The University of Manchester

MANCHESTER 1824

# Economics Discussion Paper Series EDP-0923

# Learning and Stability of the Bayesian – Walrasian Equilibrium

Marialaura Pesce
Nicholas C. Yannelis

December 2009

Economics
School of Social Sciences
The University of Manchester
Manchester M13 9PL

# Learning and Stability of the Bayesian - Walrasian Equilibrium*

November 30, 2009

**Marialaura Pesce[1] and Nicholas C. Yannelis[2]**

**Abstract:** Under the Bayesian-Walrasian Equilibrium (BWE) (see Balder and Yannelis (2009)), agents form price estimates based on their own private information, and in terms of those prices they can formulate estimated budget sets. Then, based on his/her own private information, each agent maximizes interim expected utility subject to his/her own estimated budget set. From the imprecision due to the price estimation it follows that the resulting equilibrium allocation may not clear the markets for every state of nature, i.e., exact feasibility of allocations may not occur. This paper shows that if the economy is repeated from period to period and agents refine their private information by observing the past BWE, then in the limit all agents will obtain the same information and market clearing will be reached. The converse is also true. The analysis provides a new way of looking into the Arrow-Debreu "state contingent model" which is based on a statistical foundation.

*We are deeply indebted to Erik J. Balder for his comments and useful suggestions. We also thank the audience of the IX SAET Conference in Ischia for several valuable comments.

[1]Dipartimento di Matematica e Statistica,
Universitá di Napoli Federico II, Napoli 81026, ITALY
e-mail: marialaura.pesce@unina.it

[2]Department of Economics
University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA and
Economics - School of Social Sciences
The University of Manchester, Oxford Road, Manchester M13 9PL, UK
e-mail: nyanneli@illinois.edu, nicholasyannelis@manchester.ac.uk

# 1 Introduction

It is by now well known that a Rational Expectations Equilibrium (REE), as introduced in Radner (1979) does not exist. It only exists in a generic sense and not universal. Moreover, it fails to be fully Pareto optimal and incentive compatible and it is not implementable as a perfect Bayesian equilibrium of an extensive form game (Glycopantis and Yannelis (2005, 2009)). The difficulty with the rational expectations equilibrium is that agents are maximizing interim expected utility conditioned on their own private information and on the information that the equilibrium prices have generated, i.e., agents are acting as knowing the equilibrium prices.

Balder and Yannelis (2009) introduced a new concept called Bayesian-Walrasian Equilibrium (BWE) which has Bayesian features. In particular, agents form price estimates based on their own private information[3], and in terms of those prices they can formulate estimated budget sets. Then, based on his/her own private information, each agent maximizes interim expected utility subject to his/her own estimated budget set. From the imprecision due to the price estimation it follows that the resulting equilibrium allocation may not clear the markets for every state of nature, i.e., exact feasibility of allocations may not occur. This new concept exists under the standard continuity and concavity assumptions on the utility function and furthermore, it was shown in Balder-Yannelis (2009) that in the example of the non-existence of a REE of Kreps (1977), a BWE exists, but markets do not clear exactly. The non-market clearing of the BWE, is of course not desirable as in equilibrium there may be excess supply or demand. However, Balder and Yannelis (2009) do show that in the case of perfect foresight (correct price estimates) and also in the case of symmetric information, a BWE exists and the exact feasibility of allocations holds.

The main purpose of this paper is to provide a dynamic rationalization of the BWE and show that exact market clearing will always hold in the limit. In particular, we show that if the economy is repeated from period to period and agents refine their private information by observing the past BWE, then in the limit all agents will obtain the same information and market clearing will be reached.[4]

[3]This is contrary to the rational expectations equilibrium, when agents act as knowing the equilibrium prices.

[4]Such a conjecture was made in Balder-Yannelis [2009, p. 396] and it was left as an open question if it is true. Theorem 6.2 in section 6, provides an affirmative answer to the Balder-Yannelis conjecture.

In other words, in the limit there is no asymmetric information among agents, and all markets are cleared for every state of nature. Furthermore, we show that the limit BWE always exist, it is Pareto optimal and obviously incentive compatible. Hence, the limit BWE has desirable properties, i.e., it exists under the standard assumptions and it is efficient. This is quite interesting because, even if in each period we have an excess demand or supply, repeating the asymmetric information economy, agents eventually will learn all the information (i.e., become symmetrically informed) and in that limit economy the BWE satisfies the exact feasibility.

It is rather surprising that the converse is also true, i.e., starting from a BWE in the limit symmetric economy we can construct a sequence of approximate BWE or epsilon-BWE which converges to it. This means that even if agents make mistakes along the sequence by optimizing approximately their conditional expected utility subject to the estimated budget set, and even if market clearing doesn't not hold in the sequence of the epsilon-BWE, still in the limit agents will reach the exact BWE, where exact optimization and exact feasibility hold. This result can be viewed as a stability property of the BWE.

The above results have several implications.

First, the BWE concept where agents are symmetrically informed and obviously all markets are cleared can be viewed as the limit of an asymmetric BWE where the asymmetrically informed agents maximize interim utility based on their own information subject to an estimated budget constraint and market clearing doesn't exist. Agents by observing past BWE learn the information of others and the asymmetries on the private information disappear in the limit.

Second, since all decisions are interim, the symmetric BWE provides an interim way of looking at the state contingent contacts of Arrow-Debreu. Recall, that the state contingent model of Arrow-Debreu explains how agents make contracts under uncertainty in an ex-ante way. In particular, in the Arrow-Debreu model, agents have state dependent utility functions and initial endowments and make contracts ex-ante, before the state of nature is realized, i.e., no signaling is considered. Ex-post, once the state of nature is realized the contract which was made ex-ante is executed and consumption takes place. In the symmetric BWE, agents make similar contracts in an interim stage, i.e., after they have received a signal as to what is the event which contains the realized state of nature. Hence, our new framework provides an interim version of the state contingent model of Arrow-Debreu.

Third, since the BWE is based on estimated prices and thus on estimated budget constraints, our new modeling provides a statistical foundation of our equilibrium concept which seems to be suitable to econometric applications.

Finally, we have provided a new way to examine the Walrasian equilibrium concept under uncertainty, which is not subjected to the criticism of the REE. Our new concept universally exists and it is efficient.

The paper is organized as follows: Section 2 introduces the reader to the BWE and its relationship to the REE. Section 3 is discusses the dynamics and the idea of learning and proves results on the convergence of private information and section 4 introduces formally the limit symmetric BWE and section 5 provides the conditions under which the BWE convergences to the symmetric BWE. Section 6 shows that any symmetric BWE can be reached as the limit of an approximate sequence of BWE. Some concluding remarks are collected in section 7.

# 2 Bayesian-Walrasian Equilibria

We consider an **exchange economy with asymmetric information**

$$\mathcal{E} = \left\{(\Omega, \mathcal{F}, \mathbb{P});\ (X_i, \mathcal{F}_i, u_i, e_i)_{i \in I=\{1,\ldots,n\}}\right\},$$

where,

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability measure space describing the **exogenous uncertainty**, i.e., $\Omega$ is the countable set of all possible states of nature and $\mathcal{F}$ is the $\sigma$-field that represents the set of all events.
2. $\left(X_i, \mathcal{F}_i, u_i, e_i\right)_{i \in I}$ is the set of **agent's characteristics**. Each economic agent $i \in I$ is characterized by:
   - $X_i : \Omega \to 2^{\mathbb{R}^\ell_+}$ is agent $i$'**random consumption set** of each agent.
   - $\mathcal{F}_i$ is a measurable partition[5] of $(\Omega, \mathcal{F})$ denoting **the private information** of agent $i$. The interpretation is as usual: if $\omega \in \Omega$ is the state of nature that is going to be realized, agent $i$ observes $E^{\mathcal{F}_i}(\omega)$ the element of $\mathcal{F}_i$ which contains $\omega$.

[5] By an abuse of notation we will still denote by $\mathcal{F}_i$ the $\sigma$-algebra that the partition $\mathcal{F}_i$ generates.

- a **random utility function** representing his preferences:

$$\begin{array}{rcl} u_i : \ \Omega \times I\!\!R^\ell_+ & \rightarrow & I\!\!R \\ (\omega, x) & \rightarrow & u_i(\omega, x) \end{array}$$

  Through the paper, we will assume that for all $i \in I$ and $\omega \in \Omega$, $u_i(\omega, \cdot)$ is strongly monotone.

- a **random initial endowment** of physical resources represented by the function

$$e_i : \ \Omega \rightarrow I\!\!R^\ell_+$$

  We assume that the function $e_i$ is $\mathcal{F}_i$-measurable and $e_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$.

- $\mathbb{P}$ is **the common prior**.

Suppose that for all $\omega \in \Omega$, $\mathbb{P}(\omega) > 0$. Upon the realization of $\omega \in \Omega$, which agent $i$ perceives through $E^{\mathcal{F}_i}(\omega)$, she forms the **conditional probability** given by[6]

$$\mathbb{P}_i(\omega'|\omega) = \begin{cases} \frac{\mathbb{P}(\omega')}{\mathbb{P}(E^{\mathcal{F}_i}(\omega))} & \text{if } \omega' \in E^{\mathcal{F}_i}(\omega), \\ 0 & \text{otherwise.} \end{cases}$$

Denote by

$$\begin{array}{rcl} \bar{L}_{X_i} & = & \{x_i : \Omega \rightarrow I\!\!R^\ell_+ \text{ such that } x_i(\omega) \in X_i(\omega) \text{ for all } \omega \in \Omega\} \\ L_{X_i} & = & \{x_i \in \bar{L}_{X_i} \text{ such that } x_i(\cdot) \text{ is } \mathcal{F}_i\text{−meaurable}\}. \end{array}$$

Let $\bar{L}_X = \prod_{i=1}^{n} \bar{L}_{X_i}$ and $L_X = \prod_{i=1}^{n} L_{X_i}$.

An **allocation** $x$ for an economy $\mathcal{E}$ is a function

$$\begin{array}{rcl} x : \ I \times \Omega & \rightarrow & I\!\!R^\ell_+ \\ (i, \omega) & \rightarrow & x_i(\omega), \quad \text{such that } x \in L_X. \end{array}$$

An allocation $x$ is said to be **feasible** if for all $\omega \in \Omega$,

$$\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega).$$

[6]Notice that $\displaystyle\sum_{\omega' \in E^{\mathcal{F}_i}(\omega)} \frac{\mathbb{P}(\omega')}{\mathbb{E}^{\mathcal{F}_i}(\omega)} = 1.$

Agent $i$'s **interim expected utility** is the function $v_i : \Omega \times L_{X_i} \to I\!R$, defined by

$$v_i(x_i|\mathcal{F}_i)(\omega) = \mathbb{E}(u_i(x_i)|\mathcal{F}_i)(\omega) = \sum_{\omega' \in \Omega} u_i(\omega', x_i(\omega'))\mathbb{P}_i(\omega'|\omega).$$

We can also express the interim expected utility using the conditional probability as

$$v_i(x_i|\mathcal{F}_i)(\omega) = \sum_{\omega' \in E^{\mathcal{F}_i}(\omega)} u_i(\omega', x_i(\omega')) \frac{\mathbb{P}(\omega')}{\mathbb{P}(E^{\mathcal{F}_i}(\omega))}.$$

A **random price vector** is an $\mathcal{F}$-measurable, non-zero function $p : \Omega \to \Delta$, where $\Delta = \{p \in \mathbf{R}^{\ell}_{+} : \sum_{j=1}^{\ell} p^j = 1\}$. Given a random price vector $p$, every agent $i$ adopts the following conditional expectation

$$\hat{p}_i(\omega) = \sum_{\omega' \in \Omega} p(\omega')\mathbb{P}_i(\omega'|\omega) = \sum_{\omega' \in E^{\mathcal{F}_i}(\omega)} p(\omega') \frac{\mathbb{P}(\omega')}{\mathbb{P}(E^{\mathcal{F}_i}(\omega))}. \tag{1}$$

In other words, $\hat{p}_i(\omega)$ is agent $i$'s **Bayesian price estimate** of the random price vector $p$, given that the state $\omega$ has been realized[7].

Using this natural estimate for the price, given her information $\sigma$-algebra, agent $i$ forms the following **estimated budget set**

$$\hat{B}_i(\omega, p) = \{x_i \in X_i(\omega) : \ \hat{p}_i(\omega) \cdot x_i \leq \hat{p}_i(\omega) \cdot e_i(\omega)\} =$$

$$\left\{ x_i \in X_i(\omega) \ : \ \sum_{\omega' \in \Omega} p(\omega')\mathbb{P}_i(\omega'|\omega) \cdot x_i \leq \sum_{\omega' \in \Omega} p(\omega')\mathbb{P}_i(\omega'|\omega) \cdot e_i(\omega) \right\}.$$

We now extend definition of BWE, introduced by Balder-Yannelis (2009), to differential information economies with a countable set of states of nature.

[7]Since we consider countably many states of nature, then for each $i \in I$ and each good $h \in \{1, \ldots, \ell\}$, $x_i^h$ and $e_i^h$ are real sequences (i.e., $(x_i^h(\omega_1), x_i^h(\omega_2), \ldots)$ and $(e_i^h(\omega_1), e_i^h(\omega_2), \ldots)$). In the same spirit of Peleg and Yaari (1970), we do not restrict the choice of equilibrium prices to the dual space, but we look at them as functions defined from $\Omega$ to the $I\!R^{\ell}$-simplex, and hence for each good $h$, $p^h$ is a real sequence as well as $x_i^{\ell}$ and $e_i^h$ are. Therefore, our approach is not exactly the infinite one, but it is closer to the finite-dimensional case. The reason is that we consider price-quantity inner products point wise $\hat{p}_i(\omega) \cdot x_i(\omega)$, where for each $\omega$ and each $i$, $\hat{p}_i(\omega)$ and $x_i(\omega)$ are elements of $I\!R^{\ell}_{+}$ (see for example the budget set below). We consider the topology of point wise convergence, that is the sequence $x^t$ converges to $x^*$ if and only if $\lim_{t \to \infty} x^t(\omega) = x^*(\omega)$ for all $\omega \in \Omega$. Similarly, a sequence of price $p^t$ converges to $p^*$ if and only if $\lim_{t \to \infty} p^t(\omega) = p^*(\omega)$ for all $\omega \in \Omega$. Moreover, notice that even if (1) is a countable sum, since for each $\omega$, $p(\omega) \in \Delta$, then for all $i \in I$ and each $\omega \in \Omega$, $\hat{p}_i(\omega) \leq 1$.

**Definition 2.1.** *An* **Asymmetric Bayesian-Walrasian Equilibrium (BWE)** *of the differential information exchange economy* $\mathcal{E}$ *is a pair* $(p, x)$ *such that*

$$\begin{array}{ll} (i) & p : \Omega \to \triangle \textit{ is a random price vector}, \\ (ii) & x = (x_i)_{i\in I} \in L_X \textit{ is an allocation}, \\ (iii) & x_i(\omega) \in argmax_{y_i \in \hat{B}_i(\omega,p)} v_i(y_i|\mathcal{F}_i)(\omega) \textit{ for all } \omega \in \Omega \textit{ and every } i \in I, \\ (iv) & p(\omega) \cdot \sum_{i\in I}(x_i(\omega) - e_i(\omega)) = \max_{1\le h\le \ell} \sum_{i\in I}(x_i(\omega) - e_i(\omega))_h \quad \textit{for all } \omega \in \Omega. \end{array}$$

Condition $(iii)$ states that for all state of nature agent $i$ maximizes his/her interim expected utility based on his/her own private information subject to his/her estimated budget set. Condition $(iv)$ is an unusual substitute for the classical feasibility property of Walrasian equilibria. As noted by Balder and Yannelis (see Proposition 2.1 p. 389), condition $(iv)$ must be considered the price that has to be paid when one wishes to avoid the unrealistic hypothesis surrounding the REE. However, we obtain the exact feasibility, i.e.,

$$(iv) \quad \sum_{i\in I} x_i(\omega) = \sum_{i\in I} e_i(\omega) \qquad \text{for all } \omega \in \Omega,$$

provided that

$$\begin{array}{llll} (a) & \mathcal{F}_i = \mathcal{F}_j & \text{for all } i, j \in I & \text{(symmetric information)} \\ (b) & \hat{p}_i = p & \text{for all } i \in I & \text{(correct price estimate or perfect forecasting of price } p), \end{array}$$

as the following proposition shows (see proposition 2.1 in Balder-Yannelis (2009) for the case of finite states of nature).

**Proposition 2.2.** *If* $(p, x)$ *is a BWE as in Definition 2.1, then free-disposal feasibility*

$$\sum_{i\in I} x_i(\omega) \le \sum_{i\in I} e_i(\omega) \qquad \textit{for all } \omega \in \Omega$$

*holds in each of the following cases:*

$$\begin{array}{llll} (a) & \mathcal{F}_i = \mathcal{F}_j & \textit{for all } i, j \in I & (\textit{symmetric information}) \\ (b) & \hat{p}_i = p & \textit{for all } i \in I & (\textit{correct price estimate or perfect forecasting of price } p). \end{array}$$

*Moreover, if in additional* $u_i(\omega, \cdot)$ *is strongly monotonic on* $X_i(\omega)$ *for all* $i \in I$ *and for all* $\omega \in \Omega$*, then the above feasibility condition sharpens into*

$$\sum_{i\in I} x_i(\omega) = \sum_{i\in I} e_i(\omega) \qquad \textit{for all } \omega \in \Omega.$$

**Proof:** Similar arguments to the ones used in Balder-Yannelis (2009, Proposition 2.1) can be adopted to complete the proof. □

Thus, we are interested in finding a framework which guarantees that either condition $(a)$ or condition $(b)$ are satisfied, and hence in the BWE exact feasibility of allocations is obtained, as the definition below shows.

**Definition 2.3.** *A pair $(p^*, x^*)$ is said to be a* **symmetric Bayesian-Walrasian Equilibrium** *if*

$$
\begin{array}{ll}
(i^*) & p^* : \Omega \to \triangle \; is\ a\ random\ price\ vector, \\
(ii^*) & x^* = (x_i^*)_{i\in I} \in \bar{L}_X \; is\ an\ allocation\ \mathcal{F}^*-measurable, \\
. & where\ \mathcal{F}^*\ is\ the\ symmetric\ information\,(i.e.,\ for\ all\ i \in I, \mathcal{F}_i = \mathcal{F}^*.) \\
(iii^*) & x_i^*(\omega) \in argmax_{y_i \in \hat{B}_i(\omega,p)} v_i(y_i|\mathcal{F}^*)(\omega)\ for\ all\ \omega \in \Omega\ and\ every\ i \in I, \\
(iv^*) & \sum_{i\in I} x_i^*(\omega) = \sum_{i\in I} e_i^*(\omega) \qquad for\ all\ \omega \in \Omega.
\end{array}
$$

We will consider a dynamic framework, in which agents learn by observing the past BWE and refining their private information. This allows us to get in the limit a symmetric information economy and under certain additional assumptions a full information economy (i.e., all agents know everything, i.e., $\mathcal{F}_i = \mathcal{F}$ for all $i \in I$). Therefore, we will prove not only that in the limit a BWE satisfies the exact feasibility, but also that it exists, it is (interim) Pareto optimal and obviously incentive compatible.

Let $\sigma(p)$ be the smallest sub-$\sigma$-algebra of $\mathcal{F}$ for which $p$ is measurable and let $\mathcal{G}_i = \sigma(p) \vee \mathcal{F}_i$ denote the smallest $\sigma$-algebra[8] containing both $\sigma(p)$ and $\mathcal{F}_i$. The notion below is due to Radner (1979), (see also Allen (1981)).

**Definition 2.4.** *A pair $(p, x)$, where $p$ is a price system and $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is an allocation, is a* **rational expectations equilibrium** *(REE) for the economy $\mathcal{E}$, denoted by* REE($\mathcal{E}$) *if*

[8]The field of events discernable by every player is the **"coarse"** $\sigma$-field $\bigwedge_{i\in I} \mathcal{F}_i$, which is the largest $\sigma$-algebra contained in each $\mathcal{F}_i$. While, agents by pooling their information, discern the events in the **"fine"** $\sigma$-field $\bigvee_{i\in I} \mathcal{F}_i$, which denotes the smallest $\sigma$-algebra containing all $\mathcal{F}_i$.

(*i*) *for all $i$ the consumption function $x_i(\cdot)$ is $\mathcal{G}_i$-measurable.*

(*ii*) *for all $i$ and for all $\omega$ the consumption function maximizes*

$$v_i(x_i|\mathcal{G}_i)(\omega) = \sum_{\omega' \in E^{\mathcal{G}_i}(\omega)} u_i(\omega', x_i(\omega')) \frac{\mathbb{P}(\omega')}{\mathbb{P}\big(E^{\mathcal{G}_i}(\omega)\big)},$$

*(where $E^{\mathcal{G}_i}(\omega)$ is the event in $\mathcal{G}_i$ which contains $\omega$ and $\mathbb{P}(E^{\mathcal{G}_i}(\omega)) > 0$) subject to*

$$p(\omega) \cdot x_i(\omega) \leq p(\omega) \cdot e_i(\omega)$$

*i.e. the budget set at state $\omega$, and*

(*iii*) $\sum_{i=1}^{n} x_i(\omega) = \sum_{i=1}^{n} e_i(\omega)$ *for all $\omega$.*

The REE and BWE are different concepts. For example, as it is shown in Balder-Yannelis (2009, p.390), in the Kreps example the REE does not exist, but the BWE exists. However, in the case that the private information of each agent in a BWE is singletons and if the REE is full revealing[9], then both notions become ex-post Walrasian equilibrium.

# 3 Learning and convergence of private information

Let $T = \{1, 2, \ldots\}$ be the discrete set of time horizon and $\mathcal{E}^t$ be a differential information economy with initial endowment $e^t$ for each $t \in T$. Denote by $\sigma(u_i, e_i^1, X_i)$ the $\sigma$-algebra generated by random utility function, the random initial endowment, and the random strategy set of agent $i$ at beginning, that is $\mathcal{F}_i^1 = \sigma(X_i, u_i, e_i^1)$. At time $t$, there is additional information available to agent $i$ acquired by observing past BWE prices and allocations. We can express each agent's private information recursively by

$$\mathcal{F}_i^{t+1} = \mathcal{F}_i^t \vee \sigma(p^t, x^t),$$

[9] A REE price $p$ is said to be full revealing if the information it generates is singletons, that is $\sigma(p) = \mathcal{F}$, and hence for each agent $i$, $\mathcal{G}_i = \mathcal{F}$.

where $\sigma(p^t, x^t)$ is the information that the BWE generates at period $t$, i.e., the smallest $\sigma$-algebra for which the Bayesian-Walrasian equilibrium in period $t$ is measurable, and $\mathcal{F}_i^t \vee \sigma(p^t, x^t)$ is the join (i.e., the smallest $\sigma$-algebra containing $\mathcal{F}_i^t$ and $\sigma(p^t, x^t)$). Thus, for all $i \in I$,

$$\begin{array}{lllcl} \text{if } t = 1 & \text{then,} & \mathcal{F}_i^1 & = & \sigma(X_i, u_i, e_i^1) \\ \text{if } t = 2 & \text{then,} & \mathcal{F}_i^2 & = & \mathcal{F}_i^1 \vee \sigma(x^1, p^1) \\ \text{if } t = 3 & \text{then,} & \mathcal{F}_i^3 & = & \mathcal{F}_i^2 \vee \sigma(x^2, p^2) \\ & & & = & \mathcal{F}_i^1 \vee \sigma(x^1, p^1) \vee \sigma(x^2, p^2) \\ & & & \vdots & \end{array}$$

Therefore, the private information of agent $i$ at time $t$ is given by

$$\mathcal{F}_i^t = \sigma(u_i, e_i^1, X_i, (p^{t-1}, x^{t-1}), (p^{t-2}, x^{t-2}), \ldots, \sigma(p^{t-(t-1)} x^{t-(t-1)})) = \mathcal{F}_i^1 \vee \bigvee_{k=1}^{t-1} \sigma(p^k, x^k),$$

where $(p^{t-1}, x^{t-1}), (p^{t-2}, x^{t-2}), \ldots, (p^{t-(t-1)}, x^{t-(t-1)})$ are past Bayesian-Walrasian equilibria. Clearly, for each agent $i$ and each time period $t$ we have

$$\mathcal{F}_i^t \subseteq \mathcal{F}_i^{t+1} \subseteq \mathcal{F}_i^{t+2} \subseteq \cdots$$

The above expression represents a **learning process** for player $i$ and it generates a sequence of exchange economies $\mathcal{E}^t = \{(X_i, \mathcal{F}_i^t, u_i, e_i^t) : i = 1, \ldots, n\}$.

We suppose that at beginning (i.e., $t = 1$) there are at least two agents who are asymmetrically informed, where for each agent $i$, the private information $\mathcal{F}_i^1$ is the information generated by the initial endowment, the utility function and the consumption set, that is $\mathcal{F}_i^1 = \sigma(X_i, u_i, e_i^1)$. We call this assumption a **non trivial differential information economy**.

Notice that agents do not form expectations over the entire horizon but only for the current period, i.e., each agent's interim expected utility is based on his/her current period private information. Obviously, since the private information set of each agent becomes finer over time, the interim expected utility of each agent is changing as well. The information gathered at a given time $t$ will affect the outcome in periods $t+1, t+2, \ldots$. Furthermore notice that for each agent $i \in I$ and each time $t \in T$,

$$\mathcal{F}_i^t = \mathcal{F}_i^1 \vee \left( \bigvee_{k=1}^{t-1} \sigma(p^k, x^k) \right),$$

that is for all $i$ and for all $t$, $\mathcal{F}_i^t = \mathcal{F}_i^1 \vee G^t$, where for all $t \in T$, $G^t = \bigvee_{k=1}^{t-1} \sigma(x^k, p^k)$ is the information generated by all past Bayesian-Walrasian equilibria until time $t$.

Thus, the information of each player $i$, after $t$ periods, has two components. The former is given by her initial private information $\mathcal{F}_i^1$ which contributes to the asymmetric part and the latter is given by the information generated by the past period BWE allocations which creates the common part of information since all agents see the same BWE outcome in each period. In the limit economy, the private information of each agent $i$, $\mathcal{F}_i^*$, is the initial private information information of $i$, $\mathcal{F}_i^1$, together with the information generated by all past periods BWE, that is

$$\mathcal{F}_i^* = \mathcal{F}_i^1 \vee \bigvee_{k=1}^{\infty} \sigma(x^k, p^k) = \mathcal{F}_i^1 \vee G^\infty.$$

Notice that even in the limit economy, agents may have different private information. Consider, for example, the case in which in each period $t \in T$, the information generated by the equilibria is coarser than the information of each trader, that is $\sigma(x^t, p^t) \subseteq \mathcal{F}_i^t$ for all $i \in I$. This means that no agent learns anything, that is

$$\mathcal{F}_i^{t+1} = \mathcal{F}_i^t \qquad \text{for all } i \in I \text{ and for all } t \in T.$$

Therefore, since in each period agents learn nothing, even in the limit economy the information of each agent $i$ is the initial one. Hence, it might be that in each period the same economy is replicated and therefore the equilibria remains the same. This makes the convergence results trivial, because a constant sequence of identical equilibria converges always to itself and vice versa. Clearly, we want to avoid this trivial case. In other words, we want the learning process to make sense. Formally, we need that at least one agent learns something in at least one period. This is guaranteed by the following assumption which we call **"non trivial learning"**.

$$(A.0) \quad \mathcal{F}_i^1 \subseteq G^\infty = \bigvee_{k=1}^{\infty} \sigma(x^k, p^k) \quad \text{for all } i \in I.$$

Assumption $(A.0)$ states that for each agent $i$, the pooled information generated by all past equilibria is at least as fine as agent $i$'s initial information.

**Remark 3.1.** Notice that the "no trivial learning assumption" does not mean that all agents will learn something, it only implies that at least one agent learns something in some period. Indeed, if nobody can learn, that is

$$\text{for all } i \in I \text{ and for all } t \in T \quad \mathcal{F}_i^t = \mathcal{F}_i^{t+1},$$

then for all $i \in I$, $\mathcal{F}_i^1 = \mathcal{F}_i^* = \mathcal{F}_i^1 \vee G^\infty$, and therefore by $(A.0)$, $\mathcal{F}_i^1 = G^\infty$. This means that already at the beginning, (i.e. $t = 1$) all agents have the same information, which is a contradiction to the assumption of the non trivial differential information economy.

**Remark 3.2.** We notice that, under the non-trivial learning assumption $(A.0)$, as time goes to infinity each agent gets the same information, given by the information generated by all the past periods Bayesian-Walrasian equilibria, i.e., $G^\infty = \bigvee_{k=1}^{\infty} \sigma(p^k, x^k)$. In other words, the private information $\mathcal{F}_i^t$ of each player $i$ converges to $G^\infty$. This follows from the fact that in the limit, the information of each agent $i$ is given by

$$\mathcal{F}_i^* = \mathcal{F}_i^1 \vee G^\infty,$$

which is finer than $G^\infty$. On the other hand, by the non trivial learning assumption $(A.0)$, for all $i \in I$, $\mathcal{F}_i^1 \subseteq G^\infty$. Hence, for each agent $i$, $\mathcal{F}_i^*$ coincides with $G^\infty$. This means that as time goes on, the common part of information becomes prevalent and in the limit all agents will have the same information generated by all the past periods BWE. Therefore, in the limit we reach an economy with symmetric information in which, if utility functions are strongly monotone, a BWE satisfies the exact feasibility as indicated in Section 2, condition $(a)$. Notice that $(A.0)$ is also a necessary condition for $\mathcal{F}_i^* = G^\infty$ for all $i \in I$. In fact, from if for all $i \in I$, $\mathcal{F}_i^* = G^\infty$, then $(A.0)$ holds.

**Remark 3.3.** If $\Omega$ is finite, since agents can not learn forever, from remark 3.2 it follows that there exists a period $s \in T$ from which all agents will have the same information $G^\infty$, that is, there exists $s \in T$ such that for all $i \in I$,

$$\mathcal{F}_i^t = G^\infty \qquad \text{for all } \ t > s.$$

**Remark 3.4.** Notice that we have stated that in the limit we reach an economy in which all agents have the same information. However, this does not mean that in the limit agents will know everything (i.e, $G^\infty$ may not be singletons). Indeed, it might be the case that $G^\infty \subset \mathcal{F}$. However, if there exists a period $t$ in which the pooled information of all agents is the full information $\mathcal{F}$, then after that period, agents will learn everything. Formally,

$$\text{if there exists } k \in T \ \text{ such that } \quad \bigvee_{i \in I} \mathcal{F}_i^k = \mathcal{F}, \ \text{ then } G^\infty = \mathcal{F}. \tag{2}$$

Indeed, for all $i \in I$, $\mathcal{F}_i^k \subseteq G^\infty$ (by assumption $(A.0)$), thus

$$\mathcal{F} = \bigvee_{i \in I} \mathcal{F}_i^k \subseteq G^\infty \subseteq \mathcal{F},$$

hence $G^\infty = \mathcal{F}$.

**Remark 3.5.** Typically, it is assumed that in the starting period $t = 1$, $\bigvee_{i \in I} \mathcal{F}_i^1 = \mathcal{F}$. As noted above, this condition is a particular case of (2) (i.e., $k = 1$). This means that in the limit the information of each agent will be generated by the partition of only singletons. Therefore, in the limit the measurability assumption will play no role, since the full information BWE will be an ex-post Walrasian equilibrium. Our more general framework includes the above case, where $\bigvee_{i \in I} \mathcal{F}_i^1 = \mathcal{F}$, as a special case[10].

# 4 Limit Symmetric Bayesian-Walrasian Equilibrium

We define the **limit symmetric information economy**

$$\mathcal{E}^* = \{(\Omega, \mathcal{F}, \mathbb{P}), \ (X_i, \mathcal{F}_i^*, u_i, e_i^*)_{i \in I}\}$$

[10]As it was pointed out in Section 2, in the special case of the full revealing REE, the BWE and REE coincide as they lead to an ex-post Walrasian equilibrium.

as follows: $(\Omega, \mathcal{F}, \mathbb{P})$, $X_i$ and $u_i$ are defined as in Section 2. While,

$$\mathcal{F}_i^* = G^\infty = \bigvee_{k=1}^{\infty} \sigma(x^k, p^k) \qquad \text{for all } i = 1, \ldots, n$$

and $e_i^*$ is the initial endowment of agent $i$ which is assumed to be $\mathcal{F}_i^*$-measurable and such that $e_i^*(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$. Obviously, all agents in this economy have the same private information generated by the Bayesian-Walrasian Equilibria in all periods (i.e., $\mathcal{F}_i^* = G^\infty = \bigvee_{k=1}^{\infty} \sigma(x^k, p^k)$).

**Remark 4.1.** Since in the limit all agents will get the same information (i.e., $\mathcal{F}_i^* = G^\infty$ for all $i \in I$), then they will have also the same Bayesian estimate price (i.e., $\hat{p}_i(\omega) = \hat{p}(\omega)$ for all $\omega \in \Omega$), and therefore, if the utility functions are strongly monotone, then the limit symmetric BWE satisfies the exact feasibility.

For each $i$, define the set[11]

$$L_{X_i}^\infty = \{x_i \in \bar{L}_{X_i} : \quad x_i(\cdot) \text{ is } G^\infty\text{−measurable}\}.$$

Let $L_X^\infty = \prod_{i=1}^n L_{X_i}^\infty$.

The BWE($\mathcal{E}^*$) is defined for the economy $\mathcal{E}^*$ analogously with the economy $\mathcal{E}^t$.

**Definition 4.2.** *A pair $(p^*, x^*)$ is said to be a* **limit symmetric Bayesian-Walrasian Equilibrium** *of the limit symmetric information economy $\mathcal{E}^*$, denoted by BWE($\mathcal{E}^*$), if*

$$\begin{array}{ll} (i^*) & p^* : \Omega \to \triangle \textit{ is a random price vector}, \\ (ii^*) & x^* \in L_X^\infty, \\ (iii^*) & x_i^*(\omega) \in argmax_{y_i \in \hat{B}_i(p^*,\omega)} v_i(y_i | G^\infty)(\omega) \quad for\ all\ i\ and\ for\ all\ \omega. \\ (iv^*) & \displaystyle\sum_{i \in I} x_i^*(\omega) = \sum_{i \in I} e_i^*(\omega) \qquad for\ all\ \omega \in \Omega. \end{array}$$

Condition $(iii^*)$ indicates that agents maximize interim expected utility subject to a point wise budget set conditioned on an event which is an element of the pooled

[11]Notice that since for all $i \in I$, $\mathcal{F}_i^t \subseteq \mathcal{F}_i^{t+1}$ for all $t$, then $L_{X^t} \subseteq L_X^\infty$ for all $t$.

information that all past BWE prices and allocations have generated. Notice that this event is the same for each agent, as the private information of each agent $\mathcal{F}_i$ is the same for each individual, i.e., $\mathcal{F}_i = G^\infty$ for all $i \in I$. Condition $(iv^*)$ is the standard feasibility condition.

Definition 4.2 can be viewed as an interim version of the state contingent model of Arrow-Debreu. Although, in this economy all agents have the same information, it may be still the case that the pooled information over the entire horizon may not be singletons. Therefore, the $G^\infty$ measurability condition of allocations makes sense. However, if the pooled information over the entire horizon is singletons, definition 4.2 reduces to an ex-post Walrasian equilibrium.

# 5 Convergence of the Bayesian-Walrasian Equilibria to the symmetric one

In the following section we will provide conditions under which any sequence $(p^t, x^t) \in BWE(\mathcal{E}^t)$ will have a subsequence that converges to $(p^*, x^*) \in BWE(\mathcal{E}^*)$ in the limit symmetric information economy $\mathcal{E}^*$. The following assumptions will be needed for our results.

(A.1) For every $i \in I$, $X_i : \Omega \to 2^{\mathbb{R}^\ell_+}$ is a convex and closed-valued correspondence[12].

(A.2) $u_i(\omega, \cdot)$ is continuous, strongly monotone and bounded for all $\omega \in \Omega$ and for each $i \in I$.

(A.3) For every $i \in I$, $t \in T$ and for all $\omega \in \Omega$, the initial endowment $e_i^t(\omega)$ is strictly positive.

(A.4) $\{e_i^t, \mathcal{F}_i^t\}_{t \in T}$ is a martingale.

(A.5) $\{\sum_{i=1}^n e_i^t, \wedge_{i=1}^n \mathcal{F}_i^t\}_{t \in T}$ is a martingale.

[12]Notice that $X_i$ is also a non-empty correspondence, since we assume that it contains the initial endowment, i.e., $e_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$.

**Remark 5.1.** Notice that the above assumptions may not guarantee the existence of a BWE when $\Omega$ is countable. On the other hand, if $\Omega$ is finite and $u_i(\omega, \cdot)$ is concave for all $i$ and $\omega$, from assumptions $(A.1)$, $(A.2)$ and $(A.3)$ it follows that in each period a BWE exists (see Theorem 4.1 in Balder-Yannelis (2009)).

**Remark 5.2.** From assumption (A.4) it follows that $e_i^t(\omega)$ converges to $e_i^*$ (recall the Martingale convergence Theorem), and $(A.5)$ implies that the total initial endowment also converges [13].

Before to prove the next Theorem we need the following Lemma.

**Lemma 5.3.** *Let $a_n^1, a_n^2, \ldots, a_n^d$ be $d$ sequences that converge respectively to $a^1, a^2, \ldots, a^d$. Then*

$$\lim_{n\to+\infty} \max_{1\le j\le d} a_n^j = \max_{1\le j\le d} \lim_{n\to+\infty} a_n^j.$$

**Proof:** Denote $b_n = \max_{1\le j\le d} a_n^j$ and notice that for all $n \in I\!N$, there exists $j \in \{1, \ldots, d\}$ such that $b_n = a_n^j$. We need to prove that

$$\lim_{n\to\infty} b_n = \max_{1\le j\le d} a^j.$$

Directly from hypothesis, i.e., for each $j \in \{1, \ldots, d\}$ the sequence $a_n^j$ converges to $a^j$, it follows that for each $j \in \{1, \ldots, d\}$,

$$\forall \epsilon > 0 \qquad \exists n_\epsilon^j \text{ such that } \qquad |a_n^j - a^j| < \epsilon \qquad \forall n > n_\epsilon^j.$$

Let $n_\epsilon = \max_{1\le j\le d} n_\epsilon^j$, then

$$\forall \epsilon > 0 \qquad \exists n_\epsilon \text{ such that } \qquad a^j - \epsilon < a_n^j < a^j + \epsilon \qquad \forall n > n_\epsilon \qquad \text{and} \qquad \forall j.$$

In particular for all $\epsilon > 0$, there exists $n_\epsilon$ such that for all $j \in \{1, \ldots, d\}$,

$$a_n^j < a^j + \epsilon \le \max_{1\le j\le d} a^j + \epsilon \qquad \forall n > n_\epsilon, \quad \text{thus}$$

$$\forall \epsilon > 0 \qquad \exists n_\epsilon \text{ such that } \qquad b_n = \max_{1\le j\le d} a_n^j < \max_{1\le j\le d} a^j + \epsilon \qquad \forall n > n_\epsilon. \tag{3}$$

[13] Similar assumptions were made in Koutsougeras-Yannelis (1999).

On the other hand, for all $\epsilon > 0$, there exists $n_\epsilon$ such that for all $j \in \{1, \ldots, d\}$,

$$\max_{1 \le j \le d} a_n^j \ge a_n^j > a^j - \epsilon \qquad \forall n > n_\epsilon, \quad \text{thus}$$

$$\forall \epsilon > 0 \qquad \exists n_\epsilon \text{ such that} \qquad b_n = \max_{1 \le j \le d} a_n^j > \max_{1 \le j \le d} a^j - \epsilon \qquad \forall n > n_\epsilon. \tag{4}$$

Hence from (3) and (4), it follows that

$$\lim_{n \to +\infty} \max_{1 \le j \le d} a_n^j = \max_{1 \le j \le d} \lim_{n \to +\infty} a_n^j.$$

□

We are now ready to state the following Theorem.

**Theorem 5.4.** *Let $\{\mathcal{E}^t : t \in T\}$ be a sequence of differential information economies satisfying assumptions $(A.0) - (A.4)$ and let $(p^t, x^t) \in \mathrm{BWE}(\mathcal{E}^t)$. Then, there exists a subsequence $(p^{t_n}, x^{t_n})$ which converges to $(p^*, x^*) \in \mathrm{BWE}(\mathcal{E}^*)$.*

**Proof:** Let $(p^t, x^t)$ be a sequence of BWE, we need to find a subsequence $(p^{t_n}, x^{t_n})$ that converges to $(p^*, x^*)$. Let $Z = \{(p^t, x^t) : t \in T\}$ where $(p^t, x^t) \in \mathrm{BWE}(\mathcal{E}^t)$ for each $t$. Let $L_\Delta$ be the space of all functions $f : \Omega \to \Delta$. Define $L_{P,X} = L_\Delta \times L_X$. We may take the random consumption set of each agent to be $X_i(\omega) = [0, \sup_t \sum_{i=1}^n e_i^t(\omega)]$. Such a set is clearly compact, convex and nonempty, therefore $L_X$ is compact. Since $L_\Delta$ is defined on a compact set $\Delta$, it is also compact. Therefore by Tychonoff's theorem, their product $L_{P,X}$ is compact. Thus, the closure of the set $Z$, $\bar{Z}$, is compact, and hence there exists a subsequence $\{(p^{t_n}, x^{t_n}) \in L_{P,X} : n = 1, 2, \ldots\}$ which converges to $(p^*, x^*)$.

We must show that $(p^*, x^*)$ is a Bayesian-Walrasian equilibrium for the limit symmetric information economy $\mathcal{E}^*$, i.e., $(p^*, x^*) \in \mathrm{BWE}(\mathcal{E}^*)$. It is clear that $p^*$ is a price system. Since for all $t$ and for all $i$, $\mathcal{F}_i^t \subseteq G^\infty$ and $x_i^t$ is $\mathcal{F}_i^t$-measurable, then $x_i^t$ is $G^\infty$-measurable for all $t$ and $i$ (recall $(A.0)$). Therefore, $x^* \in L_X^\infty$. Thus the conditions $(i^*)$ and $(ii^*)$ hold.

We want now to prove that condition $(iii^*)$ also holds. Suppose otherwise that $x^*$ violates $(iii^*)$, then there exist an agent $i$, a state $\omega^*$, and an allocation $y \in L_X^\infty$ such that

$$v_i(y_i | G^\infty)(\omega^*) > v_i(x_i^* | G^\infty)(\omega^*) \tag{5}$$

and

$$\hat{p}^*(\omega^*) \cdot y_i(\omega^*) \leq \hat{p}^*(\omega^*) \cdot e_i^*(\omega^*).$$

Notice that if $\omega^*$ is such that $\hat{p}^*(\omega^*) \cdot y_i(\omega^*) = \hat{p}^*(\omega^*) \cdot e_i^*(\omega^*)$, then since the conditional expected utility is continuous, from assumption $(A.3)$, we can find $\tilde{y}_i$ in the neighborhood of $y_i$ such that $v_i(\tilde{y}_i|G^\infty)(\omega^*) > v_i(x_i^*|G^\infty)(\omega^*)$ and $\hat{p}^*(\omega^*) \cdot \tilde{y}_i(\omega^*) < \hat{p}^*(\omega^*) \cdot e_i^*(\omega^*)$. Without loss of generality, let $\tilde{y}_i$ equal to $y_i$, then it follows from above that

$$\hat{p}^*(\omega^*) \cdot y_i(\omega^*) < \hat{p}^*(\omega^*) \cdot e_i^*(\omega^*). \tag{6}$$

For any $i$, $\mathcal{F}_i^{t_n}$ is an increasing sequence of private information, then for any time $t$, it is coarser than its limit $G^\infty$, and thus $E^{G^\infty}(\omega^*) \subseteq E^{\mathcal{F}_i^{t_n}}(\omega^*)$. Let us define $y_i^{t_n}$ by

$$y_i^{t_n}(\omega) = \begin{cases} y_i(\omega) & \text{for } \omega \in E^{G^\infty}(\omega^*) \\ x_i^{t_n}(\omega) & \text{for } \omega \notin E^{G^\infty}(\omega^*). \end{cases}$$

Then,

$$\begin{aligned} v_i(y_i^{t_n}|\mathcal{F}_i^{t_n})(\omega^*) &= \sum_{\omega \in E^{\mathcal{F}_i^{t_n}}(\omega^*)} u_i(\omega, y_i^{t_n}(\omega)) \frac{\mathbb{P}(\omega)}{\mathbb{P}\big(E^{\mathcal{F}_i^{t_n}}(\omega^*)\big)} \\ = \sum_{\omega \in E^{G^\infty}(\omega^*)} u_i(\omega, y_i^{t_n}(\omega)) \frac{\mathbb{P}(\omega)}{\mathbb{P}\big(E^{\mathcal{F}_i^{t_n}}(\omega^*)\big)} &+ \sum_{\omega \in E^{\mathcal{F}_i^{t_n}}(\omega^*) \setminus E^{G^\infty}(\omega^*)} u_i(\omega, y_i^{t_n}(\omega)) \frac{\mathbb{P}(\omega)}{\mathbb{P}\big(E^{\mathcal{F}_i^{t_n}}(\omega^*)\big)} \\ = \sum_{\omega \in E^{G^\infty}(\omega^*)} u_i(\omega, y_i(\omega)) \frac{\mathbb{P}(\omega)}{\mathbb{P}\big(E^{\mathcal{F}_i^{t_n}}(\omega^*)\big)} &+ \sum_{\omega \in E^{\mathcal{F}_i^{t_n}}(\omega^*) \setminus E^{G^\infty}(\omega^*)} u_i(\omega, x_i^{t_n}(\omega)) \frac{\mathbb{P}(\omega)}{\mathbb{P}\big(E^{\mathcal{F}_i^{t_n}}(\omega^*)\big)} \\ = v_i(x_i^{t_n}|\mathcal{F}_i^{t_n})(\omega^*) &+ \big[v_i(y_i|G^\infty)(\omega^*) - v_i(x_i^{t_n}|G^\infty)(\omega^*)\big] \frac{\mathbb{P}\big(E^{G^\infty}(\omega^*)\big)}{\mathbb{P}\big(E^{\mathcal{F}_i^{t_n}}(\omega^*)\big)}. \end{aligned}$$

Since $v_i(y_i|G^\infty)(\omega^*) > v_i(x_i^*|G^\infty)(\omega^*)$ (see (5)) and $x_i^{t_n}$ converges to $x_i^*$, by the continuity of the conditional expected utility $v_i( \cdot |G^\infty)(\omega^*)$, there exists large enough $t$ such that $v_i(y_i|G^\infty)(\omega^*) > v_i(x_i^{t_n}|G^\infty)(\omega^*)$. Therefore, we have

$$v_i(y_i^{t_n}|\mathcal{F}_i^{t_n})(\omega^*) > v_i(x_i^{t_n}|\mathcal{F}_i^{t_n})(\omega^*).$$

Then,

$$\hat{p}_i^{t_n}(\omega^*) \cdot y_i^{t_n}(\omega^*) > \hat{p}_i^{t_n}(\omega^*) \cdot e_i^{t_n}(\omega^*)$$

or equivalently by the definition of $y_i^{t_n}$ in the state $\omega^*$,

$$\hat{p}_i^{t_n}(\omega^*) \cdot y_i(\omega^*) > \hat{p}_i^{t_n}(\omega^*) \cdot e_i^{t_n}(\omega^*). \tag{7}$$

Notice that, since $E^{\mathcal{F}_i^t}(\omega^*) \supseteq E^{\mathcal{F}_i^{t+1}}(\omega^*) \supseteq E^{G^\infty}(\omega^*)$ for all $i$ and $t$, $\mathbb{P}(E^{\mathcal{F}_i^t}(\omega^*))$ converges to $\mathbb{P}(E^{G^\infty}(\omega^*))$, then $\mathbb{P}(\omega^*|\mathcal{F}_i^t)$ converges to $\mathbb{P}(\omega^*|G^\infty)$. Moreover, since $p^t$ converges to $p^*$, then $\hat{p}_i^t(\omega^*)$ converges to $\hat{p^*}(\omega^*)$ for all $i \in I$. Therefore, assumption $(A.4)$ and the condition (7) imply that

$$\hat{p}^*(\omega^*) \cdot y_i(\omega^*) \geq \hat{p}^*(\omega^*) \cdot e_i^*(\omega^*),$$

which contradicts (6). Hence, also the third condition $(iii^*)$ holds. To complete the proof, we must show that condition $(iv^*)$ holds as well.

By the definition of BWE$(\mathcal{E}^{t_n})$, for all $\omega \in \Omega$,

$$p^{t_n}(\omega) \cdot \sum_{i \in I}(x_i^{t_n}(\omega) - e_i^{t_n}(\omega)) = \max_{1 \leq h \leq \ell} \sum_{i \in I}(x_i^{t_n}(\omega) - e_i^{t_n}(\omega))_h,$$

by the continuity of the inner product and by Lemma 5.3 we have that

$$p^*(\omega) \cdot \sum_{i \in I}(x_i^*(\omega) - e_i^*(\omega)) = \max_{1 \leq h \leq \ell} \sum_{i \in I}(x_i^*(\omega) - e_i^*(\omega))_h.$$

Therefore, $(p^*, x^*)$ must be a Bayesian-Walrasian equilibrium of the limit symmetric information economy and hence by Proposition 2.2, it follows that $\sum_{i=1}^n x_i^*(\omega) = \sum_{i=1}^n e_i^*(\omega)$ for all $\omega \in \Omega$. This means that $(x^*, p^*) \in BWE(\mathcal{E}^*)$. □

**Corollary 5.5.** *If $\Omega$ is finite and $u_i(\omega, \cdot)$ is concave for all $i$ and $\omega$, then under assumptions $(A.0) - (A.4)$, there exists a limit symmetric Bayesian Walrasian equilibrium $(p^*, x^*)$ of the limit economy $\mathcal{E}^*$.*

**Proof:** Theorem 4.1 in Balder-Yannelis (2009) guarantees in each period $t$ the existence of a Bayesian Walrasian equilibrium $(x^t, p^t)$ for the economy $\mathcal{E}^t$. From Theorem 5.4 it follows that a limit symmetric Bayesian Walrasian equilibrium exists. □

The pair $(x^*, p^*)$ is trivially incentive compatible because in the economy $\mathcal{E}^*$ all agents have the same information.

Moreover a limit symmetric Bayesian-Walrasian Equilibrium of the economy $\mathcal{E}^*$ is also (interim) Pareto optimal as shown by the next proposition. We first recall the notion of an (interim) Pareto optimal allocation.

**Definition 5.6.** *An allocation $x \in L_X^\infty$ is said to be interim Pareto optimal if the following is not true: there exist $y \in L_X^\infty$ and a state $\omega \in \Omega$, such that*

$$\begin{array}{ll} (i) & v_i(y_i|G^\infty)(\omega) > v_i(x_i|G^\infty)(\omega) \quad for\ all\ i \in I \\ (ii) & \sum_{i\in I} y_i(\omega) = \sum_{i\in I} e_i^*(\omega). \end{array}$$

The proof of the Proposition below is standard.

**Proposition 5.7.** *Any limit symmetric BWE of the limit symmetric information economy $\mathcal{E}^*$ is (interim) Pareto optimal.*

# 6 Stability of the symmetric Bayesian-Walrasian equilibrium

Let us now show the converse of Theorem 5.4, i.e., any limit symmetric information BWE can be reached by a sequence of approximate BWE outcomes. We can view this result as a stability property of Bayesian Walrasian equilibrium, in the sense that we can always construct a route to reach the limit symmetric information BWE.

Before we state the result, we first define the notion of an approximate (or $\varepsilon$-) BWE where each agent optimizes the conditional expected utility within a small error $\varepsilon > 0$ in each state. Since we allow the possibility of making errors in maximization, agents are considered to be bounded rational.

**Definition 6.1.** *A pair $(p, x)$ is said to be an* $\varepsilon$-**Bayesian-Walrasian Equilibrium** ($\varepsilon$-**BWE**) *of the differential information exchange economy $\mathcal{E}$ if*

$$\begin{array}{ll} (i) & p : \Omega \to \triangle \ is\ a\ random\ price\ vector, \\ (ii) & x \in L_X \ is\ an\ allocation, \\ (iii) & for\ all\ i\ and\ for\ all\ \omega\ the\ consumption\ function \\ x_i & maximizes\ the\ conditional\ expected\ utility\ within\ \varepsilon > 0,\ i.e., \end{array}$$

$$\begin{array}{rcl} v_i(x_i|\mathcal{F}_i)(\omega) & \geq & max\{v_i(y_i|\mathcal{F}_i)(\omega) : \ y_i(\omega) \in \hat{B}_i(\omega, p)\} - \varepsilon \\ and \qquad x_i(\omega) & \in & \hat{B}_i(\omega, p), \end{array}$$

*(iv)* $p(\omega) \cdot \sum_{i \in I}(x_i(\omega) - e_i(\omega)) = \max_{1 \leq h \leq \ell} \sum_{i \in I}(x_i(\omega) - e_i(\omega))_h$ *for all* $\omega \in \Omega$.

We now show that a BWE in the limit can be reached by a sequence of approximate BWE allocations. In other words, given a price-consumption pair which is a BWE for the limit symmetric economy, and obviously exact feasibility holds, we can construct a sequence of $\varepsilon$-BWE that converges to it. Notice that in the sequence, agents maximize approximately their interim utility subject to their estimate budget set and also market clearing does not hold. Therefore, we can conclude that the BWE is stable.

**Theorem 6.2.** *Let* $\{\mathcal{E}^t : t \in T\}$ *be a sequence of differential information exchange economies satisfying assumptions (A.0)-(A.5) and let* $(p^*, x^*) \in \mathrm{BWE}(\mathcal{E}^*)$. *Then, for any* $\varepsilon > 0$, *there exists a sequence* $(p^t, x^t) \in \mathrm{BWE}_\varepsilon(\mathcal{E}^t)$ *such that* $(p^t, x^t)$ *converges to* $(p^*, x^*)$.

**Proof:** Let $x_i^t = \mathbf{E}[x_i^* | \wedge_{j=1}^n \mathcal{F}_j^t]$ and $p^t = \mathbf{E}[p^* | \wedge_{j=1}^n \mathcal{F}_j^t]$. Then, since $\{\wedge_{j=1}^n \mathcal{F}_j^t\}$ is monotone increasing, we have

$$x_i^t = \mathbf{E}[x_i^* | \wedge_{j=1}^n \mathcal{F}_j^t] = \mathbf{E}[\mathbf{E}[x_i^* | \wedge_{j=1}^n \mathcal{F}_j^{t+1}] | \wedge_{j=1}^n \mathcal{F}_j^t] = \mathbf{E}[x_i^{t+1} | \wedge_{j=1}^n \mathcal{F}_j^t],$$

$$p^t = \mathbf{E}[p^* | \wedge_{j=1}^n \mathcal{F}_j^t] = \mathbf{E}[\mathbf{E}[p^* | \wedge_{j=1}^n \mathcal{F}_j^{t+1}] | \wedge_{j=1}^n \mathcal{F}_j^t] = \mathbf{E}[p^{t+1} | \wedge_{j=1}^n \mathcal{F}_j^t].$$

Hence, by construction $\{(p^t, x^t), \wedge_{j=1}^n \mathcal{F}_j^t\}$ is a martingale and by the martingale convergence theorem $(p^t, x^t)$ converges to $(p^*, x^*) \in \mathrm{BWE}(\mathcal{E}^*)$.

To complete the proof, we must show that $(p^t, x^t)$ lies in $\mathrm{BWE}_\varepsilon(\mathcal{E}^t)$ for all but finite $t$'s. That is, the set

$$K = \{t \in T \ : \ (p^t, x^t) \notin BWE_\varepsilon(\mathcal{E}^t)\}$$

should be finite. Suppose, by way of contradiction, $(p^t, x^t) \notin \mathrm{BWE}_\varepsilon(\mathcal{E}^t)$ for infinitely many $t$'s.

By using the feasibility $(iv^*)$ of $x^*$ and assumption (A.5), we have

$$\begin{aligned}
\sum_{i=1}^{n} x_i^t &= \sum_{i=1}^{n} \mathbf{E}[x_i^* | \wedge_{j=1}^{n} \mathcal{F}_j^t] = \mathbf{E}[\sum_{i=1}^{n} x_i^* | \wedge_{j=1}^{n} \mathcal{F}_j^t] \\
&= \mathbf{E}[\sum_{i=1}^{n} e_i^* | \wedge_{j=1}^{n} \mathcal{F}_j^t] = \sum_{i=1}^{n} \mathbf{E}[e_i^* | \wedge_{j=1}^{n} \mathcal{F}_j^t] \\
&= \sum_{i=1}^{n} e_i^t \qquad \text{by } (A.5).
\end{aligned}$$

Thus, for all $\omega \in \Omega$ condition $(iv)$ is satisfied. In fact,

$$p^t(\omega) \cdot \sum_{i \in I} (x_i^t(\omega) - e_i^t(\omega)) = 0 = \max_{1 \leq h \leq \ell} \sum_{i \in I} (x_i^t(\omega) - e_i^t(\omega))_h.$$

Since $x_i^t$ is defined as the conditional expectation on $\wedge_{j=1}^{n} \mathcal{F}_j^t$, it is $\mathcal{F}_i^t$-measurable, and hence the conditions $(i)$ and $(ii)$ are satisfied.

Then, from the above assumption, there exist an agent $i$ and a state $\omega^* \in \Omega$ such that, for infinitely many $t$'s,

$$x_i^t(\omega^*) \notin \operatorname{argmax}_{y_i^t \in \hat{B}_i(p^t, \omega^*)} v_i(y_i^t | \mathcal{F}_i^t)(\omega^*) - \varepsilon. \tag{8}$$

Therefore, there exists a sequence $y_i^t \in L_{X_i^t}$ such that

$$\begin{aligned}
v_i(y_i^t | \mathcal{F}_i^t)(\omega^*) &> v_i(x_i^t | \mathcal{F}_i^t)(\omega^*) + \varepsilon, \text{ and} \\
\hat{p}_i^t(\omega^*) \cdot y_i^t(\omega^*) &\leq \hat{p}_i^t(\omega^*) \cdot e_i^t(\omega^*).
\end{aligned} \tag{9}$$

Since $y_i^t \in L_{X_i^t}$, which is a compact set as noted in the proof of the theorem 5.4, then we can find a subsequence $\{y_i^{t_n}\}$ of $\{y_i^t\}$ which converges to $y_i$.

Since for all[14] $i$, $\hat{p}_i^{t_n}$ converges to $\hat{p}^*$, $y_i^{t_n}$ to $y_i$, and $e_i^{t_n}$ to $e_i^*$ (recall assumption $(A.4)$), then from (9) it follows that

$$\hat{p}^*(\omega^*) \cdot y_i(\omega^*) \leq \hat{p}^*(\omega^*) \cdot e_i^*(\omega^*).$$

[14]See the proof of theorem 5.4.

Moreover taking limits in the expression below

$$v_i(y_i^{t_n}|\mathcal{F}_i^{t_n})(\omega^*) > v_i(x_i^{t_n}|\mathcal{F}_i^{t_n})(\omega^*) + \varepsilon,$$

it follows that

$$v_i(y_i|G^\infty)(\omega^*) \geq v_i(x_i^*|G^\infty)(\omega^*) + \varepsilon > v_i(x_i^*|G^\infty)(\omega^*),$$

which is a contradiction to the fact that $(p^*, x^*) \in \mathrm{BWE}(\mathcal{E}^*)$. Thus $K$ must be a finite set, i.e., $(p^t, x^t) \notin BWE_\epsilon(\mathcal{E}^t)$ for only finite number of $t$'s. Therefore, there exists a sequence $(p^t, x^t) \in BWE_\epsilon(\mathcal{E}^t)$ that converges to $(p^*, x^*) \in BWE(\mathcal{E}^*)$, as it was to be shown. □

# 7 Conclusions

We introduced the concept of a symmetric BWE. This concept is free of the undesirable property of the asymmetric BWE which does not allow for market clearing. Most importantly, the symmetric BWE can provide a dynamic foundation for the asymmetric one since by repetition, agents observe the equilibrium outcomes, learn the information of the others and in the limit symmetric information prevails. Thus, despite the fact that from period to period the agent's price estimates are not accurate and market clearing doesn't hold, nonetheless, in the limit the correct estimates will prevail and market clearing will hold. We also showed that the converse is true, i.e., given a symmetric BWE, we can always construct a sequence of approximate BWE which converges to the symmetric BWE.

The concept of the symmetric BWE not only it is Pareto optimal and it exists universally (compare with the REE which exists only generically), but also, it provides a new interim way to explain the Arrow-Debreu state contingent contacts. This new way has a statistical foundation as it is based on price estimates and on an estimated budget set, which makes our modeling attractive to empirical analysis.

# References

[1] Allen, B.: Generic existence of completely revealing equilibria with uncertainty, when prices convey information. *Econometrica* **49**, 1173-1199 (1981)

[2] Balder, E.J., Yannelis, N.C.: Bayesian-Walrasian equilibria: beyond the rational expectations equilibrium. *Economic Theory* **38**, 385-397 (2009). Doi 10.2007/s00199-007-0288-6 1)

[3] Glycopantis, D., Muir, A., Yannelis, N.C.: Non-implementation of rational expectations as a pefect Bayesian equilibrium. *Economic Theory* **26**, 765-791 (2005)

[4] Glycopantis, D., Muir, A., Yannelis, N.C.: On non-revealing rational expectations equilibrium. *Economic Theory* **38**, 351-369 (2009)

[5] Koutsougeras, L., Yannelis, N.C.: Bounded rational learning in differential information economies: Core and Value, *Journal of Mathematical Economics* **31**, 373-391 (1999)

[6] Kreps, M.D.: A note on the fulfilled expectations equilibria, *Journal of Economic Theory* **14**, 32-43 (1977)

[7] Peleg, B., Yaari, M.E.,: Markets with countable many commodities, *International Economic Review* **11**, 369-377 (1970)

[8] Radner, R.: Rational expectation equilibrium: generic existence and information revealed by prices, *Econometrica* **47**, 655-678 (1979)