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# Agreeing to spin the subjective roulette wheel: Bargaining with subjective mixtures


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# Agreeing to Spin the Subjective Roulette Wheel: Bargaining with Subjective Mixtures

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**Abstract**

In this paper the Bargaining Problem of Nash (1950) is reconsidered for deterministic outcomes. Unlike the case of risky lotteries, the use of cardinal utility in this domain is not immediately justified. To remedy this, we introduce the notion of a *Conditional Bargaining Problem*. This supposes that individuals can use some agreed-upon event to assess 'subjective preference mixtures' (Ghirardato, Maccheroni, Marinacci and Siniscalchi, 2003). We use the language of subjective mixtures to define *Subjectively Fair Compromise* and *Subjectively Fair Efficiency* of a Conditional Bargaining Solution. When combined with *Individual Rationality*, these axioms uniquely identify a non-trivial solution to the Conditional Bargaining Problem for a general class of 'locally cardinal' preferences. Furthermore, if preferences do exhibit a general cardinal utility representation, then the solution is independent of the assumed event and is precisely the unconditional solution proposed by Kalai and Smorodinsky (1975).

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# 1 Introduction

Situations where people could, by reaching some agreement, enjoy mutual benefits are pervasive in economic life. Typically, in such situations, there is more than one agreement that could be chosen. The question of how an agreement is reached is known as the *Bargaining Problem*. Since Nash (1950), there has been extensive research on axiomatic approaches to the Bargaining Problem. A recent survey of this literature is Thomson (2009).

In this paper we consider a Bargaining Problem where: there is a rich set of possible alternatives, and the outcomes involved are *deterministic*. Risky lotteries over the outcomes are not feasible. An example of such a problem is: how should £100 be split between two people? A basic assumption might be that each individual has preferences for money represented by an ordinal utility function. The well known result of Shapley (1969) states, roughly, that it is impossible to find a sensible solution to this problem. To define a sensible solution to a Bargaining Problem requires further knowledge about the structure of preferences.

To justify the use of cardinal utility, it is typical in the bargaining literature to extend the problem to include all risky lotteries over the feasible alternatives. The first contribution of this paper is the notion of a *Conditional Bargaining Problem*. A Conditional Bargaining Problem is a Bargaining Problem where the individuals have agreed on one event. Preferences for betting on this event are then considered. The event could, for example, be "the coin is heads"[1] We view the Conditional Bargaining Problem as: the simplest extension of a Bargaining Problem that allows us to measure cardinal information about the individuals' preferences.

The second contribution of this paper is that we bring modern techniques of non-expected utility theory to bear on the Bargaining Problem. In a recent pa-

[1]Alternatively, the event could be "the day is one year from now" and so consider event-contingent payoffs to be intertemporal payoffs.

per, Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003) defined the concept of a *Subjective Mixture*. A Subjective Mixture of two outcomes is defined using preferences for betting on some event. For a wide class of preferences, *Biseparable Preferences*, Subjective Mixtures are a method of measuring utility mixtures. That is, provided preferences are Biseparable, Subjective Mixtures reveal cardinal information about utility. The authors used this observation to construct a 'Subjective Mixture Space' and used this to translate utility representation theorems from the Anscombe-Aumann to the Savage setting.

In this paper we use Subjective Mixtures as a conceptual tool to axiomatise a solution to the Conditional Bargaining Problem. Using the language of Subjective Mixtures we define two new axioms: *Subjectively Fair Compromise* and *Subjectively Fair Efficiency*. The importance of 'fairness' in bargaining has been widely recognised. Empirical evidence contradicting the equilibrium based predictions in the noncooperative case is often explained by some notion of fairness[2]. Our new axioms capture two simple ideas that we hope a solution satisfies: if we cannot have exactly we want then any compromise should be 'fair', and there should be no 'fair' way that we could do better. When combined with *Individual Rationality*, a unique solution to Conditional Bargaining Solution exists for very general preferences. If preferences are Biseparable, then the solution to the Conditional Bargaining Problem is independent of the event used to take the required measurements. Furthermore, the solution coincides with that of Kalai and Smorodinsky (1975). We take this as a solution to the (unconditional) Bargaining Problem.

[2]For an overview of bargaining and ultimatum experiments, see Roth (1995).

## 2 Conditional Bargaining Problems

The tuple $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ is a *Conditional Bargaining Problem.* Here, $A \subseteq X_1 \times X_2$ is the *feasible set of alternatives.* Elements of $X_i$ are $x_i$, called *outcomes.* So an alternative is a pair $(x_1, x_2)$ where each individual receives an outcome. $d = (d_1, d_2)$ is the *disagreement alternative.* We assume the Conditional Bargaining Problem satisfies the following structural assumption:

**Structural Assumption 2.1.** *$X_1$ and $X_2$ are compact, connected and separable*[3] *in some topologies, $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. $A$ is compact in the product topology $\mathscr{T}_1 \times \mathscr{T}_2$. The disagreement alternative $d$ is contained in $X_1 \times X_2$.*

It is useful to introduce, briefly, the Savage (1954) framework. There is a set of *states*, $\mathscr{S} = \{\ldots, s, \ldots\}$. The individuals do not know which state will obtain, but that only one will. Subsets of $\mathscr{S}$ are *events*, the set of which is $\mathscr{E} = 2^{\mathscr{S}} = \{\ldots, E, E', \ldots\}$. Acts are functions from states to outcomes $f : \mathscr{S} \to X_i$.

We write $x_i E y_i$ for the act with outcome $x_i$ if $s \in E$ and $y_i$ otherwise. For a fixed event $E$, the $x_i E y_i$ is an *E-Binary Act.* The set of $E$-Binary Acts over individual $i$'s outcomes is $X_i^2(E)$. Preferences $\succcurlyeq_i$ are defined over $X_i^2(E)$. Outcomes $x_i \in X_i$ are naturally identified with constant $E$-Binary Acts, $x_i E x_i$, and the restriction of preferences to outcomes is also written $\succcurlyeq_i$. We make the simplifying assumption that there are no two distinct outcomes, for either individual, such that $x_i \sim_i x_i'$.[4]

For a fixed $E$, an act $x_i E y_i$ with $x_i, y_i \in X_i$ and $x_i \succcurlyeq_i y_i$ is an $E$-bet. The set of $E$-bets is $X^2_{\succcurlyeq_i}(E) := \{x_i E y_i : x_i, y_i \in X_i, x_i \succcurlyeq_i y_i\}$. It is a rank-ordered subset of the product set $X_i^2$, with the rank-ordering agreeing with $\succcurlyeq_i$. The set of $E$-bets is endowed with the (restriction of the) product topology $\mathscr{T}_i \times \mathscr{T}_i$.[5]

[3] See Fishburn (1970, p.35,62-64) for brief definitions.

[4] Alternatively one may assume that we have already 'passed to the quotient'.

[5] Connectedness and separability of $X^2_{\succcurlyeq_i}(E)$ follow from Wakker (1989) Lemmas 7.2 and 7.3.

For the Conditional Bargaining Problem, an event $E$ is chosen in advance. The event $E$ is *essential* for both individuals, $x_i \succ_i x_iEy_i \succ_i y_i$ for some $x_i, y_i \in X_i$. We suppose that each individual has preferences satisfying the following basic axioms:

**A1** (Weak Ordering) Preferences for $E$-bets are complete and transitive.

**A2** (Dominance) If $x_i \succ x_i'$ and $y_i \succ y_i'$ then $x_iEy_i \succ x_i'Ey_i'$.

**A3** (Continuity) For $x_iEy_i \in X^2_{\succcurlyeq_i}(E)$, the lower and upper preference sets

$$\{x_i'Ey_i' \in X^2_{\succcurlyeq_i}(E) : x_iEy_i \succ x_i'Ey_i'\} \ \& \ \{x_i'Ey_i' \in X^2_{\succcurlyeq_i}(E) : x_i'Ey_i' \succ x_iEy_i\}$$

are open.

**A4** ($E$-Tradeoff Consistency) The following implication holds for $E$-bets:

$$x_iEy_i \sim x_i'Ey_i' \ \& \ z_iEy_i \sim z_i'Ey_i' \ \& \ w_iEx_i \sim w_i'Ex_i' \ \Rightarrow w_iEz_i \sim w_i'Ez_i'$$

The following theorem is due to Köbberling and Wakker (2003, p.403):

**THEOREM 2.2.** *Let preferences $\succcurlyeq_i$ be defined over $X^2_{\succcurlyeq_i}(E)$ and satisfy axioms A1-A4. Then there is an real-valued, increasing function $u_i^E$ on $X_i$ that is continuous in the topology $\mathscr{T}_i$ and a number $0 < \rho_i(E) < 1$ such that for any $x_iEy_i, x_i'Ey_i' \in X_i^2(E)$:*

$$\begin{aligned} x_iEy_i &\succcurlyeq_i x_i'Ey_i' \\ \Leftrightarrow \ \rho_i(E)u_i^E(x_i) + (1-\rho_i(E))u_i^E(y_i) &\geqslant \rho_i(E)u_i^E(x_i') + (1-\rho_i(E))u_i^E(y_i') \end{aligned}$$

*The utility function $u_i^E$ is cardinally unique.*

Preferences satisfying the above axioms are *E-Locally Biseparable.* Such preferences are very general. In particular, the existence of the cardinal utility index for outcomes is only established for a fixed, essential event $E$. Without imposing further restrictions on preferences, it is quite possible that representations obtained with different events would yield various utilities that are not affinely equivalent. Let $V_i^E(x_i E y_i) \equiv \rho_i(E) u_i^E(x_i) + (1-\rho_i(E)) u_i^E(y_i)$. As $u_i^E(x_i) \equiv V_i^E(x_i E x_i)$, the dominance axiom ensures that preferences for outcomes are represented by $u_i$. We may define the *certainty equivalent* function $c_i^E : X^2_{\succcurlyeq_i}(E) \to X$ as $c_i^E(x_i E y_i) := \{z_i \in X_i : u_i^E(z_i) = V_i^E(x_i E y_i)\}$. Given the structure and preferences here, the certainty equivalent function is a well-defined, continuous and $\succcurlyeq_i$-increasing function. Note that $c_i^E$ may depend on the choice of $E$. Let $M_i(A)$ be the $\succcurlyeq_i$-maximal outcome in $A$. Such outcomes exist as preferences are continuous and $A \subseteq X_1 \times X_2$ is compact. For the rest of the paper we fix a Conditional Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$, so we write $M_i$ instead of $M_i(A)$.

The basic task of bargaining theory is to identify a feasible alternative as a solution to the problem. A *Solution* is a function $S_E : \langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle \to A$ that assigns a unique, feasible alternative to any Conditional Bargaining Problem. Since we fix a Conditional Bargaining Problem for the rest of the paper, any statements we make about the properties of $S_E$ are taken to hold for any Conditional Bargaining Problem satisfying Structural Assumption 2.1.

# 3 Subjective Mixtures

In this section we outline the notion of *Subjective Mixtures* due to Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003) (GMMS from here on). GMMS introduced this theory in order to bring a mixture space-type structure to the purely subjective framework of Savage. In doing so, a tool was developed by which results derived in the classic Anscombe-Aumann framework can be immediately translated to that of Savage.

GMMS begin with the notion of a *E-preference average*, defined as follows:

**Definition 3.1** ($E$-Preference Average)**.** *Given two outcomes $x_i \succ_i y_i$, the $E$-*

preference average *of $x_i$ and $y_i$ (given $E$) is an outcome $z_i$ satisfying $x_i \succ_i z_i \succ_i y_i$ and,*

$$x_i E y_i \sim_i c_i^E(x_i E z_i) E c_i^E(z_i E y_i)$$

GMMS outline several justifications for the use of the term '$E$-preference average'. Firstly, for any $z_i', z_i'' \in X_i$ with $x_i \succcurlyeq_i \{z_i', z_i''\} \succcurlyeq_i y_i$ it can be shown that $E$-Locally Biseparable preferences necessarily imply $c_i^E(x_i E z_i') E c_i^E(z_i'' E y_i) \sim_i c_i^E(x_i E z_i'') E c_i^E(z_i' E y_i)$. GMMS interpret this to mean that the inner outcomes $z_i', z_i''$ of the compound acts play a *symmetric* role when the individual evaluates these bets. Since we identify $x_i E x_i$ with the outcome $x_i = c_i^E(x_i E x_i)$ we may rewrite the condition as $c_i^E(x_i E x_i) E c_i^E(y_i E y_i) \sim_i c_i^E(x_i E z_i) E c_i^E(z_i E y_i)$. The term $E$-preference average is justified then observing the inner $x_i$ and $y_i$ play a symmetric role in the evaluation of the $E$-bets *and* replacing $x_i$ and $y_i$ with $z_i$ retains the indifference. In short, $z_i$ implies the kind of conditions we would expect of any general 'average' of $x_i$ and $y_i$.

The second justification for the term Preference Average is seen by substituting the $E$-locally biseparable representation obtained in Theorem 2.2. GMMS show in their Proposition 1 that $z_i$ is a $E$-preference average of $x_i$ and $y_i$ iff:

$$u_i^E(z_i) = \frac{1}{2} u_i^E(x_i) + \frac{1}{2} u_i^E(y_i)$$

For the considered preferences, $E$-preference averages precisely identify *utility midpoints*. Note that the class of $E$-locally biseparable preferences includes: Biseparable preferences (Ghirardato and Marrinaci, 2001), Subjective Expected Utility preferences (Savage, 1954), Choquet Expected Utility preferences (Schmeidler, 1987; Gilboa, 1987) and Multiple-Prior preferences (Gilboa and Schmeidler, 1989). So the above holds for most of the popular models of choice under uncertainty currently used in economics. We first note that $E$-preference averages always exist:

**Lemma 3.2.** *For any preference relation $\succcurlyeq_i$ over $X^2_{\succcurlyeq_i}(E)$ satisfying axioms A1, A2 and A3 and outcomes $x_i \succ y_i$, a unique $E$-preference average of $x_i$ and $y_i$ exists.*

*Proof.* Preferences over $X_i^2$ are represented by a continuous function $V_i$. Fix any $x_i \succ y_i$ and define a function $f$ so that $f(t) = V_i(c_i^E(x_iEt)Ec_i^E(tEy_i))$ for all $t \in X_i$. $f$ is clearly continuous, being the composition of continuous functions. By the dominance axiom, $f(x) > V_i(x_iEy_i) > f(y)$. Then, since $f$ is continuous on a connected set $X_i$, there is a $z_i$ so that $f(z_i) = V_i(x_iEy_i)$ equivalent to the sought after indifference. One can show $x_i \succ z_i \succ y_i$ and that $z_i$ is unique using the dominance axiom. □

Denote the $E$-preference average of $x_i$ and $y_i$ as $\frac{1}{2}x_i \oplus_i^E \frac{1}{2}y_i$. It is then possible to define $\frac{3}{4}x_i \oplus_i^E \frac{1}{4}y_i$ as the $E$-preference average of $x_i$ and $\frac{1}{2}x_i \oplus_i^E \frac{1}{2}y_i$. Proceeding in this way, it is possible to define $E$-subjective mixtures for any dyadic rational and, appealing to the continuity of preferences, to construct any $\alpha : (1-\alpha)$ *$E$-Subjective Mixture* of $x_i$ and $y_i$, denoted $\alpha x_i \oplus_i^E (1-\alpha)y_i$. GMMS proved the following:

**Lemma 3.3.** *For any preference relation $\succcurlyeq_i$ over $X^2_{\succcurlyeq_i}(E)$ satisfying axioms A1, A2, A3 and A4:*

$$z_i = \alpha x_i \oplus_i^E (1-\alpha)y_i \quad \Leftrightarrow \quad u_i^E(z_i) = \alpha u_i^E(x_i) + (1-\alpha)u_i^E(y_i)$$

It is clear that $\alpha M_i \oplus_i^E (1-\alpha)d_i$ equals $M_i$ when $\alpha = 1$ and equals $d_i$ when $\alpha = 0$. The following monotonicity condition also follows immediately: $\alpha M_i \oplus_i^E (1-\alpha)d_i \succ \beta M_i \oplus_i^E (1-\beta)d_i$ whenever $\alpha > \beta$. In view of this monotonicity, the continuity of the $\oplus_i^E$ operation, connectedness, and the assumption that no two outcomes are indifferent, every outcome $x_i \in X_i$ with $M_i \succcurlyeq_i x_i \succcurlyeq_i d_i$ is a $\alpha : 1-\alpha$ $E$-Subjective Mixture for a unique $\alpha \in (0,1)$.

# 4 Subjectively Fair Conditions

In this section we propose a solution to the Conditional Bargaining Problem. The aim is to model how individuals, or perhaps some impartial arbitrator, could decide on an alternative. We use the language of subjective mixtures to define two axioms, capturing ideas of fair compromise and efficiency, that we insist on our solution satisfying. In return, these two axioms, in combination with individual rationality, will determine a unique and non-trivial alternative for any Conditional Bargaining Problem. Recall: for a Conditional Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ we denote the solution $S_E$.

The first axiom we insist that our solution must satisfy is *Subjectively Fair Compromise*. The basic idea is this: Each individual has a most preferred outcome $M_i$. The disagreement outcome $d_i$ is, presumably, the worst outcome that an individual would deem relevant. The ideal alternative $(M_1, M_2)$ is not typically available, but there is often an alternative better than $(d_1, d_2)$. So a compromise has to be reached. Then, the solution $S_E$ satisfies Subjectively Fair Compromise if each individual is using the same $\alpha : (1-\alpha)$ subjective mixture of their best and disagreement outcome. Presumably, 1 would have good reason to object to any solution that systematically assigned 2 an outcome 'closer' to their ideal, or assigned 1 an outcome 'closer' to his worst outcome. The language of subjective mixtures allows us to formalise this:

**Axiom S1 (Subjectively Fair Compromise)**

If $(x_1, x_2) \in S_E$ then,

$$x_1 = \alpha M_1 \oplus_1^E (1-\alpha) d_1 \quad \Leftrightarrow \quad x_2 = \alpha M_2 \oplus_2^E (1-\alpha) d_2$$

The second axiom that we impose on our solution is *Subjectively Fair Efficiency*. This axiom attempts to formalise the notion: there should be no 'fair' way of doing better. Suppose the solution assigns an alternative $(x_1, x_2)$ where one individual 1 gets an outcome $x_1 = \alpha M_1 \oplus_1^E (1-\alpha)d_1$. Then this solution is Subjectively Fair Efficient if there is no other feasible alternative that is 'subjectively fair', they get the same $\beta : 1-\beta$ subjective mixture, *and* makes both individuals better off. Presumably, if this were not the case, then *both* individuals would have good reason to object. Formally, we have:

**Axiom S2 (Subjectively Fair Efficiency)**

> If $(x_1, x_2) \in S_E$ and $x_i = \alpha_i M_i \oplus_i^E (1-\alpha_i)d_i$, then there is no alternative $(\beta M_1 \oplus_1^E (1-\beta)d_1, \beta M_2 \oplus_2^E (1-\beta)d_2) \in A$ with $\beta > \alpha_i$ for $i = 1, 2$.

The final axiom we insist that the solution satisfy, *Individual Rationality*, states simply that no individual receives an outcome worse than their disagreement outcome. Individual Rationality has appeared extensively in the literature on Bargaining. It captures, in the simplest way, the fact that we are modelling the cooperative behaviour of individuals seeking to enjoy mutual gains:

**Axiom S3 (Individual Rationality)**

> If $(x_1, x_2) \in S_E$ then $x_i \succcurlyeq_i d_i$, $i = 1, 2$.

We are now set to state the main theorem of this paper:

**Theorem 4.1.** *For any Conditional Bargaining Problem satisfying Assumption 2.1, with preferences satisfying axioms A1-A4, the Subjectively Fair Compromise, Subjectively Fair Efficiency and Individual Rationality axioms jointly identify a unique, non-trivial solution.*

*Proof.* Theorem 2.2 applies, so preferences are $E$-Locally Biseparable. Individual Rationality eliminates all alternatives where one individual receives an outcome worse than their disagreement outcome. So, only $(x_1, x_2)$ with $M_i \succcurlyeq_i x_i \succcurlyeq_i d_i$

are permissible. Consider individual 1. All the permissible outcomes $x_1$ that individual 1 could receive satisfy $M_1 \succcurlyeq_1 x_1 \succcurlyeq_1 d_1$. Then the permissible outcomes are all $\alpha : 1-\alpha$ $E$-Subjective Mixtures of $M_1$ and $d_1$, each for a unique $\alpha \in (0,1)$. Subjectively Fair Compromise then insists that for any $x_1 = \alpha M_1 \oplus_1^E (1-\alpha)d_1$, the only permissible outcome for individual 2 to receive is $x_2 = \alpha M_2 \oplus_2^E (1-\alpha)d_2$. Equivalently, by Lemma 3.3, the solution in utility space lies somewhere in on the line segment $[(u_1^E(d_1), u_2^E(d_2)), ((u_1^E(M_1), u_2^E(M_2))]$. Continuity of the utilities obtained ensures that the image of $A$ in utility space is compact. The Subjectively Fair Efficiency axiom then selects the unique feasible point where this line segment closest to $(M_1, M_2)$ $\square$

The unique solution $S_E$ satisfying the Subjectively Fair Compromise, Subjectively Fair Efficiency and Individual Rationality axioms selects the alternative that, in utility space, is the optimal point on the line segment connecting the disagreement alternative with the 'utopia' alternative. The solution, therefore, is a conditional version of the well-known solution of Kalai and Smorodinsky (1975).

The use of Subjective Mixtures, in the above formulation, serves two purposes. Consider the problem of splitting £100. A naive notion of fair compromise is to give each person £50, that is, giving each the same amount of money. The immediate response to this suggestion is that this does not reflect the preferences of each person. The appropriate currency should correct for different preferences, so utility units are more appropriate. Therefore, in view of Lemma 3.3, *the first purpose of Subjective Mixtures is to correct for differences in preferences.* Having corrected for differing preferences, one may then consider a fair outcome to be one that assigns each individual the same utility. One then arrives at the well known *Egalitarian Solution* (Kalai, 1977). This suggestion, however, is highly questionable as it involves interpersonal comparisons of utility. The utility numbers derived here are not completely determined, they are unique only up changes in scale and location.

Subjective Mixtures are, however, uniquely determined. Therefore, *the second purpose of Subjective Mixtures is to provide an appropriate unit for interpersonal comparisons.* We then naturally arrive at the Kalai-Smorodinsky solution, a scale invariant version of the Egalitarian solution (Thomson, 2009, p.9).

## 4.1 The Biseparable Case

A solution to the Conditional Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ can be interpreted as a *Conditional Solution* to the Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2\rangle$. To resolve the Bargaining Problem with deterministic outcomes, we suggested that preference averages are measured using some essential event $E$. The main problem with this proposal is that the solution may depend on the chosen event. Thus, we replace the problem of finding a suitable alternative with that of finding a suitable event. Although in general this is problematic, for a large class of preferences it turns out be no problem at all. Suppose there is a set of essential events $\mathscr{E}^*$. A *Capacity* is a normalised and monotone set function, that is, $E \subseteq E'$ implies $\rho(E) \leqslant \rho(E')$, $\rho(\mathscr{S}) = 1$ and $\rho(\emptyset) = 0$. Preferences are *Biseparable* if there exists a utility function $u_i : X_i \to \mathbb{R}$ and a capacity $\rho : \mathscr{E}^* \to (0, 1)$ so that:

$$\begin{aligned} x_i E y_i &\succcurlyeq_i x_i' E' y_i' \\ \Leftrightarrow \quad \rho_i(E)u_i(x_i) + (1 - \rho_i(E))u_i(y_i) &\geqslant \rho_i(E')u_i(x_i') + (1 - \rho_i(E'))u_i(y_i') \end{aligned}$$

Biseparable preferences were first defined and axiomatised by Ghirardato and Marinacci (2001). The key difference between Biseparable and $E$-Locally Biseparable preferences is that the cardinal utility index obtained is now independent of the event considered. Biseparable preferences are still general enough to include: Expected Utility, Choquet Expected Utility and Multiple Priors preferences. It should clear, that for Biseparable preferences, and any $E, E' \in \mathscr{E}^*$:

$$S_E \equiv S_{E'} \equiv S$$

For Biseparable preferences, the choice of the essential event used to construct the Conditional Bargaining problem is irrelevant; the same alternative will be selected as the solution. It cannot be controversial, then, to consider any conditional solution $S_E$, with $E \in \mathscr{E}^*$, as a solution to the Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2 \rangle$.

## 4.2 The Extension to $n \geqslant 2$ Players

When preferences are Biseparable, the Subjectively Fair Compromise, Subjectively Fair Efficiency and Individual Rationality axioms uniquely determine a solution that coincides with the Kalai-Smorodinsky (KS) solution. The KS solution, in utility space, selects the point where each individual is assigned the best alternative subject to the constraint that each is rewarded the same proportion of their highest possible utility. That is, the Pareto Efficient alternative on the line segment $[(u_1(d_1), u_2(d_2)), (u_1(M_1), u_2(M_2))]$. Kalai and Smorodinsky stated four attractive axioms the jointly characterise the KS solution: *Pareto Efficiency*, *Symmetry*, *Affine Invariance* and *Restricted Monotonicity*, axioms K1-K4 respectively.

In terms of the rule "find the best point on the line", the KS solution appears very simple to generalise to the $n$-player case. The existence of this point in $n$ dimensions is not problematic. The problem, as shown by Roth (1979), is that such a point in more than two dimensions need not be Pareto Efficient. In fact, Roth (1979) showed that there is no solution satisfying the natural $n$-player versions of K1-K4. The Subjectively Fair Compromise, Subjectively Fair Efficiency and Individual Rationality axioms are straightforwardly extended to the $n$-player case. In utility space, the resulting solution is still the point on the line segment described above. Every Microeconomist knows that allocations can be Pareto Efficient, yet still be drastically 'unfair'. The $n$-player solution here is 'subjectively fair' but not Pareto Efficient. This captures a situation where someone may benefit without harming anyone else, but deems it 'unfair' to take the opportunity.

# 5 Closing Comments

In this paper we defined a Conditional Bargaining Problem and, using the subjective mixture techniques of GMMS, introduced axioms capturing subjective notions of fair compromise and efficiency. These axioms in turn, when combined with Individual Rationality, uniquely determined a solution to the problem. For Biseparable preferences, this solution to the Conditional Bargaining Problem is independent of the choice of event used to measure subjective mixtures. This is not the case for general $E$-locally biseparable preferences. For the latter case, the outcomes for each individual may change with the choice of event. How the individuals, or perhaps an impartial arbitrator, should choose the event in such cases is an interesting problem for further research.

A potential criticism of our *punctual* axioms is the explicit use made of best and worst outcomes. In the characterisation of Kalai and Smorodinsky (1975), the importance of the utopia and disagreement alternatives emerges purely as a consequence of *relational* axioms.[6] We counter this criticism in two ways. Firstly, it seems inherent in the definition of compromise that the best outcomes should be discussed. They are the outcomes one would choose, after all, if one were a dictator. Secondly, the best and worst outcomes of any decision are known to be more salient, psychologically, than other outcomes (Lopes, 1987; Cohen and Jaffray, 1988). Simple generalisations of expected utility to incorporate this have been successful in explaining a variety of economic anomalies (Cohen, 1992; Chateauneuf, Eichberger and Grant, 2007). This makes the explicit reference to these outcomes less arbitrary.

The Subjective Mixture techniques used in this paper begin with the notion of

[6] An axiom is punctual if it refers only to the (Conditional) Bargaining Problem being considered. An axiom is relational if it specifies how characteristics of the solution change, or do not change, as the data of the problem changes.

a Preference Average. In Section 3 we outlined the definition of a Preference Average, introduced by GMMS. The main justification for the term Preference Average was that such outcomes are *utility midpoints*. We should point out that there are several other techniques for measuring utility midpoints. Most recently, Baillon, Driesen and Wakker (2009) presented a particularly simple method. Starting with their method, or any of the references contained there concerning utility midpoint elicitation, one can proceed to construct Subjective Mixtures as described in Section 3.

Another topic of future research is to study alternative solutions to the Conditional Bargaining Problem. In this paper we begin with preferences over outcomes, obtaining cardinal utility by extending the problem and considering preferences for $E$-bets. We are not the first to suggest working with preferences and alternatives, as opposed to utilities from the outset. An earlier paper that stressed the benefits of the preferences and alternatives approach is Rubinstein, Safra and Thomson (1992), RST. Translating the *Ordinal Nash Solution* of RST gives an analogue in the language of subjective mixtures, we call *No Subjectively Fair Deviations*:

**Axiom S4 (No Subjectively Fair Deviations)**

$(y_1, y_2) \in S_E$ if and only if for any player $i = 1, 2$, $j \neq i$, any alternative $(x_1, x_2) \in A$ and any $\alpha \in [0, 1]$,

$$y_i \prec \alpha x_i \oplus_i^E (1-\alpha) d_i \Longrightarrow x_j \preccurlyeq_j \alpha y_j \oplus_j^E (1-\alpha) d_j$$

The interpretation of the No Subjectively Fair Deviations axiom is straightforward: Suppose the candidate $(y_1, y_2)$ is on the table and, for the sake of clarity, that $\alpha = 1/2$. Player $i$ views his proposed outcome as 'not even half as good' as another and suggests this instead. The other player sees the newly suggested outcome as 'not even half as good' as sticking with solution. As neither individual's claim is stronger, in terms of intensity of preference, such changing from the solution could be deemed 'unfair' to at least one individual. The axiom states any remaining challenges to the solution can be similarly dismissed. Whenever a solution satisfying S4 is well defined and preferences are biseparable, it follows by Proposition 1 of RST that the solution is the alternative which maximises the product of utility differences from the disagreement alternative. Thus a Nash-type solution can be obtained directly from RST applied to our problem. The interpretation of S4 is quite different to the RST setting of risky lotteries. We prefer Subjectively Fair Compromise and Subjectively Fair Efficiency primarily because, in this setting, they always imply a well defined solution. We hope, however, that the techniques outlined here can stimulate more interesting conditions.

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