times [0,Y_L] \longrightarrow \mathbb{R}$ by

$$\xi_i(\sigma_1,\ldots,\sigma_{n-1},Y) = \zeta_i(\sigma_i) - Y$$

and $\xi_n : \Delta \times [0,Y_L] \longrightarrow \mathbb{R}$ by

$$\xi_n(\sigma_1,\ldots,\sigma_{n-1},Y) = \zeta_n\left(1 - \sum_{j=1}^{n-1}\sigma_j\right) - Y.$$

We claim that no zero $\left(\widehat{\sigma}_1,\ldots,\widehat{\sigma}_{n-1},\widehat{Y}\right)$ of $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)$ can lie on the boundary of its domain. To see this, first note that $\widehat{Y} < Y_L$, for $\boldsymbol{\xi}(\sigma_1,\ldots,\sigma_{n-1},Y_L) = \mathbf{0}$ would lead to $\sigma_j < 1/(n+1)$ for $j = 1,\ldots,n-1$ and $1-\sum_{j=1}^{n-1}\sigma_j < 1/(n+1)$, but these inqualities are inconsistent. This also means that we cannot have $\widehat{\sigma}_j = 0$ for some $j = 1, \ldots, n-1$, for this would imply $\widehat{Y} = \zeta_j(0) = Y_L$. Similarly, $\sum_{j=1}^{n-1} \widehat{\sigma}_j = 1$ would imply $\widehat{Y} = \zeta_n(0) = Y_L$. Finally, $\widehat{Y} = 0$ is ruled out because this would mean $\zeta_n(\widehat{\sigma}_n) = 1$, where $\widehat{\sigma}_n = 1 - \sum_{j=1}^{n-1} \widehat{\sigma}_j$, and Proposition 2.2 would imply $\widehat{\sigma}_n = 0$, contradicting $\widehat{\sigma}_j > 0$ for $j = 1, \ldots, n-1$.

It follows from $\widehat{Y} < Y_L$ that $\widehat{Y} = \zeta_i(\widehat{\sigma}_i) = \varphi_i(\widehat{\sigma}_i)$, so $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_n,)$ and $\widehat{Y}$ satisfy (4). This means we can demonstrate existence of a Nash equilibrium by showing that $\boldsymbol{\xi}$ has at least one zero and we will do this using topological degree theory.

Note that $\boldsymbol{\xi}$ depends on $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ and in this and the next paragraphs we make the dependence explicit by writing it as $\boldsymbol{\xi}(\sigma_1, \ldots, \sigma_{n-1}, Y; \boldsymbol{\theta})$. Note that the Jacobian of $\boldsymbol{\xi}$ with respect to $(\sigma_1, \ldots, \sigma_{n-1}, Y)$ is

$$\mathbf{J} = \begin{pmatrix} \varphi_1' & \cdots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \varphi_{n-1}' & -1 \\ -\varphi_n' & \cdots & -\varphi_n' & -1 \end{pmatrix}$$

and this has determinant

$$\det \mathbf{J} = -\sum_{j=1}^{n} \prod_{k \neq j} \varphi_k'.$$

If $\theta_j = 1$, it follows from Proposition 2.2 that $\varphi_j' < 0$. If this holds for all j, the sign of $\det \mathbf{J}$ is $(-1)^n$. In particular, the degree of $\boldsymbol{\xi}(\cdot; \mathbf{e})$ is non-zero, where $\mathbf{e} = (1, \ldots, 1)$.

Now, for $t \in [0,1]$, let $\boldsymbol{\theta}(t) = t\boldsymbol{\theta} + (1-t)\mathbf{e}$ and note that $\boldsymbol{\xi}(\sigma_1, \ldots, \sigma_{n-1}, Y; \boldsymbol{\theta}(t))$ is a continuous function of $(\sigma_1, \ldots, \sigma_{n-1}, Y, t)$. Homotopy equivalence (see Lloyd [13] for example) implies that the degree of $\boldsymbol{\xi}(\sigma_1, \ldots, \sigma_{n-1}, Y; \boldsymbol{\theta}(t))$ is invariant to the value of t since $\boldsymbol{\xi}$ has no zeroes on the boundary of the domain $\Delta \times [0, Y_L]$. It follows that $\boldsymbol{\xi}(\sigma_1, \ldots, \sigma_{n-1}, Y; \boldsymbol{\theta})$ has non-zero degree and hence has at least one zero.

To extend this conclusion to the case where some or all r_i take the value 1, consider a sequence $(r_1^m, \ldots, r_n^m)$ which satisfies $0 < r_j^m < 1$ for all $j = 1, \ldots, n$ and $m = 1, 2, \ldots$ and $r_j^m \longrightarrow r_j$ as $m \longrightarrow \infty$. We have shown that there is an equilibrium $(\widetilde{y}_1^m, \ldots, \widetilde{y}_n^m)$ for each m. Such an equilibrium must satisfy $0 \leq \widetilde{y}_j^m \leq R^{u_j}$ for all j (strategies failing the upper inequality are strictly dominated for contestant j) and compactness implies that $(\widetilde{y}_1^m, \ldots, \widetilde{y}_n^m)$ must have a subsequence convergent to $(\widetilde{y}_1, \ldots, \widetilde{y}_n)$, say. Using the fact that the payoff $\widetilde{\pi}_j$ is continuous in y_j and r_j for all j, a simple limiting argument allows us to conclude that $(\widetilde{y}_1, \ldots, \widetilde{y}_n)$ satisfies the equilibrium condition for parameter set $(r_1, \ldots, r_n)$, completing the existence proof. ■