The University of Manchester

MANCHESTER 1824

Economics Discussion Paper Series EDP-0915

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September 2009

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Acknowledgements: The authors are grateful to Heather Anderson, Alastair Hall and David Harvey for their thoughtful comments, which have helped to shape this paper. Responsibility for any errors and omissions, however, lies entirely with the authors.

Abstract

While a number of unit root testing procedures have been developed to account for nonlinearity under the alternative hypothesis of stationarity, almost all available tests assume a linear DGP under the unit root null hypothesis. This paper establishes some theoretical results relating to the inclusion of nonlinear terms in an ADF regression and proposes two new unit root tests that allow the process to be nonlinear under the null hypothesis. More specifically, block and model-based bootstrap procedures are developed for smooth transition threshold models. Simulations show that the latter is preferred and the model-based bootstrap test delivers a good size performance across all specifications, including linear and effectively abrupt transition models. The model-based test also dominates the standard ADF test in terms of power and an application to the US unemployment rate shows that it can overturn conclusions based on an erroneous linearity assumption.

Keywords: Unit root tests, nonlinearity, smooth transition models, US unemployment

***JEL* codes:** C12, C32, E24

1. Introduction

Recent years have witnessed an explosion of interest in the use of nonlinear models for the analysis of economic time series, with regime-switching specifications being widely applied due to their ability to replicate key characteristics of observed series. Although many varieties of such models are available, the threshold autoregressive (TAR) model due to Tsay (1989) and its smooth transition autoregressive (STAR) generalization promoted by Teräsvirta (1994) are popular choices for empirical analyses. Alongside empirical applications, well-developed tests are now available to detect the presence of this type of nonlinearity; see, for example, Hansen (1996), Harvey and Leybourne (2007), and Luukkonen, Saikkonen and Teräsvirta (1988). With the exception of Harvey and Leybourne (2007), who provide a test for nonlinearity of a STAR form when the order of integration is unknown, these tests are based on the assumption that the series under analysis is stationary. Indeed, the assumption of stationarity is crucial in this context, since the asymptotic distributions of commonly applied nonlinearity test statistics differ when the data are generated by a unit root process rather than a stationary one (Caner and Hansen 2001, Kılıç, 2004).

Due to the fundamental role of the order of integration for modelling time series data and for the conduct of statistical inference, the first step of empirical time series modelling in economics is almost invariantly an examination of the integration order of the series. While the popular Augmented Dickey-Fuller (ADF) test performs relatively well for linear time series, a number of studies show that its power decreases dramatically for stationary nonlinear series (for example, Balke and Fomby, 1997, Pippenger and Goering, 1993). Consequently, an important strand of the recent literature has developed unit root tests that are applicable in the context of threshold and smooth transition models, with contributions including Enders and Granger (1998), Sollis, Leybourne and Newbold (2002), Kapetanios, Shin and Snell (2003) and Seo (2008). However, these studies allow for such nonlinearity only under the alternative hypothesis of stationarity and hence do not account for any effects that may arise when the process is both nonstationary and nonlinear. To our knowledge, the only paper that allows for this case is Caner and Hansen (2001), who show that the asymptotic distribution of Wald tests for a unit root depend on nuisance parameters when threshold nonlinearity is present under the null. Although they examine the performance of asymptotic and bootstrap tests for this case, Caner and Hansen (2001) find that these tests do not have good size. Therefore, no satisfactory approach is currently available to test the unit root null hypothesis when the nonstationary process may exhibit threshold or smooth transition nonlinearity.

The present study extends the existing literature by developing tests that are well-sized for the null hypothesis of nonstationarity allowing for the presence of STAR-type nonlinearity of unknown (logistic or exponential) form. The testing methodology uses modified ADF regressions, which are corrected for the nonlinearity under consideration. Although we examine modifications based on the inclusion of terms arising from Taylor series approximations to the nonlinearity, these do not yield good size when the process under the null is nonlinear. Consequently, we develop a model-based bootstrap procedure, which delivers excellent size properties in all cases we consider. In terms of power as well as size, this test performs well not only for STAR-type nonlinear processes, but also for TAR-type nonlinear and linear processes. In addition to providing Monte Carlo evidence, an application to the monthly US unemployment rate shows that accounting for nonlinearity plays an important role for the decision on the appropriate level of integration for this series.

The remainder of the paper is organised as follows. After reviewing the literature that jointly considers nonstationarity and nonlinearity in Section 2, Section 3 develops our approach to testing for unit root in the presence of possible nonlinearity. Monte Carlo experiments, which examine empirical size and power characteristics of the new tests, are provided in Section 4. The empirical application to US unemployment is presented in Section 5 and Section 6 concludes. Proofs are provided in the Appendix.

2. Literature Review

Growing dissatisfaction with the performance of the standard ADF test in the presence of nonlinearity has recently led to a number of studies focusing on the interactions between nonlinearity and nonstationarity, especially when the (possible) nonlinearity is of the TAR or STAR form.

An early and seminal contribution to this literature is Enders and Granger (1998), who propose a two-step procedure to test the null hypothesis of a linear unit root against the alternative of a stationary two-regime threshold model. Although their test recognises that the threshold value is not identified under the null hypothesis, nevertheless it is less powerful than the standard ADF test even in the presence of substantial asymmetry. Rather than a two-stage approach, a number of recent studies, including Bec, Ben Salem and Carrasco (2004), Kapetanios and Shin (2006) and Bec, Guay and Guerre (2008), embed a search for the threshold(s) within the unit root test procedure, with this approach generally delivering power improvements over the linear ADF test. Rather than autoregressive augmentation, Seo (2008) proposes the use of a first-order test regression with residual autocorrelation taken into account through a residual-based block bootstrap, and finds that this approach yields power improvements over the ADF test against stationary TAR processes.

Again in a two-regime TAR context, Caner and Hansen (2001) use a model-based bootstrap procedure to test the null hypothesis of nonstationarity in a two-regime threshold model, where nonlinearity may apply under either the null or alternative hypothesis. The asymptotic null distribution of the test statistic then depends on whether a threshold effect is present or not. For this reason, they define two bootstrap procedures, one assuming a threshold effect is present (identified threshold bootstrap) and the other under linearity (unidentified threshold bootstrap), both for the null hypothesis of a unit root process. However, because their Monte Carlo experiments indicate that the size distortions of the unidentified threshold bootstrap are less than those of the identified threshold bootstrap even in the presence of nonlinearity, they recommend that the unidentified threshold bootstrap method should be employed regardless of the extent of nonlinearity under the unit root null hypothesis.

Another group of studies focuses on stationary STAR models under the alternative hypothesis, with the null hypothesis being a linear unit root. Kapetanios, Shin and Snell (2003) propose a test statistic against a globally stationary exponential STAR (ESTAR) process with the nuisance parameter problem which arises under the linear and nonstationary null hypothesis overcome through a Taylor series approximation. Sollis (2009) applies a similar approach for an asymmetric ESTAR process under the alternative, while Eklund (2003) considers a stationary logistic STAR (LSTAR) process.

Although Taylor series approximations are popular when the process has a STAR form under the alternative, Sollis, Leybourne and Newbold (2002) take a different approach by testing the unit root null hypothesis after estimating the model under the alternative by nonlinear least squares. In a similar vein, Kılıç (2003) uses a supremum test statistic, with this obtained by searching over relevant transition function parameters that apply under the alternative hypothesis. Park and Shintani (2005) allow for a general form of nonlinear regime-switching model under the alternative hypothesis, including TAR, ESTAR and LSTAR forms. Their asymptotic theory is based on setting a limit parameter space for the parameters of the transition function, based on the empirical values of the transition variable, with the asymptotic distribution of their supremum test statistic depending only on the type of transition function and the limit parameter space. Finally, Rothe and Sibbersten (2006) take a semi parametric approach to allow for ESTAR nonlinearity under the alternative.

Many of the above studies include Monte Carlo analyses that favour their particular approach, but Choi and Moh (2007) undertake a comparison across a range of tests and a variety of nonlinear data generating processes (DGPs). They conclude that the distance from nonstationarity is the main determinant of power for all tests considered, with the ADF test generally having relatively good power when the sample size is small (50 or 100 observations). For larger samples, the test of Park and Shintani (2005) performs best overall, which emphasizes the importance of allowing flexibility when considering the nature of possible nonlinearity.

Nevertheless, all the tests proposed except that of Caner and Hansen (2001) assume linearity under the null hypothesis of nonstationarity. Although Caner and Hansen (2001) establish that the asymptotic distribution of the Wald unit root test changes in the presence of TAR nonlinearity under the null and propose a bootstrap unit root test for this case, this test suffers from substantial size distortions and they recommend the use of a bootstrap procedure based on a linear DGP. Consequently, from a practical perspective, they do not provide a useful unit root test for the case when nonlinearity is present under the unit root null. Further, the good performance of the Park and Shintani (2005) approach in the study of Choi and Moh (2007) points to the benefits of flexibility about the nature of potential nonlinearity.

The present paper develops bootstrap testing methodologies that permit nonlinearity under the null hypothesis of nonstationarity. Although the model-based identified bootstrap of Caner and Hansen (2001) suffers size distortions in this case, our approach overcomes these. Further, in common with Park and Shintani (2005), our preferred model-based bootstrap is flexible about the nature of nonlinearity, allowing for ESTAR and LSTAR forms, including TAR models as a limit form of the latter.

3. Methodology

After the nonlinear DGP is outlined in subsection 3.1, subsection 3.2 discusses a unit root test approach based on further augmenting the ADF regression for nonlinearity. In the light of the theoretical issues discussed in subsection 3.3, the final subsection proposes two bootstrap unit root tests to take account of this nonlinear DGP under the null hypothesis.

3.1 The DGP

Our primary DGP is the first-order LSTAR process given by

$$\Delta y_t = \alpha_1 \Delta y_{t-1}\left(1 - F_L\left(\Delta y_{t-1}; \gamma_L, c_L\right)\right) + \alpha_2 \Delta y_{t-1} F_L\left(\Delta y_{t-1}; \gamma_L, c_L\right) + \varepsilon_t \tag{1}$$

where F_L is the logistic function

$$F_L\left(\Delta y_{t-1}\right) = \left(1 + \exp\left(-\gamma_L\left(\Delta y_{t-1} - c_L\right)\right)\right)^{-1}, \quad \gamma_L > 0 \tag{2}$$

with transition variable Δy_{t-1} and parameters γ_L and c_L governing its slope and location, respectively. As in Caner and Hansen (2001), the transition variable in (1) is a lagged difference to ensure stationarity of this variable under both the null and alternative hypotheses. The intercept is zero in this DGP, while $\varepsilon_t \sim iid\ N(0, 1)$.

Alongside the logistic specification of (2), the exponential transition function can be considered, where F_E replaces F_L in (1) and F_E is defined by

$$F_E\left(\Delta y_{t-1}\right) = 1 - \exp\left(-\gamma_E\left(\Delta y_{t-1} - c_E\right)^2\right), \quad \gamma_E > 0\,. \tag{3}$$

Although our main focus is (2), the discussion below can be suitably modified for the ESTAR case, which is explicitly examined for the model-based bootstrap in subsection 3.4 below. In either case, extensions to higher order dynamics in (1), or to a different delay parameter in (2) or (3), are straightforward.

Clearly, the extent of nonlinearity in the DGP of (1) will depend on the transition function parameters in (2). In particular, for very small values of γ_L, such as 0.01 or 0.05, the transition function is effectively flat and (1) is near-linear. However, the logistic function acquires the familiar *S*-shape as the slope parameter increases and eventually approaches an indicator function for very large values of γ_L, which we capture by γ_L = 50. Hence, increasing the slope while holding the other parameters constant allows us to consider near-linearity, STAR and TAR type nonlinearity within the framework of (1) and (2).

However, the extent of nonlinearity is also affected by the threshold location parameter, c_L [1]. More specifically, holding other parameters constant, an increase in c_L relative to the unconditional mean of the process causes it to remain longer in the lower regime (where $F_L \approx 0$) and to move less frequently to the upper regime ($F_L \approx 1$). Consequently, as the threshold increases, nearly all observations may fall into the lower regime and for practical purposes the model approaches a single-regime one[2]. In order to capture these effects, our simulations employ c_L = 0.0, 0.5, 1.1, in addition to a range of values for γ_L.

3.2 Linear and Augmented Linear Test Regressions

The standard ADF test is frequently applied prior to considering nonlinearity. Hence, for data generated by (1) and (2), we consider the performance of the ADF test based on the *t*-ratio of ρ in the regression

$$\Delta y_t = \delta_0 + \rho y_{t-1} + \sum_{i=1}^{p} \delta_i \Delta y_{t-i} + u_t \tag{4}$$

where the lag order p is specified by a data-based technique and u_t is assumed to be iid (0, σ^2). Since the standard ADF test does not allow for nonlinearity, residuals from (4) may inherit nonlinear features from the DGP and hence violate the white noise assumption, leading to the unreliability of this test in the Monte Carlo experiments of Pippenger and Goering (1993), Balke and Fomby (1997) and others.

As noted in Section 2, a strand of the literature that tests the linear unit root null hypothesis against the alternative hypothesis of a stationary STAR process proceeds by approximating the nonlinearity through a Taylor series expansion. An analogous approach when such nonlinearity is present under the null hypothesis is to augment the linear ADF regression using Taylor series terms to take account of the nonlinearity under the null. Therefore, following Luukkonen *et al.* (1988), the LSTAR form for (1) can be considered through a third-order Taylor

[1] The variance of the error terms also affects the form of the true process; see Pippenger and Goering (1993), who consider TAR type nonlinearity but it is quite straightforward to interpret their results for STAR type nonlinearity. Throughout our study, the disturbance variance is, without loss of generality, set to 1.

[2] Similarly, as the threshold progressively decreases in relation to the unconditional mean, ultimately virtually all observations will fall in the upper regime. As these cases are symmetric, we consider only increasing c_L.

series approximation around $\gamma_L = 0$. Similarly, this augmentation may also capture nonlinearities of the ESTAR form. Assuming it is known that autoregressive augmentation of $p = 1$ is appropriate, this leads us to the unit root test regression

$$\Delta y_t = \beta_0 + \rho_1 y_{t-1} + \beta_1 \Delta y_{t-1} + \beta_2 \Delta y_{t-1}{}^2 + \beta_3 \Delta y_{t-1}{}^3 + \beta_4 \Delta y_{t-1}{}^4 + u_{1t}\,. \tag{5}$$

Clearly, the aim of including the Taylor series approximation in (5) is to account for nonlinearity in regression (1) and hence to obtain disturbances u_{1t} with an approximate *iid*(0, $\sigma_1{}^2$) structure. However, the Taylor approximation is derived for $\gamma_L \to 0$, which is associated with weak nonlinearity. Consequently, it may be less adequate as γ_L increases, corresponding to stronger nonlinearity. To illustrate the nature of the approximation, the logistic transition function (2) and its Taylor series approximation around $\gamma_L = 0$ are graphed in Figure 1 for $c_L = 0$ and various slope parameter values[3]. It is evident from this figure that the difference between the logistic function and its approximation is nearly zero for small γ_L values, irrespective of the value of the transition variable Δy_{t-1}. Further, across all γ_L considered, the approximation is also good for Δy_{t-1} in the neighbourhood of $c_L = 0$ (where $F_L = 0.5$). However, for values of the slope parameter γ_L substantially larger than 0, the two functions differ substantially when Δy_{t-1} falls far below c_L and hence the true value of F_L is close to its lower bound of zero. To a lesser extent, this also applies when Δy_{t-1} is large relative to c_L. Therefore, correcting the standard ADF regression through the inclusion of terms arising from a third-order Taylor series approximation of the logistic function around $\gamma_L = 0$ may leave nonlinearity effects in the test regression disturbance term u_{1t}.

In addition to (5), we consider the performance of the ADF test modified by the inclusion of the true transition function, namely

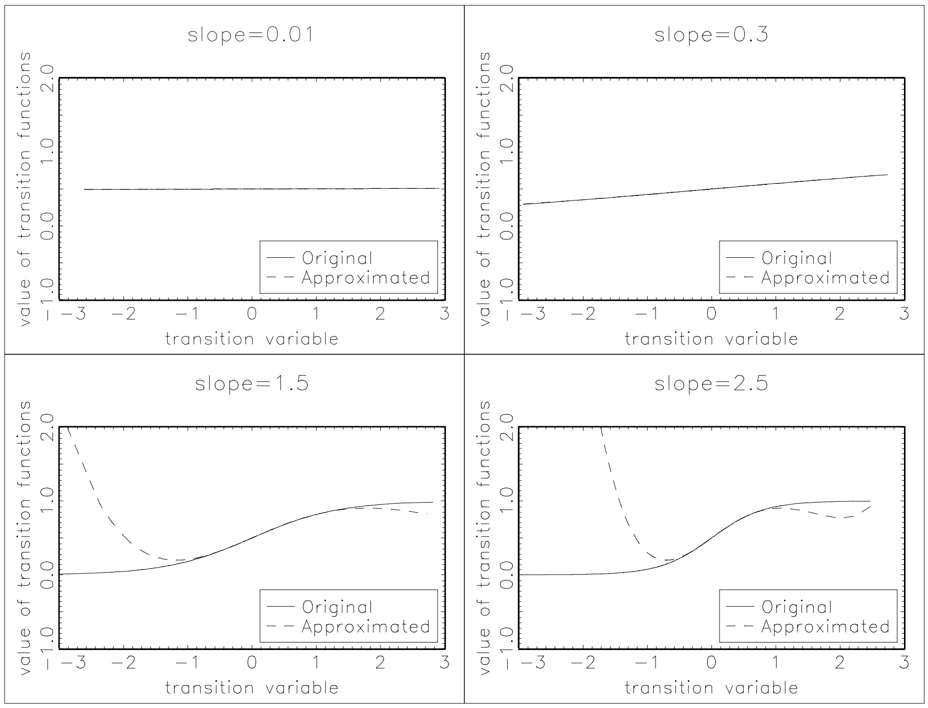
$$\Delta y_t = \alpha_0 + \rho_2 y_{t-1} + \alpha_1 \Delta y_{t-1}\left(1 - F_L\left(\Delta y_{t-1}; \gamma_L, c_L\right)\right) + \alpha_2 \Delta y_{t-1} F_L\left(\Delta y_{t-1}; \gamma_L, c_L\right) + u_{2t} \tag{6}$$

where u_{2t} is assumed to be *iid*(0, $\sigma_2{}^2$). This situation is, of course, unrealistic in practice, since the transition function is unknown. However, (6) allows us to investigate (in an ideal scenario)

[3] The values graphed are obtained analytically, with the ones of the approximate transition function given by the third order Taylor series approximation $0.5 + 0.25(\Delta y_{t-1} - c_L)\gamma_L - (1/48)(\Delta y_{t-1} - c_L)^3 \gamma_L^3$.

how the performance of the ADF test is affected by nonlinearity, when due account is taken of this nonlinearity.

Figure 1: Exact (Original) and Taylor Series Approximation Transition Functions



Notes: All transition functions have c_L = 0, with a third-order Taylor series approximation applied around $\gamma_L = 0$.

Against this background, subsection 3.3 discusses theoretical results related to the use of (5) and (6) in conjunction with the asymptotic Dicker-Fuller distribution that applies for a linear unit root test regression.

3.3 Asymptotic Distributions

Few theoretical results exist in the literature relating to the asymptotic distribution of unit root tests allowing for STAR-type nonlinearity. Indeed, the only results relate to regressions of the form

$$\Delta y_t = \rho y_{t-1}\left(1-\exp\left(-\gamma y_{t-1}{}^2\right)\right)+\varepsilon_t, \tag{7}$$

which is used by Kapetanios *et al*. (2003) in order to develop a unit root test (using the computed *t*-ratio statistic for ρ = 0) to allow for ESTAR nonlinearity under the alternative hypothesis. For implementation, Kapetanios *et al*. (2003) approximate the exponential transition function in (7) through a Taylor series approximation. Kılıç (2003) develops this approach by allowing Δy_{t-1} to be the transition variable in (7) and employing a supremum approach that searches over the unknown (and, under the null hypothesis, unidentified) slope parameter γ. These authors establish the asymptotic distributions of their test statistics, which differ from the familiar Dickey-Fuller one.

However, those authors consider nonlinearity only under the alternative hypothesis, whereas our concern is to allow for such nonlinearity under the unit root null hypothesis. The following Theorem sheds light on this issue.

Theorem. *Assume that y_t is generated by the integrated linear AR*(1) *process*

$$\Delta y_t = \beta_1 \Delta y_{t-1} + u_t \tag{8}$$

in which $y_0 = 0$, $|\beta_1| < 1$, $u_t \sim iid(0, \sigma^2)$. *Further, when* (5) *is employed, it is assumed that*

$$E[u_t^8] = \mu_8 < \infty$$

while for (6) *it is assumed that transition function parameters* γ_L *and* c_L *are given. Then*:

$$t_{\hat{\rho}_i} \xrightarrow{L} \frac{1}{2} \frac{\left[(W(1))^2 - 1\right] - W(1)\int_0^1 W(r)dr}{\sqrt{\int_0^1 (W(r))^2 dr - \left(\int_0^1 W(r)dr\right)^2}}, \quad i = 1, 2 \tag{9}$$

where $\xrightarrow{L}$ *indicates convergence in distributuion,* $W(r)$ *is standard Brownian motion,* $t_{\hat{\rho}_i}$ (i = 1, 2) *is the t-ratio statistic computed for the null hypothesis* $\rho_i = 0$ (i = 1, 2) *in* (5) *or* (6)*, as appropriate*.

This theorem, which is proved in the Appendix, establishes that the inclusion of additional variables in the form of either Taylor series expansion terms or values arising from an (arbitrary) transition function do not affect the unit root asymptotic distribution when, in fact, these terms are irrelevant because the DGP is a linear integrated AR(1) process.

This theorem does not, however, show that the asymptotic distribution of (9) applies when the true DGP is a nonlinear STAR process in Δy_t. Indeed, to our knowledge, and notwithstanding their widespread use in empirical applications, the asymptotics of integrated STAR processes, such as y_t in (1), are not yet known. However, Caner and Hansen (2001) study the related case of an integrated threshold DGP, which is a limit (as $\gamma_L \to \infty$) of the LSTAR model of (1) and (2). Employing unit root tests formed from two computed *t*-ratio statistics, one corresponding to each regime, they show (Caner and Hansen, 2001, Theorem 6) that the asymptotic distribution of each *t*-ratio is a mixture of a normal and a Dickey-Fuller distribution. The implication is that the unit root distribution for their nonlinear DGP is shifted to the right compared with the Dickey-Fuller case, with the ADF distribution of (9) providing an upper bound to the true asymptotic rejection probability under the null.

The result of Caner and Hansen (2001) suggests that the Dickey-Fuller distribution may also provide a bound for the asymptotic distributions of our test statistics in a STAR DGP, particularly when the test regression of (6) is employed and hence the nature of the nonlinearity is explicitly taken into account (as in Caner and Hansen, 2001, for their DGP). It is also compatible with our result, established in the above Theorem, that the Dickey-Fuller distribution applies for a linear DGP. Since it is reasonable to anticipate that the true asymptotic distribution in the nonlinear STAR case will depend on unknown parameters in the DGP (again, as in Caner and Hansen, 2001), a bootstrap approach may be required in practical applications in order to approximate this distribution. To this end, the next subsection proposes two bootstrap approaches.

3.4 Bootstrap Unit Root Tests

The bootstrap tests proposed here are designed to account for the impact of any nonlinearity on the null distribution of the test statistics. Consequently, these tests aim to account fully for the nonlinear dynamics of the DGP (1) in finite samples. Our approach is related to the identified threshold bootstrap method proposed by Caner and Hansen (2001), although the test statistic we use and the nature of the nonlinearity differ from their case. Two bootstrap procedures are examined, with one relating to the use of a third-order Taylor series approximation to the transition function, as in (5), and the other based on the ADF regression modified by the transition function itself, namely (6).

3.4.1 Block Bootstrap

Accounting for nonlinearity through the Taylor series approximation in (5) is convenient in a bootstrap procedure because it does not require the parameters of the transition function to be known or explicitly estimated. However, Figure 1 emphasizes the inadequacy of the Taylor series approximation for representing a true logistic transition function when the nonlinearity is reasonably strong, suggesting that some nonlinear dependence may remain in u_{1t} of (5). Residual-based bootstrap procedures that employ random resampling of single residuals are consequently inappropriate since they rely on residuals $\tilde{u}_{1t}$ being iid. However, the block bootstrap may be able to replicate such patterns of dependence by random sampling of blocks of consecutive observations and hence improve the performance of the test.

Being nonstationary under the null of $\rho_1 = 0$, the sequence y_t cannot be resampled directly. Instead, we can resample blocks of Δy_t or residuals $\tilde{u}_{1t}$, with the level form of the bootstrap DGP then being generated recursively using the resampled blocks. Paparoditis and Politis (2003) apply both the difference and residual-based block bootstraps in the context of the standard ADF test and show that the latter approach is more powerful. Very recently, Seo (2008) extends the residual-based block bootstrap approach to test for a linear unit root against a stationary TAR alternative. Such an application, however, is not feasible in our case due to the existence of the higher order terms of Δy_t in (5), which can lead to explosive bootstrap DGPs. Moreover, the inadequacy of the Taylor series approximation for representing a true logistic transition function in the case of strong nonlinearity is still a deterrent for the recursion, as the resultant bootstrap DGP would not mimic the true DGP.

Therefore, we apply the difference-based block bootstrap approach by resampling blocks of Δy_t and obtain the level form of the bootstrap DGP recursively. The bootstrap algorithm proceeds as follows:

i) Wrap observations $\{\Delta y_t\}_{t=2}^{T}$ around a circle, and use the overlapping blocking scheme of Künsch (1989) to resample the sequence $\{\Delta y_t\}_{t=2}^{T}$. That is, for a given block length $b < T-1$, construct $T-1$ blocks as $\{\Delta y_2, ..., \Delta y_b, \Delta y_{b+1}\}$, $\{\Delta y_3, ..., \Delta y_{b+1}, \Delta y_{b+2}\}$, ……, $\{\Delta y_{T-b+1}, ...\Delta y_{T-1}, \Delta y_T\}$, $\{\Delta y_{T-b+2}, ..., \Delta y_T, \Delta y_2\}$, ….., $\{\Delta y_T, ..., \Delta y_{b-1}, \Delta y_b\}$. Then draw

$k = \dfrac{T+100}{b}$ blocks with replacement from the $T-1$ blocks and paste these end-to-end to form the block bootstrap sample $\left\{\Delta y_t^{block}\right\}_{t=2}^{T+100}$ [4].

ii) Generate the level series of the bootstrap DGP, $\left\{y_t^*\right\}_{t=1}^{T+100}$, recursively using the block bootstrap sample Δy_t^{block}, namely $y_t^* = y_{t-1}^* + \Delta y_t^{block}$, $t = 2,\ldots\ldots,T+100$ where $y_1^* = 0$ [5].

iii) Cut the first 100 observations of y_t^* and estimate the modified ADF regression (5) with the sequence $\left\{\Delta y_t^*\right\}_{t=2}^{T}$.

iv) Calculate the bootstrap *t*-statistic for the null hypothesis $\rho_1 = 0$.

The use of the block bootstrap enables us to generate the bootstrap DGP Δy_t^* from the block bootstrap sample Δy_t^{block}, which is obtained by direct sampling of realizations from the true DGP of (1) and hence is anticipated to mimic this DGP. Nevertheless, Bühlman (2002) and Horrowitz (2003) indicate that the block bootstrap sample might not mimic the true DGP as it may generate dependence artifacts where resampled blocks are linked together, resulting in corrupted dependence in the bootstrap DGP observations Δy_t^*.

3.4.2 Model-Based Bootstrap

Rather than using the Taylor series expansion, consider the ADF regression modified using the nonlinear transition function as in (6). While the true transition function is unknown in practice, this case provides a useful starting point for considering the model-based approach. Also, inclusion of the known parameter case in the subsequent Monte Carlo analysis provides a benchmark for the performance under other, more realistic, assumptions.

When all parameters of (1) are assumed known, the model-based bootstrap procedure for a given data realisation is described by the following algorithm:

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[4] Wrapping the data around a circle removes the effect of the first and last *b*-1 observations of Δy_t appearing in fewer blocks than the remaining observations and hence ensures that each observation has an equal chance of appearing in the block bootstrap sample.

[5] Note that the mean of the block bootstrap DGP is not adjusted here. To analyze the effect of mean adjustment, Δy_t^{block} is replaced with its mean-adjusted form $\Delta\tilde{y}_t^{block} = \Delta y_t^{block} - \left(\Delta\overline{y}^{block} - \Delta\overline{y}\right)$ where $\Delta\overline{y}^{block}$ and $\Delta\overline{y}$ represent means of the block bootstrap and true DGPs, respectively. Nearly identical results to those reported for the case of unadjusted mean point to the insensitivity of the block bootstrap procedure to mean adjustment.

i) Imposing the null hypothesis of $\rho_2 = 0$, estimate the remaining parameters of the modified ADF regression (6) by OLS conditional on (γ_L, c_L) to obtain the residuals.

ii) Draw a random sample with replacement from the residuals and obtain $\{u_{2t}^*\}_{t=3}^{T+100}$.

iii) Using the true parameters, generate the level series for the bootstrap DGP, $\{y_t^*\}_{t=1}^{T+100}$, recursively, as

$$y_t^* = y_{t-1}^* + \alpha_1 \Delta y_{t-1}^* \left(1 - F_L\left(\Delta y_{t-1}^*; \gamma_L, c_L\right)\right) + \alpha_2 \Delta y_{t-1}^* F_L\left(\Delta y_{t-1}^*; \gamma_L, c_L\right) + u_{2t}^* \tag{10}$$

for $t = 3, \ldots\ldots, T+100$ where $y_1^* = y_2^* = 0$.

iv) Drop the first 100 bootstrap observations and estimate the modified ADF regression (6) by OLS conditional on (γ_L, c_L):

$$\Delta y_t^* = \delta_0 + \rho\, y_{t-1}^* + \delta_1 \Delta y_{t-1}^* \left(1 - F_L(\gamma_L, c_L)\right) + \delta_2 \Delta y_{t-1}^* F_L(\gamma_L, c_L) + u_t \tag{11}$$

and calculate the *t*-ratio statistic for $\rho = 0$.

v) Repeat steps ii) to iv) to generate *B* bootstrap replications and use these to calculate the bootstrap critical value of the *t*-statistic for the null hypothesis $\rho = 0$.

In more realistic scenarios, neither the true lag order nor the parameters or even the nonlinear functional form are known. To investigate the case of unknown lag order and parameters, we assume that the researcher considers the model under the unit root null hypothesis to have the form

$$\Delta y_t = \left(\alpha_{01} + \sum_{i=1}^{p} \alpha_{i1}\right)\left(1 - F_L(\Delta y_{t-1}; \gamma_L, c_L)\right) + \left(\alpha_{02} + \sum_{i=1}^{p} \alpha_{i2}\right)\left(1 - F_L(\Delta y_{t-1}; \gamma_L, c_L)\right) + u_t \tag{12}$$

with a regime-dependent intercept included to allow for any starting value effects, while the lag order *p* and all parameters are unknown[6]. Following a common practice in empirical analysis, the lag order *p* in (12) is selected from the data using the Schwartz criterion (SBC) in a linear autoregressive specification with a maximum of 8 lags, with this lag order maintained for the subsequent nonlinear analysis. Using this *p*, the parameters of (12) are then estimated by nonlinear least squares (NLS). Moreover, in order to guarantee global stationarity of the bootstrap DGP, we impose the restrictions that the roots of the characteristic equations in both regimes F_L =

[6] We also investigated the scenario where $p = 1$ is known, but the transition function parameters are unknown. The results are very similar to those reported for the more realistic case of unknown *p*.

0 and F_L = 1 are less than one in absolute value[7]. To ensure sufficient observations are available in each regime of the bootstrap DGP for reliable estimation of other parameters, $\hat{c}_L$ is restricted to be lie between the 5th and 95th percentiles of the transition variable $\Delta y_{t\text{-}1}$ when the sample size is reasonably large, with this range restricted to the 10th to 90th percentiles of $\Delta y_{t\text{-}1}$ when a relatively small sample (T = 100) is employed.

The generalization of the bootstrap algorithm above to this case is then relatively straightforward. Specifically, the bootstrap DGP in step iii) is generated using the estimates $(\hat{\alpha}_{01}, \hat{\alpha}_{1i}, \hat{\alpha}_{02}, \hat{\alpha}_{2i}, \hat{\gamma}_L, \hat{c}_L)$, while the modified ADF regression employed in step iv) becomes

$$\Delta y_t^* = \rho\, y_{t-1}^* + \left(\delta_{01} + \sum_{i=1}^{p} \delta_{1i} \Delta y_{t-i}\right)\left(1 - F_L(\hat{\gamma}_L, \hat{c}_L)\right) + \left(\delta_{02} + \sum_{i=1}^{p} \delta_{2i} \Delta y_{t-i}\right) F_L(\hat{\gamma}_L, \hat{c}_L) + u_t \qquad (11')$$

which is estimated by OLS conditional on $(\hat{\gamma}_L, \hat{c}_L)$. Full nonlinear estimation in each bootstrap replication, which would include the transition function parameters to yield estimates $\hat{\gamma}_{L,boot}$ and $\hat{c}_{L,boot}$, is not performed due to the computational cost involved. Davidson and MacKinnon (1999) also find the cost of full re-estimation to be prohibitive for the bootstrap in a nonlinear context and adopt an approximation. In our case, a comparison of empirical distributions for the bootstrap unit root test statistic obtained using $(\hat{\gamma}_L, \hat{c}_L)$ and $(\hat{\gamma}_{L,boot}, \hat{c}_{L,boot})$ in (11′) for p = 1 indicated only trivial differences[8] and hence the reported results apply the nonlinear estimates obtained from the original data in the B bootstrap replications.

Finally, we consider the situation where the form of the transition function is unknown, with both logistic and exponential cases, namely (2) and (3) respectively, considered plausible. In order to capture the data-based decision undertaken in such cases, and following Kesriyeli, Osborn and Sensier (2006), we select between these transition functions based on the minimum residual sum of squares (SSR). Therefore, for a given data series, $\{y_t\}_{t=1}^{T}$, and after selection of the AR lag order p via SBC as above, two-dimensional grid searches are undertaken for both parameter sets $(\gamma_L,\ c_L)$ and $(\gamma_E,\ c_E)$ of (2) and (3) respectively, with the lowest SSR over these

7 These conditions are sufficient to ensure stationarity; to our knowledge, necessary and sufficient conditions for stationarity of the LSTAR model are not yet available in the literature. In some rare cases, especially when T = 100, a Monte Carlo realization yields initial values for the AR coefficients failing the stationarity conditions. Such a realization is discarded. With the stationarity restrictions imposed, NLS estimation is carried out using the Newton-Raphson optimization algorithm in the CML subroutine library of GAUSS 5.0.

8 Simulations were undertaken for T = 100, 300 using 5000 replications setting α_1 = 0.5, α_2 = -0.1 and c_L = 0, with a range of slope coefficients.

yielding the selected transition function form. This form and the corresponding parameters estimated by NLS are then employed in the bootstrap.

Alongside the possibility of an exponential transition function being selected in the application of the bootstrap unit root test, we also examine a stationary ESTAR(1) data generating process. The procedure allowing unknown parameters, lag order and functional form[9] of the transition function is also applied for this ESTAR DGP, where the global stationarity condition used is that the characteristic roots of the outer regime (F_E = 1) are less than one in absolute value, allowing a local unit root or even an explosive behaviour in the inner regime $\left(F_E \approx 0\right)$.

4. Monte Carlo Results

This section analyzes the size and power properties of the tests discussed in Section 3.

4.1 Size Analysis

Empirical sizes of the tests are investigated using the LSTAR data generating process of (1) and (2), for sample sizes T = 100, 300 with transition function parameters c_L = {0, 0.5, 1.1} and γ_L = {0.01, 0.1, 0.3, 0.9, 1.5, 2.5, 50}. These slope values capture near-linear ($\gamma_L = 0.01, 0.1$) and TAR-type processes ($\gamma_L = 50$), in addition to logistic STAR nonlinearity. The same location and slope parameter are also applied to investigate the model-based bootstrap test for an ESTAR DGP. The coefficients α_1 and α_2 are chosen to guarantee global stationarity of Δy_t, while also varying the strength and nature of the implied AR dynamics. The disturbance variance is set to unity and the nominal test size is 5%.

4.1.1 Augmented Linear Procedures

Table 1 analyzes the empirical sizes for the linear and augmented test regressions discussed in Section 3.2, with the DGP being an LSTAR(1) process. To investigate the adequacy of the Taylor series approximation to nonlinearity in (5) and the performance of the modified ADF test in (6),

[9] We also investigated the case where the functional form of the ESTAR DGP is known, but the parameters and the lag order are unknown. Since the results are very similar to those reported for the case of unknown functional form, they are not reported.

which includes the true transition function, the lag order of one is assumed known. However, a data-dependent lag specification is employed for the standard ADF test, in order to allow the regression to capture serial correlation arising from the linear misspecification of the nonlinear DGP. In this case, the lag order is chosen using SBC from linear autoregressive models to a maximum order of $p = 12$, with the adequacy of the order verified by a Lagrange multiplier (LM) test for serial correlation applied at the 5% level. If significant autocorrelation is detected, the lag order is increased until the test is passed. The critical values employed are based on a linear random walk DGP.

The empirical rejection frequencies reported in Table 1 employ the linear Dickey-Fuller critical values and are obtained using 50,000 replications for sample sizes of $T = 100$ and $T = 300$. In addition to the empirical sizes of the tests, the final column provides an indicator of the power of the nonlinearity under consideration. This power measure is the empirical rejection frequency obtained by applying the Luukkonen *et al.* (1988) test for the null hypothesis of linearity at a nominal size of 5%, over the 50,000 replications.

According to Table 1, regardless of the other parameters, the standard ADF test has good size when the slope parameter takes a small value, namely 0.01, 0.1 or (when $T = 100$) 0.3. In such cases, as illustrated in Figure 1, the LSTAR process is close to linear, with this also indicated in Table 1 by the power of the nonlinearity being close to the nominal test size. However, size distortions appear as γ_L increases, pointing to stronger nonlinearities, with the ADF test always being undersized when substantial nonlinearity is present. Indeed, in a number of cases the empirical size is around half or less of the nominal size, so that the conventional ADF test attributes nonlinearity to nonstationarity. This is most marked for the larger sample size of $T = 300$, where the empirical size is around 1% when nonlinearity is evident (that is, when the power of nonlinearity is very close to unity.

As a rule of thumb, the size distortions for the ADF test become substantive (say, outside the range 0.04 to 0.06) when the power of nonlinearity is above around 44%. However, except for cases of near-linearity, size distortions are worse when $T = 300$ than for $T = 100$, implying that these distortions are an asymptotic issue and do not disappear with larger sample sizes.

The third-order Taylor series approximation does not improve the size performance of the ADF test, at least when the true AR lag order $p = 1$ is employed for the former. This comment also applies when the true transition function is used. Indeed, the Taylor series approximation and the transition function based unit root tests deliver very similar empirical size, with both showing a mildly stronger tendency to under-reject the unit root null hypothesis compared to the standard ADF test. Consequently, modifying the ADF regression to take account of nonlinearity in the DGP is not sufficient to solve the problem of under-rejection, despite this being a consequence of the nonlinearity itself.

These results confirm the anticipated asymptotic distributions for the nonlinear unit root tests, as discussed in subsection 3.3. In particular, when the DGP is effectively linear the Dickey-Fuller distribution continues to apply for the test regressions (5) and (6). However, the presence of nonlinearity affects the null distribution of the unit root test statistic, which shifts further to the right (leading to greater under-sizing compared with the nominal significance level) as the extent of nonlinearity increases. This indicates that the distribution depends on nuisance parameters, namely the parameters driving the nonlinearity in the DGP. The smaller empirical size that is evident in Table 1 for all three tests when $T = 300$ is compared with $T = 100$ is a consequence of the stronger evidence of nonlinearity that this larger sample provides, with this nonlinearity rendering the asymptotic distribution of the test statistic obtained for a linear DGP less relevant.

4.1.2 Bootstrap Procedures

We next turn to the empirical sizes of the bootstrap tests, proposed in Section 3.4, to investigate their performance in the presence of STAR nonlinearity. The empirical rejection probabilities, calculated from 5000 Monte Carlo replications and 400 bootstrap replications, are reported in Tables 2 and 3 for the block bootstrap and the model-based bootstrap, respectively. Due to substantial computational costs, this investigation considers the range of slope parameter values {0.1, 0.9, 1.5, 2.5, 50}, and hence examines fewer cases of mild nonlinearity compared with Table 1. The slope parameter value of 0.1 captures near-linearity in Tables 2 and 3, with the remaining values representing increasing degrees of nonlinearity; see the power of nonlinearity values in Table 1 for the logistic transition function case.

The block bootstrap uses the ADF regression modified by the Taylor series approximation, namely (5). An important aspect in applying block bootstrap methods is the determination of the block length, b. Like Seo (2008), we do not use a data-dependent method to determine an optimal b, but rather experiment with different values to investigate whether the test performance depends on the block length. For this purpose, we use 5 and 8 for the sample size of $T = 100$ and 5 and 10 for $T = 300$[10].

The first inference from Table 2 is the relative insensitivity of the rejection probabilities to the block length b, although the longer block length perhaps performs a little better overall. Secondly, the results show substantial improvement over the size distortions evident in Table 1 for the true process shows substantial nonlinearity with $\gamma_L = \{0.9, 1.5, 2.5, 50\}$. Nevertheless, some distortions remain for both sample sizes[11], perhaps due to the poor fit of the Taylor series approximation and the corrupted dependence in the bootstrap DGP resulting from the use of the block bootstrap. Further, regardless of the other parameters, the test has good size in the case of near-linearity, where $\gamma_L = 0.1$. Table 3 shows that the benchmark model-based bootstrap test, which assumes the parameters of the DGP (6) are known, works nearly as well as the ADF test in Table 1 when there is little nonlinearity ($\gamma_L = 0.01, 0.1$). Indeed, for $\gamma_L = 0.1$, the empirical size for the benchmark case in Table 3 ($T = 300$) is comparable to that of the linear ADF test in Table 1. Moreover, the benchmark model-based bootstrap test corrects the under-rejections exhibited by all non-bootstrap tests in Table 1 for moderate to strong nonlinearity.

Next, the assumptions of known parameters, known order of the AR process and the form of the transition function are progressively removed, with the relevant empirical sizes reported in two further sets of results for the LSTAR DGP. With both the lag order and parameter values unknown, the test continues to perform well overall, although there is a tendency for over-rejection in the case of near-linearity ($\gamma_L = 0.1$). This is especially noticeable when $T = 100$ and may be due to the estimation of (effectively) unidentified transfer function parameters. In general, the good size performance is maintained even when the investigator has to discriminate between LSTAR and ESTAR transition functions[12]. Finally, the empirical sizes are reported for the case

[10] As underlined by Davidson and MacKinnon (2006), if the selected b value is too small, then the block bootstrap samples cannot mimic the dependence structure of the original data because of the high number of corruptions that occur whenever one block ends and the next starts. On the other hand, if the block length is too large, then the block bootstrap samples might be excessively affected by the characteristics of the actual sample. Seo (2008) reports results for his block bootstrap test against a stationary TAR alternative with $b = 6$.

[11] Employing a fourth order Taylor series approximation results in no substantial size improvement above those reported.

[12] Some experiments were also conducted using the sequential hypothesis testing approach of Teräsvirta (1994) for the selection between LSTAR and ESTAR models. The findings were, however, nearly identical findings to the reported based on overall minimum SSR.

where the true DGP is ESTAR(1). Although the mild over-rejections become more frequent compared to the LSTAR(1) case, in general the bootstrap test continues to work well.

Therefore, the results imply that both bootstrap procedures are able to deliver good approximations to the asymptotic distributions of the unit root test statistics of (5) and (6) in the presence of nonlinearity. However, although the block bootstrap procedure based on (5) does relatively well, the model-based bootstrap using (6) is even better. Implementation of data-based procedures for lag selection and to discriminate between logistic and exponential transition functions, as well as estimation of the transition function parameters, results in relatively mild deterioration of this performance compared with the benchmark case of where the form of the model is known. These results imply that the model-based bootstrap is able to closely replicate the null distribution of the unit root test statistic not only in the presence of substantial nonlinearity, but even when such nonlinearity is effectively absent[13].

4.2 Power Analysis

The power properties of the tests are analyzed using a stationary LSTAR DGP:

$$\Delta y_t = \rho y_{t-1} + \alpha_1 \Delta y_{t-1}\left(1 - F_L\left(\Delta y_{t-1};\gamma_L, c_L\right)\right) + \alpha_2 \Delta y_{t-1} F_L\left(\Delta y_{t-1};\gamma_L, c_L\right) + v_t \tag{13}$$

where v_t is *iid* $N(0, 1)$. In this specification, it is assumed that $\rho < 0$ and y_t follows a nonlinear stationary process, with short-term momentum-type dynamics driven by Δy_{t-1}. The two extreme regimes $F_L\left(\Delta y_{t-1};\gamma_L, c_L\right) \approx 0$ and $F_L\left(\Delta y_{t-1};\gamma_L, c_L\right) \approx 1$ are then characterised by two different stationary autoregressive processes, where the roots of the characteristic equations in both regimes are less than one in absolute value. However, depending on the closeness of the roots to one, the degree of persistency can differ across regimes, with persistent but stationary (near-unit root) process in one regime and a less persistent process in the other regime. To illustrate these aspects of the DGPs, Table 4 includes the roots of the characteristic polynomials for each of the two regimes implied when the given α_1 or α_2 is combined with ρ = -0.05 and ρ = -0.1.

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[13] Indeed, the performance of the model-based bootstrap test was also investigated for a number of strictly linear cases where $\gamma_L = 0$, with the estimation issue that arises due to the transition function being constant confronted by employing the Moore-Penrose generalized inverse, as in Leybourne, Newbold and Vougas (1996) and Park and Shintani (2005). Although detailed results are not reported, the bootstrap test in such cases shows a mildly stronger tendency to over-reject the null compared to near-linear cases. Nevertheless, it continues to work well with empirical sizes in the range {0.054, 0.062}. This indicates that the model-based bootstrap test is applicable when nonlinearity is considered to be possible but has not been established.

The values of $(\alpha_1, \alpha_2, \gamma_L, c_L)$ and the sample sizes are identical to those employed in the size analysis. Since the power will increase as the process under consideration moves away from the null of a unit root, we consider only DGPs in the neighbourhood of the null, with ρ = -0.05, -0.1. To control for the size distortions reported in Tables 1 to 3, size-adjusted power is reported. Indeed, since the ADF test is under-sized in the presence of substantial nonlinearity, the size-adjusted power values reported will be higher than those obtained using a nominal 5% size in such cases. Nevertheless, it is also obvious that substantive size distortions render the power unreliable in practical situations. Consequently, the power analysis examines only cases for which the empirical size is in the range {0.04, 0.06}.

Since the modified ADF test employing the Taylor series approximation provides no improvement in size over the standard ADF test in Table 1, while the test using the true transition function is infeasible in practice, we examine the power only for the standard ADF test, together with the block bootstrap and model-based bootstrap tests. The same data-dependent lag specification described in subsection 4.1.1 is employed for the standard ADF test. As the empirical sizes of the block bootstrap tests in Table 2 are generally a little better for the longer block length, the reported results use b = 8 and 10 for sample sizes of 100 and 300, respectively. However, the results are qualitatively unchanged for b = 5. Finally, the power of the model-based bootstrap test is simulated only for the most realistic case, where the parameters, the lag order and the form of the transition function are unknown. The power analysis employs the same number of Monte Carlo and bootstrap replications as employed for size.

Table 4 reports the results. As may be anticipated, the power of the standard ADF test, which assumes linearity, decreases as the nonlinearity under consideration gets stronger. The block bootstrap test generally provides lower power than the standard ADF test regardless of the distance from nonstationarity. This is in line with results of Paparoditis and Politis (2003) for the difference-based block bootstrap approach, and the power loss may be due to sampling random blocks of Δy_t, which suffers from overdifferencing, and hence a corrupted dependence structure when y_t is a (nonlinear) stationary process as in (13). It should also be noted that little comparison between these tests can be made from Table 4 for T = 300, because of the poor size performance of the ADF test with this larger sample, except in the effective absence of nonlinearity.

However, the model-based bootstrap test outperforms both the standard ADF test and the block bootstrap test for all cases, whether the DGP is near-linear, or has a nonlinear STAR (or TAR) form. It is particularly notable that the model-based bootstrap test does not suffer a power loss compared with the ADF test when the process is near-linear, since this is effectively the situation for which the ADF test is designed. The clear pattern of the results is that the model-based bootstrap test has the highest power, while the power of the block bootstrap test is generally lower compared to those of the standard ADF test and the model-based bootstrap test.

In all cases, power depends on the roots of the characteristic equation, with the smallest power being obtained for the first DGP, which has α_1 = -0.4, α_2 = 0.1. Note, in particular, that when ρ = -0.05, the larger root in each regime for this DGP is close to unity. Although all other DGPs in Table 4 have one regime with a root of at least 0.94 when ρ = -0.05, higher power apparently results in these cases because the other regime is further from the nonstationarity boundary. It is, of course, unsurprising that power is always substantially larger for T = 300 than for T = 100 and (because of its effect on the characteristic roots) when ρ = -0.10 compared with ρ = -0.05.

5. Application to US Unemployment Rate

To investigate the validity of the natural rate hypothesis, the model-based bootstrap unit root test is applied to the US unemployment rate[14]. The business cycle asymmetry of unemployment has received great attention in the literature, with steep increases during recessions followed by more gradual declines during expansions. Recognizing this cyclical asymmetry as a nonlinear phenomenon, Bianchi and Zoega (1998), Koop and Potter (1999), Skalin and Teräsvirta (2002), Panagiotidis and Pelloni (2003), amongst others, utilize various nonlinear models to examine the unemployment rate dynamics. These studies, however, either employ standard unit root tests or simply assume stationarity (nonstationarity) based on the natural rate (hysteresis) hypothesis prior to their nonlinear analysis. Other studies, including Leybourne *et al.* (1998), Park and Shintani (2005), Gustavson and Österholm (2006), Yılancı (2008) and Franchi and Ordonez

[14] The two competing viewpoints about the persistence of the unemployment rate are the natural rate hypothesis due to Phelps (1967) and Friedman (1968) and the hysteresis hypothesis introduced by Blanchard and Summers (1986). Under the natural rate hypothesis, the unemployment rate is a mean-reverting process, with short-term deviations from a constant natural rate being temporary, whereas it is a nonstationary process under the hysteresis hypothesis with persistent short-term deviations.

(2008), focus on the possibility that the unemployment rate follows a stationary TAR or STAR type nonlinear process and apply unit root tests that allow for the alternative of nonlinear mean reversion. Only Caner and Hansen (2001), however, allow the possibility that the unemployment rate may be nonlinear under the null hypothesis of a unit root.

Like Caner and Hansen (2001), we examine the US unemployment rate. Specifically, we consider the unemployment rate (seasonally adjusted) among males aged 20 and over[15], at the monthly frequency over the period January 1963 to January 2009. Although a visual inspection of the data in Figure 2 does not reveal clear evidence for stationarity, it does suggest the presence of nonlinearity, with steep increases ending in sharp peaks and gradual declines.

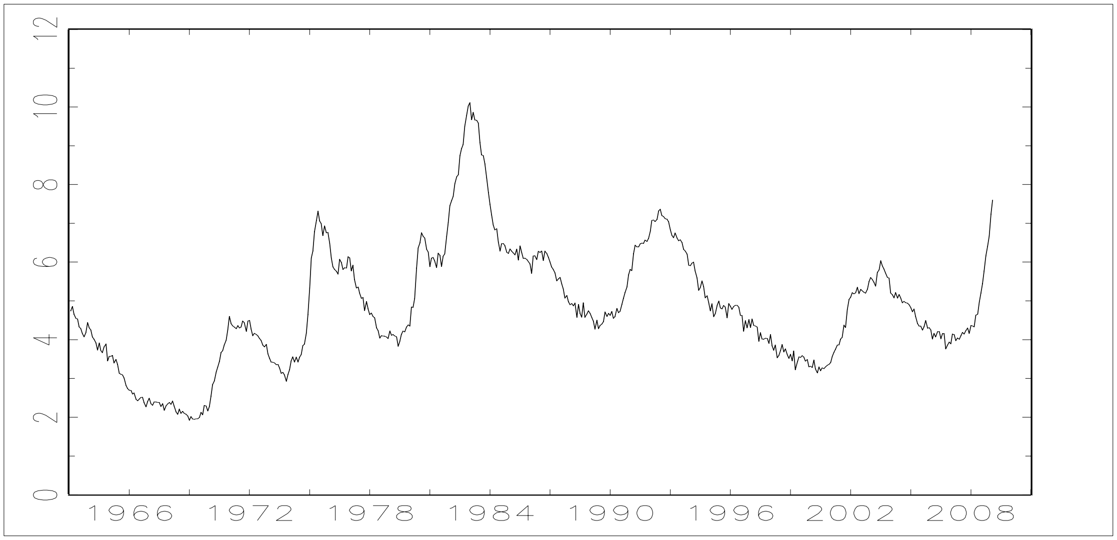
The standard ADF test fails to reject a unit root in the US male unemployment rate at the 10% significance level or lower, with a test statistic of -2.534 obtained using augmentation of 12 lags[16]. However, since the ADF test may be misleading in the presence of nonlinearities, as shown in the previous section, our analysis relies on the model-based bootstrap test. When computing the model-based bootstrap test statistic, the lag length for the autoregressive model of the unemployment rate is set equal to that of the ADF test, namely 12.

Since the transition variable is unknown, the model-based bootstrap unit root test of subsection 3.4 is applied separately for each transition variable Δy_{t-d} where values $d = 1,.....,d^{\max}$ are considered for the delay parameter d and $d^{\max}$ is equal to the selected order p of the autoregression. For each d, the functional form of the transition function is also treated as unknown, with logistic and exponential forms considered. The test provides evidence of stationarity (at the 5% significance level) for delay parameters 1, 2, 3 and 10, all of which indicate a LSTAR transition, suggesting different unemployment rate dynamics in expansions and recessions. Of these, the strongest evidence of stationarity is provided by $d = 1$, which yields a bootstrap test statistic of -3.971 and p-value of 0.002[17]. Hence, allowing for nonlinearity reverses the result of the ADF test. Application of the LM-type linearity test of Luukkonen *et al.* (1988) reinforces the presence of nonlinearity of LSTAR form, with linearity of the unemployment rate rejected with a p-value of 0.001. These findings indicate the

[15] As in many other studies, the series is constructed as the ratio of the unemployment level to the civilian labour force, both obtained from the *Bureau of Labor Statistics*.
[16] Augmentation was determined using the SBC criterion, to a maximum lag of 12, and checked using the Lagrange multiplier (LM) test for residual autocorrelation (at 5% significance level) to order 12.
[17] While this procedure follows much of the literature, nevertheless it should be noted that the use of multiple testing here implies that the quoted p-value for the test is unreliable.

inappropriateness of the standard ADF test for examining unemployment rates and favour the natural rate hypothesis, in line with Caner and Hansen (2001).

Figure 2: US Adult Male Unemployment Rate



6. Conclusions

This paper contributes to the literature that jointly analyzes nonstationarity and nonlinearity by developing a new unit root testing methodology that allows nonlinearity under the null hypothesis of a unit root. This case has previously been considered only by Caner and Hansen (2001) in the context of a two-regime TAR model; the present paper considers the broader class of smooth transition nonlinear autoregressive (STAR) processes, which encompasses the process considered in Caner and Hansen (2001) as a limiting case.

We provide three substantive results. Firstly, we prove that the addition of terms (either from a Taylor series expansion or a transition function with given parameters) to an ADF regression to account for possible nonlinearity leaves the asymptotic unit root distribution unaffected under the unit root null hypothesis when the true data generating process is a linear process. Secondly, our simulations show that this does not carry over when the true process is nonlinear, with the use of Dickey-Fuller critical values leading to very substantial under-sizing in the presence of strong nonlinearity. These results indicate that the true unit root distribution may depend on nuisance parameters, namely the (typically unknown) transition function parameters. Thirdly, in the light of these findings, we provide a bootstrap testing methodology that delivers correctly sized unit-root tests for STAR processes.

Two bootstrap approaches, the block bootstrap and a model-based bootstrap, are proposed in order to replicate the true null distribution. Although the block bootstrap test has better size properties than the standard ADF test, the model-based bootstrap dominates these. It not only delivers a test with reliable empirical size, but also has higher power than either the block bootstrap or the ADF test. Further, it is flexible in allowing (effectively) linear as well as either logistic or exponential STAR nonlinear processes. Our Monte Carlo results indicate that it performs well in all cases considered, which include ones that approximate TAR processes.

An application to the monthly U.S. male unemployment rate indicates that accounting for nonlinearities is important, with the model-based bootstrap test providing empirical support for the natural rate hypothesis while the standard ADF test fails to do so.

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Appendix: Proof of Theorem 1

We consider the linear $I(1)$ process

$$\Delta y_t = \beta_1 \Delta y_{t-1} + u_t, \quad |\beta| < 1 \tag{A.1}$$

where u_t is iid with mean zero and variance σ^2. Equivalently, we can then write

$$\eta_t = \frac{u_t}{1-\beta_1 L} = \sum_{j=0}^{\infty} \beta_1^j u_{t-j} \tag{A.2}$$

with $\sum_{j=0}^{\infty} \left|\beta_1^j\right| < \infty$. Since u_t is iid, then η_t defined by (A.2) is a strictly stationary and ergodic process with mean zero and variance $\gamma_0 = \sigma^2/\left(1-\beta_1^2\right)$. Further, denote $\xi_t = \eta_1 + \eta_2 + \ldots + \eta_t$ for $t = 1, 2, \ldots, T$ with $\xi_0 = 0$ and, therefore $y_t = \xi_t + y_0$. However, all test regressions we consider include an intercept, which takes account of y_0, and purely for expositional simplicity we assume below that $y_0 = 0$ and hence $y_t = \xi_t$.

<u>A.1 Regression Augmented with Taylor Series Expansion</u>

Consider first the unit root test in (5) for the DGP (A.1). Given the assumption $E[u_t^8] = \mu_8 < \infty$, and applying the Cauchy-Schwartz inequality, it follows that η_t in (A.2) also has finite eighth moment. Then re-write (5) as

$$\Delta y_t = \mathbf{x}_t{}'\boldsymbol{\beta} + u_t \tag{A.3}$$

where $\boldsymbol{\beta} = \left(\beta_0\ \rho_1\ \beta_1\ \beta_2\ \beta_3\ \beta_4\right)'$, $\mathbf{x_t} = \left(1\ y_{t-1}\ \Delta y_{t-1}\ \Delta y_{t-1}^2\ \Delta y_{t-1}^3\ \Delta y_{t-1}^4\right)'$. The deviation of the OLS estimator $\hat{\boldsymbol{\beta}}_{\mathbf{T}}$ in (A.3) from the true $\boldsymbol{\beta}$ = (0 0 β_1 0 0 0)' is

$$\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta} = \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t{}'\right)^{-1} \sum_{t=1}^{T} \mathbf{x}_t u_t \tag{A.4}$$

and the asymptotic distributions of the elements of $\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t{}'$ and $\sum_{t=1}^{T} \mathbf{x}_t u_t$ can be obtained as follows.

1) Elements of $\sum_{t=1}^{T} \mathbf{x_t} \mathbf{x_t}'$:

   i) Hamilton (1994, pp.505-506) demonstrates that

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\xrightarrow{L}\frac{\sigma}{1-\beta_1}\int_0^1 W(r)\,dr \tag{A.5}$$

$$T^{-1/2}\sum_{t=1}^{T}\eta_{t-1}\xrightarrow{L}\frac{\sigma}{1-\beta_1}W(1) \tag{A.6}$$

$$T^{-2}\sum_{t=1}^{T}\xi_{t-1}{}^{2}\xrightarrow{L}\frac{\sigma^2}{(1-\beta_1)^2}\int_0^1 (W(r))^2\,dr \tag{A.7}$$

$$T^{-1}\sum_{t=1}^{T}\xi_{t-1}\eta_{t-1}\xrightarrow{L}(1/2)\left(\frac{\sigma^2}{(1-\beta_1)^2}W(1)^2-\gamma_0\right)+\gamma_0 \tag{A.8}$$

ii) Since $E\left(\eta_t^{8}\right)<\infty$, the law of large numbers (LLN) implies that

$$T^{-1}\sum_{t=1}^{T}\eta_{t-1}{}^{2}\xrightarrow{p}\gamma_0 \tag{A.9}$$

and

$$T^{-1}\sum_{t=1}^{T}\eta_{t-1}{}^{i}\xrightarrow{p}E(\eta_t^{i}),\ i=2,3,....8 \tag{A.10}$$

where $\xrightarrow{p}$ indicates convergence in probability.

iii) The asymptotic distribution of $T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\eta_{t-1}{}^{2}$ is derived as follows. Firstly, substracting and adding terms in $\gamma_0=E\left(\eta_{t-1}^2\right)$, we have

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\eta_{t-1}{}^{2}=T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\left(\eta_{t-1}{}^{2}-\gamma_0\right)+T^{-3/2}\gamma_0\sum_{t=1}^{T}\xi_{t-1}.$$

Using $\xi_{t-1}=\xi_{t-2}+\eta_{t-1}$, the first term on the right-hand side of this expression can be written as

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\left(\eta_{t-1}{}^{2}-\gamma_0\right)=T^{-3/2}\sum_{t=1}^{T}\xi_{t-2}\left(\eta_{t-1}{}^{2}-\gamma_0\right)+T^{-3/2}\left(\sum_{t=1}^{T}\eta_{t-1}{}^{3}-\gamma_0\sum_{t=1}^{T}\eta_{t-1}{}^{2}\right)$$

and the results of (ii) then imply that

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\left(\eta_{t-1}{}^{2}-\gamma_0\right)=T^{-3/2}\sum_{t=1}^{T}\xi_{t-2}\left(\eta_{t-1}{}^{2}-\gamma_0\right)+o_p(1). \tag{A.11}$$

Next, let $k_t=\left(\eta_{t-1}\ \ \eta_{t-1}{}^{2}-\gamma_0\right)'$ and $K_t=\sum_{s=1}^{t}k_s$. Given that η_{t-1} is a zero mean, strictly stationary and ergodic process, so is $\left(\eta_{t-1}{}^{2}-\gamma_0\right)$. It then follows from Phillips (1988) that the sum $T^{-1}\sum_{t=1}^{T} K_{t-1}k_t'$ converges to a stochastic integral. The convergence of $T^{-1}\sum_{t=1}^{T} K_{t-1}k_t'$ requires convergence of all elements to some stochastic integrals. Since

$$T^{-1}\sum_{t=1}^{T} K_{t-1}k_t' = T^{-1}\sum_{t=1}^{T}\begin{pmatrix}\sum_{s=1}^{t-1}\eta_{s-1} \\ \sum_{s=1}^{t-1}\left(\eta_{s-1}{}^2-\gamma_0\right)\end{pmatrix}\left(\eta_{t-1} \quad \eta_{t-1}{}^2-\gamma_0\right)$$

$$= T^{-1}\sum_{t=1}^{T}\begin{pmatrix}\xi_{t-2} \\ \sum_{s=1}^{t-1}\left(\eta_{s-1}{}^2-\gamma_0\right)\end{pmatrix}\left(\eta_{t-1} \quad \eta_{t-1}{}^2-\gamma_0\right),$$

this implies that

$$T^{-1}\sum_{t=1}^{T}\xi_{t-2}\left(\eta_{t-1}{}^2-\gamma_0\right)=O_p(1)$$

and, therefore,

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-2}\left(\eta_{t-1}{}^2-\gamma_0\right)=o_p(1). \tag{A.12}$$

Consequently, combining (A.5) and (A.12) through (A.11), then

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\Delta y_{t-1}{}^2 \xrightarrow{L} \frac{\sigma}{1-\beta_1}\gamma_0\int_0^1 W(r)\,dr. \tag{A.13}$$

iv) Generalizing the arguments in (iii) yields

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\eta_{t-1}{}^i \xrightarrow{L} \frac{\sigma}{1-\beta_1}E\left(\eta_t{}^i\right)\int_0^1 W(r)\,dr,\ i=3,4. \tag{A.14}$$

2) Elements of $\sum_{t=1}^{T}\mathbf{x}_t u_t$:

Standard results yield

$$T^{-1/2}\sum_{t=1}^{T}u_t \xrightarrow{L} N(0,\sigma^2)=\sigma W(1) \tag{A.15}$$

$$T^{-1}\sum_{t=1}^{T}u_t\xi_{t-1} \xrightarrow{L} (1/2)\frac{\sigma^2}{1-\beta_1}\left(W(1)^2-1\right). \tag{A.16}$$

Since u_t is iid and $\eta_{t-1} = g(u_{t-1}, u_{t-2}, u_{t-3}, ...)$, where $g(.)$ is a continuous function, $u_t\eta_{t-1}$ is a martingale difference sequence with a finite variance $\sigma^2 E\left(\eta_t^{\,2}\right)$. Hence, from the martingale difference sequence CLT,

$$T^{-1/2}\sum_{t=1}^{T} u_t\eta_{t-1}^{i} \xrightarrow{L} N\left(0, \sigma^2 E\left(\eta_t^{2i}\right)\right) = \sigma\left[E\left(\eta_t^{2i}\right)\right]^{1/2} W(1), \quad i = 1,2,3,4\,. \qquad \text{(A.17)}$$

Noting that $\mathbf{x}_t = \left(1\ y_{t-1}\ \Delta y_{t-1}\ \Delta y_{t-1}^2\ \Delta y_{t-1}^3\ \Delta y_{t-1}^4\right)' = \left(1\ \xi_{t-1}\ \eta_{t-1}\ \eta_{t-1}^2\ \eta_{t-1}^3\ \eta_{t-1}^4\right)'$, it is straightforward to see from (A.5) to (A.17) that $\hat{\boldsymbol{\beta}}_{\mathbf{T}} - \boldsymbol{\beta} = o_p(1)$ and hence $\hat{\boldsymbol{\beta}}_{\mathbf{T}}$ is consistent. Further, the convergence rates in those expressions imply that the appropriate scaling matrix is $\mathbf{D_T} = diag\left(T^{1/2}\ \ T\ \ T^{1/2}\ \ T^{1/2}\ \ T^{1/2}\ \ T^{1/2}\right)$.

Using these results,

$$\mathbf{D_T}(\hat{\boldsymbol{\beta}}_{\mathbf{T}} - \boldsymbol{\beta}) = \begin{pmatrix} T^{1/2}\hat{\beta}_{0T} \\ T\hat{\rho}_{1t} \\ T^{1/2}\left(\hat{\beta}_{1T} - \beta_1\right) \\ T^{1/2}\hat{\beta}_{2T} \\ T^{1/2}\hat{\beta}_{3T} \\ T^{1/2}\hat{\beta}_{4T} \end{pmatrix} = \left(\mathbf{D_T^{-1}}\sum_{t=0}^{T}\mathbf{x_t}\mathbf{x_t}'\mathbf{D_T^{-1}}\right)\left(\mathbf{D_T^{-1}}\sum_{t=0}^{T}\mathbf{x_t}\,u_t\right) \qquad \text{(A.18)}$$

$$\xrightarrow{L} \mathbf{V^{-1}q} = \begin{bmatrix} \mathbf{V_1} & \mathbf{V_2} \\ \mathbf{V_2}' & \mathbf{V_3} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q_1} \\ \mathbf{q_2} \end{bmatrix}$$

where

$$\mathbf{V_1} = \begin{pmatrix} 1 & \dfrac{\sigma}{1-\beta_1}\displaystyle\int_0^1 W(r)\,dr & 0 \\ \dfrac{\sigma}{1-\beta_1}\displaystyle\int_0^1 W(r)\,dr & \dfrac{\sigma^2}{(1-\beta_1)^2}\displaystyle\int_0^1 \left(W(r)\right)^2 dr & 0 \\ 0 & 0 & \dfrac{\sigma^2}{1-\beta_1^{\,2}} \end{pmatrix},$$

$$\mathbf{V}_2 = \begin{pmatrix} \dfrac{\sigma^2}{1-\beta_1^{\ 2}} & E\left(\eta_t^{\ 3}\right) & E\left(\eta_t^{\ 4}\right) \\ \dfrac{\sigma}{1-\beta_1} \dfrac{\sigma^2}{1-\beta_1^{\ 2}} \int_0^1 W(r)\,dr & \dfrac{\sigma E\left(\eta_t^{\ 3}\right)}{1-\beta_1} \int_0^1 \left(W(r)\right) dr & \dfrac{\sigma E\left(\eta_t^{\ 4}\right)}{1-\beta_1} \int_0^1 \left(W(r)\right) dr \\ E\left(\eta_t^{\ 3}\right) & E\left(\eta_t^{\ 4}\right) & E\left(\eta_t^{\ 5}\right) \end{pmatrix},$$

$$\mathbf{V_3} = \begin{pmatrix} E\left(\eta_t^{\ 4}\right) & E\left(\eta_t^{\ 5}\right) & E\left(\eta_t^{\ 6}\right) \\ E\left(\eta_t^{\ 5}\right) & E\left(\eta_t^{\ 6}\right) & E\left(\eta_t^{\ 7}\right) \\ E\left(\eta_t^{\ 6}\right) & E\left(\eta_t^{\ 7}\right) & E\left(\eta_t^{\ 8}\right) \end{pmatrix}$$

$$\mathbf{q}_1 = \begin{pmatrix} \sigma W(1) \\ \dfrac{\sigma^2}{2(1-\beta_1)}\left(W(1)^2 - 1\right) \\ \left(\sigma^2 \gamma_0\right)^{1/2} W(1) \end{pmatrix}, \ \mathbf{q_2} = \begin{pmatrix} \left(\sigma^2 E(\eta_t^{\ 4})\right)^{1/2} W(1) \\ \left(\sigma^2 E(\eta_t^{\ 6})\right)^{1/2} W(1) \\ \left(\sigma^2 E(\eta_t^{\ 8})\right)^{1/2} W(1) \end{pmatrix}.$$

Therefore, $T\hat{\rho}_{1T} \xrightarrow{L} \mathbf{v_2 q}$ where $\mathbf{v_2}$ is the second row of $\mathbf{V}^{-1}$ and, after some matrix algebra, $\mathbf{v_2}$ can be shown to equal

$$\begin{pmatrix} \dfrac{(1-\beta_1)\int_0^1 W(r)\,dr}{\sigma\left(\left(\int_0^1 W(r)\,dr\right)^2 - \int_0^1 (W(r))^2\,dr\right)} & \dfrac{-(1-\beta_1)^2}{\sigma^2\left(\left(\int_0^1 W(r)\,dr\right)^2 - \int_0^1 (W(r))^2\,dr\right)} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently,

$$T\hat{\rho}_{1T} \xrightarrow{L} (1-\beta_1) \frac{(1/2)\left(W(1)^2 - 1\right) - W(1)\int_0^1 W(r)dr}{\int_0^1 (W(r))^2\,dr - \left(\int_0^1 W(r)dr\right)^2}. \tag{A.19}$$

The OLS t test statistic for $\rho_1 = 0$ in (5) can be written as

$$t_{\hat{\rho}_{1T}} = \frac{T\hat{\rho}_{1T}}{\sqrt{s_T^{\ 2}\mathbf{e_T}\left(\mathbf{D_T}^{-1}\sum_{t=1}^{T}\mathbf{x_t}\mathbf{x_t}'\mathbf{D_T}^{-1}\right)^{-1}\mathbf{e_T}'}} \tag{A.20}$$

where $\mathbf{e_T}' = \begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0\end{pmatrix}$ and $s_T{}^2 = \left(T-6\right)^{-1}\left(\eta_t - \mathbf{x_t}'\hat{\boldsymbol{\beta}}_\mathbf{T}\right)'\left(\eta_t - \mathbf{x_t}'\hat{\boldsymbol{\beta}}_\mathbf{T}\right)$. From the continuous mapping theorem and $\left(\hat{\boldsymbol{\beta}}_\mathbf{T} - \boldsymbol{\beta}\right) = o_p(1)$, it is easy to show that $s_T{}^2 \xrightarrow{p} \sigma^2$. Hence,

$$\sqrt{s_T{}^2\mathbf{e_T}\left(\mathbf{D_T}^{-1}\sum_{t=1}^{T}\mathbf{x_t x_t}'\mathbf{D_T}^{-1}\right)^{-1}\mathbf{e_T}'} \xrightarrow{p} \sqrt{\frac{\left(1-\beta_1\right)^2}{\int_0^1 \left(W(r)\right)^2 dr - \left(\int_0^1 W(r)dr\right)^2}}. \tag{A.21}$$

Using (A.17) and (A.21) in (A.20) immediately yields the result that $t_{\hat{\rho}_{1T}}$ asymptotically follows the ADF distribution of (9).

A.2 Regression Augmented with Transition Function

In order to obtain the asymptotic null distribution of the $t_{\hat{\rho}_{2T}}$ statistic in the test regression (6), which includes an arbitrary logistic transition function $F_L(\gamma_{0L}, c_{0L})$, re-parameterize this regression in the form of (A.3), but now with $\boldsymbol{\beta} = \left(\beta_0\ \rho_2\ \beta_1\ \beta_2^*\right)' = \left(0\ 0\ \beta_1\ 0\right)'$ and $\mathbf{x_t} = \left(1\ y_{t-1}\ \ \Delta y_{t-1}\ \ \Delta y_{t-1}F_t\right)'$, in which $F_t = F_L\left(\gamma_{0L}, c_{0L}\right)$ is defined as in (2) for given parameters (γ_{0L}, c_{0L}). This parameterization of the test regression is convenient for notational purposes, while leaving the unit root test coefficient ρ_2 unaffected. The true process is again given by (A.1), with $y_0 = 0$. Since F_t is a continuous and bounded function of $\Delta y_{t-1} = \eta_{t-1}$, it is strictly stationary and ergodic with $0 \leq F_t \leq 1$ and $0 \leq E\left(F_t^i\right) \leq 1$, $i = 1, 2, 3, 4$.

The deviations of the OLS estimator $\hat{\boldsymbol{\beta}}_\mathbf{T}$ from the true $\boldsymbol{\beta}$ again has the form of (A.4), with elements as follows.

1) Elements of $\sum_{t=1}^{T}\mathbf{x_t x_t}'$:

The asymptotic distributions of (A.5) to (A.7) continue to apply, while the Law of Large Numbers (LLN) and boundness of F_t imply that

$$T^{-1}\sum_{t=1}^{T}F_t\eta_{t-1} \xrightarrow{p} C_0 \tag{A.22}$$

$$T^{-1}\sum_{t=1}^{T}F_t\ \eta_{t-1}^2 \xrightarrow{p} C_1 \tag{A.23}$$

$$T^{-1}\sum_{t=1}^{T} F_t^2 \eta_{t-1}^2 \xrightarrow{\ p\ } C_2 \tag{A.24}$$

where

$C_0 = E\left(F_t\,\eta_{t-1}\right) \le \left(E\left(F_t^2\right)\right)^{1/2}\left(E\left(\eta_{t-1}^2\right)\right)^{1/2} < \infty\,, \quad C_1 = E\left(F_t\,\eta_{t-1}^2\right) \le \left(E\left(F_t^2\right)\right)^{1/2}\left(E\left(\eta_{t-1}^4\right)\right)^{1/2} < \infty$

and $C_2 = E\left(F_t^2\,\eta_{t-1}^2\right) \le \left(E\left(F_t^4\right)\right)^{1/2}\left(E\left(\eta_{t-1}^4\right)\right)^{1/2} < \infty$.

Further, to derive the asymptotic distribution of $T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\,F_t\,\eta_{t-1}$, we re-write it as

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\,F_t\,\eta_{t-1} = T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\left(F_t\,\eta_{t-1} - E\left(F_t\,\eta_{t-1}\right)\right) + T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\,E\left(F_t\,\eta_{t-1}\right)$$

and from (A.22) it becomes

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\,F_t\,\eta_{t-1} = T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\left(F_t\,\eta_{t-1} - C_0\right) + T^{-3/2}C_0\sum_{t=1}^{T}\xi_{t-1} \tag{A.25}$$

Consider the first term on the right-hand side of this expression, and again noting that $\xi_{t-1} = \xi_{t-2} + \eta_{t-1}$, re-write it as

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\left(F_t\,\eta_{t-1} - C_0\right) = T^{-3/2}\sum_{t=1}^{T}\xi_{t-2}\left(F_t\,\eta_{t-1} - C_0\right) + T^{-3/2}\left(\sum_{t=1}^{T}F_t\,\eta_{t-1}{}^2 - C_0\sum_{t=1}^{T}\eta_{t-1}\right).$$

Then inferences from (A.6) and (A.23) imply that

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-2}\left(F_t\,\eta_{t-1} - C_0\right) = T^{-3/2}\sum_{t=1}^{T}\xi_{t-2}\left(F_t\,\eta_{t-1} - C_0\right) + o_p\left(1\right).$$

Now setting $k_t = \left(\eta_{t-1} \;\; F_t\eta_{t-1} - C_0\right)'$ and following analogous steps to those used to obtain (A.13) above, it can be seen that

$$T^{-3/2}\sum_{t=1}^{T}\xi_{t-1}\,F_t\,\eta_{t-1} \xrightarrow{\ L\ } \frac{\sigma}{1-\beta_1}C_0\int_0^1 W(r)\,dr\,. \tag{A.26}$$

2) Elements of $\sum_{t=1}^{T}\mathbf{x_t}u_t$:

The results in (A.15) to (A.17) also apply in this case, while the martingale difference sequence CLT reveals that

$$T^{-1/2}\sum_{t=1}^{T}F_t\,\eta_{t-1}\,u_t \xrightarrow{\ L\ } \left(\sigma^2\,C_2\right)^{1/2}W(1) \tag{A.27}$$

From the convergence rates of the elements of $\left(\hat{\boldsymbol{\beta}}_{\mathbf{T}}-\boldsymbol{\beta}\right)$, the form of the scaling matrix is $\mathbf{D_T} = diag\left(T^{1/2}\ \ T\ \ T^{1/2}\ \ T^{1/2}\right)$. Therefore, analogously to (A.18),

$$\mathrm{D_T}(\hat{\beta}_\mathrm{T}-\beta) = \left(\mathrm{D_T^{-1}}\sum_{t=1}^{T}\mathrm{x_t x_t{}'D_T^{-1}}\right)^{-1}\mathrm{D_T^{-1}}\sum_{t=1}^{T}\mathrm{x_t}\,u_t \xrightarrow{L} \begin{bmatrix}\mathbf{V_1} & \mathbf{V_2}\\ \mathbf{V_2}' & \mathbf{V_3}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{h_1}\\ \mathbf{h_2}\end{bmatrix} \qquad \text{(A.28)}$$

where

$$\mathbf{V_1} = \begin{pmatrix} 1 & \dfrac{\sigma}{1-\beta_1}\int_0^1 W(r)\,dr \\ \dfrac{\sigma}{1-\beta_1}\int_0^1 W(r)\,dr & \dfrac{\sigma^2}{(1-\beta_1)^2}\int_0^1 \left(W(r)\right)^2 dr \end{pmatrix},\ \mathbf{V}_2 = C_0\begin{pmatrix} -1 & 1 \\ \dfrac{-\sigma}{1-\beta_1}\int_0^1 W(r)\,dr & \dfrac{\sigma}{1-\beta_1}\int_0^1 W(r)\,dr \end{pmatrix},$$

$$\mathbf{V_3} = \begin{pmatrix} \dfrac{\sigma^2}{1-\beta_1^2}-2C_1+C_2 & C_1-C_2 \\ C_1-C_2 & C_2 \end{pmatrix},$$

$\mathbf{h_1} = \begin{pmatrix} \sigma W(1) \\ \dfrac{\sigma^2}{2(1-\alpha_1)}\left(W(1)^2-1\right) \end{pmatrix}$ and $\mathbf{h}_2 = \begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix}$, in which $h_{21} \sim \left(\sigma^2\left(\dfrac{\sigma^2}{1-\beta_1^2}-2C_1+C_2\right)\right)^{1/2} N(0,1)$

and $h_{22} \sim \left(\sigma^2 C_2\right)^{1/2} N(0,1)$.

From (A.28), and after some matrix algebra, it can be seen that

$$T\hat{\rho}_{2T} \xrightarrow{\mathrm{L}} \begin{pmatrix} \dfrac{(1-\beta_1)\int_0^1 W(r)dr}{\sigma((\int_0^1 W(r)dr)^2-\int_0^1 (W(r))^2 dr} & \dfrac{-(1-\beta_1)^2}{\sigma^2((\int_0^1 W(r)dr)^2-\int_0^1 (W(r))^2 dr} & 0 & 0 \end{pmatrix}\mathbf{h}.$$

Therefore, $T\hat{\rho}_{2T}$ follows the asymptotic distribution of (A.19) and the ADF distribution of (9) follows for the *t*-ratio in a straightforward way.

Table 1: Empirical Size of ADF Test Augmented for Nonlinearity

α_1 α_2	γ	Linear ADF		Taylor Series Approximation		True Transition Function		Power of Nonlinearity	
		T = 100	T = 300	T = 100	T = 300	T = 100	T = 300	T = 100	T = 300
	0.01	0.0500	0.0512	0.0480	0.0499	0.0491	0.0486	0.042	0.048
-0.4 0.1	0.1	0.0527	0.0502	0.0467	0.0502	0.0489	0.0476	0.046	0.056
	0.3	0.0499	0.0500	0.0460	0.0464	0.0484	0.0459	0.065	0.127
	0.9	0.0459	0.0344	0.0407	0.0311	0.0424	0.0303	0.174	0.518
c = 1.1	1.5	0.0430	0.0272	0.0372	0.0263	0.0387	0.0260	0.259	0.725
	2.5	0.0412	0.0251	0.0370	0.0236	0.0382	0.0223	0.329	0.833
	50	0.0409	0.0249	0.0357	0.0227	0.0365	0.0213	0.424	0.917
	0.01	0.0538	0.0518	0.0526	0.0529	0.0535	0.0529	0.035	0.041
0.1 0.8	0.1	0.0521	0.0512	0.0528	0.0512	0.0544	0.0490	0.041	0.062
	0.3	0.0501	0.0436	0.0486	0.0397	0.0493	0.0390	0.084	0.285
	0.9	0.0447	0.0223	0.0369	0.0162	0.0386	0.0154	0.436	0.968
c = 0	1.5	0.0393	0.0174	0.0289	0.0106	0.0280	0.0111	0.640	0.998
	2.5	0.0357	0.0151	0.0222	0.0086	0.0237	0.0084	0.714	0.999
	50	0.0290	0.0110	0.0171	0.0074	0.0185	0.0077	0.718	0.999
	0.01	0.0521	0.0519	0.0509	0.0519	0.0514	0.0515	0.036	0.041
0.6 0.1	0.1	0.0519	0.0511	0.0507	0.0514	0.0522	0.0499	0.039	0.051
	0.3	0.0520	0.0467	0.0503	0.0471	0.0511	0.0462	0.058	0.129
	0.9	0.0450	0.0318	0.0425	0.0284	0.0442	0.0296	0.158	0.568
c = 1.1	1.5	0.0432	0.0278	0.0395	0.0238	0.0417	0.0237	0.230	0.759
	2.5	0.0433	0.0250	0.0394	0.0214	0.0408	0.0212	0.287	0.851
	50	0.0425	0.0241	0.0383	0.0207	0.0385	0.0186	0.366	0.926
	0.01	0.0526	0.0520	0.0495	0.0513	0.0509	0.0486	0.038	0.042
0.7 -0.2	0.1	0.0504	0.0497	0.0489	0.0498	0.0529	0.0482	0.044	0.070
	0.3	0.0502	0.0376	0.0455	0.0361	0.0474	0.0364	0.106	0.362
	0.9	0.0353	0.0167	0.0300	0.0133	0.0315	0.0120	0.559	0.990
	1.5	0.0313	0.0121	0.0212	0.0085	0.0218	0.0086	0.772	0.999
c = 0.5	2.5	0.0256	0.0111	0.0172	0.0071	0.0169	0.0073	0.857	1.000
	50	0.0212	0.0103	0.0136	0.0061	0.0138	0.0062	0.906	1.000
	0.01	0.0512	0.0518	0.0490	0.0502	0.0523	0.0512	0.038	0.044
0.5 -0.1	0.1	0.0506	0.0508	0.0482	0.0513	0.0522	0.0498	0.040	0.055
	0.3	0.0506	0.0453	0.0461	0.0430	0.0514	0.0437	0.065	0.161
	0.9	0.0426	0.0246	0.0381	0.0236	0.0406	0.0203	0.250	0.766
c = 0	1.5	0.0350	0.0162	0.0308	0.0136	0.0318	0.0137	0.442	0.948
	2.5	0.0300	0.0123	0.0244	0.0102	0.0259	0.0091	0.535	0.984
	50	0.0242	0.0092	0.0189	0.0076	0.0191	0.0082	0.576	0.993

Notes: All DGPs are LSTAR(1) processes, with true AR and location parameters given in the first column and transition function slope parameters in the second column. Columns three to eight report empirical rejection frequencies for the (linear) ADF test and this test augmented for nonlinearity, obtained using 50,000 replications at the 5% nominal significance level for sample sizes of T = 100, 300. The ADF test employs a data-based lag selection criterion with the true lag order employed in other cases. Power of nonlinearity is the proportion of replications for which nonlinearity is rejected by the test of Luukkonen *et al.* (1988) at the 5% level.

Table 2: Empirical Size of Block Bootstrap Unit Root Test

α_1 α_2	γ	Shorter Block Length T = 100	Shorter Block Length T = 300	Longer Block Length T = 100	Longer Block Length T = 300
-0.4 0.1	0.1	0.0474	0.0524	0.0450	0.0462
	0.9	0.0446	0.0430	0.0384	0.0396
	1.5	0.0400	0.0414	0.0346	0.0472
c = 1.1	2.5	0.0410	0.0438	0.0368	0.0394
	50	0.0414	0.0406	0.0416	0.0376
0.1 0.8	0.1	0.0540	0.0518	0.0523	0.0510
	0.9	0.0598	0.0530	0.0568	0.0482
	1.5	0.0594	0.0606	0.0574	0.0570
c = 0	2.5	0.0598	0.0658	0.0640	0.0550
	50	0.0638	0.0598	0.0582	0.0550
0.6 0.1	0.1	0.0564	0.0538	0.0522	0.0513
	0.9	0.0546	0.0540	0.0530	0.0494
	1.5	0.0540	0.0522	0.0518	0.0487
c = 1.1	2.5	0.0560	0.0522	0.0545	0.0465
	50	0.0554	0.0512	0.0532	0.0520
0.7 -0.2	0.1	0.0522	0.0548	0.0464	0.0512
	0.9	0.0530	0.0488	0.0550	0.0506
	1.5	0.0548	0.0566	0.0564	0.0560
c = 0.5	2.5	0.0590	0.0582	0.0574	0.0558
	50	0.0582	0.0560	0.0560	0.0562
0.5 -0.1	0.1	0.0522	0.0568	0.0474	0.0516
	0.9	0.0448	0.0500	0.0455	0.0496
	1.5	0.0422	0.0450	0.0480	0.0426
c = 0	2.5	0.0488	0.0488	0.0444	0.0456
	50	0.0430	0.0524	0.0480	0.0476

Notes: Empirical rejection frequencies are reported using a nominal size 5% with 5000 Monte Carlo replications and 400 bootstrap replications for an LSTAR(1) DGP. The shorter block length b is 5 for both sample sizes, while the longer block length is b = 8 for T = 100 and b = 10 for T = 300. The autoregressive and transition function parameters of the DGP are given in the first two columns of the table. The analysis assumes the true lag order of one is known, and the block bootstrap is applied to an ADF regression augmented by a third order Taylor series approximation to the transition function.

Table 3: Empirical Size of Model-Based Bootstrap Unit Root Test

α_1 α_2	γ	LSTAR Data Generating Process: Known Lags & Parameters, T = 100	Known Lags & Parameters, T = 300	Unknown Lags & Parameters; Known Form, T = 100	Unknown Lags & Parameters; Known Form, T = 300	Unknown Lags, Parameters & Form, T = 100	Unknown Lags, Parameters & Form, T = 300	ESTAR Data Generating Process: Unknown Lags, Parameters & Form, T = 100	Unknown Lags, Parameters & Form, T = 300
-0.4 0.1	0.1	0.0452	0.0529	0.0540	0.0516	0.0572	0.0524	0.0546	0.0474
	0.9	0.0468	0.0469	0.0480	0.0474	0.0490	0.0462	0.0492	0.0432
	1.5	0.0494	0.0482	0.0498	0.0460	0.0512	0.0454	0.0528	0.0452
c = 1.1	2.5	0.0482	0.0462	0.0538	0.0450	0.0524	0.0434	0.0510	0.0434
	50	0.0479	0.0482	0.0526	0.0446	0.0478	0.0448	0.0516	0.0504
0.1 0.8	0.1	0.0465	0.0505	0.0564	0.0548	0.0562	0.0554	0.0552	0.0530
	0.9	0.0498	0.0502	0.0502	0.0434	0.0508	0.0502	0.0546	0.0554
	1.5	0.0488	0.0499	0.0447	0.0484	0.0488	0.0440	0.0578	0.0560
c = 0	2.5	0.0506	0.0487	0.0427	0.0492	0.0440	0.0472	0.0542	0.0568
	50	0.0508	0.0506	0.0426	0.0498	0.0414	0.0514	0.0598	0.0548
0.6 0.1	0.1	0.0454	0.0494	0.0574	0.0494	0.0558	0.0542	0.0524	0.0472
	0.9	0.0504	0.0514	0.0519	0.0496	0.0498	0.0464	0.0492	0.0456
	1.5	0.0502	0.0474	0.0478	0.0472	0.0514	0.0488	0.0440	0.0434
c = 1.1	2.5	0.0510	0.0490	0.0467	0.0470	0.0528	0.0450	0.0522	0.0440
	50	0.0490	0.0480	0.0468	0.0482	0.0526	0.0440	0.0556	0.0524
0.7 -0.2	0.1	0.0472	0.0528	0.0555	0.0506	0.0568	0.0556	0.0526	0.472
	0.9	0.0518	0.0498	0.0471	0.0472	0.0472	0.0458	0.0496	0.444
	1.5	0.0502	0.0508	0.0453	0.0566	0.0528	0.0466	0.0458	0.452
c = 0.5	2.5	0.0492	0.0513	0.0468	0.0476	0.0438	0.0464	0.0460	0.426
	50	0.0489	0.0516	0.0451	0.0504	0.0534	0.0484	0.0560	0.540
0.5 -0.1	0.1	0.0488	0.0496	0.0552	0.0546	0.0582	0.0554	0.0532	0.526
	0.9	0.0494	0.0503	0.0482	0.0440	0.0466	0.0422	0.0586	0.548
	1.5	0.0498	0.0500	0.0448	0.0454	0.0430	0.0466	0.0564	0.552
c = 0	2.5	0.0504	0.0487	0.0432	0.0438	0.0448	0.0472	0.0556	0.532
	50	0.0496	0.0499	0.0458	0.0466	0.0456	0.0462	0.0544	0.532

Notes: All DGPs are STAR(1) processes, with true AR and location parameters given in the first column and transition function slope parameters in the second column. The remaining columns report empirical rejection frequencies for the model-based bootstrap test of the unit root null hypothesis, obtained using 5,000 Monte Carlo replications, 400 bootstrap replications and a nominal significance level of 5% for sample sizes of T = 100, 300. Lags refers to the number of autoregressive lags included in the estimated model, while Form refers to logistic versus exponential transition function. Except for the results in the third and fourth columns, the transition function parameters are estimated.

Table 4: Power Analysis

Parameters (Roots)		γ	Linear ADF Test: T = 100, ρ = -0.05	Linear ADF Test: T = 100, ρ = -0.1	Linear ADF Test: T = 300, ρ = -0.05	Linear ADF Test: T = 300, ρ = -0.1	Block Bootstrap: T = 100, ρ = -0.05	Block Bootstrap: T = 100, ρ = -0.1	Block Bootstrap: T = 300, ρ = -0.05	Block Bootstrap: T = 300, ρ = -0.1	Model-Based Bootstrap: T = 100, ρ = -0.05	Model-Based Bootstrap: T = 100, ρ = -0.1	Model-Based Bootstrap: T = 300, ρ = -0.05	Model-Based Bootstrap: T = 300, ρ = -0.1
α_1 = -0.4	α_2 = 0.1	0.1	0.114	0.246	0.471	0.959	0.065	0.128	0.350	0.890	0.170	0.329	0.514	0.954
(0.96, -0.41)	(0.94, 0.11)	0.9	0.095	0.214	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.144	0.287	0.454	0.934
(0.93, -0.43)	(0.89, 0.11)	1.5	0.090	0.204	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.342	0.874	0.147	0.293	0.448	0.931
		2.5	0.080	0.204	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.151	0.303	0.438	0.930
c = 1.1		50	0.086	0.211	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.061	0.123	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.141	0.282	0.474	0.947
α_1 = 0.1	α_2 = 0.8	0.1	0.268	0.709	0.970	0.999	0.202	0.525	0.932	1.0	0.349	0.721	0.971	1.0
(0.94, 0.11)	(0.88 ± 0.19i)	0.9	0.224	0.640	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.263	0.641	0.955	1.0	0.375	0.761	0.979	1.0
(0.89, 0.11)	(0.85 ± 0.28i)	1.5	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.271	0.647	0.975	1.0	0.373	0.782	0.982	1.0
		2.5	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{High}}$	NA$^{\text{High}}$	0.976	1.0	0.371	0.776	0.985	1.0
c = 0		50	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.269	0.652	0.971	1.0	0.352	0.755	0.986	1.0
α_1 = 0.6	α_2 = 0.1	0.1	0.216	0.602	0.926	0.999	0.155	0.402	0.863	1.0	0.305	0.660	0.926	1.0
(0.80, 0.75)	(0.94, 0.11)	0.9	0.215	0.635	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.192	0.538	0.914	1.0	0.321	0.708	0.955	1.0
(0.75 ± 0.19i)	(0.89, 0.11)	1.5	0.216	0.623	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.190	0.517	0.923	1.0	0.340	0.730	0.967	1.0
		2.5	0.203	0.613	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.215	0.538	0.914	0.999	0.336	0.737	0.954	1.0
c = 1.1		50	0.186	0.578	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.206	0.518	0.918	1.0	0.340	0.721	0.958	1.0
α_1 = 0.7	α_2 = -0.2	0.1	0.167	0.488	0.840	0.997	0.108	0.282	0.755	0.996	0.252	0.573	0.860	0.999
(0.83 ± 0.14i)	(0.96, -0.21)	0.9	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.160	0.432	0.855	0.999	0.267	0.619	0.906	1.0
(0.80 ± 0.24i)	(0.92, -0.22)	1.5	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.179	0.488	0.892	1.0	0.297	0.660	0.935	1.0
		2.5	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.191	0.485	0.903	1.0	0.280	0.627	0.927	1.0
c = 0.5		50	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.175	0.457	0.870	1.0	0.282	0.646	0.916	1.0
α_1 = 0.5	α_2 = -0.1	0.1	0.158	0.434	0.790	0.998	0.102	0.249	0.692	0.992	0.237	0.517	0.804	0.998
(0.89, 0.56)	(0.95, -0.10)	0.9	0.137	0.365	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.108	0.260	0.690	0.995	0.212	0.481	0.754	0.999
(0.70 ± 0.10i)	(0.91, -0.11)	1.5	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.106	0.291	0.686	0.994	0.209	0.460	0.804	0.999
		2.5	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.103	0.268	0.705	0.996	0.199	0.476	0.812	1.0
c = 0		50	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	NA$^{\text{Low}}$	0.110	0.285	0.711	0.996	0.214	0.494	0.793	0.999

Note: All DGPs are stationary STAR(1) processes, written as in equation (13) of the text, with true parameters, α_1, α_2 and location parameter c given in the first column and transition function slope parameters in the second column. Under each value α_i (i = 1, 2), the roots of the characteristic equation are given when this value is used in (13) in conjunction with ρ = -0.05 (first row) and ρ = -0.10 (second row). Size-adjusted empirical rejection probabilities are reported in the remaining columns, using 50,000 Monte Carlo replications for the standard ADF test, 5,000 Monte Carlo replications and 400 bootstrap replications for the bootstrap-based unit root tests. The ADF test employs a data-based lag selection criterion. The block bootstrap uses an ADF regression augmented with one autoregressive lag and a third order Taylor series approximation to the transition function, using bootstrap length b = 8 when T = 100 and b = 10 for T = 300. The model-based bootstrap assumes unknown parameters, lag order and LSTAR/ESTAR form. NA$^{\text{Low}}$ indicates that the power is not computed due to undersizing (empirical size less than 0.04), while NA$^{\text{High}}$ indicates the ones that are not computed due to oversizing (empirical size more than 0.06), in relation to the nominal size of 5 %.