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**Economics
Discussion Paper Series
EDP-0918**

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September 2009

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September, 2009

Abstract

Consider a market in which there are n firms producing a homogeneous good. We make \bar{t} observations of this market and each observation includes firms' prices, outputs and profits. We provide a set of necessary and sufficient conditions for each observation to be rationalized as a pure strategy Bertrand equilibrium with each firm having the same cost function across the observations, but different market demands across observations. Moreover, we show that when firms' profits are unobservable the Bertrand model can still be refuted.

JEL Classification: C60, C72, D43

Keywords: Testable implications; rationalizability; Bertrand competition

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1 Introduction

A basic requirement of a scientific theory is that it produces propositions which can be tested against empirical observations. In this paper we examine the testable implications of the canonical Bertrand oligopoly model. The question we aim to answer is conceptually very simple: given a set of observations of oligopoly competition what restrictions must the observations satisfy if they are to be explained as the firms playing a Bertrand equilibrium in each observation. Despite its simplicity, the question is an important one. Being able to define the Bertrand model in terms of prices, outputs and profits, permits greater variety than when we state specific primitives (cost functions and market demand) and analyze the implications. Furthermore, it means we can define the Bertrand model in terms of what is observable, rather than stating primitives which cannot be directly observed.

1.1 Related Literature

In economics, theories of individual choice are well developed, but the testable implications of these theories are less well developed.¹ One area where this is not true is consumer theory. The theory of consumer choice, derived from the maximization of a rational preference relation has well-known testable implications. If we observe a set of prices, wealth levels and Walrasian demands which fail to satisfy the weak axiom of revealed preference (WARP) then there does not exist a rational preference relation, the maximization of which, would produce the observed demands. However, it is also known that if a set of observations satisfy the weak axiom then this is not sufficient for the existence of a rationalizing preference relation.² Mas-Colell et al. (1995, p.35) present an example, by John R. Hicks, of a three-good economy where individual Walrasian demands satisfy the weak axiom but exhibit intransitive choice.

Afriat (1967) analyzed the problem of finding a rational preference relation when we

¹A survey of the recent developments of the testable implications of economic models is provided by Carvajal et al. (2004).

²Although observations which satisfy the weak axiom also satisfy the weak weak axiom (WWA) and can be rationalized by a complete, but not necessarily transitive, preference relation (Jerison and Quah, 2008).

make a finite set of observations of prices and consumption choices. A number of different equivalent conditions can be stated which are sufficient for the existence of a rational preference relation, and it was shown that the utility function could be constructed as a piecewise-affine function. These conditions are equivalent to the strong axiom of revealed preference (SARP) which is a strengthening of the weak axiom to rule out intransitive choices. The discovery of the strong axiom as necessary and sufficient for the existence of a rational preference relation was an important result, and it might be expected that this would be repeated for other economic models.

The canonical general equilibrium model was thought to have no testable implications because of the Debreu-Mantel-Sonnenschein (DMS) results. These stated that given any continuous function, defined on a compact subset of the normalized price space, which is homogeneous of degree zero and satisfies Walras law, we can define an economy, with at least as many consumers as goods, for which the function is the aggregate excess demand of the economy. The results suggested that other than the properties of continuity, homogeneity of degree zero and Walras law the aggregate excess demand of an economy was arbitrary. However, recent research has suggested that these negative conclusions are not the full story. Balasko (2009, p.14) notes that if one examines the aggregate excess demand for an economy on the full normalized price space, instead of a compact subset, then the excess demand functions have well-defined properties, and are not in any sense arbitrary. Specifically, the aggregate excess demands are of topological degree one whereas an arbitrary function can have a degree equal to any integer.

A seminal paper by Brown and Matzkin (1996) examined whether the general equilibrium model has any testable implications. Instead of looking for testable restrictions on the aggregate excess demand functions they assumed we make a finite set of observations on the equilibrium manifold of a pure exchange economy. That is, the graph of the Walrasian correspondence (mapping the endowment space into equilibrium price vector). They constructed a set of polynomial inequalities, similar to the Afriat inequalities, together with market clearing conditions, and showed that the model had testable implications. Moreover, they exhibited simple examples of exchange economies which could not be rationalized as Walrasian equilibria. In addition, they provided testable implications for economies in which agents have homothetic preferences and for Robinson

Crusoe economies. Carvajal (2004) looked for testable implications of general equilibrium with random preferences. If we make a set of observations of endowments, and for each set of endowments we observe a probability measure over the price distribution, then the observations are said to be rationalizable if there exists a probability measure over the set of possible preferences which induces the observed measure over prices via Walrasian equilibrium. In this case, the model has testable implications and examples can be constructed, similar to the pure exchange economies presented by Brown and Matzkin, which cannot be rationalized by a probability measure over preferences.

Bachmann (2006) studied the core of an exchange economy. It was assumed we make a finite set of observations of initial endowments and final allocations. The observations are rationalizable if there exists a set of rational preference relations, one for each agent, such that the final allocation provides a least as high a utility as the initial endowment, and no proper coalition, or the grand coalition, can improve upon the final allocations. The core was found to be characterized by a set of polynomial inequalities, and these inequalities could be violated which showed that the core is a falsifiable concept.

The first paper to examine the testable implications of game theoretical models was Sprumont (2000). This paper analyzed normal form games with finite action spaces, and looked at two different rationalization concepts: Nash and Pareto rationalization. It is assumed we make observations of players making choices from subsets of their strategy sets. Then the observations are said to be Nash rationalizable if there exist rational preference relations over all possible outcomes such that the observed choices are Nash equilibria in each observation. The observations are Pareto rationalizable if the outcomes are Pareto optimal. The paper identified necessary and sufficient conditions for a set of observations to be rationalized as a Nash equilibrium, and showed that Nash rationalizability implies Pareto rationalizability (although the converse is not true).

Recently, Carvajal and Quah (2008)³ have extended the research agenda on the testable implications of economic models by analyzing the Cournot oligopoly model. They assumed that we make a finite set of observations of oligopoly competition, and each observation includes the market price, firms' outputs and profits. They provided a set of necessary and sufficient conditions for the observations to be rationalized as a Cournot

³Herein referred to as simply CQ.

equilibrium. Generally speaking, a set of oligopoly observations is Cournot rationalizable if there exist cost functions for each firm, which are unchanged across the observations, and a set of market demands, such that the observed outcomes are a Cournot equilibrium.

1.2 Overview

As noted at the beginning, in this paper we aim to extend the testable implications of oligopoly models by analyzing the Bertrand model. It is assumed that we make a finite set of observations of oligopoly competition, and each observation includes firms' prices (possibly different), outputs and profits. We then identify the conditions which this data set is to satisfy if each observation is to be rationalized as a Bertrand equilibrium. We define Bertrand rationalizability in an analogous way to CQ: the observations are Bertrand rationalizable if there exist cost functions for each firm, which are unchanged across the observations, and market demands for each observation, such that the observed prices, in each observation, are a Bertrand equilibrium.

The main result in this paper (Theorem 1) is a set of necessary and sufficient conditions for a data set to be rationalized as a pure strategy Bertrand equilibrium. We also demonstrate, by a simple example, that the Bertrand model can be refuted even when firms' costs can not be inferred from the data set. The rest of the paper is organized as follows. The next section sets out the Bertrand model, defines Bertrand rationalizability and presents the main results. In section 3 we present some example observations and apply the results obtained. The example data sets demonstrate that the Bertrand model is refutable: there exist data sets which cannot be rationalized as Bertrand equilibria. In section 4 we discuss the results obtained and consider a possible extension of the work presented here.

2 Rationalizability- The Bertrand Model

2.1 Bertrand Equilibrium

Before addressing the problem of rationalizing a set of observations, we define precisely what is meant by a Bertrand equilibrium. The interested reader can find more detailed

discussions of the Bertrand model in Tirole (1988) and Vives (1999). Consider a market in which there are $N = \{1, \dots, n\}$, $n \geq 2$, firms which produce a homogeneous good. The firms compete by simultaneously and independently setting prices. That is to say, each firm chooses a $P_i \in \mathfrak{R}_+$. The firms commit to supplying all the demand forthcoming at any price. This means that there is no rationing of consumers, and output is demand-determined. Each firm has a cost function $C_i(Q)$ which is strictly increasing and satisfies $C_i(0) = 0$. The last condition on the cost function rules out the existence of sunk costs, so that no production is costless.⁴ The market demand is given by $D(P)$ and is strictly decreasing in price when output is strictly positive. The profit accruing to the firms can be summarized as follows. If any firm is undercut it receives zero demand and its profit is zero. If a firm has the unique minimum price then it receives all the demand and its profit is $P_i D(P_i) - C_i(D(P_i))$. If a firm ties with other firms at the lowest price then a sharing rule describes how the market output is shared. The conventional sharing rule is the equal sharing rule where firms tying at the minimum price share the demand equally. If firm i ties at the minimum price, P_i , with $m - 1$ other firms, then it receives $\frac{1}{m} D(P_i)$ demand and makes profit of $\frac{1}{m} P_i D(P_i) - C_i(\frac{1}{m} D(P_i))$. Given a vector of prices (P_i, P_{-i}) the payoff to firm i can be summarized as:

$$\pi_i(P_i, P_{-i}) = \begin{cases} P_i D(P_i) - C_i(D(P_i)), & \text{if } P_i < P_j \ \forall j \neq i; \\ \frac{1}{m} P_i D(P_i) - C_i(\frac{1}{m} D(P_i)) & \text{if } i \text{ ties with } m - 1 \text{ firms at the lowest price;} \\ 0 & \text{if } P_i > P_j \text{ for some } j. \end{cases} \quad (1)$$

Definition 1 *A pure strategy Bertrand equilibrium is a Nash equilibrium of the game with payoffs defined by equation (1). That is, a vector of prices (P_i^*, P_{-i}^*) such that $\pi_i(P_i^*, P_{-i}^*) \geq \pi_i(P_i, P_{-i}^*)$ for all $P_i \in \mathfrak{R}_+$ and $i \in N$.*

⁴A comprehensive analysis of equilibrium existence in the Bertrand model with sunk costs has been provided by Coloma and Saporiti (2008). However, as this is not central to the problem addressed here we do not pursue this issue.

2.2 Bertrand Rationalization

Now suppose we do not know the economic primitives (cost functions and market demand), but instead we observe the outcome of oligopoly competition in a homogenous product market. There is a set of firms $N = \{1, \dots, n\}$, $n \geq 2$, and a set of observations $T = \{1, \dots, \bar{t}\}$. In each period we observe firms' prices, outputs and profits. We can summarize the data set as $(P_{it}, Q_{it}, \Pi_{it})_{i \in N, t \in T}$. Define the following $P_t^* = \min_{i \in N} P_{it}$, $Q_t^* = \sum_{i \in N} Q_{it}$ and $A_t = \{i \in N : P_{it} = P_t^*\}$. In words, P_t^* is the minimum price in observation t , Q_t^* is the aggregate output in observation t , and A_t is the set of firms which tie at the minimum price in observation t . Given that the possible set of observations of oligopoly competition could be extremely varied we shall focus upon data sets which satisfy some basic requirements. We shall refer to observations which satisfy these requirements as being generic homogeneous-good market data sets. The next definition presents the formal requirements.

Definition 2 *A set of oligopoly observations $(P_{it}, Q_{it}, \Pi_{it})_{i \in N, t \in T}$ is a generic homogeneous-good market data set if it satisfies the following conditions:*

- i) $P_{it} > 0$, $Q_{it} \geq 0$, $\Pi_{it} \geq 0$, $P_{it}Q_{it} \geq \Pi_{it}$ and $Q_{it} \neq Q_{it'}$ whenever $t \neq t'$.*
- ii) If $P_{it} > P_{jt}$, for some $j \neq i$, then $Q_{it} = 0$.*
- iii) If $P_{it} = P_{jt} = P_t^*$ then $Q_{it} = Q_{jt}$.*
- iv) $|A_t| \geq 2$.*

The first part of the definition states that prices are strictly positive, outputs and profits are non-negative, revenue is greater than or equal to profit, and firms' outputs are different in each observation. The requirements in (ii)-(iii) ensure that the observations are consistent with firms competing in a homogeneous good market where consumers always prefer to buy from the firms with the minimum price and firms tying at the minimum price receive an equal share of the market demand. The condition in (iv) means the data set is such that at least two firms tie at the minimum price in each observation. This rules out atypical observations where we may observe some firm acting as a monopolist. It should be clear from the definition of the Bertrand model that any data set not satisfying at least (ii) and (iii) could not be explained by the Bertrand model. Although we can not directly observe firms' cost functions we can infer costs at the observed output levels

by calculating $C_{it} = P_{it}Q_{it} - \Pi_{it}$.

The data set will be said to be *Bertrand rationalizable* if there exist smooth, strictly increasing cost functions, which are unchanged across the observations, and a smooth, strictly decreasing market demand for each observation, which are consistent with the observed prices, outputs and profits. Moreover, given the cost and demand functions, the observed prices constitute a pure strategy Bertrand equilibrium in each observation. The next definition formalizes this notion.

Definition 3 *The set of observations $(P_{it}, Q_{it}, \Pi_{it})_{i \in N, t \in T}$ is Bertrand rationalizable if there exist smooth functions, $\bar{C}_i(x)$ for each $i \in N$, $\bar{D}_t(x)$ for each $t \in T$ such that:*

- i) $\bar{C}_i(0) = 0$ and $\bar{C}'_i(x) > 0$ for all $x \geq 0$.*
- ii) $\bar{D}_t(x) \geq 0$ and $\bar{D}'_t(x) \leq 0$ for all $x \geq 0$ with the latter inequality holding strictly whenever $\bar{D}_t(x) > 0$.*
- iii) $\bar{C}_i(Q_{it}) = C_{it}$ and $\bar{D}_t(P_t^*) = Q_t^*$.*
- iv) The set of prices (P_{1t}, \dots, P_{nt}) is a Bertrand equilibrium in pure strategies for each $t \in T$.*

We shall introduce some notation which helps organize the observations. Define $R_i(t) = \{t' \in T : Q_{it'} \geq Q_{it}^*\}$ and $r_i(t) = \{t' \in R_i(t) : Q_{it'} < Q_{it} \forall t' \in R_i(t)\}$. The set $R_i(t)$ contains the observations when the output of firm i is greater than, or equal, to the *aggregate* output in observation t and $r_i(t)$ is the observation belonging to $R_i(t)$ when the output of firm i is the minimum of its output levels corresponding to observations in $R_i(t)$.

Define $S_i(t) = \{t' \in T : Q_{it'} < Q_{it}\}$ and $s_i(t) = \{t' \in S_i(t) : Q_{it'} > Q_{it} \forall t' \in S_i(t)\}$. The set $S_i(t)$ contains the observations when the output of firm i is strictly less than its output in observation t , and $s_i(t)$ is the observation the output of firm i is the maximum of its output levels in $S_i(t)$.

Define $\hat{Q}_t = Q_t^*/(|A_t| + 1)$. In words, \hat{Q}_t is the aggregate market output in observation t divided by the cardinality of A_t plus one. Let $M_i(t) = \{t' \in T : Q_{it'} \geq \hat{Q}_t\}$ and $m_i(t) = \{t' \in M_i(t) : Q_{it'} < Q_{it} \forall t' \in M_i(t)\}$. The set $M_i(t)$ is the set of observations when the output of firm i is greater than or equal to \hat{Q}_t , and $m_i(t)$ is the observation which minimizes output across observations in $M_i(t)$.

We now introduce three conditions which collectively will characterize the set of ob-

servations which can be rationalized by the Bertrand model.

Definition 4 *The set of observations satisfy the increasing cost condition (ICC) if, whenever defined, $C_{it} - C_{is_i(t)} > 0$.*

The economic interpretation of the increasing cost condition is straightforward: it states that whenever we observe firms produce higher outputs then their costs increase. It should be clear that any data set which did not satisfy this condition could not be Bertrand rationalized because it would not be possible to construct a cost function which explains the observe costs and is strictly increasing. Hence, ICC is necessary for Bertrand rationalization.

Definition 5 *The set of observations satisfy the monopoly deviation condition (MDC) if, whenever defined, $P_t^*Q_{it} - C_{it} \geq P_t^*Q_t^* - C_{ir_i(t)}$ with the inequality holding strictly whenever $Q_{ir_i(t)} > Q_t^*$.*

The monopoly deviation condition states that the observed profits of the firms are no less than the profits which they could obtain from supplying the entire market at the existing price and incurring a cost at least as large as that required to meet the market demand at the minimum price. The economic intuition is that when the condition is satisfied we can construct cost and demand functions such that no firm has a profitable deviation by undercutting the market. The condition rules out potential monopoly deviations. If this condition were not satisfied then some firm would have a profitable deviation by slightly undercutting the existing market price. Therefore, MDC is necessary for Bertrand rationalization.

Definition 6 *The set of observations satisfy the tie deviation condition (TDC) if, whenever defined, $P_t^*\hat{Q}_t - C_{im_i(t)} \leq 0$ for all $i \in N \setminus A_t$ with the inequality holding strictly whenever $Q_{im_i(t)} > \hat{Q}_t$.*

The tie deviation condition states that whenever a firm does not tie at the minimum price then the profit which it could achieve from tying at the minimum price, and incurring a cost at least as large as that required to meet the demand forthcoming, results in profit which is no greater than zero. If the condition is satisfied then we can construct cost and demand functions such that firms not tying at the minimum price have no profitable deviation by joining the price tie at the minimum price. If the condition is not satisfied then for any cost and demand functions we construct, some firm would always

have a profitable deviation by joining a price tie. Hence, TDC is necessary for Bertrand rationalization.

To recall, the problem we wanted to address was whether the Bertrand equilibrium could be stated in terms of prices, outputs and profits only. We now present our main result which is a necessary and sufficient condition for a set generic observations to be rationalized as pure strategy Bertrand equilibrium.

Theorem 1 *A generic homogeneous-good market data set $(P_{it}, Q_{it}, \Pi_{it})_{i \in N, t \in T}$ is Bertrand rationalizable if, and only if, it satisfies ICC, MDC and TDC.*

Proof. See the Appendix. ■

2.3 Refutability when Costs are Unobservable

A further question which is relevant is how much information is needed to be able to refute the Bertrand model. In the next section we present some simple examples of data sets which illustrate that when we observe firms' prices, outputs and profits the Bertrand model is refutable. That is to say, there exist data sets which cannot be Bertrand rationalized. However, the requirement that we observe firms' prices, outputs and profits is quite demanding. Suppose we cannot observe the profits of the firms. Then we cannot infer firms' costs and the data set which we observe reduces to $(P_{it}, Q_{it})_{i \in N, t \in T}$. Given that we only observe market prices and outputs is the Bertrand model refutable? That is, can any data set $(P_{it}, Q_{it})_{i \in N, t \in T}$ be Bertrand rationalized?

Proposition 1 *The set of observations $(P_{it}, Q_{it})_{i \in N, t \in T}$ can be refuted as being Bertrand rationalizable.*

Proof. We shall show that there is an example data set which cannot be Bertrand rationalized despite not being able to observe firms' costs. Consider the following data set of three firms ($n = 3$) and two observations ($\bar{t} = 2$). As above, the first number is the price, the second is output. In the first observation, firms 1 and 2 serve the market

	1	2	3
t=1	(1, 3)	(1, 3)	(2, 0)
t=2	($\frac{1}{2}$, 2)	(1, 0)	($\frac{1}{2}$, 2)

and firm 3 sets a strictly higher price. In the second observation, firms 1 and 3 serve the market and firm 2 sets a strictly higher price. The profit of firm 3 in observation 2 is given by $\Pi_{32} = 1 - C_3(2)$. In the first observation firm 3 must not have a profitable deviation by joining the price tie at the minimum price. The profit which firm 3 would obtain by joining the price tie is $\Pi_{31} = 2 - C_3(2)$ (firms share the market demand equally). However, for any cost function we construct for firm 3 we must have $\Pi_{31} = 2 - C_3(2) > \Pi_{32} = 1 - C_3(2)$. Which means firm 3 would always strictly prefer to join the price tie in observation one, and which contradicts the observed prices being a Bertrand equilibrium. ■

The example used to prove Proposition 1 can be explained in a standard “revealed preference” manner: in observation one, $-C_3(0)$ is revealed at least as good as $\Pi_{31} = 2 - C_3(2)$ for firm 3. In observation two, $\Pi_{32} = 1 - C_3(2)$ is revealed at least as good as $-C_3(0)$ for firm 3. However, $\Pi_{31} = 2 - C_3(2) > \Pi_{32} = 1 - C_3(2)$ means that firm 3’s price choices are “revealed” incompatible with Bertrand equilibrium.

Providing a complete characterization of which data sets, consisting of just prices and quantities, can be Bertrand rationalized remains an open and interesting question for future research.

3 Example Data Sets

Having presented the main results we present some example data sets to which we apply the conditions. We focus on simple examples of duopolies ($n = 2$) with two observations ($\bar{t} = 2$). As presented above the numbers represent prices, outputs and profits respectively. We also note that some data sets which cannot be Bertrand rationalized can be rationalized by the Cournot model, and *vice versa*.⁵ Therefore, both models are able to explain a greater variety of observations than each individually.

	1	2
t=1	$(1, 1, \frac{1}{2})$	$(1, 1, \frac{1}{2})$
t=2	$(2, 2, 2)$	$(2, 2, 2)$

Table 1: Example 1

3.1 Example 1

Consider the first example data set. As both firms set the same price in each observation the tie deviation condition is trivially satisfied, and all that remains to be checked is the the increasing cost condition and the monopoly deviation condition are satisfied. As the observations are symmetric we need only check one firm's observations. In observation one firm 1 incurs a cost of $C_{11} = (1)(1) - \frac{1}{2} = \frac{1}{2}$. In observation two, firm 1 produces higher output and incurs a cost of $C_{12} = (2)(2) - 2 = 2$ and the increasing cost condition is satisfied. In observation two, firm 1 produces an output equal to the aggregate output in observation one so the monopoly deviation condition requires $\frac{1}{2} \geq (1)(2) - 2 = 0$ which is satisfied. As all the conditions are satisfied we know the observations can be Bertrand rationalized.

To see formally that we can construct market demands and cost functions which explain each observation as a pure strategy Bertrand equilibrium consider the following: suppose each firm's cost function is given by $C(Q) = \frac{1}{2}Q^2$ this is smooth, strictly increasing, and satisfies $C(0) = 0$, $C(1) = \frac{1}{2}$ and $C(2) = 2$. Suppose that the market demand in observation one is the piecewise-affine function $\bar{D}_1(P_1) = \max\{0, 4 - 2P_1\}$. This is smooth,⁶ strictly decreasing and satisfies $\bar{D}_1(1) = 2$. Given these market primitives the monopoly profit of firms is given by:

$$\pi(P) = P(4 - 2P) - \frac{1}{2}(4 - 2P)^2$$

⁵The reader is referred to the paper by CQ for more details on the Cournot model, but essentially, the condition which has to be checked for Cournot rationalizability, in addition to ICC, is what CQ describe as the "discrete marginal condition". This condition requires that $P_t^*Q_{it'} - C_{it'} < P_t^*Q_{it} - C_{it}$ for all $t' \in S_i(t)$.

⁶The kink in the function which occurs when quantity demanded becomes zero could be smoothed out on a small interval without affecting the analysis.

The tied profit which the firms would obtain at any price is given by:

$$\pi(P, 2) = \frac{1}{2}P(4 - 2P) - \frac{1}{8}(4 - 2P)^2$$

In Figure 1 we show the graphs of these two functions. The dashed function is the tied profit function and the continuous function is the monopoly profit. It can be seen that the market price in observation one, which is $P = 1$, is a Bertrand equilibrium given the constructed functions because the tied profit is above the monopoly profit so that no firm can profitably undercut the market. Now the variation in the data set has to be

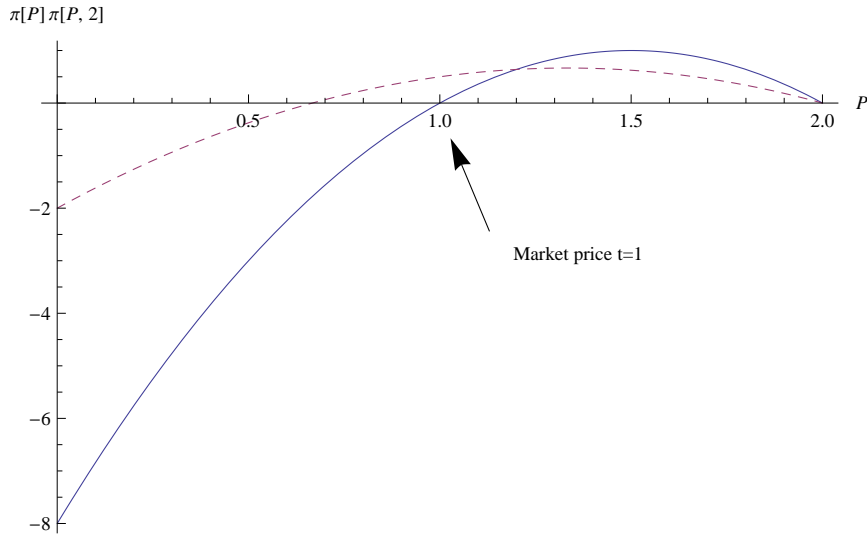


Figure 1: Bertrand rationalization of Example 1 t=1

explained by changes in the market demand. Suppose that in observation two the market demand is again a piecewise-affine function given by $\bar{D}_2(P_2) = \max\{0, 8 - 2P_2\}$. This is smooth, strictly decreasing and satisfies $\bar{D}_2(2) = 4$. Given that the firms' cost functions are fixed across the observations the monopoly profit is then given by:

$$\pi(P) = P(8 - 2P) - \frac{1}{2}(8 - 2P)^2$$

The tied profit is then given by:

$$\pi(P, 2) = \frac{1}{2}P(8 - 2P) - \frac{1}{8}(8 - 2P)^2$$

In Figure 2 we show the graphs of these two functions. It can be seen that the change in the market demand explains the variation in the data as the market price in observation two, which is $P = 2$, is now a Bertrand equilibrium where undercutting to obtain monopoly profits is not a profitable deviation.

It is worth noting that this example can also be rationalized by the Cournot model as it satisfies the discrete marginal condition. Therefore, the observations could be explained by firms choosing prices with a commitment to supply all the demand forthcoming, or by choosing quantities and letting the market determine the price.

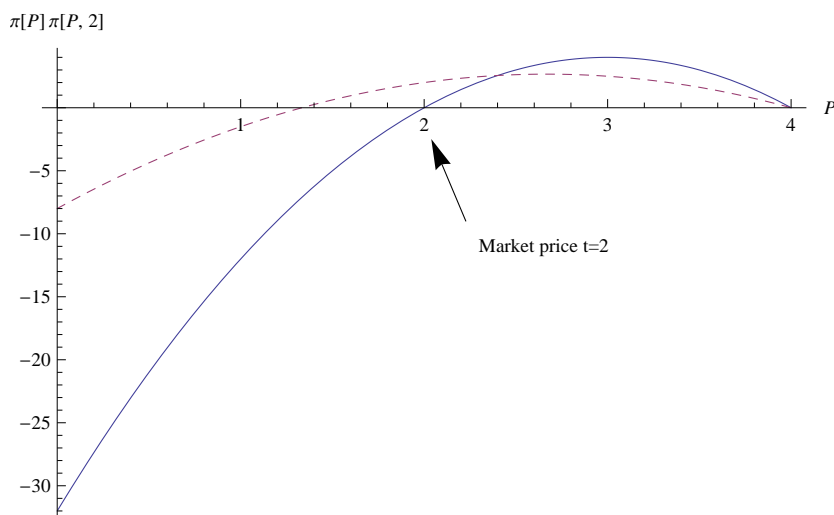


Figure 2: Bertrand rationalization of Example 1 t=2

3.2 Example 2

Now consider the data set in Example 2. As both firms tie in each observation the tie deviation condition is satisfied. All that remains is to check the increasing cost condition and the monopoly deviation condition. In observation one, firm 1 incurs a cost of $C_{11} = (1)(1) - \frac{1}{2} = \frac{1}{2}$. In observation two, firm 1 produces a higher output and incurs a cost of $C_{12} = (\frac{3}{2})(2) - 1 = 2$ and the increasing cost condition is satisfied. In observation two, firm 1 produces output equal to the aggregate output in observation 1. The monopoly deviation condition requires that $\frac{1}{2} \geq (1)(2) - 2 = 0$ which is satisfied. We can conclude that the observations can be Bertrand rationalized. Formally, the cost function $C(Q) = \frac{1}{2}Q^2$ satisfies all the requirements for Bertrand rationalization, and the market demands in observations one and two can be constructed as $\bar{D}_1(P_1) = \max\{0, 4 - 2P_1\}$ and $\bar{D}_2(P_2) = \max\{0, 8 - \frac{8}{3}P_2\}$ respectively.

However, it is interesting that despite the simplicity of the example, this data set *cannot* be rationalized by the Cournot model. To gain some intuition as to why this is

	1	2
t=1	$(1, 1, \frac{1}{2})$	$(1, 1, \frac{1}{2})$
t=2	$(\frac{3}{2}, 2, 1)$	$(\frac{3}{2}, 2, 1)$

Table 2: Example 2

the case note that in observation two each firm produces 2 units of output. If a firm were instead to produce only 1 unit of output they would obtain a market price greater than $\frac{3}{2}$ as the aggregate market quantity would decrease. The firm would incur a cost of only $\frac{1}{2}$. Therefore, the profit from producing only 1 unit of output is strictly greater than $\frac{3}{2} - \frac{1}{2} = 1$. The result is that a firm could profitably deviate by reducing its quantity in the second observation and which rules out rationalizing the observations as Cournot equilibria.⁷

3.3 Example 3

Now consider the data set in Example 3. As both firms tie at the same price in each observation the tie deviation condition is satisfied. Checking whether the increasing cost condition is satisfied we can note that in observation one firm 1 incurs a cost of $C_{11} = (1)(1) - \frac{3}{4} = \frac{1}{4}$. In the observation two, firm 1 produces a higher output and incurs a cost of $C_{12} = (\frac{2}{3})(2) - \frac{1}{3} = 1$ and the increasing cost condition is satisfied. In observation two, firm 1 produces a quantity equal to the aggregate market output in observation one. For the monopoly deviation condition to be satisfied we require $\frac{3}{4} \geq (1)(2) - 1 = 1$ which is not satisfied. We can conclude that this set of observations cannot be Bertrand rationalized.

To gain more intuition behind why the data set cannot be Bertrand rationalized suppose we construct a cost function which is smooth, strictly increasing and passes through the observed costs. For example, the cost function $C(Q) = \frac{1}{4}Q^2$ satisfies $C(0) = 0$, $C(1) = \frac{1}{4}$ and $C(2) = 1$. Supposing we also construct a market demand which is smooth, strictly decreasing and satisfies $\bar{D}_1(1) = 2$ for observation one. If a firm were to

⁷Formally, the failure of the data set to be Cournot rationalizable is because it does not satisfy the discrete marginal condition.

undercut the market in observation one and post a price $1 - \epsilon$ they would obtain profit of $\bar{D}_1(1 - \epsilon)(1 - \epsilon) - \frac{1}{4}(\bar{D}_1(1 - \epsilon))^2$. The limit as $\epsilon \rightarrow 0$ of this profit function, which is well defined because cost and demand functions are smooth, is $(2)(1) - 1 = 1$ which is greater than the observed profit in observation one. For any smooth cost function and market demand we construct, both firms will have profitable deviations. This is why the observations cannot be Bertrand rationalized.

What is also interesting about this example is that it cannot be rationalized by the Cournot model either. To see this, note that in observation two each firm produces 2 units of output. Suppose one firm were to deviate and produce only 1 unit of output instead. Then, the market price for their output would be greater than $\frac{2}{3}$ because the aggregate market quantity reduces, and the firm incurs a cost of $\frac{1}{4}$. The profit which the firm receives is then greater than $(\frac{2}{3})(1) - \frac{1}{4} = \frac{5}{12}$, which is greater than the observed profit of $\frac{1}{3}$ in observation two. For this reason, the observations cannot be Cournot rationalized.

	1	2
t=1	$(1, 1, \frac{3}{4})$	$(1, 1, \frac{3}{4})$
t=2	$(\frac{2}{3}, 2, \frac{1}{3})$	$(\frac{2}{3}, 2, \frac{1}{3})$

Table 3: Example 3

3.4 Example 4

Consider the data set in Example 4. As both firms set the same price in both observations the tie deviation condition is satisfied. In observation one, firm 1 incurs a cost of $C_{11} = (2)(1) - 1 = 1$. In observation two, firm 1 produces higher output and incurs a cost of $C_{12} = (2\frac{1}{2})(2) - 2\frac{1}{2} = 2\frac{1}{2}$ and the increasing cost condition is satisfied. In observation two, firm 1 produces output equal to the aggregate output in observation one. Checking whether the monopoly deviation condition is satisfied requires $1 \geq (2)(2) - 2\frac{1}{2} = 1\frac{1}{2}$ which is not satisfied. We can conclude that this set of observations cannot be Bertrand rationalized. However, this set of observations satisfy the discrete marginal condition and can be rationalized by the Cournot model.

	1	2
t=1	(2, 1, 1)	(2, 1, 1)
t=2	$(2\frac{1}{2}, 2, 2\frac{1}{2})$	$(2\frac{1}{2}, 2, 2\frac{1}{2})$

Table 4: Example 4

4 Conclusion

An essential requirement of a scientific theory is that it can be tested against empirical observations. In this paper we have identified the testable implications of the Bertrand model. A set of necessary and sufficient conditions has been proposed for a data set to be rationalized as pure strategy Bertrand equilibrium. Despite the generality of the results presented here we should like to note a direction in which the results could be extended. When we make observations of oligopoly competition which do not satisfy the conditions for either Bertrand or Cournot rationalizability, such as Example 3 above, how are we to explain the observations? It is well known that the Nash equilibrium solution is not the only outcome which could result from firms being rational and rationality being common knowledge. Is it possible for us to characterize the set of oligopoly observations which are consistent with firms revealing some form of rationality which is perhaps weaker than the Nash equilibrium solution? In the context of consumer theory, Manzini and Mariotti (2009) have characterized weaker forms of rationality than that described by the weak axiom of revealed preference, and they show that observations which violate the weak axiom can sometimes still be explained as individuals optimizing some objective function.⁸ Extending these weaker forms of rationality to oligopoly competition remains an open and interesting question.

⁸They present an example which they call the ‘frugal consumer’ who minimizes an objective function which is strictly increasing and convex in consumption goods and show that the choices of this individual violate the weak axiom of revealed preference but can be explained by a weaker form of rationality known as discard-rationality.

5 Appendix

5.1 Proof of Theorem 1

First we prove the necessity part of the result, and second the sufficiency part.

Necessity Start by assuming that the observations are Bertrand rationalizable. Suppose that ICC is not satisfied. Then for some i and t we have $C_{it} - C_{is_i(t)} = \int_{Q_{is_i(t)}}^{Q_{it}} \bar{C}'_i(x) dx \leq 0$, which contradicts $\bar{C}'_i(x) > 0$.

Second, suppose that MDC is not satisfied. Then for some i and t we have $P_t^* Q_{it} - C_{it} \leq P_t^* Q_t^* - C_{ir_i(t)}$ and $Q_{ir_i(t)} > Q_t^*$. As the observations are rationalizable we have $P_t^* Q_{it} - \bar{C}_{it}(Q_{it}) \leq P_t^* \bar{D}_t(P_t^*) - \bar{C}_i(Q_{ir_i(t)})$. However, firm i could set a price $P_t^* - \epsilon$, and by choosing ϵ sufficiently small, we have $\bar{D}_t(P_t^* - \epsilon) < Q_{ir_i(t)}$ and $\bar{C}_i(\bar{D}_t(P_t^* - \epsilon)) < \bar{C}_i(Q_{ir_i(t)})$. Which means firm i could obtain a strictly higher profit than that observed and which contradicts the observed prices being a Bertrand equilibrium.⁹

Finally, suppose TDC is not satisfied. Then there is an i and t such that $i \in N \setminus A_t$ and $P_t^* \hat{Q}_t - C_{im_i(t)} > 0$. As the observations are rationalizable we have $P_t^* \hat{Q}_t - \bar{C}(Q_{im_i(t)}) > 0$. Given that $i \in N \setminus A_t$ we know that $P_{it} > P_t^*$, $Q_{it} = 0$ and $\Pi_{it} = 0$. However, firm i could set its price equal to P_t^* , obtain demand of \hat{Q}_t , and profit of $P_t^* \hat{Q}_t - \bar{C}(Q_{im_i(t)}) > 0$, which contradicts the observed prices being a Bertrand equilibrium. Therefore, ICC, MDC and TDC are all necessary conditions for Bertrand rationalizability.

Sufficiency To show that the conditions are sufficient for Bertrand rationalizability we proceed in three steps. First, we show how the firms' cost functions are constructed. Second, we show how the market demands are constructed. Finally, we show that these constructs satisfy all the requirements of Definition 2.

Step 1: Construction of the firms' cost functions. First, as the observed costs satisfy ICC we can construct a smooth, strictly increasing, cost function with the properties that $\bar{C}_i(Q_{it}) = C_{it}$ and $\bar{C}_i(0) = 0$. Second, we impose the following three restrictions upon the cost function.

- 1) For all $t \in T$ such that $R_i(t)$ is not empty and $Q_{ir_i(t)} > Q_t^*$ choose the cost function such that $\bar{C}_i(Q_t^*) > \max\{P_t^*(Q_t^* - Q_{it}) + C_{it}, C_{is_i(t)}\}$.
- 2) Define $V_i = \{t' \in T : R_i(t') = \emptyset\}$, $v_i = \{t' \in V_i : Q_{t'}^* \leq Q_t^* \forall t' \in V_i\}$ and $w_i =$

⁹The same argument can be used when $P_t^* Q_{it} - C_{it} < P_t^* Q_t^* - \bar{C}_{ir_i(t)}$ and $Q_{ir_i(t)} = Q_t^*$.

$\max_{t \in V_i} P_t^*(Q_t^* - Q_{it}) + C_{it}$. Then we can choose the cost functions such that $\bar{C}_i(Q_{v_i}^*) > w_i$.
 3) For any $i \in N \setminus A_t$ such that $M_i(t) \neq \emptyset$ and $Q_{im_i(t)} > \hat{Q}_t$ choose the cost function so that $\bar{C}_i(\hat{Q}_t) > P_t^* \hat{Q}_t$. Similarly, if $i \in N \setminus A_t$ and $M_i(t) = \emptyset$ then we can choose the cost function such that $\bar{C}_i(\hat{Q}_t) > P_t^* \hat{Q}_t$.

The restriction in (1) can be satisfied because as MDC holds we have that $P_t^* Q_{it} - C_{it} > P_t^* Q_t^* - C_{ir_i(t)}$ and we can choose the cost function so that $\bar{C}_i(Q_t^*) = C_{ir_i(t)} - \epsilon$, provided ϵ is sufficiently small. The restriction in (2) can always be satisfied because it states that for those observations when we do not observe firm i producing an output at least as great as the aggregate market output we can choose the cost function so that MDC is satisfied. These restrictions ensure that for every firm and observation the cost functions satisfy the inequality $P_t^* Q_{it} - \bar{C}_i(Q_{it}) \geq P_t^* Q_t^* - \bar{C}_i(Q_t^*)$. The restriction in (3) can always be satisfied because as TDC is satisfied for all $i \in N \setminus A_t$ we have $P_t^* \hat{Q}_t - C_{im_i(t)} < 0$ whenever $Q_{im_i(t)} > \hat{Q}_t$ and we can choose the cost function so that $\bar{C}_i(Q_{im_i(t)}) = C_{im_i(t)} - \epsilon$.

Step 2: Construction of the market demands Consider the following piecewise-affine market demand $\bar{D}_t(P_t) = \max\{0, 2Q_t^* - Q_t^* P_t / P_t^*\}$. This is smooth,¹⁰ strictly decreasing and satisfies $\bar{D}_t(P_t^*) = Q_t^*$. Also, the revenue (as a function of price) from this market demand is $R(P_t) = 2Q_t^* P_t - Q_t^* P_t^2 / P_t^*$ whenever demand is positive. The marginal revenue is $R'(P_t) = 2Q_t^* - 2Q_t^* P_t / P_t^*$. Hence, the marginal revenue at P_t^* is $R'(P_t^*) = 0$. That is, revenue is maximized at P_t^* . This means $P_t \bar{D}_t(P_t) < P_t^* \bar{D}_t(P_t^*)$ whenever $P_t \neq P_t^*$. This property will be used in Step 3.

Step 3: Sufficiency for Bertrand rationalizability. Having described how we can construct the cost functions we now show that these constructs satisfy the requirements of Definition 2 and that ICC, MDC and TDC are sufficient for Bertrand rationalizability. First, we constructed cost functions in Step 1 which were smooth, strictly increasing and satisfied $\bar{C}_i(Q_{it}) = C_{it}$ for each $i \in N$. Second, we constructed smooth, strictly decreasing market demands satisfying $\bar{D}_t(P_t^*) = Q_t^*$ for each observation. Hence, conditions (i)-(iii) of Definition 2 have been satisfied, and all that remains to be shown is that given the constructed cost functions and market demands the observed prices constitute a Bertrand equilibrium.

¹⁰The kink which occurs when quantity demanded becomes zero could be smoothed out on a small interval without affecting the analysis.

There are three possibly profitable price deviations:(i) a firm raises its price, (ii) a firm with a price strictly greater than the minimum price could tie at the minimum price, (iii) a firm could undercut the market a post a strictly lower price. We consider these cases separately.

Case 1: For any $i \in A_t$ increasing their price results in receiving zero demand and making zero profit because $|A_t| \geq 2$. For any $i \in N \setminus A_t$ increasing their price also results in receiving zero demand and making zero profit. Therefore, no firm can profitably deviate by increasing its price.

Case 2: By construction, the cost function of any firm $i \in N \setminus A_t$ satisfies $P_t^* \hat{Q}_t - \bar{C}_i(\hat{Q}_t) \leq 0$. If firm i set its price equal to the minimum it would receive demand of \hat{Q}_t , by the equal sharing rule, and less than, or equal to zero, profit. Hence, this is not a profitable deviation.

Case 3: A firm undercuts the market and posts a price $P_t^* - \epsilon$. The profit would then be $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^* - \epsilon))$. As we constructed the market demand to achieve the revenue maximum at P_t^* we have $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) < P_t^* \bar{D}_t(P_t^*) = P_t^* Q_t^*$. As the market demand is strictly decreasing, and the cost functions strictly increasing we have $\bar{C}_i(\bar{D}_t(P_t^* - \epsilon)) > \bar{C}_i(Q_t^*)$. Combining these gives $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^* - \epsilon)) < P_t^* Q_t^* - \bar{C}_i(Q_t^*)$. As the constructed cost functions satisfy $P_t^* Q_{it} - \bar{C}_i(Q_{it}) \geq P_t^* Q_t^* - \bar{C}_i(Q_t^*)$ we have $P_t^* Q_{it} - \bar{C}_i(Q_{it}) > (P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^* - \epsilon))$. We can conclude that no firm can profitably undercut the market, and the set of observed market prices is a Bertrand equilibrium. ■

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