The University of Manchester

Economics
Discussion Paper Series
EDP-1115

On the Nonparametric Tests of Univariate GARCH Regression Models

Wasel Shadat

June 2011

Economics
School of Social Sciences
The University of Manchester
Manchester M13 9PL

On the Nonparametric Tests of Univariate GARCH Regression Models*

Wasel B Shadat†‡
University of Manchester

June 20, 2011

Abstract

This paper investigates simultaneous consistent nonparametric testing of the conditional mean and conditional variance structure of univariate GARCH, or UGARCH, regression models. The approach is developed from the Integrated Generalized Spectral (IGS) and Projected Integrated Conditional Moment (PICM) procedures proposed recently by Escanciano (2008 and 2009, respectively) for time series models. Extending Escanciano (2008), a new and simple wild bootstrap procedure is proposed to implement these tests. A Monte Carlo study compares the performance of these nonparametric tests and four parametric tests of nonlinearity and/or asymmetry under a wide range of alternatives. The simulation results demonstrate the proposed bootstrap scheme's ability to control the size extremely well and therefore the power comparison seems justified. The simulation exercise also presents the new evidence of the effect of conditional mean misspecification on various parametric tests of conditional variance. The testing procedures are also illustrated with the help of the S&P 500 data.

JEL classification: C12, C32

Keywords: Univariate GARCH models; Nonparametric consistent testing; Projected integrated conditional moments tests; Integrated generalized spectral tests; Monte Carlo experiment

---

*Earlier versions of this paper were presented in the Manchester Doctoral conference 2010 and Econometrics and Applied Economics Workshops 2010/11, University of Manchester. Our thanks to workshop participants for their insightful comments and discussions. We are grateful for the insightful comments of Chris Orme, Alastair Hall, Len Gill and Ralf Becker, which greatly improved the exposition of this paper. The standard disclaimer applies.

†Address correspondence to Wasel Shadat, Economics, School of Social Science, University of Manchester, Manchester M13 9PL, UK; e-mail: wbshadat@yahoo.com.

‡This research is a part of the author's PhD thesis and was supported by a Commonwealth Scholarship and Fellowship Plan and a Manchester School Award; both of which are gratefully acknowledged.

1 Introduction

In many economic and financial time series applications, such as portfolio selection and asset pricing, deciding whether the dynamics are determined by the conditional mean and/or conditional variance has significant implications. The widespread application of the univariate ARCH (UARCH) or its extension, univariate generalized ARCH (UGARCH), in financial econometrics is a clear indication of the popularity and success of these models.[1] Specification testing procedures of these volatility models, particularly parametric ones, have rightfully received considerable attention in the literature and almost all of them implicitly assume a correct specification for the conditional mean. A separate stream of literature deals with the problem of diagnostic testing of the mean function in the presence of heteroskedascity. On the other hand, only a few studies consider simultaneously testing both the mean and variance function.

To test for the presence of the ARCH effects Engle (1982) provided a LM type testing procedure.[2] Despite the singularity problem of the block of information matrix required to construct a LM test for the GARCH disturbances against white noise disturbances (Bollerslev, 1986), Lee (1991) bypassed this problem by showing that for the linear regression model LM tests for the GARCH and ARCH disturbances are the same. A number of diagnostic testing procedures have also been proposed in the literature; for example, a portmanteau type test to test the null that the squared standardized error process is serially uncorrelated (Li and Mak, 1994), score-type tests for testing a GARCH specification against a higher order GARCH (Bollerslev, 1986), for asymmetry (Engle and Ng, 1993) and for (a) no remaining ARCH effects in standardized errors, (b) linearity or symmetry against a smooth transition GARCH, and (c) parameter constancy against smoothly changing parameters (Lundbergh and Teräsvirta, 2002). Halunga and Orme (2009) provided a unifying framework based on a Conditional Moment (CM) principle and proposed "new" tests for asymmetry and non-linearity.

The parametric CM tests, however, are not necessarily consistent against all possible alternatives (see, for example, Bierens, 1982; Holly, 1982; Newey, 1985; Tauchen, 1985) as they only employ a finite number of moment restrictions implied by the model; e.g.,

$$\mathrm{E}\left[e\left(\theta_0\right)|Z\right] = 0 \text{ a.s. for some } \theta_0 \in \Theta \subset \Re^p, \tag{1}$$

where $e\left(\theta_0\right)$ is the regression error and Z is the conditioning set. Further we want to reemphasize that all above mentioned tests are developed assuming a correct mean specification. It is worthwhile to quote Engle's (1982, pp.990) third interpretation of the ARCH regression model which says "an approximation to a more complex regression which has non-ARCH disturbances. The

[1] To allow for asymmetry and/or nonlinearity in the ARCH/GARCH process, several extensions, such as exponential GARCH (EGARCH), threshold GARCH (TGARCH), smooth transition GARCH (STGARCH) to name a few, have been proposed in the subsequent literature; for a survey of univariate ARCH/GARCH models, see Bollerslev et al. (1994).

[2] This test has a simple form as TR^2 and under the null of no ARCH effect (i.e., white noise disturbances) has a χ^2 distribution.

ARCH specification might then be picking up the effect of variables omitted from the estimated model. The existence of an ARCH effect would be interpreted as evidence of misspecification, either by omitted variables or through structural change. If this is the case, ARCH may be a better approximation to reality than making standard assumptions about the disturbances, but trying to find the omitted variable or determine the nature of the structural change would be even better". Noting the possible spill-over effect from neglected misspecification of conditional mean on testing the conditional variance and the inconsistency of the parametric CM tests, this paper aims to test both joint (mean and variance) specification of a UGARCH regression model and its marginal components simultaneously applying a consistent nonparametric approach.

If the conditional mean is misspecified, existing tests in the literature may perform poorly by over-rejecting the correct conditional variance specification. A number of studies examined the effect of misspecified mean (e.g., omitted variables, structural change parameter instability, noisy chaotic function) on the diagnostic tests of variance specification. For example, Lumsdaine and Ng (1999) found that under mean misspecification the ARCH-LM test, in general, over-rejects the null of conditional homoskedasticity and suggested using recursive residuals or some functions of them to robustify the ARCH-LM test.[3] For a noisy chaotic (highly non-linear) conditional mean model with homoskedastic errors, Kyrtsou (2008) showed that the ARCH-LM (Engle, 1982) and the McLeod and Li (1983) tests for non-linearity in the second moment may exhibit spurious heteroskedasticity due to inappropriate filtering of neglected non-linearity.[4] Blake and Kapetanios (2007) also found that misspecification of the conditional mean may lead to the spurious rejection of the null of no ARCH and suggested a new testing procedure based on an artificial neural network (ANN) which is robust to the presence of neglected non-linearity.[5] On the other hand there is also the problem of testing the conditional mean specification in the presence of (G)ARCH error. To address this problem a number of techniques have been suggested in the literature; e.g., using a heteroskedasticity-consistent covariance matrix estimator or correctly specifying ARCH process (Lee et al., 1993), using a heteroskedastic consistent auxiliary regression together with the wild bootstrap (Becker and Hurn, 2009).

The above evidence suggests that it is desirable to test the conditional mean and variance specification simultaneously. There are some suggestions of se-

[3] In their empirical application with the S&P 500 returns, which was also used by Bera and Higgins (1997), they found that the standard ARCH-LM test rejects the null hypothesis of conditional homoskedasticity while using recursive residuals and their squares cannot reject the null.

[4] Kyrtsou's noisy chaotic model with homoskedastic error is given by:

$$Y_t = \alpha \frac{Y_{t-t'}}{1+Y_{t-t'}^c} - \beta Y_{t-1} + \gamma Y_{t-j}\left(1 - Y_{t-j}\right) + \varepsilon_t;\ \varepsilon_t \backsim N(0,1)$$

where α, β and γ are parameters; t' is the delay, and c is constant.

[5] In their empirical application with bilateral exchange rate, they again found that standard ARCH-LM test rejects the null quite often (18 out of 39 series) while using new tests only 10 series reject the null of no ARCH.

quential testing; i.e., the conditional mean first and then testing the conditional variance with correctly specified mean function. If sole interest lies in the conditional variance specification, another possibility is to estimate the conditional mean by some nonparametric method and then use the residuals in the variance diagnostic test (cf. Blake and Kapetanios, 2007). Although the problem of testing many conditional moment restrictions has been considered in the literature (see, for example, Chen and Fan, 1999 for mixing data or Delgado, Dominguez and Lavergne, 2006 for independent data), the literature on joint diagnostic tests of conditional mean and variance is very limited. Ngatchou-Wandji (2005) suggested a Wald-type test based on χ^2-discrepancy measures which is, however, not consistent against a wide class of alternatives to the correct specification. The aim of this paper is to develop tests of both specifications jointly by employing consistent testing procedures.

Bierens (1982) first proposed a consistent testing procedure, an Integrated CM (ICM) test, for non-linear parametric regression models involving independent and identically distributed (i.i.d.) data. Since then a vast amount of literature has addressed the issue of consistent testing in both the i.i.d. and time series contexts. For time series, a few prominent examples are Bierens (1984), de Jong (1996), Bierens and Polberger (1997), Koul and Stute (1999), Hong and Lee (2005), Escanciano (2006a, 2006b, 2007a, 2008). This can be broadly categorized into two classes of tests: namely tests based on a local approach and tests based on an integrated approach. The first approach uses nonparametric smoothing estimators of a local measure of dependence $\mathrm{E}\left[e\left(\theta_0\right)|Z\right]$. The local approach requires smoothing of the data which leads to a less precise fit and for high (or even moderate) dimension of Z, this approach suffers from "curse of dimensionality"; i.e., considerable bias, even for large sample (see, for example, Section 7.1 of Fan and Gijbels, 1996). But these tests have standard asymptotic null distributions, though finite-sample distributions depend on the choice of the bandwidth and on the nonparametric estimator. On the other hand, ICM tests use integrated (or cumulative) measures of dependence and avoid the smoothing by converting conditional orthogonality conditions of (1) to uncountably many unconditional (parametric) orthogonality moment restrictions; i.e.,

$$\mathrm{E}\left[e\left(\theta_0\right)|Z\right] = 0 \text{ a.s.} \Leftrightarrow \mathrm{E}\left[e\left(\theta_0\right) w\left(Z, x\right)\right] = 0, \text{ a.e. in } \Pi \subset \Re^q, \qquad (2)$$

where the parametric family $\{w\left(., x\right) : x \in \Pi\}$ is such that (2) holds and $\Pi \subset \Re^q$ is a properly chosen space. More details are provided in Section 3, also see e.g., Bierens and Polberger (1997), Stinchcombe and White (1998), Escanciano (2006a). However, in this case the asymptotic null distribution depends on the Data Generating Process (DGP) and null specification. Hence critical values can not be tabulated for general cases and a bootstrap procedure is required to implement the test.

We can, in principle, use the classical ICM tests in our case. However, for GARCH regression model bootstrapping is a complex and problematic operational issue. One solution is to use a *feasible projected* version of the classical ICM test (Escanciano, 2009), which we shall term as the Projected ICM (PICM) test, to address our problem of simultaneous testing of joint and marginal hypothesis. Although this procedure also requires bootstrap, it does not need to estimate the parameters in the bootstrap world, therefore making it much simpler to implement. To the best of our knowledge there is no study which examines this approach in detail in the context of GARCH regression models.

Escanciano (2008) proposed an Integrated Generalized Spectral (IGS) test which is a consistent joint and marginal testing procedure for conditional mean and conditional variance models. This test is based on a pair-wise generalized spectral approach. To calculate the critical values of the IGS tests, the author suggested and theoretically justified a Fixed Design Wild Bootstrap (FDWB) procedure which requires estimating the parameters in each bootstrap replication. The theoretical null conditional heteroskedastic model considered by Escanciano indeed include GARCH regression models, however the null DGPs in his Monte Carlo experiments do not include GARCH with conditional mean. The two null DGPs considered in his simulation design can be consistently estimated by the OLS regression, both in real and bootstrap world (see Remark 1). Further the validity of the OLS approach was not checked. This FDWB method, as we will see later, involves generating two separate bootstrap data samples: one provides only the conditional mean structure (and does not include any information about conditional variance structure) while the other provides conditional variance structure. Therefore, for GARCH regression models the QML estimation is not possible in the bootstrap world (since we need a single dependent variable containing both mean and variance structure) whereas we still require the QMLE in the real world to estimate conditional variance parameters. This implies that Escanciano's procedure is not directly applicable in our case (see Section 5 for more on this topic).

1.1 Contributions and Plan of the Paper

The first major contribution of this paper is to identify a problem of Escanciano's (2008) Fixed Design Wild Bootstrap (FDWB) in the context of UGARCH regression model, implying that a modification of this procedure is required to accommodate this model. Since a full parametric GARCH bootstrap procedure for IGS tests using the QMLE is operationally problematic, a simple alternative bootstrap procedure for IGS tests using least squares estimation has been proposed. Asymptotic analysis suggests that it does not strictly satisfy the sufficient conditions identified in the previous literature (Escanciano 2007b) which however does not necessarily imply that our procedure is asymptotically invalid. Nonetheless, due to the simplicity of the procedure a Monte Carlo study is conducted to evaluate its ability to control for size of the IGS tests. Our simulation study demonstrates excellent size property for the proposed bootstrap procedure which raises questions about the restrictiveness of the conditions set by Escanciano (2007b) and probably the IGS tests can be implemented under a weaker set of conditions. Moreover the ability of our procedure to control the size very well makes power comparisons justified. The specification testing literature reveals that asymptotically valid tests often display poor size properties leading to inconsequential power comparisons. For example, asymptotically valid tests of stationarity against the alternative of a unit root process is known to lead to over-rejections in finite samples when the considered process is stationary but highly persistent (see, e.g., Lanne and Saikkone, 2003; Caner and Kilian, 2001). Similarly, Orme (1990) showed that several asymptotically valid variants of the IM test statistics, such as 'TR^2' variant proposed by Chesher (1983) and Lancaster (1984), demonstrate extremely poor size. Shi (2011) found that the Vuong (1989) test for nonnested model comparison over-reject the null hypothesis when the null hypothesis is true.

Secondly, we illustrate in detail how to execute the PICM tests for this model. Thirdly, an extensive Monte Carlo study is conducted to compare the performance of these two nonparametric tests and four parametric CM tests of nonlinearity and/or asymmetry considered in Halunga and Orme (2009), Engle and Ng (1993) and Lundbergh and Teräsvirta (2002). This simulation exercise also provides us with the opportunity to investigate the effect of conditional mean misspecification on various parametric LM and CM tests of conditional variance in the regression context which has not been done in the literature before. Finally, we illustrate the testing procedures with the help of the S&P 500 data.

The remainder of this paper is organized as follows. In Section 2 we provide the null model and moment conditions considered here. The PICM testing framework is discussed next which is followed by the Escanciano's IGS test and wild bootstrap scheme suggested by him. In Section 5 the limitations of Escanciano's FDWB procedure while applying for our GARCH regression model are pointed out and we put forward a modified bootstrap scheme. The parametric CM tests of the conditional variance are briefly introduced in Section 6. Finally, we present the simulation evidence in Section 7 and an empirical application in Section 8. All proofs are relegated to the appendix. In the remaining A^c is the complex conjugate of A, $\|A\|_M$ denotes the weighted norm $A'MA^c$ for a positive definite matrix M and a complex vector A.

2 The Null Model and Moment Conditions

Consider a $\{(y_t, X_t')'\}_{t\in\mathbb{Z}}$ be a strictly stationary and ergodic time series process on a probability space $(\Omega, \mathcal{F}, P)$ where y_t is the dependent variable and $Z_{t-1} = (y_{t-1}, X_t')' \in \Re^{1+m}, m \in \mathbb{N}$, is the explanatory random vector containing lagged values of y_t and possibly other variables. Suppose $\mathcal{I}_{t-1} = \left(Z_{t-1}', Z_{t-2}', \cdots\right)'$ is the information set at time $t-1$ and $\mathcal{F}_{t-1} = \sigma\left(Z_{t-1}', Z_{t-2}', \cdots\right)$ is the $\sigma-$field generated by the past information up to and including time $t-1$. Then define the conditional mean and variance $m\left(\mathcal{I}_{t-1}\right) = \mathrm{E}\left[y_t | \mathcal{I}_{t-1}\right]$, and $h\left(\mathcal{I}_{t-1}\right) = Var\left[y_t | \mathcal{I}_{t-1}\right]$, respectively, and standardized errors $\zeta_t = \frac{(y_t - m(\mathcal{I}_{t-1}))}{\sqrt{h(\mathcal{I}_{t-1})}}, t \in \mathbb{Z}$.

We consider the following parametric model

$$\begin{aligned} y_t &= m\left(\mathcal{I}_{t-1,q}, \varphi_0\right) + \varepsilon_{0t}, \\ \varepsilon_{0t} &= \sqrt{h\left(\mathcal{I}_{t-1,q}, \eta_0\right)}\zeta_t, \end{aligned} \tag{3}$$

where ζ_t is i.i.d. $(0,1)$, $\mathcal{I}_{t-1,q} = \{Z_s\}_{s=1}^{t-q}$, $q < \infty$, $q \in \mathbb{N}$ and $\theta_0 = (\varphi_0', \eta_0')' \in \Theta \subset \Re^p$ where φ_0 and η_0 represent the true conditional mean and conditional variance parameters, respectively. This specification is quite general and includes linear ARMA-ARCH, ARMA-GARCH as well as nonlinear conditional mean (e.g., GARCH-in-Mean, bilinear) and non-linear and asymmetric variance models (e.g., GJR GARCH, EGARCH, STGARCH).

To focus our discussion we will consider the AR(1)-GARCH(1, 1) process, where

$$\begin{aligned} m\left(\mathcal{I}_{t-1,q}, \varphi_0\right) &= W_t\varphi_0 = \varphi_{00} + \varphi_{01}y_{t-1}, \\ h\left(\mathcal{I}_{t-1,q}, \eta_0\right) &= \eta_0' s_{0,t-1} = \alpha_{00} + \alpha_{01}\varepsilon_{0,t-1}^2 + \beta_{01}h_{0,t-1}, \end{aligned} \tag{4}$$

with $W_t = (1, y_{t-1})$, $\varphi' = (\varphi_0, \varphi_1)$, $s_{t-1} = \left(1, \varepsilon_{t-1}^2, h_{t-1}\right)'$, $\eta = (\alpha_0, \alpha_1, \beta_1)'$. For notational convenience we write $m_t \equiv m\left(\mathcal{I}_{t-1,q}, \varphi\right)$ and $h_t \equiv h\left(\mathcal{I}_{t-1,q}, \eta\right)$. Under correct model specification, $\{\varepsilon_{0t}\}$ is MDS wrt $\mathcal{F}_{t-1}$, with zero mean and conditional variance h_t. That is the correct joint specification is tantamount to saying

$$H_0 : \mathrm{E}\left[e_{0,1t}|\mathcal{I}_{t-1}\right] = 0 \text{ a.s. and } \mathrm{E}\left[e_{0,2t}|\mathcal{I}_{t-1}\right] = 0 \text{ a.s. for some } \theta_0 \in \Theta, \tag{5}$$

where $e_{0,1t} \equiv \varepsilon_{0t} = Y_t - m_{0t}$ and $e_{0,2t} \equiv \varepsilon_{0t}^2 - h_{0t}$. Or more compactly

$$H_0 : \mathrm{E}\left[e_{0,t}|\mathcal{I}_{t-1}\right] = 0 \text{ a.s. for some } \theta_0 \in \Theta, \tag{6}$$

where $e_{0,t} \equiv e_t(\theta_0) = (e_{0,1t}, e_{0,2t})'$. It is important to note that first conditional moment restriction (CMR) corresponds to adequacy of the conditional mean whereas both CMRs are necessary for correct specification of conditional variance.

We assume that our model satisfies the following regularity conditions:

Assumption 2.1 $\{y_t, X_t\}_{t\in\mathbb{Z}}$ *is a strictly stationary and ergodic process.*

Assumption 2.2 $\mathrm{E}\left[e_{0,1t}^2\right] = \mathrm{E}\left[\left(y_t - m_{0t}\right)^2\right] < \infty$, *and*
$\mathrm{E}\left[e_{0,2t}^2\right] = \mathrm{E}\left[e_{0,1t}^2 - h_{0t}\right]^2 < \infty$.

Assumption 2.3 *Let* Θ_0 *be a small convex neighborhood of* θ_0. *The functions* $m\left(\mathcal{I}_{t-1,q}, .\right)$ *and* $h\left(\mathcal{I}_{t-1,q}, .\right)$ *are twice continuously differentiable (a.s.) wrt* $\theta \in \Theta_0$. *Also,* $\mathrm{E}\left[\sup_{\theta\in\Theta_0}\left\|g_{jt}'(\theta)\right\|\right] < \infty$, $j = 1, 2$ *where* $g_{1t}'(\theta) = \frac{\partial e_{1t}}{\partial\theta'} = \frac{\partial m_t}{\partial\theta'}$, $g_{2t}'(\theta) = \frac{\partial e_{2t}}{\partial\theta'} = 2e_{1t}(\theta)\, g_{1t}'(\theta) - \frac{\partial h_t}{\partial\theta'}$.

Assumption 2.4 *The parameter space* Θ *is compact in* $\Re^p$ *and* θ_0 *belongs to the interior of* Θ.

Assumption 2.5 *The observed information set at time t, $\widehat{\mathcal{I}}_t$, may contain some initial values and satisfies*

$$\left(\sum_{t=1}^{T}\left[\mathrm{E}\sup_{\theta\in\Theta_0}\left\|m\left(\widehat{\mathcal{I}}_{t-1},\theta\right)-m\left(\mathcal{I}_{t-1},\theta\right)\right\|^2\right]^{1/2}\right)^2 = o(T),$$

$$\left(\sum_{t=1}^{T}\left[\mathrm{E}\sup_{\theta\in\Theta_0}\left\|\left(Y_t-m\left(\widehat{\mathcal{I}}_{t-1},\theta\right)\right)^2-\left(Y_t-m\left(\mathcal{I}_{t-1},\theta\right)\right)^2\right\|^2\right]^{1/2}\right)^2 = o(T),$$

$$\left(\sum_{t=1}^{T}\left[\mathrm{E}\sup_{\theta\in\Theta_0}\left\|h\left(\widehat{\mathcal{I}}_{t-1},\theta\right)-h\left(\mathcal{I}_{t-1},\theta\right)\right\|^2\right]^{1/2}\right)^2 = o(T).$$

These assumptions are fairly general and considerably weaker compared to the related conditions made in the literature. Assumptions 2.1 and 2.2 do not involve any mixing or asymptotic independence assumption as opposed to the mixing Assumption A.1 in Hong and Lee (2003), yet they allow for a long memory process. Assumption 2.3 and 2.4 are standard in the literature see, e.g., Escanciano (2006a) and satisfied in our model. Finally, Assumption 2.5 is a start-up value condition and similar in spirit to Assumption A4 in Hong and Lee (2003), which ensures that the impacts of initial values are asymptotically negligible. This condition holds for many time series models including ARMA-GARCH models; see Francq and Zakoian (2004).

3 The Projected ICM (PICM)Test

3.1 Test Statistics and Limit Distribution

The underlying idea of the ICM tests is to characterize the CMR under consideration by an infinite number of unconditional moment restrictions. More specifically, for a moment condition of the form $\mathrm{E}\left[e_t(\theta_0)\middle|Z_{t-1}\right]$, we have (by the law of iterated expectation)

$$\mathrm{E}\left[e_t(\theta_0)\,w(Z_{t-1},x)\right]=\int_{(-\infty,x]}\mathrm{E}\left[e_t(\theta_0)\middle|Z_{t-1}=x\right]\mathrm{d}P_z,\ x\in\Re^{1+m}, \quad (7)$$

where P_z is a stationary probability measure of Z_{t-1} and $w(Z_{t-1},x)$ is some weighting function such that the above equivalence holds (see Lemma 1 for the sufficient conditions for the weight function). Now from (7) and Billingsley (1995, Theorem 16.10iii), we have

$$\mathrm{E}\left[e_t(\theta_0)\middle|Z_{t-1}\right]\equiv 0 \text{ a.s.} \Leftrightarrow \mathrm{E}\left[e_t(\theta_0)\,w(Z_{t-1},x)\right]\equiv 0.$$

Within this framework, the null hypothesis can be written as:

$$H_0:\mathrm{E}\left[e_t(\theta_0)\,w(Z_{t-1},x)\right]=0,\ \forall x\in\Re^{1+m} \text{ and some } \theta_0\in\Theta,$$

against the class of nonparametric alternatives

$$H_A : P_z \left[\mathrm{E} \left[e_t \left(\theta_0 \right) w \left(Z_{t-1}, x \right) \right] \neq 0 \right] > 0, \ \forall \ \theta \in \Theta.$$

Lemma 1 *Let $\mathcal{C}_b \left(\Re^{1+m} \right)$ be the space of all bounded, continuous, complex valued functions on $\Re^{1+m}$. Then, any of the following conditions are sufficient for the class of function $\mathcal{W} = \left\{ w \left(Z, x \right) : x \in \Pi \subset \left[-\infty, \infty \right]^s \right\}$, Z is a random variable with the same dimension as Z_t, $t \in \mathbb{Z}$, Π is the nuisance parameter space with dimension s which depends on the particular family $\mathcal{W}$ used to satisfy the equivalence in (7):*

1. *$\mathcal{W} \subset \mathcal{C}_b \left(\Re^{1+m} \right)$ is a vector lattice that contains the constant functions and separate points of $\mathbb{R}^{1+m}$.*
2. *$\mathcal{W} \subset \mathcal{C}_b \left(\Re^{1+m} \right)$ is a algebra that contains the constant functions and separate points of $\Re^{1+m}$.*
3. *$\mathcal{W} = \left\{ w \left(x'Z \right) : x \in \Pi \right\}$ and is a non-polynomial analytical function.*
4. *$\mathcal{W} = \left\{ 1 \left(Z \in \mathcal{B}_x \right) : x \in \Pi \right\}$ and $\left\{ \mathcal{B}_x \right\}_{x \in \Pi}$ is a separating class of Borel sets of $\Re^{1+m}$.*

The most commonly used examples of w are the indicator weight function $w \left(Z_{t-1}, x \right) = 1 \left(Z_{t-1} \leq x \right)$ with $x \in \Pi_{ind} = \left[-\infty, \infty \right]^{m+1}$ (Stute and Zhu, 2002) and the complex exponential function $w \left(Z_{t-1}, x \right) = \exp \left(i x' Z_{t-1} \right)$ with $i = \sqrt{-1}$ and $x \in \Pi_{\exp} = \Re^{m+1}$ (Bierens, 1982). Different families w have different power properties. "Optimal" choices for w depend on the true alternative at hand and the function used to measure orthogonality restrictions.

Now define the classical marked empirical process as

$$R_T \left(x, \theta \right) = T^{-1/2} \sum_{t=1}^{T} \left[e_t(\theta) w \left(Z_{t-1}, x \right) \right].$$

In the field of econometrics and statistics inference, there is a long tradition of using processes like $R_T \left(x, \theta \right)$, see, e.g., Bierens (1982), Stute (1997), Koul and Stute(1999), Escanciano (2007a) among others. Note that θ_0 is a nuisance parameters in the construction of the test and most existing tests do not acknowledge this fact. Since θ_0 is unknown, deviations in the direction of the score function cannot be differentiated from local deviation of θ_0 (i.e., deviations within the parametric model) which may result in tests with low power. To address this, for any $\sqrt{T}$-consistent estimator $\hat{\theta}$, Escanciano (2009) suggested a *projected* marked empirical process as follows:

$$R_T^1 \left(x, \hat{\theta} \right) = T^{-1/2} \sum_{t=1}^{T} \left[w \left(Z_{t-1}, x \right) I_e - G' \left(x, \hat{\theta} \right) \Xi^{-1} g \left(Z_{t-1}, \hat{\theta} \right) \right] \hat{e}_t, \qquad (8)$$

where $\hat{e}_t \equiv e_t(\hat{\theta})$, I_e is the identity matrix with same dimension of $\hat{e}_t$,

$$\begin{aligned} g\left(Z_{t-1},\theta\right) &= \frac{\partial e_t(\theta)}{\partial\theta}, \\ \Xi &= \mathrm{E}\left[g\left(Z_{t-1},\theta_0\right) g\left(Z_{t-1},\theta_0\right)'\right], \text{ and} \\ G\left(x,\theta\right) &= \mathrm{E}\left[g\left(Z_{t-1},\theta\right) w\left(Z_{t-1},x\right)\right]. \end{aligned}$$

The idea is that because of the amalgamation of the score information, tests constructed based on (8) do not waste power due to the local deviation of θ_0. Note that, under some regularity conditions,

$$\sup_{x\in\Re^{1+m}} \left\| R_T^1\left(x,\hat{\theta}\right) - R_T^1\left(x,\theta_0\right)\right\| = o_p\left(1\right). \tag{9}$$

Because of this key property, implementation of the test neither requires the asymptotic distribution of the estimator nor θ to be estimated in the each bootstrap world.[6]

Then the test statistic is a continuous functional of the feasible projected marked empirical process:

$$\hat{R}_T^1\left(x\right) = T^{-1/2}\sum_{t=1}^{T}\left[w\left(Z_{t-1},x\right) I_e - \hat{G}'\left(x,\hat{\theta}\right)\hat{\Xi}^{-1} g\left(Z_{t-1},\hat{\theta}\right)\right]\hat{e}_t, \tag{10}$$

with

$$\begin{aligned} \hat{G}\left(x,\hat{\theta}\right) &= T^{-1}\sum_{t=1}^{T}\left[g\left(Z_{t-1},\hat{\theta}\right) w\left(Z_{t-1},x\right)\right], \text{ and} \\ \hat{\Xi} &: = T^{-1}\sum_{t=1}^{T}\left[g\left(Z_{t-1},\hat{\theta}\right) g\left(Z_{t-1},\hat{\theta}\right)'\right]. \end{aligned}$$

Using the Cramer-von Mises (CvM) norm, the test statistic becomes:

$$CvM_T := \int_{\Re^{1+m}} \left\|\hat{R}_T^1\left(x\right)\right\|^2 dF_{T,Z}\left(x\right), \tag{11}$$

where $F_{T,Z}$ is the Empirical Distribution Function (EDF) of $\{Z_{t-1}\}_{t=1}^{T}$. For the limit distribution of $\hat{R}_T^1\left(x\right)$, in addition to the Assumptions 2.1 - 2.5, we require the following assumptions:

Assumption 3.1 *The derivatives* $g\left(Z_{t-1},\theta\right)$ *satisfies*

$$\mathrm{E}\left[\sup_{\theta\in\Theta_0}\left\|g\left(Z_{t-1},\theta\right)\right\|^2\right] < \infty, \;\; \mathrm{E}\left[\left\|g\left(Z_{t-1},\theta_0\right) e_t\left(\theta_0\right)\right\|^2\right] < \infty,$$

and $\mathrm{E}\left[g\left(Z_{t-1},\theta\right) g'\left(Z_{t-1},\theta\right)\right]$ *is positive definite in* Θ_0.

[6] As a matter of fact, any test statistics of the form $T^{-1/2}\sum_{t=1}^{T} a\left(Z_t\right)\hat{e}_t(\hat{\theta})$, with $\mathrm{E}\left[a\left(Z_t\right) g(Z_t,\theta_0)\right] = 0$, satisfies (9).

Assumption 3.2 *$\left\|\overline{G}_t(x_1) - \overline{G}_t(x_2)\right\| \leq C_t \|x_1 - x_2\|^{s_1}$ for each $(x_1, x_2) \in \Re^{1+m} \times \Re^{1+m}$, for some $s_1 > 0$ and a generic stationary sequence C_t with $\mathrm{E}[C_t] < \infty$ where $\overline{G}_t(x_1) = \mathrm{E}\left[\mathrm{E}\left[\sup_{\theta\in\Theta_0}\left\|e_t^2(\theta_0)|Z_{t-1}\right\|\right] w(Z_{t-1}, x)|\mathcal{F}_{t-1}\right].$*

Assumption 3.3 $\sqrt{T}\left(\hat{\theta} - \theta_0\right) = O_p(1).$

Assumption 3.2 requires the existence of conditional Lipschitz moments while by Assumption 3.3 we only need any $\sqrt{T}$ consistent estimator of θ. Then following Theorem and Corollary provide the limit distribution of the process $\hat{R}_T^1(x)$ and CvM test statistics (for proof, see Escanciano, 2009):

Theorem 1 *Under the above assumptions,*

$$\sup_{x\in\Re^{1+m}}\left\|\hat{R}_T^1(x) - R_T^1(x, \theta_0)\right\| = o_p(1).$$

Corollary 1 *Under the above assumptions,*

$$\begin{aligned}\hat{R}_T^1(x) &\Rightarrow R_\infty^1,\\ CvM_T &\Rightarrow CvM_\infty := \int_{R^{1+m}}\left\|R_\infty^1(x)\right\|^2 dF_Z(x),\end{aligned}$$

where R_∞^1 is a Gaussian process with zero mean and covariance function $\mathrm{E}\left[Ke_t(\theta_0)e_t(\theta_0)'K'\right]$ with $K = w(Z_{t-1}, x)I_e - G'(.,\theta_0)\Gamma^{-1}g(Z_{t-1}, \theta_0).$

3.2 The PICM Tests of the UGARCH Regression Model

Since the asymptotic null distribution of $\hat{R}_T^1(x)$ depends on the DGP and null hypothesis in a complicated way, some approximation is required to obtain the critical values. More specifically, the unknown limiting null distribution of $CvM_T = \psi\left(\hat{R}_T^1(x)\right)$; i.e., the distribution of $\psi\left(R_\infty^1(x)\right)$, is approximated by the bootstrap distribution of $\psi\left(\hat{R}_T^{*1}(x)\right)$ where $\hat{R}_T^{*1}(x)$ is some bootstrap version of $\hat{R}_T^1(x)$. Although there are some suggestions available in the literature,[7] the most popular choice is the wild bootstrap technique which is used in a variety of problems; see, e.g., Stute et al. (1998), Whang (2000), Escanciano (2007a) among others.

Definition 1 *A wild bootstrap involves adding together an estimated predicted part, which serves as a bootstrap world conditional mean, and a bootstrap error term which allow for heteroskedasticity of unknown form. Consider the regression model: $y_t = x_t'\beta + \varepsilon_t$. A typical observation for a wild bootstrap scheme for this regression model can be written as*

$$y_t^* = x_t'\hat{\beta} + \varepsilon_t^*,$$

[7] For example, the Hansen's (1996) conditional p-value method, the Khmaladze's (1981) martingale transformation, upper bounds for the critical values (Bierens and Ploberger, 1997).

where $\hat{\beta}$ is an estimator of β and $\varepsilon_t^* = f(\hat{\varepsilon}_t) U_t$ is the bootstrap error term in which $f(\hat{\varepsilon}_t)$ is some function of OLS residuals $\hat{\varepsilon}_t = y_t - x_t'\hat{\beta}$ ($\hat{\varepsilon}_t$ can possibly be obtained using a different estimator, say $\widetilde{\beta}$, other than $\hat{\beta}$) and U_t is a mutually independent drawing from a pick distribution which itself is completely independent of the data (y_t, x_t') with $\mathrm{E}(U_t) = 0$, $Var(U_t) = 1$ and has bounded support. The two most popular choices of U_t are Mammen's (1993) i.i.d. Bernoulli variates with

$$\Pr\left(U_t = \frac{1}{2}\left(1-\sqrt{5}\right)\right) = \frac{1+\sqrt{5}}{2\sqrt{5}}, \ \Pr\left(U_t = \frac{1}{2}\left(1+\sqrt{5}\right)\right) = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}}, \tag{12}$$

and Davidson and Flachaire's (2001,2008) Rademacher distribution with

$$\Pr(U_t = 1) = \Pr(U_t = -1) = 0.5. \tag{13}$$

Note that the wild bootstrap method requires a separable moment function ε_t so that the dependent variable can be recovered in an additive manner (e.g., in a regression model, $\varepsilon_t = y_t - x_t'\hat{\beta}$ so that $y_t = x_t'\hat{\beta} + \varepsilon_t$) and involves the estimation of θ_0 in each bootstrap replication.

Escanciano (2009) suggested a simple multiplier type approximation of $\hat{R}_T^1(x)$ as

$$\hat{R}_T^{*1}(x) = T^{-1/2} \sum_{t=1}^{T} \left[w(Z_{t-1}, x) I_e - \hat{G}'\left(x, \hat{\theta}\right) \hat{\Xi}^{-1} g\left(Z_{t-1}, \hat{\theta}\right) \right] U_t \hat{e}_t(\hat{\theta}),$$

where $\{U_t\}_{t=1}^T$ is a drawing from a pick distribution. This allows the execution of the tests without estimating the parameters in the bootstrap world (see Chapter 2.9 in van der Vaart and Wellner, 1996 for a discussion on multiplier central limit Theorem). This can be seen as an extension of the conventional wild bootstrap and can be applied to possibly non-separable CMR. For any continuous functional $\psi(.)$, Escanciano (2009) proved the consistency of this bootstrap procedure by showing that $\psi\left(\hat{R}_T^{*1}(x)\right) \xrightarrow{d} \psi\left(R_\infty^1\right)$.

Escanciano (2009) does mention about GARCH regression models in his paper (see Example 2 in Escanciano, 2009), however details are not provided there. For the AR-GARCH regression model defined in (4), to obtain the marginal tests corresponding to the conditional mean and the conditional variance, we construct separate projection for each element of $\hat{e}_t = (\hat{e}_{1t}, \hat{e}_{2t})'$ and then sum the resulting test statistics to obtain the joint specification tests. That is, for $\hat{e}_t = \{\hat{e}_{jt}\}$, $j = 1, 2$ we first construct

$$\hat{R}_{Tj}^1(x) = T^{-1/2} \sum_{t=1}^{T} \left[w(Z_{t-1}, x) I_e - \hat{G}_j'\left(x, \hat{\theta}\right) \hat{\Xi}_j^{-1} g_j\left(Z_{t-1}, \hat{\theta}\right) \right] \hat{e}_{jt},$$

with

$$
\begin{aligned}
g_j\left(Z_{t-1},\hat{\theta}\right) &= \left.\frac{\partial e_{jt}}{\partial\theta}\right|_{\theta=\hat{\theta}}, \\
\hat{G}_j\left(x,\hat{\theta}\right) &= T^{-1}\sum_{t=1}^{T}\left[g_j\left(Z_{t-1},\hat{\theta}\right)w\left(Z_{t-1},x\right)\right], \text{ and} \\
\hat{\Xi}_j &= T^{-1}\sum_{t=1}^{T}\left[g_j\left(Z_{t-1},\hat{\theta}\right)g_j\left(Z_{t-1},\hat{\theta}\right)'\right].
\end{aligned}
$$

The step by step PICM testing procedure is given below:

1. Given the information set $\mathcal{I}_{t-1} = \left(Z'_{t-1}, Z'_{t-2}, \cdots\right)'$ at time $t-1$, construct the $(T \times T)$ weight matrix $\mathcal{W}$. For example, the (r,s)-th element of indicator weight matrix is $\mathcal{W}_{r,s} = 1\left(Z_{r-1} \leq Z_{s-1}\right),\ r,s = 1,\cdots,T$.

2. Estimate $\hat{\theta}$ and $\hat{e}_t = \left(\hat{e}_{1t}, \hat{e}_{2t}\right)'$ by the QMLE.

3. For $i = 1,2$; construct the matrix of derivative $\widetilde{G}_i$ with rows $g_i\left(Z_t,\hat{\theta}\right)'$ where

$$
\begin{aligned}
g_1\left(Z_t,\theta\right) &= \frac{\partial e_{1t}\left(\theta\right)}{\partial\theta} = \left(\frac{\partial e_{1t}\left(\theta\right)}{\partial\varphi'}, \frac{\partial e_{1t}\left(\theta\right)}{\partial\eta'}\right)' \\
&= \left(-W_t,\ 0\right)',
\end{aligned}
$$

and

$$
\begin{aligned}
g_2\left(Z_t,\theta\right) &= \frac{\partial e_{2t}\left(\theta\right)}{\partial\theta} = \left(\frac{\partial e_{2t}\left(\theta\right)}{\partial\varphi'}, \frac{\partial e_{2t}\left(\theta\right)}{\partial\eta'}\right)' \\
&= \left(\left(-2W_t e_{1t} - \frac{\partial h_t}{\partial\varphi'}\right),\ -\frac{\partial h_t}{\partial\eta'}\right)',
\end{aligned}
$$

where (for the GARCH(1,1) case) $\frac{\partial h_t}{\partial\varphi} = -2\alpha_1\varepsilon_{t-1}W_{t-1} + \beta_1\frac{\partial h_{t-1}}{\partial\varphi}$ and $\frac{\partial h_t}{\partial\eta} = s_{t-1} + \beta_1\frac{\partial h_{t-1}}{\partial\eta}$ which can be obtained by recursions. For the GARCH (p,q) case this can be generalized easily.

4. Regress $\hat{e}_i$ on $\widetilde{G}_i$, $i = 1,2$ (deleting the columns containing only zeros); and obtain the residuals as $\widetilde{\widetilde{e}}_i \equiv M_{\widetilde{G}_i}\hat{e}_i$, where $M_{\widetilde{G}_i} = I_T - \widetilde{G}_i\left(\widetilde{G}_i'\widetilde{G}_i\right)^{-1}\widetilde{G}_i'$ is the usual Projection matrix.

5. Calculate $CvM_{T,i} = T^{-2}\widetilde{\widetilde{e}}_i'\mathcal{W}\mathcal{W}'\widetilde{\widetilde{e}}_i = T^{-2}\hat{e}_i'PP'\hat{e}_i$, where $P \equiv M_{\widetilde{G}_i}\mathcal{W}$.

6. In bootstrap world, generate $\hat{e}_i^* = \left\{U_t\hat{e}_{it}\right\}_{t=1}^T$ where $\left\{U_t\right\}_{t=1}^T$ is a sequence of i.i.d. draws from a pick distribution and $CvM_{Ti}^* = T^{-2}\hat{e}_i'^*PP'\hat{e}_i^*$.

7. For joint test, obtain $CvM_{T,J} = \sum_{i=1}^{2} CvM_{T,i}$ and $CvM^*_{T,J} = \sum_{i=1}^{2} CvM^*_{T,i}$.
8. Reject H_0 at $100\alpha\%$ when bootstrap p-value $p^*_T < \alpha$, where

$$p^*_T = P\left(CvM^*_T \geq CvM_T \left| \{y_t, \mathcal{I}_{t-1}\}_{t=1}^T \right.\right).$$

   In practice, the bootstrap p-value is computed as

$$p^*_T = \frac{\#\left\{CvM^*_{T,b} \geq CvM_T\right\}}{B},$$

   where $\#\{A\}$ denotes the number of times that event A occurs, $CvM^*_{T,b},\ b = 1, \cdots, B$ are the bootstrap realizations of the test statistics and B is the number of bootstrap replications.

4 The Integrated Generalized Spectral (IGS) Tests

Most financial data shows highly persistent volatility suggesting that the conditioning set for variance specifications should contain long lags. Again for large lags d, classical consistent tests are affected by "curse of dimensionality". For example, de Jong's (1996) generalization of Bierens (1982) test for $d \to \infty$ as $T \to \infty$ suffers from two drawbacks: it requires numerical integration with dimension T and loss of degrees of freedom due to introducing many lags. In addition popular conditional variance models (such as GARCH, ARCH(∞)) are non-markovian. The IGS testing approach, introduced by Hong (1999) in a non-linear time series framework, is particularly useful when dealing with infinite-dimensional conditioning sets and non-Markovian processes. Hong and Lee (2003, 2005) extended this idea to test the null for processes having conditional dependence at second and higher conditional moments. Escanciano and Velasco (2006) proposed generalized spectral tests for Martingale Difference Hypothesis (MDH) which, unlike Hong and Lee tests, do not depend on kernel and bandwidth parameter and do not require the existence of fourth moment. We introduce the idea of generalized spectral density below.

Definition 2 *Generalized Spectral Density (Hong, 1999): For a strictly stationary time series $\{e_t\}$ consider the spectrum of the transformed series $\left\{e^{iue_t}\right\}$, where $i = \sqrt{-1}$, $u \in (-\infty, \infty)$. The covariance between e^{iue_t} and $e^{ive_{t-j}}$ is given by*

$$\begin{aligned} \sigma_j(u,v) &= cov\left(e^{iue_t}, e^{ive_{t-j}}\right) \quad j = 0, \pm 1, \cdots, \\ &= \mathrm{E}\left[e^{i(ue_t + ve_{t-j})}\right] - \mathrm{E}\left[e^{iue_t}\right]\mathrm{E}\left[e^{ive_{t-j}}\right], \end{aligned}$$

where the first component is the joint and second is the product of marginal characteristic functions of (e_t, e_{t-j}). Thus $\sigma_j(u,v) = 0$ for all $(u,v) \in \Re^2$ iff

e_t and e_j are independent. Assuming $\sup_{(u,v)\in\Re^2}\sum_{j=-\infty}^{\infty}\|\sigma_j(u,v)\|<\infty$, *the Fourier transform of $\sigma_j(u,v)$ exists:*

$$f(\omega,u,v)=\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\sigma_j(u,v)\,e^{-ij\omega},\quad \omega=[-\pi,\pi], \tag{14}$$

which contains the same information as contained in $\sigma_j(u,v)$ and important to note that no moment condition on e_t is required. However, when $var(e_t)$ exists, the conventional spectral density is obtained from (14) as:

$$-\frac{\partial^2 f(\omega,u,v)}{\partial u\partial v}=\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}cov(e_t,e_{t-j})\,e^{-ij\omega}.$$

For this reason Hong (1999) termed (14) as generalized spectral density. Note that when $\{e_t\}$ is i.i.d., $f(\omega,u,v)$ becomes a flat generalized spectrum: $f_0(\omega,u,v)=\frac{1}{2\pi}\sigma_0(u,v)$, $\omega=[-\pi,\pi]$. Therefore any deviation of $f(\omega,u,v)$ from $f_0(\omega,u,v)$ provides evidence of serial dependence of $\{e_t\}$.

Now we briefly outline Escanciano's (2008) joint and marginal IGS tests which apply a pair-wise generalized spectrum approach. Under H_0, we write the joint CMR as:

$$\gamma_j(\theta_0)=\mathrm{E}\left[e_t(\theta_0)\,|Z_{t-j}\right]=0 \text{ a.s. } \forall j\geq 1 \text{ for some } \theta_0\in\Theta\subset\Re^p. \tag{15}$$

By appropriately choosing a weight function $w(Z_{t-j},x)$, (15) can be written as:

$$\gamma_{j,w}(x,\theta_0)=\mathrm{E}\left[e_t(\theta_0)\,w(Z_{t-j},x)\right]=0 \text{ a.e. in } \Pi\subset[-\infty,\infty]^s, \forall j\geq 1, \tag{16}$$

where Π is the nuisance parameter space with dimension s which depends on the particular family $\mathcal{W}$ used. To consider simultaneously all dependence measures, define $\gamma_{-j,w}(.,\theta_0)=\gamma_{j,w}(.,\theta_0)$ for $j\geq 1$ and $\gamma_{0,w}(.,\theta_0)=\mathrm{E}\left[e_t(\theta_0)\,w(Z_t,x)\right]$. Then the Fourier transform of the functions $\left\{\gamma_{j,w}(.,\theta_0)\right\}_{j=-\infty}^{\infty}$ is

$$f_w(u,x,\theta_0)=\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma_{j,w}(x,\theta_0)\,e^{-iju},\ \forall u\in[-\pi,\pi],\ x\in\Pi. \tag{17}$$

Under H_0, it becomes $f_w(u,x,\theta_0)\equiv f_{0,w}(x,\theta_0)=(2\pi)^{-1}\zeta_{0,w}(x,\theta_0)$ and serves as the basis to test the hypothesis (5). To avoid the nonparametric smoothed estimation of (17) (as proposed by Hong and Lee, 2003 based on Parzen's (1957) smoothed kernel estimators), a generalized spectral distribution function is used based on the dependence measure $\left\{\zeta_{j,w}(.,\theta_0)\right\}_{j=-\infty}^{\infty}$ and the test is based on the integral of $f_w(u,x,\theta_0)$:

$$\begin{aligned}Q_w(\tau,x,\theta_0) &= 2\int_0^{\tau\pi}f_w(u,x,\theta_0)\,du,\quad \forall\tau\in[0,1],\ x\in\Pi\\ &= \gamma_{0,w}(x,\theta_0)\,\tau+2\sum_{j=1}^{\infty}\gamma_{j,w}(x,\theta_0)\,\frac{\sin j\pi\tau}{j\pi}.\end{aligned} \tag{18}$$

For a sample $\{Y_t, \mathcal{I}_{t-1,q}\}_{t=1}^T$, let $\hat{\theta}$ be a $\sqrt{T}$-consistent estimator for θ_0 (e.g., the QMLE) and let

$$\hat{e}_{1t} \equiv e_{1t}\left(\hat{\theta}\right),\ \hat{e}_{2t} \equiv e_{2t}\left(\hat{\theta}\right) \text{ and } \hat{e}_t = \left(\hat{e}_{1t}, \hat{e}_{2t}\right)'. \tag{19}$$

Then, sample analogues of (16) and (18) are, respectively,

$$\hat{\gamma}_{j,w}\left(x,\hat{\theta}\right) = \frac{1}{T_j}\sum_{t=j}^{T} \hat{e}_t w\left(Z_{t-j}, x\right),\ T_j = T - j + 1, \tag{20}$$

$$\hat{Q}_w\left(\tau, x, \hat{\theta}\right) = \hat{\gamma}_{0,w}\left(x,\hat{\theta}\right)\tau + 2\sum_{j=1}^{T} \hat{\gamma}_{j,w}\left(x,\hat{\theta}\right)\left(\frac{T_j}{T}\right)^{1/2}\frac{\sin j\pi\tau}{j\pi}, \tag{21}$$

where $\left(\frac{T_j}{T}\right)^{1/2}$ is a finite sample correction factor to put lesser weight for larger lag, which has no effect on asymptotic theory but provides better finite sample performance. Under H_0, $Q_w\left(\tau, x, \theta_0\right) = \gamma_{0,w}\left(x, \theta_0\right)\tau$ suggesting a test based on the distance between $\hat{Q}_w\left(\tau, x, \hat{\theta}\right)$ and $\hat{Q}_{0,w}\left(\tau, x, \hat{\theta}\right) = \hat{\gamma}_{0,w}\left(x, \hat{\theta}\right)\tau$. We define the marked empirical process $R_{T,w}\left(\tau, x, \hat{\theta}\right)$ as

$$\begin{aligned} R_{T,w}\left(\tau, x, \hat{\theta}\right) &= \left(\frac{T}{2}\right)^{1/2}\left[\hat{Q}_w\left(\tau, x, \hat{\theta}\right) - \hat{Q}_{0,w}\left(\tau, x, \hat{\theta}\right)\right] \\ &= \sum_{j=1}^{T} T_j^{1/2}\hat{\gamma}_{j,w}\left(x,\hat{\theta}\right)\frac{\sqrt{2}\sin j\pi\tau}{j\pi}. \end{aligned} \tag{22}$$

To evaluate the distance from $R_{T,w}$ to zero we need to consider some norms; e.g., Cramer-von Mises (CvM) and Kolmogorov-Smirnov (KS) functionals. Using the CvM norm, the joint specification test statistics are given by

$$\begin{aligned} J_{T,w}^2 &\equiv J_{T,w}^2\left(\hat{\theta}_T\right) = \int_{\Pi'}\left\|R_{T,w}\left(\tau, x, \hat{\theta}\right)\right\|_M^2 W\left(\mathrm{d}\,x\right)\mathrm{d}\,\tau \\ &= \sum_{j=1}^{T}\frac{T_j}{\left(j\pi\right)^2}\int_{\Pi}\left\|\hat{\gamma}_{j,w}\left(x,\hat{\theta}\right)\right\|_M^2 W\left(\mathrm{d}\,x\right), \end{aligned}$$

where $\Pi' = [0,1]\times\Pi$, $W(.)$ is an integrating function depending on the weight family $\mathcal{W}$ and M is a 2×2 psd matrix with rows $(m_1, 0)$ and $(0, m_2)$ to obtain marginal components from the joint test (see Assumption 4.2). For example, $m_1 = 1$, and $m_2 = 0$, leads to marginal test for mean specification. For any $\sqrt{T}$-consistent estimators $\hat{\theta}$, with indicator weight function $w\left(Z_{t-j}, x\right) = 1\left(Z_{t-j} \leq x\right)$ and $W(.) = F_T(.)$ where $F_T(.)$ is the empirical distribution function of $\{Z_{t-1}\}_{t=1}^T$, the test statistic has the following simple form:

$$J_{T,I}^2 = \sum_{j=1}^{T}\frac{T_j}{T\left(j\pi\right)^2}\sum_{t=1}^{T}\left\{m_1\hat{\sigma}_{1e}^{-2}\hat{\gamma}_{I,j,m}^2\left(Z_{t-1},\hat{\theta}\right) + m_2\hat{\sigma}_{2e}^{-2}\hat{\gamma}_{I,j,v}^2\left(Z_{t-1},\hat{\theta}\right)\right\}, \tag{23}$$

where $T_j = T - j + 1$, $\hat{\sigma}^2_{ie} = T^{-1}\sum_{t=1}^{T} \hat{e}^2_{it}$, $i = 1, 2$ and $\hat{\gamma}_{I,j} = \left(\hat{\gamma}_{I,j,m}, \hat{\gamma}_{I,j,v}\right)'$, with

$$\begin{aligned}
\hat{\gamma}_{I,j,m}\left(Z_{t-1}, \hat{\theta}\right) &= \frac{1}{T_j}\sum_{t=j}^{T} \hat{e}_{1t} w\left(Z_{t-j}, Z_{t-1}\right), \\
\hat{\gamma}_{I,j,v}\left(Z_{t-1}, \hat{\theta}\right) &= \frac{1}{T_j}\sum_{t=j}^{T} \hat{e}_{2t} w\left(Z_{t-j}, Z_{t-1}\right).
\end{aligned}$$

Note that $(m_1, m_2) = (1, 1)$ gives the joint test whilst setting $(m_1, m_2) = (1, 0)$ and $(m_1, m_2) = (0, 1)$ give the marginal tests for conditional mean $\left(D^2_{T,I,m}\right)$ and conditional variance $\left(D^2_{T,I,v}\right)$, respectively; i.e.,

$$D^2_{T,I,m} = \sum_{j=1}^{T} \frac{T_j}{T\left(j\pi\right)^2} \sum_{t=1}^{T} m_1 \hat{\sigma}^{-2}_{1e} \hat{\gamma}^2_{I,j,m}\left(Z_{t-1}, \hat{\theta}\right), \quad (24)$$

$$D^2_{T,I,v} = \sum_{j=1}^{T} \frac{T_j}{T\left(j\pi\right)^2} \sum_{t=1}^{T} m_2 \hat{\sigma}^{-2}_{2e} \hat{\gamma}^2_{I,j,v}\left(Z_{t-1}, \hat{\theta}\right). \quad (25)$$

Similarly with the complex exponential weight function $w\left(Z_{t-j}, x\right) = \exp\left(ix' Z_{t-j}\right)$ and $W\left(dx\right) = \varphi\left(x\right) dx$ where $\varphi\left(x\right)$ is the standard normal density, the test statistic can be expressed as

$$J^2_{T,C} = \sum_{j=1}^{T} \frac{T_j^{-1}}{\left(j\pi\right)^2} \sum_{t=1}^{T}\sum_{s=j}^{T} \left\{ \frac{m_1}{\hat{\sigma}^2_{1e}} \hat{e}_{1t}\hat{e}_{1s} + \frac{m_2}{\hat{\sigma}^2_{2e}} \hat{e}_{2t}\hat{e}_{2s} \right\} \exp\left(-0.5\left(Z_{t-j} - Z_{s-j}\right)^2\right), \quad (26)$$

and analogously $D^2_{T,C,m}$ and $D^2_{T,C,v}$ are defined as

$$D^2_{T,C,m} = \sum_{j=1}^{T} \frac{T_j^{-1}}{\left(j\pi\right)^2} \sum_{t=1}^{T}\sum_{s=j}^{T} \frac{m_1}{\hat{\sigma}^2_{1e}} \hat{e}_{1t}\hat{e}_{1s} \exp\left(-0.5\left(Z_{t-j} - Z_{s-j}\right)^2\right), \quad (27)$$

$$D^2_{T,C,v} = \sum_{j=1}^{T} \frac{T_j^{-1}}{\left(j\pi\right)^2} \sum_{t=1}^{T}\sum_{s=j}^{T} \frac{m_2}{\hat{\sigma}^2_{2e}} \hat{e}_{2t}\hat{e}_{2s} \exp\left(-0.5\left(Z_{t-j} - Z_{s-j}\right)^2\right). \quad (28)$$

4.1 Asymptotic Null Distribution and Bootstrap Approximation

To establish the asymptotic theory, in addition to the Assumptions 2.1 - 2.5, Escanciano (2007b, 2008) made the following assumptions:

Assumption 4.1 *Under H_0, $\hat{\theta}$ satisfies the asymptotic Bahadur expansion*

$$\sqrt{T}\left(\hat{\theta} - \theta_0\right) = T^{-1/2} \sum_{t=1}^{T} \varrho\left(\mathcal{I}_{t-1,q}, \theta_0\right) e_t\left(\theta_0\right) + o_p\left(1\right),$$

where $\varrho(.)$ *is such that* $\mathrm{E}\left[\varrho\left(\mathcal{I}_{t-1,q},\theta_0\right)e_t\left(\theta_0\right)e_t'\left(\theta_0\right)\varrho'\left(\mathcal{I}_{t-1,q},\theta_0\right)\right]$ *exists and positive definite.*

Assumption 4.2 *The integrating function* $W(.)$ *is a probability density function absolutely continuous wrt Lebesgue measure.* M *is* 2×2 *psd matrix. The weight function* $w(.)$ *is such that the equivalence between (15) and (16) holds and it is uniformly bounded on compacta. Also,* $w(.)$ *satisfies the following Uniform Law of Large Number (ULLN)*

$$\sup_{x\in\Pi_c} T^{-1}\left\|\sum_{t=1}^{n}\widetilde{\upsilon}_t w\left(\upsilon_t,x\right)-\mathrm{E}\left[\widetilde{\upsilon}_t w\left(\upsilon_t,x\right)\right]\right\|\rightarrow 0,\ \ as,$$

whenever $\{(\widetilde{\upsilon}_t,\upsilon_t),\ t=0,\pm1,\cdots\}$ *is strictly stationary and ergodic process with* $\widetilde{\upsilon}_t\in\Re$, $\upsilon_t\in\Re^{1+m}$, $\mathrm{E}\left[\gamma_1\right]<\infty$, *and* Π_c *is any compact subset of* $\Pi\subset[-\infty,\infty]^s$.

Assumption 4.1 is satisfied under mild conditions for most estimators. Conditions for the local QMLE under martingale conditions have been established in Lee and Hansen (1994). The following Lemma shows that the QMLE $\hat{\theta}$ indeed satisfies Assumption 4.1.

Lemma 2 *The QMLE* $\hat{\theta}=\left(\hat{\varphi}',\hat{\eta}'\right)'$ *of (3) satisfies*

$$\sqrt{T}(\hat{\theta}-\theta_0)=T^{-1/2}\sum_{t=1}^{T}\varrho_t(\theta_0)e_t(\theta_0)+o_p(1),$$

and $E\left[\varrho_t(\theta_0)e_t(\theta_0)e_t'(\theta_0)\varrho_t'(\theta_0)\right]$ *is finite and positive definite, where* $J_{\theta\theta}$ *is the negative of the expected Hessian and*

$$\varrho_t(\theta_0)=J_{\theta\theta}^{-1}h_{0t}^{-1}\begin{bmatrix} f_{0t} & \frac{1}{2}c_{0t}\\ 0 & \frac{1}{2}x_{0t}\end{bmatrix}.$$

with $f_t=\dfrac{\partial m_t}{\partial\varphi}$, $x_t=\dfrac{1}{h_t}\dfrac{\partial h_t}{\partial\eta}$, $c_t=\dfrac{1}{h_t}\dfrac{\partial h_t}{\partial\varphi}$ *and* $f_{0t}=f_t(\varphi_0)$, $c_{0t}=c_t(\theta_0)$, *and* $x_{0t}=x_t(\theta_0)$.

Given these assumptions the limit distribution of $J_{T,w}^2$ under H_0 can be given as $J_{T,w}^2\xrightarrow{d}J_{\infty,w}^2=\int\left|R_w\left(\tau,x,\theta_0\right)\right|_M^2 W\left(\mathrm{d}\,x\right)\mathrm{d}\,\tau$ (for details and proof, see Escanciano 2007b, 2008). To perform the IGS tests, Escanciano (2008) suggested the following FDWB procedure to approximate $R_{T,w}^{*}\left(\tau,x,\hat{\theta}^{*}\right)=\sum_{j=1}^{T}T_j^{1/2}\hat{\gamma}_j^{*}\left(x\right)\frac{\sqrt{2}\sin j\pi\tau}{j\pi}$ with

$$\begin{aligned}\hat{\gamma}_j^{*}\left(x\right)&=\left(\hat{\gamma}_{j,m}^{*}\left(x\right),\hat{\gamma}_{j,v}^{*}\left(x\right)\right)'\\ &=\left(\frac{1}{T_j}\sum_{t=j}^{T}\hat{e}_{1t}^{*}w\left(Z_{t-j},x\right),\frac{1}{T_j}\sum_{t=j}^{T}\hat{e}_{2t}^{*}w\left(Z_{t-j},x\right)\right)',\end{aligned}$$

where $\hat{e}_t^{*}=\left(\hat{e}_{1t}^{*},\hat{e}_{2t}^{*}\right)'$ are obtained from the following algorithm:

A. Estimate the original model (here by the QMLE) and obtain $\hat{\theta}$, $\hat{m}_t$, $\hat{h}_t$ and $\hat{e}_t$.

B. Generate wild bootstrap residuals as $\hat{\varepsilon}^*_{1t} = \hat{e}_{1t}U_t$, and $\hat{\varepsilon}^*_{2t} = \hat{e}_{2t}U_t$ for $1 \leq t \leq T$ where $\{U_t\}_{t=1}^T$ is a sequence of i.i.d. draws from a pick distribution.

C. Given $\hat{\theta}$, $\hat{\varepsilon}^*_{1t}$ and $\hat{\varepsilon}^*_{2t}$, generate fixed design bootstrap data according to

$$Y^*_{1t} = \hat{m}_t + \hat{\varepsilon}^*_{1t}, \quad Y^*_{2t} = \hat{h}_t + \hat{\varepsilon}^*_{2t} \text{ for } 1 \leq t \leq T.$$

D. Compute $\hat{\theta}^*$ from the bootstrap data $\left\{Y^*_{1t}, Y^*_{2t}, \mathcal{I}'_{t-1,q}\right\}_{t=1}^T$ to construct $\hat{e}^*_{1t} = Y^*_{1t} - \hat{m}^*_t$, $\hat{e}^*_{2t} = Y^*_{2t} - \hat{h}^*_t$ for $1 \leq t \leq T$ where $\hat{m}^*_t \equiv m\left(\mathcal{I}_{t-1,q}, \hat{\theta}^*\right)$ and $\hat{h}^*_t \equiv \hat{h}^*_t\left(\mathcal{I}_{t-1,q}, \hat{\theta}^*\right)$.

The consistency of this FDWB procedure is proved in Escanciano (2007b) under previously stated assumptions A1-A7 and the following conditions on $\hat{\theta}^*$:

Assumption 4.3 *The estimator $\hat{\theta}^*$ satisfies the asymptotic expansion*

$$\sqrt{T}\left(\hat{\theta}^* - \hat{\theta}\right) = T^{-1/2}\sum_{t=1}^{T} U_t \varrho\left(\mathcal{I}_{t-1,q}, \hat{\theta}\right)\hat{e}_t + o_p(1).$$

5 Problems with Escanciano's FDWB and A Modified Testing Procedure

Since we obtain $\hat{\theta}$ and $\hat{e}_t$ by the QMLE in the real world, ideally we would like to mimic the same estimation procedure in the bootstrap world. We can apply the QMLE in the bootstrap world for the UGARCH models by adapting the model based bootstrap used in Pascual et al. (2006) and Christoffersen and Gonclaves (2005). However, this will be computationally costly and can not be performed with the standard software which may discourage the applied researcher to use these tests. Unfortunately with Escanciano's FDWB scheme, which is an easier alternative to model based bootstrap, it is problematic to employ the QMLE in the bootstrap world.

To appreciate the problem associated with this procedure, note that in step C, Y^*_{1t} and Y^*_{2t} provide the conditional mean and variance structure separately in the bootstrap world; as opposed to the single variable y_t in the real world which contain both mean and variance information. The presence of two dependent variables thus restricts the use of the QMLE to compute $\hat{\theta}^*$ in step D.

Remark 1 *The nature of the null DGPs considered in Escanciano's (2008) study is the reason for the FDWB working in his simulation study. The null DGPs are:*

$$\text{DGP1:} \qquad y_t = \sqrt{h_t}u_t;\ h_t = a + by^2_{t-1}.$$
$$\text{DGP2:} \qquad y_t = ay_{t-1} + \sqrt{h_t}u_t;\ h_t = b + cy^2_{t-1}.$$

Table 1: Empirical size with normal errors using Escanciano's FDWB

	E2		
T	100	200	300
$D^2_{I,m}$	14.80	21.50	26.20
$D^2_{I,v}$	11.20	8.30	8.70
J^2_I	14.90	13.90	18.10
$D^2_{C,m}$	8.90	10.40	12.00
$D^2_{C,v}$	10.20	7.10	5.70
J^2_C	11.90	9.30	9.50

The first one is an ARCH process with no conditional mean and the second one is an AR(1) regression model with conditional heteroskedastic (1) error (in short AR(1)-CH(1) model). Note that unlike our AR-GARCH null model, neither of these DGPs involves lagged unobserved variables, such as ε_{t-1} or h_{t-1}, making it possible to estimate the conditional variance parameters by the OLS both in real and bootstrap world. The author also indicates the application of least squares estimator in his simulation study (Escanciano, 2008, p.82). In addition, we remark that it is not verified whether the OLS estimators satisfy Assumption 4.3 in his study.

Remark 2 *Escanciano (2008) further illustrates the IGS test with an empirical application to the S&P 500 data. In particular, he fits the AR (1)-GARCH(1,1) to the data and finds the evidence that the conditional mean is well specified whereas the conditional variance is misspecified. Though the author does mention that in the real world the QMLE is used to obtain parameter estimates, it is, however, not clear how the parameter estimates in the bootstrap world are obtained. Given his FDWB algorithm, we assume that the OLS is used to obtain* $\hat{\theta}^*$.

We have examined the consequences of ignoring this problem of obtaining $\hat{\theta}^*$ by the QMLE and employ Escanciano's FDWB scheme. In particular, in the real world $\hat{\theta}$ is obtained by the QMLE and sample moment conditions are obtained through these estimates. Then in bootstrap world, in step C, the OLS is applied to Y^*_{1t} on $(1,\ \hat{m}_t)$ and Y^*_{2t} on $\left(1,\ \hat{h}_t\right)$ to get bootstrapped moment conditions. Table 1 reports the size of the IGS tests with a AR(1)-GARCH(1,1) null model (for details of the DGP, see Section 7). As expected, the poor size performance of the IGS tests with this procedure is demonstrated in Monte Carlo experiments which suggests that we need some modifications in the testing procedure.

In the next subsection we propose a simple bootstrap procedure which provide a solution to this problem.

5.1 Modified Test Procedure

To implement the IGS tests for AR-GARCH regression model as defined in (4), the idea put forward in this paper is a simple one: after the QMLE estimation to obtain $\hat{\theta}$ and $\hat{h}_t$, we consider $\hat{h}_t$ as observed and introduce a set of auxiliary OLS regressions (possibly nonlinear in case of nonlinear specification for mean function) to obtain the moment conditions in the real world; and finally mimicing the same OLS regressions in the bootstrap world. In what follows "hats ($\widehat{}$)" denotes the QMLE while "tilda ($\tilde{}$)" denotes the OLS estimation. A step-by-step discussion of the proposed testing procedure for AR(1)-GARCH(1,1) is given below, which can be generalized for nonlinear mean function and higher order/extension of GARCH models in an obvious way.

5.1.1 Real World Estimation

1. Estimate the original model by the QMLE and obtain $\hat{\theta} = (\hat{\varphi}', \hat{\eta}')'$, $\hat{m}_t$, $\hat{h}_t$.

2. Since we have linear conditional mean specification (e.g., AR(1)), regress y_t on a constant and y_{t-1} to obtain

$$
\begin{aligned}
\tilde{v}_{1t} &= y_t - \tilde{\varphi}_0 - \tilde{\varphi}_1 y_{t-1} = y_t - W_t' \tilde{\varphi} \\
&\equiv y_t - \tilde{m}_t,
\end{aligned}
$$

   where $\tilde{\varphi} = (\tilde{\varphi}_0, \tilde{\varphi}_1)'$ are OLS estimators from the regression y_t on $W_t = (1, y_{t-1})'$ and $\tilde{m}_t = W_t' \tilde{\varphi}$. See Remark 3 for nonlinear mean function.

3. Define $\hat{z}_t = \left(1, \hat{h}_t\right)'$. Then obtain the second sample moment condition as the residual from an OLS regression of $\tilde{v}_{1t}^2$ on $\hat{z}_t$; i.e.,

$$
\begin{aligned}
\tilde{v}_{2t} &= \tilde{v}_{1t}^2 - \tilde{\beta}_0 - \tilde{\beta}_1 \hat{h}_t = \tilde{v}_{1t}^2 - \hat{z}_t' \tilde{\beta} \\
&\equiv \tilde{v}_{1t}^2 - \tilde{h}_t,
\end{aligned}
$$

   where $\tilde{\beta} = \left(\tilde{\beta}_0, \tilde{\beta}_1\right)'$ are OLS estimators and $\tilde{h}_t = \hat{z}_t' \tilde{\beta}$.

5.1.2 Bootstrap World

1. Generate WB residuals :

$$
\begin{aligned}
\hat{\varepsilon}_{1t}^* &= \tilde{v}_{1t} U_t, \\
\hat{\varepsilon}_{2t}^* &= \tilde{v}_{2t} U_t,
\end{aligned}
$$

   where $\{U_t\}$ a sequence of i.i.d. r.v.s with zero mean and unit variance, bounded support and independent of the sequence $\{y_t, \mathcal{I}_{t-1,q}\}_{t=1}^T$.

2. Generate bootstrap data:

$$
\begin{aligned}
Y_{1t}^* &= \tilde{m}_t + \hat{\varepsilon}_{1t}^*, \\
Y_{2t}^* &= \tilde{h}_t + \hat{\varepsilon}_{2t}^*.
\end{aligned}
$$

3. Compute $\tilde{\varphi}^* = (\tilde{\varphi}_0^*, \tilde{\varphi}_1^*)'$ from $\{Y_{1t}^*, \mathcal{I}_{t-1,q}\}$ by an OLS regression of Y_{1t}^* on W_t and subsequently first moment condition in the bootstrap world $\tilde{v}_{1t}^* = Y_{1t}^* - \tilde{\varphi}_0^* - \tilde{\varphi}_1^* y_{t-1} = Y_{1t}^* - W_t'\tilde{\varphi}^*$.

4. Compute $\tilde{\beta}^* = \left(\tilde{\beta}_0^*, \tilde{\beta}_1^*\right)'$ from $\{Y_{2t}^*, \mathcal{I}_{t-1,q}\}$ by an OLS regression of Y_{2t}^* on $\hat{z}_t$ and obtain $\tilde{v}_{2t}^* = Y_{2t}^* - \tilde{\beta}_0^* - \tilde{\beta}_1^* \hat{h}_t = Y_{2t}^* - \hat{z}_t'\tilde{\beta}^*$.

Remark 3 *For non-linear conditional mean function, one can employ a non-linear least squares (NLS) method in Step 2 to estimate $\tilde{m}_t$ and $\tilde{v}_{1t}$. Note that we need to perform the same NLS estimation in the bootstrap world for $\tilde{v}_{1t}^*$. Alternatively we can avoid the NLSE by using the $\hat{m}_t$ (obtained from the QMLE estimation at Step 1) to estimate $\tilde{v}_{1t}$ as the residual from a OLS regression of y_t on $(1, \hat{m}_t)$. And then follow the above algorithm in bootstrap world.*

Remark 4 *Imitating the GARCH process, we can adopt a slightly different specification for the auxiliary regression in step 3 (real world) to obtain $\tilde{v}_{2t}$ and subsequently in step 4 (bootstrap world) for $\tilde{v}_{2t}^*$ (all other steps remain same). In particular, in the Step 3 (real world) compute $\tilde{\beta} = \left(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2\right)'$ by an OLS regression of $\tilde{v}_{1t}^2$ on $\hat{z}_t = \left(1, \tilde{v}_{1,t-1}^2, \hat{h}_{t-1}\right)'$ and subsequently obtain the second moment condition as $\tilde{v}_{2t} = \tilde{v}_{1t}^2 - \tilde{\beta}_0 - \tilde{\beta}_1 \tilde{v}_{1,t-1}^2 - \tilde{\beta}_2 \hat{h}_{t-1}$. Note that in this case, with this new definition of $\tilde{\beta}$ and $\hat{z}_t'$, we have $\tilde{h}_t = \hat{z}_t'\tilde{\beta}$ and this will be used in Step 2 (bootstrap world) to generate Y_{2t}^*. Similarly in Step 4 (bootstrap world), compute $\tilde{\beta}^* = \left(\tilde{\beta}_0^*, \tilde{\beta}_1^*, \tilde{\beta}_2^*\right)'$ by an OLS regression of Y_{2t}^* on $\hat{z}_t = \left(1, \tilde{v}_{1,t-1}^2, \hat{h}_{t-1}\right)'$ and subsequently obtain the second moment condition as $\tilde{v}_{2t}^* = Y_{2t}^* - \hat{z}_t'\tilde{\beta}^*$. In an analogous way to Theorem 2 (below), it can be shown that in this case also $\tilde{\beta}$ satisfies the asymptotic Bahadur expansion.*

The next theorem shows that for the above testing procedure $\tilde{\varphi}$ and $\tilde{\beta}$ in the real world satisfy the asymptotic Bahadur expansion as stated in Assumption 4.1.

Theorem 2 *Under the stated regularity conditions (Assumptions 2.1-2.5),*

$$\sqrt{T}\begin{pmatrix} \tilde{\varphi} - \varphi_0 \\ \tilde{\beta} - \beta_0 \end{pmatrix} = T^{-1/2}\sum_{t=1}^{T}\begin{bmatrix} q_t(\theta_0) \\ p_t(\theta_0) \end{bmatrix} e_t(\theta_0) + o_p(1),$$

where $\tilde{\varphi} = (\tilde{\varphi}_0, \tilde{\varphi}_1)'$, $\tilde{\beta} = \left(\tilde{\beta}_0, \tilde{\beta}_1\right)'$, $\theta_0' = (\varphi_0', \eta_0')$, and the expressions for $q_t(\theta_0)$ and $p_t(\theta_0)$ are provided in the proof.

The asymptotic analysis shows that although in the bootstrap world $\tilde{\varphi}^*$ satisfies the sufficient asymptotic expansion, unfortunately $\tilde{\beta}^*$ does not meet the sufficient conditions; i.e., Assumption 4.3 (see Appendix for the details). It is found that $\tilde{\beta}^*$ would have satisfied the Bahadur expansion if we could use the true $z_{0t} = (1, h_{0t})'$ instead of $\hat{z}_t = \left(1, \hat{h}_t\right)'$. Since h_{0t} is not observable, we are forced to use $\hat{h}_t$. Therefore, strictly speaking the proposed bootstrap procedure does not satisfy the sufficient conditions. However, this does not mean the procedure is necessarily invalid. We assess the potential validity of our procedure via a simulation study. On the other hand it is an easily implementable solution as opposed to the full parametric model based bootstrap. In addition, many asymptotically valid tests display disappointing size property (see, subsection 1.1). It is, therefore, worthwhile to investigate the finite sample performance of the IGS tests with our proposed bootstrap scheme.

6 CM Tests of the GARCH Specification

In this section we will briefly discuss the parametric CM tests considered in Halunga and Orme (2009) which will be used in our simulation. Assuming the correct specification for the conditional mean, the general CM testing framework of Halunga and Orme (2009) based on the idea that under a correct GARCH specification the squared standardized residuals ζ_t^2 should be serially uncorrelated with any function of the past information:

$$H_0 : \mathrm{E}\left[\left(\zeta_t^2 - 1\right) r_t\left(\theta_0\right)\right] = 0,$$

where is $r_t(\theta_0)$ is $\mathcal{F}_{t-1}$ measurable. Then the generic CM test indicator is

$$\delta\left(\hat{\theta}\right) = \frac{1}{T}\sum_{t=1}^{T}\left[\left(\hat{\zeta}_t^2 - 1\right)\hat{r}_t\right],$$

and the generic form of the test statistic is

$$T = n\delta\left(\hat{\theta}\right)'\hat{\Sigma}_T^{-1}\delta\left(\hat{\theta}\right),$$

where $\hat{\Sigma}_T = \Sigma + o_p(1)$, Σ is the asymptotic variance-covariance matrix of $\delta\left(\hat{\theta}\right)$ which has a χ_m^2 limiting distribution under the null (see Halunga and Orme, 2009).

Halunga and Orme (2009) also analyzed the Engle and Ng (1993) asymmetry and Lundbergh and Teräsvirta (2002) non-linearity tests and showed that these tests are asymptotically invalid as these do not take account of the asymptotically non-negligible estimation effects from the correct specification of the conditional mean function. Halunga and Orme (2009) also suggested two alternative asymptotically valid tests of asymmetry and non-linearity.

In our study we consider the following four parametric CM tests for a GARCH (1,1) regression process, all of which has a χ^2 limit distribution:

1. The Engle and Ng asymmetry (negative size bias) test (T_{EN}) with $\hat{r}_{1t} = \left[1\left(\hat{\varepsilon}_{t-1} \leq 0\right)\right]\hat{\varepsilon}_{t-1}$,

2. The Lundbergh and Teräsvirta non-linearity test (T_{LT}) with $\hat{r}_{2t} = \hat{\varepsilon}^3_{t-1}$,

3. The Halunga and Orme asymmetry (T_A) test with

$$\hat{r}_{3t} = \frac{1}{\hat{h}_t} \sum_{i=0}^{t-1} \hat{\beta}_1^i \left[1\left(\hat{e}_{2,t-1-i} \leq 0\right)\right] \hat{\varepsilon}_{t-1-i},$$

4. The Halunga and Orme nonlinearity (T_N) test with $\hat{r}_{4t} = \frac{1}{\hat{h}_t} \sum_{i=0}^{t-1} \hat{\beta}_1^i \hat{\varepsilon}^3_{t-1-i}$.

The corresponding expressions of $\hat{\Sigma}_T$ for these tests and other details are presented in Halunga and Orme (2009). To be specific T_{EN} and T_{LT} employ expression given by equation (14) in their paper (Halunga and Orme, 2009, p. 375), whereas T_A and T_N employ expressions given by equation (13) and (15), respectively (Halunga and Orme, 2009, p. 375).

7 Monte Carlo Experiments

In this section the finite sample performance of previously discussed two nonparametric testing procedures (the PICM and IGS) and four parametric CM tests (T_{LT}, T_{EN}, T_A and T_N) are compared. For both nonparametric testing procedures, we consider two family of weight functions, namely, the indicator and exponential weight functions and we set $Z_{t-1} = y_{t-1}$. The joint and marginal mean and variance IGS tests based on indicator weight function J_I^2, $D_{I,m}^2$, $D_{I,v}^2$, and complex exponential weight functions J_C^2, $D_{C,m}^2$ and $D_{C,v}^2$ are given in (23) -(28). These IGS tests are constructed employing the proposed modified bootstrap scheme. The alternative specification of the conditional variance auxiliary regression (as mentioned in Remark 4) is also considered in the simulation, however the results are qualitatively similar to the former one and to save space we do not report them here.[8] The PICM joint and marginal mean and variance tests with indicator weight are denoted by $C_{I,J}^2$, $C_{I,m}^2$ and $C_{I,v}^2$, respectively, while the corresponding tests with exponential weight are denoted by $C_{C,J}^2$, $C_{C,m}^2$ and $C_{C,v}^2$, respectively. All experiments are done with 1000 Monte Carlo replications and for nonparametric tests 300 bootstrap samples are generated. We consider the sample size $T = 100$, 200, 300 and 500, after discarding the first 200 observations from the sample to offset any initial value effect. For generating the bootstrap data, we consider the Rademacher distribution given in (13).[9] All simulations are programmed in GAUSS.

7.1 Size and Robustness to Non-normality

For size experiments we consider the following AR(1)-GARCH(1,1) null models:

[8] These can be obtained from the author upon request.

[9] Similar conclusions are obtained by using the Mammen's distribution as given in (12), hence we do not report them.

Table 2: Empirical size with the Normal errors

	E1				E2			
T	100	200	300	500	100	200	300	500
T_{LT}	0.70	2.60	3.50	2.70	1.70	2.60	2.10	1.80
T_N	1.20	3.60	4.10	4.80	3.80	4.90	3.20	5.10
T_{EN}	3.40	3.70	5.90	4.90	4.50	5.80	4.70	3.70
T_A	12.50	7.70	6.30	6.80	13.00	8.20	8.70	6.60
			IGS tests					
$D^2_{I,m}$	6.80	6.30	4.90	5.30	5.40	4.50	5.60	7.00
$D^2_{I,v}$	4.80	3.80	5.90	5.70	5.60	5.60	5.50	4.40
J^2_I	4.60	4.80	5.50	5.90	5.50	6.20	5.70	4.90
$D^2_{C,m}$	7.80	6.00	4.60	5.10	5.50	5.20	6.30	7.20
$D^2_{C,v}$	4.60	3.80	4.80	5.00	5.50	5.10	4.60	3.90
J^2_C	5.00	4.00	5.60	5.30	6.20	5.30	4.30	4.90
			PICM tests					
$C^2_{I,m}$	6.30	6.30	5.50	5.20	5.60	4.10	5.20	4.40
$C^2_{I,v}$	7.50	6.70	5.30	4.50	8.50	6.80	6.60	5.80
$C^2_{I,J}$	8.00	6.80	5.30	4.90	8.40	7.10	6.80	5.90
$C^2_{C,m}$	4.80	5.10	5.40	5.60	6.00	4.90	5.10	3.40
$C^2_{C,v}$	6.80	6.60	5.40	4.60	7.50	6.90	6.50	5.30
$C^2_{C,J}$	7.30	6.70	5.60	5.10	7.60	6.60	6.80	4.70

$$\begin{aligned}
\text{E1} &: Y_t = 1 + 0.1Y_{t-1} + \varepsilon_t,\ \varepsilon_t = \sqrt{h_t}\zeta_t, \\
h_t &= 0.20 + 0.05\varepsilon_{t-1}^2 + 0.75h_{t-1},\ \zeta_t \sim N(0,1). \\
\text{E2} &: Y_t = 1 + 0.1Y_{t-1} + \varepsilon_t,\ \varepsilon_t = \sqrt{h_t}\zeta_t, \\
h_t &= 0.01 + 0.09\varepsilon_{t-1}^2 + 0.90h_{t-1},\ \ \zeta_t \sim N(0,1).
\end{aligned}$$

E1 and E2 correspond to low persistent and high persistent volatility process with $\alpha+\beta = 0.80$ and $\alpha+\beta = 0.99$, respectively, where $\zeta_t \sim N(0,1)$. To check the robustness to non-normality, we generate these two DGPs where ζ_t follows a standardized Student's $t-$distribution with 5 and 3 degrees of freedom. The parameter values are standard in the literature and are used by Halunga and Orme (2009) and Engle and Ng (1993) among others.

Table 2 displays the size of the various tests for a nominal size of 5% and for $T = 100, 200, 300$ and 500, where the null DGPs are E1 and E2 with $\zeta_t \sim N(0,1)$. The parametric CM tests perform poorly for small sample size but size distortions decrease as T increases except T_{LT}, which is the worst performer in terms of size. This finding is similar to Halunga and Orme study where even for $T = 1000$, T_{LT} is significantly undersized. On the other hand the empirical sizes of the IGS tests, for both high and low persistent GARCH process and with both weight functions, are close to the nominal level. Even for very small size, e.g., $T = 100$ and $T = 200$, these tests, in general, demonstrate excellent size property. The PICM tests are slightly oversized for smaller sample size, but they perform much better than the parametric tests.

To investigate the robustness of these tests to non-normality, Table 3 reports the size, again against 5% nominal level, for E1 and E2 where $\zeta_t \sim t(5)$ and $\zeta_t \sim t(3)$. The performances of the parametric CM tests are worse in this case compared to Gaussian error. T_{LT}, T_N and T_{EN} are undersized while T_A is significantly oversized for both DGPs, although for $T = 500$, T_A and T_N size distortions decrease, as expected. It should be noted that Halunga and Orme (2009) reports that for $T = 1000$, T_A and T_N show satisfactory size under non-normality, however our findings reveal that for smaller sample size these are less robust to non-normality. On the other hand both nonparametric tests display robust size property under non-normality, even for very small T.

7.2 Power

For the power experiments we consider 3 types of misspecified models, namely, correct specification for mean but misspecified variance (P1, P2 and P3), misspecified mean and correct specification for variance (P4 and P5) and both conditional mean and variance are misspecified (P6):

$$\begin{aligned}
\text{P1} &: \; y_t = 1 + 0.1y_{t-1} + \varepsilon_t,\; \varepsilon_t = \sqrt{h_t}\zeta_t, \\
h_t &= 0.005 + 0.28[|\varepsilon_{t-1}| - 0.23\varepsilon_{t-1}]^2 + 0.7h_{t-1},\; \zeta_t \sim N(0,1). \\
\text{P2} &: \; y_t = 1 + 0.1y_{t-1} + \varepsilon_t,\; \varepsilon_t = \sqrt{h_t}\zeta_t, \\
h_t &= -0.23 + 0.9\log(h_{t-1}) + 0.25\left[|\zeta_{t-1}| - 0.3\zeta_{t-1}\right],\; \zeta_t \sim N(0,1). \\
\text{P3} &: \; y_t = 1 + 0.1y_{t-1} + \varepsilon_t,\; \varepsilon_t = \sqrt{h_t}\zeta_t, \\
h_t &= 0.9 + 0.1y_{t-1}^2,\quad \zeta_t \sim N(0,1). \\
\text{P4} &: \; y_t = 1 + 0.1y_{t-1} + 1.5\sqrt{h_t} + \varepsilon_t,\; \varepsilon_t = \sqrt{h_t}\zeta_t, \\
h_t &= 0.2 + 0.05\varepsilon_{t-1}^2 + 0.75h_{t-1},\; \zeta_t \sim N(0,1). \\
\text{P5} &: \; y_t = 0.4y_{t-1} - 0.3y_{t-2} + 0.5y_{t-1}\varepsilon_{t-1} + \varepsilon_t,\; \varepsilon_t = \sqrt{h_t}\zeta_t, \\
h_t &= 0.2 + 0.05\varepsilon_{t-1}^2 + 0.75h_{t-1},\; \zeta_t \sim N(0,1). \\
\text{P6} &: \; y_t = 1 + 0.1y_{t-1} + 1.5\sqrt{h_t} + \varepsilon_t,\; \varepsilon_t = \sqrt{h_t}\zeta_t, \\
h_t &= 0.005 + 0.28[|\varepsilon_{t-1}| - 0.23\varepsilon_{t-1}]^2 + 0.7h_{t-1},\; \zeta_t \sim N(0,1).
\end{aligned}$$

Among the alternative models corresponding to misspecified variance but correct mean functions, P1 is the AR(1) - GJR(1,1), P2 is the AR(1) - EGARCH (1,1), and P3 is the AR(1) - Conditional heteroskedasticity (CH(1)) model. While P4 (GARCH-in-mean - Null GARCH (1,1)) and P5 (bilinear AR(2) -Null GARCH (1,1)) are the two misspecified conditional mean with correct variance function DGPs. Finally, for P6 (GARCH-in-mean - GJR (1,1) GARCH) both functions are misspecified. The parameter values are again common in the literature; e.g., Halunga and Orme (2009), Lundbergh and Teräsvirta (2002), Engle and Ng (1993), Escanciano (2009), Becker and Hurn (2009) used these alternative models and parameters in their simulation experiments.

Table 3: Empirical size with the t(5) and t(3) standardized errors

	E1				E2			
T	100	200	300	500	100	200	300	500
				$\zeta_t \sim t(5)$				
T_{LT}	0.90	1.00	0.90	1.30	1.60	0.50	1.00	0.80
T_N	2.60	2.40	2.10	4.10	1.90	2.00	2.60	4.20
T_{EN}	2.80	2.10	3.20	3.80	2.30	2.80	2.70	2.10
T_A	14.00	10.30	8.00	6.70	16.10	10.30	8.50	9.20
				IGS tests				
$D^2_{I,m}$	5.50	5.70	5.30	4.10	5.60	6.10	5.80	6.80
$D^2_{I,v}$	5.80	5.00	5.00	6.70	4.00	5.30	5.90	4.90
J^2_I	5.80	4.90	5.30	6.20	4.30	5.60	6.00	5.20
$D^2_{C,m}$	5.00	5.40	6.10	6.10	6.50	7.00	6.00	5.20
$D^2_{C,v}$	5.90	5.30	4.90	6.30	4.70	6.30	5.30	5.80
J^2_C	6.40	5.20	5.40	5.40	4.60	6.60	5.70	5.60
				PICM tests				
$C^2_{I,m}$	7.10	5.20	6.30	4.20	4.40	5.70	5.80	5.50
$C^2_{I,v}$	6.30	6.60	7.00	5.40	8.60	6.70	6.80	6.40
$C^2_{I,J}$	6.90	6.60	6.70	5.50	8.50	6.90	7.10	6.30
$C^2_{C,m}$	5.80	5.40	5.20	4.30	4.80	6.60	5.30	5.50
$C^2_{C,v}$	6.10	6.70	6.90	4.90	9.10	6.00	5.30	5.40
$C^2_{C,J}$	7.00	6.40	7.20	4.50	8.50	6.40	6.30	5.40
				$\zeta_t \sim t(3)$				
T_{LT}	0.50	0.60	0.90	0.70	0.50	0.90	0.30	0.70
T_N	0.60	2.10	1.50	1.60	1.00	2.00	1.20	1.80
T_{EN}	1.00	2.40	1.30	1.70	1.70	1.90	1.50	2.10
T_A	15.90	14.40	10.80	4.80	16.40	11.10	9.70	9.60
				IGS tests				
$D^2_{I,m}$	5.60	5.20	5.80	5.60	5.50	6.80	6.20	6.10
$D^2_{I,v}$	4.10	6.40	4.90	5.00	5.30	5.60	4.40	5.20
J^2_I	5.10	5.90	4.00	5.90	5.40	7.70	5.40	5.30
$D^2_{C,m}$	6.00	5.80	5.10	6.60	6.00	6.30	6.40	6.20
$D^2_{C,v}$	5.00	5.40	4.30	4.60	6.00	5.80	5.10	5.00
J^2_C	5.70	6.60	5.00	5.30	6.70	7.00	6.40	5.80
				PICM tests				
$C^2_{I,m}$	5.40	5.10	5.30	4.80	6.10	5.70	5.40	5.40
$C^2_{I,v}$	4.90	5.40	5.90	5.50	5.70	7.50	5.40	6.50
$C^2_{I,J}$	5.40	5.80	6.00	5.60	6.00	7.40	6.20	7.00
$C^2_{C,m}$	5.80	4.80	5.40	5.10	5.40	4.80	6.20	5.00
$C^2_{C,v}$	5.60	4.60	6.20	5.30	7.80	7.30	5.70	4.50
$C^2_{C,J}$	6.80	4.50	5.30	5.20	8.50	7.00	6.10	4.80

Table 4 reports the empirical power of the tests for misspecified variance with correct mean models where the nominal size is again 5%. A number of interesting issues to be noticed here. Firstly, in this case it is expected that for the nonparametric tests the joint and marginal variance component (i.e., J^2 and D_v^2 in case of the IGS and C_J^2 and C_v^2 in case of the PICM) would pick up the misspecification in conditional variance while this would have no impact on D_m^2 and C_m^2. The power property indeed reflect this fact as the empirical rejection frequencies of D_m^2 and C_m^2 are close to nominal level of 5% while the power of D_v^2 and C_v^2 (J^2 and C_J^2) indicate that they pick up the misspecification and increase as sample size grow. Secondly, the parametric CM tests particularly T_N and T_A, show very good power properties even with a moderate $T = 500$ and T_{LT} lacks in power in all three cases. Once again, these are supported by the results of Halunga and Orme. Thirdly, for EGARCH(1,1) and CH(1) alternative models (i.e., P2 and P3) the IGS tests demonstrate equally impressive (even better in case of P3) power compared to parametric ones. The PICM tests display slightly lower power than the IGS tests for P1 and P2. However, note that even with $T = 500$ the power of both nonparametric tests is relatively weak in case of P1 (GJR alternative); e.g., below 30%. It is worthwhile to note that the parametric CM tests also perform relatively poorly for GJR alternative. With our small to moderate sample size this is not unusual though, as Engle and Ng (1993, p. 1762) also observe weak power for small sample size and concluded "This weakness is expected as both asymmetric effect and time-varying variance are hard to detect in small samples". Important to note that the power is increasing with T and for large sample sizes (which is the case in most real life situation) we could expect that the power would increase substantially for this type of alternative. Finally, the tests based on indicator weight function generally perform slightly better compared to complex exponential weight function.

Next, Table 5 displays the simulated power, against 5% nominal level, when the data is generated by models P4 and P5 (misspecified mean but correct variance) and P6 (both mean and variance are misspecified). For P4 and P5, one would expect that D_m^2 and C_m^2 (J^2 and C_J^2) would pick up the misspecification in conditional mean; and ideally want that D_v^2 and C_v^2 to be robust to this type of misspecification. We can not, however, be sure about the rejection frequencies of marginal variance tests (i.e., D_v^2 and C_v^2) as the conditional variance specification depends on conditional mean and thereby they may pick the misspecification in mean despite the correct specification of variance. From our Monte Carlo experiments, we can see that J^2 and D_m^2 (in case of the IGS tests) and C_J^2 and C_m^2 (in case of the PICM tests) demonstrate excellent power with both weight functions. On the other hand the rejection frequencies for D_v^2 and C_v^2 increase with sample size. However, the rejection frequencies for D_v^2 and C_v^2 are well below compared to $D_m^2$and C_m^2, respectively.

The parametric CM tests inherently assume a correct specification of mean function and it is observed that all these tests (except T_{LT}) pick up the misspecification in mean and their rejection frequencies are much higher than D_v^2 and C_v^2. T_{EN} is mostly affected by the misspecification in mean followed by T_N and T_A, whereas T_{LT} surprisingly seems to be insensitive to the conditional mean misspecification. This finding confirms that the parametric CM tests are not robust to the misspecification in mean function and in the presence of mean misspecification these tests erroneously over-reject the null of correct variance specification.

Finally when both mean and variance are misspecified (P6), evidence shows that except T_{LT} all tests, parametric and nonparametric, pick up the misspecification. Since we are using a GJR alternative model for conditional variance, marginal variance tests show relatively low power compared to joint and marginal mean tests.

Table 4: Empirical power for the DGPs P1, P2 and P3

	P1				P2				P3			
T	100	200	300	500	100	200	300	500	100	200	300	500
T_{LT}	1.80	2.40	1.80	3.80	2.50	5.10	6.90	11.80	5.80	12.60	20.60	39.40
T_N	8.60	21.70	31.60	53.10	15.10	47.50	72.00	92.70	8.30	16.60	27.40	47.30
T_{EN}	4.80	9.10	11.80	18.30	17.40	42.20	62.50	87.40	18.80	37.90	54.00	78.70
T_A	16.60	24.40	34.70	52.10	15.90	26.30	36.60	53.90	23.40	34.60	48.80	70.80
IGS tests												
$D^2_{I,m}$	5.70	7.50	5.80	7.30	5.70	6.70	6.00	7.80	5.30	5.90	5.20	6.20
$D^2_{I,v}$	8.00	12.10	16.70	25.70	24.00	52.30	70.30	92.50	22.50	41.60	58.50	80.60
J^2_I	7.30	13.20	17.00	24.10	22.60	49.40	65.50	87.30	21.80	40.30	58.60	80.50
$D^2_{C,m}$	5.70	5.70	4.30	6.70	7.50	7.20	5.60	7.90	6.70	6.20	5.30	7.00
$D^2_{C,v}$	9.20	12.50	17.30	24.00	21.40	49.20	67.90	86.50	19.80	36.60	53.70	77.40
J^2_C	8.10	12.90	16.40	23.80	17.60	37.70	49.70	71.90	16.10	32.50	47.40	72.40
PICM tests												
$C^2_{I,m}$	6.00	5.80	5.30	4.50	5.70	4.30	5.60	5.20	5.90	4.90	5.60	5.60
$C^2_{I,v}$	10.10	13.60	15.70	20.10	19.10	32.50	45.50	61.90	24.90	42.40	58.00	81.10
$C^2_{I,J}$	6.40	8.90	10.20	13.20	18.80	31.70	45.30	62.20	24.60	42.20	57.30	81.20
$C^2_{C,m}$	5.20	5.20	4.80	4.30	5.90	5.00	5.60	6.00	7.10	5.60	4.70	7.00
$C^2_{C,v}$	10.10	12.40	14.30	18.50	14.00	24.30	31.30	40.80	21.30	41.70	54.80	78.20
$C^2_{C,J}$	9.50	11.80	13.10	17.10	13.30	22.80	30.80	39.90	21.50	41.50	54.20	77.60

Table 5: Empirical power for the DGPs P4, P5 and P6

	P4				P5				P6			
T	100	200	300	500	100	200	300	500	100	200	300	500
T_{LT}	1.50	1.90	3.00	3.80	0.80	1.10	1.10	0.90	2.10	4.00	8.40	11.00
T_N	17.70	34.10	48.40	65.20	11.30	20.20	30.30	49.90	7.30	17.90	29.70	51.00
T_{EN}	40.00	60.70	74.40	88.20	6.80	13.00	19.40	34.10	18.20	38.90	56.10	77.10
T_A	14.80	20.90	29.80	39.80	13.00	21.90	33.90	50.50	18.30	31.00	45.70	65.00
IGS tests												
$D^2_{I,m}$	68.30	93.00	97.40	98.90	84.30	93.10	93.80	96.40	43.00	83.50	96.40	99.90
$D^2_{I,v}$	7.00	17.80	28.20	49.50	18.70	25.10	31.50	42.40	6.60	16.60	28.90	52.60
J^2_I	39.80	79.20	94.80	98.50	55.70	88.40	93.40	97.40	20.10	57.10	84.30	99.00
$D^2_{C,m}$	74.10	94.30	98.50	99.50	65.50	89.40	94.50	97.90	32.80	69.30	90.20	99.60
$D^2_{C,v}$	16.70	36.60	57.80	76.20	18.20	24.90	28.90	39.50	8.80	18.70	32.10	51.40
J^2_C	61.90	92.70	98.10	99.60	67.10	90.60	93.90	97.50	61.90	48.80	74.80	95.40
PICM tests												
$C^2_{I,m}$	68.80	91.50	95.40	98.30	46.50	69.50	80.60	89.30	37.80	73.80	90.40	98.40
$C^2_{I,v}$	12.30	30.60	47.00	69.20	31.70	52.10	66.90	81.10	5.90	8.30	9.40	15.50
$C^2_{I,J}$	28.30	48.70	60.60	76.00	35.80	58.60	72.80	86.20	36.20	69.90	85.00	94.70
$C^2_{C,m}$	67.20	91.70	96.90	98.90	48.80	78.30	91.10	97.10	39.20	72.90	86.10	95.90
$C^2_{C,v}$	3.40	7.50	16.10	36.30	21.50	34.60	43.10	56.40	3.40	3.40	4.30	4.50
$C^2_{C,J}$	18.40	28.10	37.40	55.80	33.80	52.90	62.00	70.50	33.80	62.20	76.00	84.80

8 Empirical Illustration

In this Section, we illustrate the nonparametric and parametric testing methodology to the famous and extensively used S&P 500 daily stock index, which is also a representative set of data for which substantial number of studies used GARCH regression models (see Bollerslev, 1992 and references therein). We consider the daily index data covering the time period July 1, 2004 to July 30, 2010. Therefore we have a sample of 1531 observations. We further subdivide the whole sample period into three two-years sub-period: July1, 2004 to June 30, 2006 (505 observations), July 3, 2006 to June 30, 2008 (501 observations) and July 1, 2008 to July 30, 2010 (525 observations). The choice of these sample periods is motivated from the financial crisis of 2007.

We want to examine the dynamics of the S&P 500 by fitting an AR(1)-GARCH(1,1) model to the data (log returns) and applying our tests to make inference of the null specification. The QML estimates of the parameters along with their standard errors are presented in Table 6. Next we apply the four parametric CM tests and two nonparametric tests (i.e., the IGS and PICM tests) and the results are given in Table 7. The parametric CM tests do not test the mean specification and we can see a clear disagreement between the nonlinearity tests of Lundbergh and Teräsvirta (2002) and Halunga and Orme (2009) and between asymmetry tests of Engle and Ng (1993) and Halunga and Orme (2009). T_{LT} and T_{EN} do not reject the null of correct conditional variance specification in any of three sub-periods and full samples whereas T_A and T_N reject the null in all periods under consideration; except T_N for 2006-08 in which case we can not reject the null. The nonparametric tests, on the other hand, give us the scope to test the mean and variance specification simultaneously. It can be seen that the AR(1) specification is correct for all periods as revealed by very large p-values of D_m^2 and C_m^2. This implies that the parametric CM tests are not adversely affected from mean misspecification in this case. Secondly, this finding questions the inclination of many empirical researchers to consider a zero (or known) conditional mean specification and fitting solely a conditional variance model while modelling stock market.

In terms of marginal variance specification tests, both the IGS and PICM tests are mostly in agreement except 2006-08 period. For the overall period 2004-2010, D_v^2 and C_v^2 strongly reject the GARCH(1,1) specification with p-values 0.01 and 0.00, respectively. The joint tests also reject the correct joint specification for this period. The results indicate that a AR(1)-GARCH(1,1) model is adequate representation for the first sub-period 2004-06 (i.e., before the financial crisis of 2007) with p-values 0.11, 0.12, 0.29 and 0.21 for D_v^2, J^2, C_v^2 and C_J^2, respectively. Similarly, both the IGS and PICM tests strongly reject the null GARCH(1,1) model for 2008-10 when the financial crisis is in place. However, for 2006-08 the PICM tests reject the null of correct variance and joint specification quite strongly with p-values 0.62 and 0.68, respectively, while the corresponding p-values for the IGS tests are 0.047 (D_v^2) and .077 (J^2) which are in borderline of acceptance and rejection region.

In summary, we can see that though the AR(1) is an adequate representa-

Table 6: The QML parameter estimates

	2004-06	2006-08	2008-10	2004-10
$\hat{\varphi}_0$	0.027	0.049	0.085	0.046*
	(0.029)	(0.037)	(0.056)	(0.021)
$\hat{\varphi}_1$	-0.044	-0.105*	-0.089	-0.081*
	(0.046)	(0.048)	(0.047)	(0.027)
$\hat{\alpha}_0$	0.030	0.012*	0.026	0.013*
	(0.020)	(0.006)	(0.015)	(0.004)
$\hat{\alpha}_1$	0.039	0.060*	0.120*	0.084*
	(0.021)	(0.016)	(0.014)	(0.012)
$\hat{\beta}_1$	0.895*	0.930*	0.878*	0.907*
	(0.054)	(0.019)	(0.02)	(0.012)
T	505	501	525	1531

Note: Figures in the parenthesis are the standard errors

Table 7: The p-values of various tests

	2004-06	2006-08	2008-10	2004-10
T_{LT}	0.1486	0.7700	0.5981	0.5340
T_N	0.0002	0.1276	0.0205	0.0002
T_{EN}	0.5536	0.7025	0.7549	0.8940
T_A	0.0002	0.0000	0.0009	0.0008
		IGS Tests		
D_m^2	0.2600	0.8200	0.4667	0.4500
D_v^2	0.1100	0.0467	0.0033	0.0100
J^2	0.1200	0.0767	0.0100	0.0167
		PICM tests		
C_m	0.1367	0.6733	0.8167	0.7233
C_v	0.2933	0.6233	0.0033	0.0000
C_J	0.2133	0.6833	0.0033	0.0000

tion of the conditional mean specification for all time periods considered here, however the GARCH (1,1) fits well only in 2004-06. Note that, the nonparametric tests tell us that there is something wrong in the specification (i.e., they are omnibus tests) but do not direct us to the correct model specification. Here comes the importance of parametric test which assume a specific parametric alternative in their construction. In this sense we consider these nonparametric and parametric testing procedures as complimentary not competing. However, the contradiction among the parametric CM tests about the variance specification is indeed a confusing issue. For example, in our case the asymmetry test of Engle and Ng (1993) never reject the null with very high p-values whereas the Halunga and Orme (2009) T_A strongly rejects GARCH (1,1) in all periods. As noted by Halunga and Orme (2009) that the Engle and Ng (1993) and Lundberg and Teräsvirta (2002) tests neglect the recursive behavior of the alternative under consideration and therefore they may lack power. In this particular case, since the nonparametric tests also suggest a misspecification we need to modify the conditional variance specification.

9 Concluding Remarks

In this paper, we investigate the nonparametric simultaneous joint and marginal conditional mean and conditional variance specification testing of the null AR-GARCH model. We explicitly demonstrate how to perform the IGS and PICM tests in the model. In particular, we propose a modified wild bootstrap procedure for the IGS tests which performs well in our Monte Carlo study. A number of parametric CM tests for conditional variance, which implicitly assumes a correct conditional mean specification, are also considered and our Monte Carlo simulation confirms that these tests are indeed sensitive to misspecified mean function. The empirical application with the S&P 500 data also highlights the usefulness of the marginal and joint testing within the nonparametric framework.

Our simulation experiments reveal that both the IGS and PICM tests have satisfactory size and impressive robustness to non-normality. The parametric CM tests suffer from size distortion for small T and the distortion is greater under non-normality. Except T_{LT}, size property of other three parametric tests improve as T increase. Further research could focus on using the proposed bootstrap scheme to improve the size properties of the parametric tests.

We want to stress that the inability of our proposed bootstrap procedure to strictly satisfy the identified sufficient conditions does not necessarily mean our test is asymptotically invalid. Our Monte Carlo evidence shows that the size performance of the IGS tests outperforms the parametric tests, particularly T_{LT}, in small samples and, in absolute terms, has satisfactory significance level. The excellent size property under variety of distributional assumptions may indicate that the identified sufficient conditions are too stringent/restrictive and there exists a weaker set of conditions under which the tests are implementable. Searching such less restrictive necessary and sufficient conditions for this test is left for future research.

The power analysis indicates that for correct mean but misspecified variance models, the marginal nonparametric tests demonstrate the ability to identify the source of misspecification with the rejection frequencies of marginal mean tests (D_m^2 and C_m^2) close to the nominal significance level whereas the joint and marginal variance tests pick up the misspecification. The parametric CM tests (barring T_{LT}) also show excellent power in this case. We also note that for GJR alternative (P1) tests have relatively low power, the IGS tests in general demonstrate better power compared to the PICM tests and the indicator weight function performs slightly better than exponential weight function.

For misspecified mean and correct variance, the parametric CM tests incorrectly but unsurprisingly over-reject the null of correct variance specification. In case of the nonparametric tests, D_m^2 and C_m^2 (and their corresponding joint tests) display excellent empirical power. As expected the marginal variance tests pick some of the misspecification through the channel of conditional mean and the rejection frequencies increase as T increase. The PICM tests, in general, suffer more with relatively higher rejection rate. However, in this case D_m^2 and C_m^2 reject significantly more often than D_v^2 and C_v^2. Our suggestion is that whenever D_m^2 or C_m^2 rejects the null one has to revise the mean specification first until the tests provide evidence against mean misspecification and then examine the variance specification.

Finally, the IGS tests can easily be applied to check the adequacy of other extensions of GARCH models without any further modifications. The PICM tests are much quicker and easy to implement for our AR-GARCH model, however for extensions of GARCH models one needs to find the first partial derivatives of the moment conditions under consideration.

Appendix

.1 Proof of Lemma 2

Proof. Write $\theta' = (\varphi', \eta')$, and let $\hat{\theta}$ be the QMLE for θ_0. Define $f_t = \dfrac{\partial m_t}{\partial \varphi}$, $x_t = \dfrac{1}{h_t}\dfrac{\partial h_t}{\partial \eta}$ and $c_t = \dfrac{1}{h_t}\dfrac{\partial h_t}{\partial \varphi}$. Halunga and Orme (2009) showed that

$$\sqrt{T}(\hat{\theta} - \theta_0) = J_{\theta\theta}^{-1}\sqrt{T}D_{\theta T}(\theta_0) + o_p(1), \tag{29}$$

where $D_{\theta T}(\theta_0) = (D_{\varphi T}(\theta), D_{\eta T}(\theta))'$, $D_{\varphi T}(\theta) = T^{-1}\sum_{t=1}^{T}\left\{\dfrac{\varepsilon_t f_t}{h_t} + \dfrac{1}{2}\left(\dfrac{\varepsilon_t^2}{h_t} - 1\right)c_t\right\}$ and $D_{\eta T}(\theta) = T^{-1}\dfrac{1}{2}\sum_{t=1}^{T}\left(\dfrac{\varepsilon_t^2}{h_t} - 1\right)x_t$.

Given regularity conditions, $\hat{\theta} \xrightarrow{p} \theta_0$ and

$$\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} N\left(0, J_{\theta\theta}^{-1}\Omega_{\theta\theta}J_{\theta\theta}^{-1}\right),$$

where $J_{\theta\theta}$ and $\Omega_{\theta\theta}$ are both finite and positive definite, and are defined in Halunga and Orme (2009, Theorem 1).

Define $e_{1t}(\varphi) \equiv \varepsilon_t = y_t - m_t$, $e_{2t} \equiv \varepsilon_t^2 - h_t = e_{1t}^2(\varphi) - \eta' s_{t-1}(\theta)$, and $e_t(\theta) = (e_{1t}(\theta), e_{2t}(\theta))'$, so that

$$\begin{aligned} D_{\varphi T}(\theta) &= T^{-1}\sum_{t=1}^{T} h_t^{-1}\left(f_t, \frac{1}{2}c_t\right)e_t(\theta), \text{ and} \\ D_{\eta T}(\theta) &= T^{-1}\sum_{t=1}^{T} h_t^{-1}\left(0, \frac{1}{2}x_t\right)e_t(\theta). \end{aligned}$$

Now, it is straightforward to show that

$$\sqrt{T}(\hat{\theta} - \theta_0) = T^{-1/2}\sum_{t=1}^{T}\varrho_t(\theta_0)e_{0t} + o_p(1), \tag{30}$$

and $E\left[\varrho_{0t}e_{0t}e_{0t}'\varrho_{0t}'\right] = J_{\theta\theta}^{-1}\Omega_{\theta\theta}J_{\theta\theta}^{-1}$ is finite and positive definite, where $e_{0t} = e_t(\theta_0)$ and $\varrho_{0t} \equiv \varrho_t(\theta_0) = J_{\theta\theta}^{-1}h_{0t}^{-1}\begin{bmatrix} f_{0t} & \frac{1}{2}c_{0t} \\ 0 & \frac{1}{2}x_{0t}\end{bmatrix}$, $f_{0t} = f_t(\varphi_0)$, $c_{0t} = c_t(\theta_0)$, and $x_{0t} = x_t(\theta_0)$. ■

.2 Proof of Theorem 2

Proof. For simplicity of exposition, we will assume a linear mean specification so that $m(w_t;\varphi) = w_t'\varphi$. In the test procedures, the estimators $\tilde{\varphi}$ and $\tilde{\beta}$ are used as follows:

(a) $\tilde{\varphi}$ is obtained from a (possibly non-linear) least squares regression of y_t on w_t' and residuals $\tilde{\upsilon}_{1t} \equiv \upsilon_{1t}(\tilde{\varphi}) = y_t - w_t'\tilde{\varphi}$ are obtained and used in the construction of the various test statistics (rather than $e_{1t}(\hat{\varphi})$), for the construction of $\tilde{\beta}$, next, and when implementing a wild bootstrap scheme (see Section .3, below).

(b) Let $\hat{h}_t = h_t(\hat{\theta})$ be constructed using the QMLE, $\hat{\theta}$, and define $\hat{z}_t' = \left(1, \hat{h}_t\right)$. Then $\tilde{\beta}$ is obtained as the (2×1) least squares parameter estimator following a regression of $\tilde{\upsilon}_{1t}^2$ on $\hat{z}_t'$. Following this regression the residuals $\tilde{\upsilon}_{1t}^2 - \hat{z}_t'\tilde{\beta}$ are obtained and used in the construction of the various test statistics (rather than $e_{2t}(\hat{\theta})$), and when implementing a wild bootstrap scheme (see Section .3, below).

For the above estimators, the corresponding "true" parameter values are φ_0 and $\beta_0 \equiv (0,1)'$, respectively. Ideally, to obtain $\tilde{\beta}$, we would like to regress $\tilde{\upsilon}_{1t}^2$ on $z_{0t}' = (1, h_{0t})$, but h_{0t} is unobservable so we use $\hat{h}_t$ instead. Because of this, the residual associated with the estimation of $\tilde{\beta}$ depends upon $\hat{\theta}$, through $\hat{h}_t$, so this must be taken into account. In addition, the moment errors must be defined in terms of the parameters being estimated in the least squares procedures. Accordingly, let $\lambda = \left(\varphi', \beta'\right)'$ and define the following "second moment" error

$$\upsilon_{2t}(\lambda) = \upsilon_{1t}^2(\varphi) - \hat{z}_t'\beta = \upsilon_{1t}^2(\varphi) - z_{0t}'\beta - (\hat{z}_t - z_{0t})'\beta.$$

The corresponding residual would then be

$$\upsilon_{2t}(\tilde{\lambda}) = \upsilon_{1t}^2(\tilde{\varphi}) - \hat{z}_t'\tilde{\beta} = \upsilon_{1t}^2(\tilde{\varphi}) - z_{0t}'\tilde{\beta} - (\hat{z}_t - z_{0t})'\tilde{\beta}.$$

It is important to make the distinction between the υ_{jt} used here and the e_{jt} defined previously, $j = 1, 2$, because although it is true that $\upsilon_{1t}(\varphi) \equiv e_{1t}(\varphi)$, for all φ, it is not true that $\upsilon_{2t}(\lambda) = \upsilon_{1t}^2(\varphi) - \hat{z}_t'\beta$ is the same as $e_{2t}(\theta)$. In the ensuing analysis it will be useful to define $\upsilon_t(\lambda) = (\upsilon_{1t}(\varphi), \upsilon_{2t}(\lambda))'$.

First consider $\tilde{\varphi}$. The least squares regression of y_t on w_t' yields $\tilde{\varphi}$, which satisfies

$$\begin{aligned} \sqrt{T}(\tilde{\varphi} - \varphi_0) &= \hat{Q}_T^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_t \upsilon_{1t}(\varphi_0) + o_p(1) \\ &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T} q_t \upsilon_t(\lambda_0) + o_p(1), \end{aligned} \tag{31}$$

where $\hat{Q}_T = T^{-1}\sum_{t=1}^{T} w_t w_t'$, $q_t = Q^{-1}[w_t,\ 0]$, $Q = \mathrm{E}[w_t w_t']$, and $\mathrm{E}[q_t \upsilon_t(\lambda_0)\upsilon_t'(\lambda_0) q_t'] = Q^{-1}\,\mathrm{E}\left[\varepsilon_{0t}^2 w_t w_t'\right] Q^{-1}$ is finite and positive definite. From this, residuals $\tilde{\upsilon}_{1t} \equiv \upsilon_{1t}(\tilde{\varphi}) = y_t - w_t'\tilde{\varphi}$ are obtained and used in the construction of the various test statistics, for the construction of $\tilde{\beta}$.

Next consider $\tilde{\beta}$. Now regress $\tilde{\upsilon}_{1t}^2 = (y_t - w_t'\tilde{\varphi})^2$ on $\hat{z}_t'$ to give

$$\tilde{\beta} = \hat{V}_T^{-1} T^{-1}\sum_{t=1}^{T} \hat{z}_t \tilde{\upsilon}_{1t}^2, \tag{32}$$

where $\hat{V}_T = T^{-1}\sum_{t=1}^{T}\hat{z}_t\hat{z}_t'$.

Note that:

$$\tilde{\upsilon}_{1t}^2 = \upsilon_{1t}^2(\varphi_0) + \tilde{\xi}_t,\ \text{ where } \tilde{\xi}_t = 2w_t'\upsilon_{1t}(\varphi_0)\left(\tilde{\varphi} - \varphi_0\right) + \left(\tilde{\varphi} - \varphi_0\right)' w_t w_t'\left(\tilde{\varphi} - \varphi_0\right),$$

and

$$\upsilon_{1t}^2(\varphi_0) = \hat{z}_t'\beta_0 + \upsilon_{2t}(\lambda_0) = \hat{z}_t'\beta_0 + \left(\upsilon_{1t}^2(\varphi_0) - h_{0t}\right) - \left(\hat{z}_t - z_{0t}\right)'\beta_0,$$

where, recall, $\beta_0 \equiv (0,1)'$, and $h_{0t} = z_{0t}'\beta_0$. Using these in (32), we obtain

$$\begin{aligned}\tilde{\beta} &= \hat{V}_T^{-1}T^{-1}\sum_{t=1}^{T}\hat{z}_t\left(\upsilon_{1t}^2(\varphi_0) + \tilde{\xi}_t\right) \\ &= \beta_0 + \hat{V}_T^{-1}T^{-1}\sum_{t=1}^{T}\hat{z}_t\left(\upsilon_{1t}^2(\varphi_0) - z_{0t}'\beta_0 - \left(\hat{z}_t - z_t\right)'\beta_0 + \tilde{\xi}_t\right),\end{aligned}$$

so that

$$\begin{aligned}\sqrt{T}(\tilde{\beta} - \beta_0) &= \hat{V}_T^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\hat{z}_t\left(\upsilon_{1t}^2(\varphi_0) - z_{0t}'\beta_0 - \left(\hat{z}_t - z_{0t}\right)'\beta_0\right) + o_p(1) \\ &= \hat{V}_T^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\hat{z}_t\upsilon_{2t}(\lambda_0) + o_p(1), \qquad (33)\end{aligned}$$

which exploits $\sqrt{T}$ consistency of $\hat{\varphi}$ and $\tilde{\varphi}$ and a ULLN which ensures that $\hat{V}_T = O_p(1)$ and $T^{-1}\sum_{t=1}^{T}\hat{z}_t\tilde{\xi}_t = o_p(1)$.

However, $\upsilon_{2t}(\lambda_0) = \upsilon_{1t}^2(\varphi_0) - \hat{z}_t'\beta_0$ depends on $\hat{\theta}$, through $\hat{z}_t$, and this must be taken into account when analyzing the asymptotic sampling distribution of $\sqrt{T}(\tilde{\beta} - \beta_0)$.

Firstly, then, since $\hat{h}_t - h_{0t} = \bar{h}_t\left(\bar{c}_t',\ \bar{x}_t'\right)\left(\hat{\theta} - \theta_0\right)$, where here a "bar" indicates evaluation at the mean value $\bar{\theta}$, we have

$$\begin{aligned}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\hat{h}_t\upsilon_{2t}(\lambda_0) &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}h_{0t}\upsilon_{2t}(\lambda_0) + T^{-1}\sum_{t=1}^{T}\upsilon_{2t}(\lambda_0)\bar{h}_t\left(\bar{c}_t',\ \bar{x}_t'\right)\sqrt{T}\left(\hat{\theta} - \theta_0\right) \\ &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}h_{0t}\upsilon_{2t}(\lambda_0) + o_p(T^{-1/2}),\end{aligned}$$

since by consistency $\hat{\theta}$ and a ULLN, $T^{-1}\sum_{t=1}^{T}\upsilon_{2t}(\lambda_0)\bar{h}_t\left(\bar{c}_t',\ \bar{x}_t'\right) = o_p(1)$. Thus

$$T^{-1/2}\sum_{t=1}^{T}\hat{z}_t\upsilon_{2t}(\lambda_0) = T^{-1/2}\sum_{t=1}^{T}z_{0t}\upsilon_{2t}(\lambda_0) + o_p(1),$$

and substituting this into (33) yields$\sqrt{T}(\tilde{\beta}-\beta_0)=\hat{V}_T^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}z_{0t}\upsilon_{2t}(\lambda_0)$, so that

$$\begin{aligned}\sqrt{T}(\tilde{\beta}-\beta_0) &= \hat{V}_T^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}z_{0t}\left(\upsilon_{1t}^2(\varphi_0)-h_{0t}\right) \\ &\quad -\hat{V}_T^{-1}\sqrt{T}\left(\hat{V}_T-T^{-1}\sum_{t=1}^{T}\hat{z}_t z_{0t}'\right)\beta_0+o_p(1). \qquad (34)\end{aligned}$$

Now, consider how we might express the second term. We have

$$\begin{aligned}\sqrt{T}\left(\hat{V}_T-T^{-1}\sum_{t=1}^{T}\hat{z}_t z_{0t}'\right)\boldsymbol{\beta}_0 &= \left(T^{-1}\sum_{t=1}^{T}\hat{z}_t\left(\hat{z}_t-z_{0t}\right)'\right)\begin{pmatrix}0\\1\end{pmatrix} \\ &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\hat{z}_t\left(\hat{h}_t-h_{0t}\right) \\ &= \left(T^{-1}\sum_{t=1}^{T}\hat{z}_t\bar{h}_t\left(\bar{c}_t',\ \bar{x}_t'\right)\right)\sqrt{T}\left(\hat{\theta}-\theta_0\right) \\ &= A\sqrt{T}\left(\hat{\theta}-\theta_0\right)+o_p(1),\end{aligned}$$

where consistency of $\hat{\theta}$ and a ULLN will ensure that $T^{-1}\sum_{t=1}^{T}\hat{z}_t\bar{h}_t\left(\bar{c}_t',\ \bar{x}_t'\right)=A+o_p(1)$. In addition, given sufficient regularity $\hat{V}_T=V+o_p(1)$, where $V=\mathrm{E}\left[z_{0t}z_{0t}'\right]$.

Substituting these results and (30) into (34) we obtain the following expression

$$\begin{aligned}\sqrt{T}(\tilde{\beta}-\beta_0) &= V^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}z_{0t}\left(\upsilon_{1t}^2(\varphi_0)-h_{0t}\right)-V^{-1}A\sqrt{T}\left(\hat{\theta}-\theta_0\right)+o_p(1) \\ &= V^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}z_{0t}\left(\upsilon_{1t}^2(\varphi_0)-h_{0t}\right)-V^{-1}A\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varrho_{0t}e_{0t}+o_p(1).\end{aligned}$$

Then, we have the following asymptotic expansion

$$\sqrt{T}\begin{pmatrix}\tilde{\varphi}-\varphi_0\\ \tilde{\beta}-\beta_0\end{pmatrix}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{\begin{bmatrix}q_t\\ \overline{p}_t\end{bmatrix}\begin{bmatrix}\upsilon_{1t}(\varphi_0)\\ \upsilon_{1t}^2(\varphi_0)-z_{0t}'\beta_0\end{bmatrix}\right\}-\nu_T(\theta_0)+o_p(1), \qquad (35)$$

where $\overline{p}_t=V^{-1}(0,z_{0t})$ and $\nu_T(\theta_0)=\begin{bmatrix}0\\ V^{-1}A\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varrho_{0t}e_{0t}\end{bmatrix}$. This expansion is crucial when comparison is made to the corresponding bootstrap expansion.

Finally, however, since $\upsilon_{1t}(\varphi_0)\equiv e_{1t}(\varphi_0)$ and $\upsilon_{1t}^2(\varphi_0)-z_{0t}'\beta_0\equiv e_{2t}(\theta_0)$, we can write

$$\sqrt{T}\begin{pmatrix}\tilde{\varphi}-\varphi_0\\ \tilde{\beta}-\beta_0\end{pmatrix}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{\begin{bmatrix}q_t\\ p_t\end{bmatrix}\begin{bmatrix}e_{1t}(\varphi_0)\\ e_{2t}(\theta_0)\end{bmatrix}\right\}+o_p(1)$$

where $p_t = V^{-1}\left([0, z_{0t}] - A\varrho_{0t}\right)$. ■

.3 Analysis of the Wild Bootstrap Scheme

The bootstrap estimator, denoted $\tilde{\lambda}^*$, is obtained from bootstrap data generated as follows.

Following the least squares procedures, defined above, which yield $\tilde{\varphi}$ and $\tilde{\beta}$, respectively, generate the following bootstrap data, where U_t are random variables with zero mean and unit variance:

1. $Y_{1t}^* = w_t'\tilde{\varphi} + \varepsilon_{1t}^*$, where $\varepsilon_{1t}^* = U_t\tilde{\upsilon}_{1t}$ and $\tilde{\upsilon}_{1t} = \upsilon_{1t}(\tilde{\varphi})$.
2. $Y_{2t}^* = \hat{z}_t'\tilde{\beta} + \varepsilon_{2t}^*$, where $\varepsilon_{1t}^* = U_t\tilde{\upsilon}_{2t}$ and $\tilde{\upsilon}_{2t} = \upsilon_{2t}(\tilde{\lambda}) = \upsilon_{1t}^2(\tilde{\varphi}) - \hat{z}_t'\tilde{\beta}$.

To obtain $\tilde{\varphi}^*$ regress Y_{1t}^* on w_t to obtain

$$\begin{aligned}
\sqrt{T}\left(\tilde{\varphi}^* - \tilde{\varphi}\right) &= \left(T^{-1}\sum_{t=1}^{T} w_t w_t'\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_t\varepsilon_{1t}^* \\
&= \hat{Q}_T^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_t U_t\upsilon_{1t}(\tilde{\varphi}), \qquad (36)
\end{aligned}$$

where we have used the fact that conditionally on the sample data, in the bootstrap world $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_t\varepsilon_{1t}^*$ is bounded in probability. In particular if $\mathrm{E}^*(.)$ and $var^*(.)$ denote expectation and variance in the bootstrap world, respectively, conditional on the sample data, then $\mathrm{E}^*\left(w_t\varepsilon_{1t}^*\right) = 0$, because $\mathrm{E}^*(U_t) = 0$, and

$$var^*\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_t\varepsilon_{1t}^*\right) = \frac{1}{T}\sum_{t=1}^{T}\upsilon_{1t}^2(\tilde{\varphi})w_t w_t'.$$

Note that the expansion (36) agrees with the expansion for $\sqrt{T}(\tilde{\varphi} - \varphi_0)$ but with $U_t\upsilon_{1t}(\tilde{\varphi})$ replacing $\upsilon_{1t}(\varphi_0)$ in the right hand side.

To obtain $\tilde{\beta}^*$, regress Y_{2t}^* on $\hat{z}_t$ to obtain

$$\begin{aligned}
\sqrt{T}\left(\tilde{\beta}^* - \tilde{\beta}\right) &= \hat{V}_T^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\hat{z}_t\varepsilon_{2t}^* \\
&= \hat{V}_T^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\hat{z}_t U_t\upsilon_{2t}(\tilde{\lambda}), \qquad (37)
\end{aligned}$$

where, again, conditionally on the sample data, in the bootstrap world $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_t\varepsilon_{2t}^*$ is bounded in probability, with

$$var^*\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\hat{z}_t\varepsilon_{2t}^*\right) = \frac{1}{T}\sum_{t=1}^{T}\upsilon_{2t}^2(\tilde{\lambda})\hat{z}_t\hat{z}_t'.$$

Therefore, although (36) is of the same form as (33) but with $\tilde{\lambda}$ replacing λ_0, in the bootstrap world $\hat{z}_t$ will be regarded as fixed, in the definition of $v_{2t}(\tilde{\lambda})$ for each simulation of U_t, rather than varying as it is in the definition of $v_{2t}(\lambda_0) = v_{1t}^2(\varphi_0) - \hat{z}_t'\beta_0$ employed in (33).

If we were able to observe h_{0t} and so could employ z_{0t} rather than $\hat{z}_t$ there would be no such discrepancy. However, as we do use $\hat{z}_t$ strictly speaking the bootstrap procedure does not satisfy the sufficient conditions.

References

[1] Becker, R. and Hurn, S., 2009. Testing for nonlinearity in mean in the Presence of heteroskedasticity. Economic Analysis and Policy, 39(2), 311-326.

[2] Bera, A.K. and Higgins, M.L., 1997. ARCH and bilinearity as competing models for nonlinear dependence. Journal of Business & Economic Statistics, 15(1), 43–50.

[3] Bierens, H. J., 1982. Consistent model specification tests. Journal of Econometrics, 20(1), 105-134.

[4] Bierens, H.J., 1984. Model specification testing of time series regressions. Journal of Econometrics, 26(3), 323-353.

[5] Bierens, H.J. and Ploberger, W., 1997. Asymptotic theory of integrated conditional moment test. Econometrica, 65(5), 1129–1151.

[6] Billingsley, P., 1995. Probability and Measure (3rd ed.). Wiley New York.

[7] Blake, A.P. and Kapetanios, G. , 2007. Testing for ARCH in the presence of nonlinearity of unknown form in the conditional mean. Journal of Econometrics 137(2), 472–488.

[8] Bollerslev, T., 1986. Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 31(3), 307–327.

[9] Bollerslev, T., Engle, R.F. and Nelson, D.B., 1994. ARCH models. In: Engle, R. and McFadden, D., eds. Handbook of Econometrics, North Holland Press: Amsterdam.

[10] Caner, M. and Kilian, L., 2001. Size distortions of tests of the null hypothesis of stationarity: Evidence and implications for the PPP debate. Journal of International Money and Finance, 20(5), 639–657.

[11] Chen, X. and Fan, Y., 1999. Consistent hypothesis testing in semiparametric and nonparametric models for econometric time series. Journal of Econometrics, 91(2), 373–401.

[12] Chesher, A., 1983. The information matrix test : Simplified calculation via a score test interpretation. Economics Letters, 13(1), 45-48.

[13] Christoffersen, P. and Goncalves, S., 2005. Estimation Risk in Financial Risk Management. Journal of. Risk, 7(3), 1-28.

[14] Davidson, R. and Flachaire, E., 2001. The wild bootstrap, tamed at last. Working Papers 1000, Queen's University, Department of Economics.

[15] Davidson, R. and Flachaire, E., 2008. The wild bootstrap, tamed at last. Journal of Econometrics, 146(1), 162-169.

[16] de Jong, R., 1996. The Bierens test under data dependence. Journal of Econometrics, 72(1), 1-32.

[17] Delgado, M., Dominguez, M. and Lavergne, P., 2006. Consistent tests of conditional moment restrictions. Annales d'Economie et de Statistique, 81(1), 33-67.

[18] Engle, R.F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica 50(4), 987-1007.

[19] Engle, R.F and Ng, V.K., 1993. Measuring and testing the impact of news on volatility. Journal of Finance, 48(5), 1749–1778.

[20] Escanciano, J.C., 2006a. Goodness-of-fit tests for linear and non-linear time series models, Journal of the American Statistical Association, 101(474), 531–541.

[21] Escanciano, J.C., 2006b.Consistent diagnostic test for regression models using projections. Econometric Theory, 22(6), 1030–1051.

[22] Escanciano, J.C., 2007a. Model checks using residual marked empirical processes, Statistica Sinica, 17(1), 115–138.

[23] Escanciano, J.C., 2007b. Joint and marginal diagnostic tests for conditional mean and variance specifications. CAEPR Working Paper No. 2007-009.

[24] Escanciano, J.C., 2008. Joint and marginal specification tests for conditional mean and variance models. Journal of Econometrics, 143(1), 74-87.

[25] Escanciano, J.C., 2009. Simple bootstrap tests for conditional moment restrictions. Mimeo, available at http://www.ecares.org/ecaresdocuments/seminars0809/escanciano.pdf.

[26] Escanciano, J.C. and Velasco, C., 2006. Generalized spectral tests for the martingale difference hypothesis. Journal of Econometrics, 134(1), pp. 151–185.

[27] Fan, J. and Gijbels, I., 1996. Local polynomial modelling and its applications: Monographs on Statistics and Applied Probability 66 (Chapman & Hall/CRC Monographs on Statistics & Applied Probability). Chapman & Hall, London, New York.

[28] Francq, C. and Zakoian, J.M., 2004. Maximum likelihood estimation of pure GARCH and ARMA-GARCH. Bernoulli, 10(4), 605–637.

[29] Halunga. A. and Orme, C.D., 2009. First-order asymptotic theory for misspecification tests of GARCH models. Econometric Theory, 25(2), 364-410.

[30] Hansen, B.E., 1996. Inference when a nuisance parameter is not identified under the null hypothesis. Econometrica, 64(2), 413–430.

[31] Holly, A., 1982. A remark on Hausman's specification test. Econometrica, 50(3), 749-759.

[32] Hong, Y., 1999. Hypothesis testing in time series via the empirical characteristic function, Journal of the American Statistical Association, 94 (448), 1201–1220.

[33] Hong, Y., and Lee, T.H., 2003. Diagnostic checking for adequacy of nonlinear time series models. Econometric Theory, 19(6), 1065–1121.

[34] Hong, Y., and Lee, Y. J., 2005. Generalized spectral tests for conditional mean models in time series with conditional heteroskedasticity of unknown form. Review of Economic Studies, 72(2), 499-541.

[35] Khmaladze, E.V., 1981. Martingale approach in the theory of goodness-of-fit tests. Theory of Probability and its Applications, 26 (2), 240–257.

[36] Koul, H.L. and Stute, W., 1999. Nonparametric model checks for time series. Annals of Statistics, 27 (1), 204–236.

[37] Kyrtsou, C., 2008. Re-examining the sources of heteroskedasticity: The paradigm of noisy chaotic models. Physica A, 387(27), 6785-6789.

[38] Lancaster, T., 1984. The covariance matrix of the information matrix test. Econometrica 52 (4), 1051-1053.

[39] Lanne, M. and Saikkone, P., 2003. Reducing size distortions of parametric stationarity tests. Journal of Time Series Analysis, 24(4), 423–439.

[40] Lee, J., 1991. A Lagrange multiplier test for GARCH models. Economics Letter, 37(3), 265-271.

[41] Lee, S. and Hansen, B. E., 1994. Asymptotic theory for the GARCH(1,1) quasi-maximum likelihood estimator. Econometric Theory, 10(1), 29-52.

[42] Lee, T., White, H. and Granger, C.W.J., 1993. Testing for neglected nonlinearity in time series models. Journal of Econometrics, 56(3), 269-290.

[43] Li, W.K. and Mak, T.K., 1994. On the squared residual autocorrelations in non-linear time series with conditional heteroskedasticity. Journal of Time Series Analysis, 15(6), 627–636.

[44] Lumsdaine, R.L. and Ng, S., 1999. Testing for ARCH in the presence of a possibly misspecified mean. Journal of Econometrics, 93(2), 257–279.

[45] Lundbergh, S. and Teräsvirta, T., 2002. Evaluating GARCH models. Journal of Econometrics, 110(2), 417–435.

[46] Mammen, E., 1993. Bootstrap and wild bootstrap for high-dimensional linear models. Annals of Statistics, 21(1), 255-285.

[47] McLeod, A. I. and Li, W. K., 1983. Diagnostic checking ARMA time series models using squared residual autocorrelations. Journal of Time Series Analysis, 4(4), 269–273.

[48] Newey, W.K., 1985. Maximum likelihood specification testing and conditional moment tests. Econometrica, 53(5), 1047–1070.

[49] Ngatchou-Wandji, J., 2005. Checking nonlinear heteroscedastic time series models. Journal of Statistical Planning and Inference, 133(1), 33–68.

[50] Orme, C., 1990. The small-sample performance of the Information-Matrix test. Journal of Econometrics, 46(3), 309-331.

[51] Parzen, E., 1957. On consistent estimates of the spectrum of a stationary time series. The Annals of Mathematical Statistics, 28(2), 329–348.

[52] Pascual, L., Romo, J. and Ruiz, E., 2006. Bootstrap prediction for returns and volatilities in GARCH models, Computational Statistics & Data Analysis, 50 (9), 2293–2312.

[53] Shi, X., 2011. Size distortion and modification of classical Vuong tests. mimeo, University of Wisconsin-Madison. (available at http://www.ssc.wisc.edu/~xshi/research/One_Step_Vuong_Test.pdf).

[54] Stinchcombe, M. and White, H., 1998. Consistent specification testing with nuisance parameters present only under the alternative. Econometric Theory, 14(3), 295–325.

[55] Stute, W., 1997. Nonparametric model checks for regression. Annals of Statistics, 25(2), 613–641.

[56] Stute, W., Gonzalez-Manteiga, W. and Presedo-Quindimil, M., 1998. Bootstrap approximations in model checks for regression. Journal of the American Statistical Association, 93(441), 141–149.

[57] Stute, W. and Zhu, L.X., 2002. Model checks for generalized linear models. Scandinavian Journal of Statistics, 29(3), 535-545.

[58] Tauchen, G., 1985. Diagnostic testing and evaluation of maximum likelihood models. Journal of Econometrics, 30(1-2), 415-443.

[59] van der Vaart, A. and Wellner, J., 1996. Weak convergence and empirical processes (with applications to statistics). Springer-Verlag, New York.

[60] Vuong, Q. H., 1989. Likelihood ratio tests for model selection and non-nested hypotheses. Econometrica, 57(2), 307-33.

[61] Whang, Y., 2000. Consistent bootstrap tests of parametric regression functions. Journal of Econometrics, 98(1), 27-46.