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The role of demand uncertainty in the two-stage Hotelling model

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Abstract

This paper re-examines the Hotelling two-stage mill-pricing duopoly game, with the firms uncertain of the exact location of the demand at the time of choosing locations. A model is proposed that allows changes in the degree of demand uncertainty while preserving the average demand across all states of nature. This adjustment leads to strikingly different comparative statics results from those present in the existing literature. The effect of uncertainty is found to be similar to that of price discrimination in the 'certainty' model, as it leads to a decrease of product differentiation, profit reduction and social welfare improvement, with the standard 'certainty' results appearing as limiting cases.

Keywords: Hotelling, Location, Uncertainty

JEL Classification: C72, D43, D81, L13, R32

1 Introduction

Since the publication of Hotelling's original paper [7], spatial product differentiation has been a long debated issue in economic literature. First questioned on the grounds of the "purely spatial" model [4], the initial "Minimum Differentiation Principle" was eventually overturned by d'Aspremont et al. [3]. They showed that in a two stage location-then-price duopoly with quadratic transportation costs and mill-pricing the firms maximize product differentiation to the largest possible extent. Two well-established extensions of the latter framework form the baseline of this paper.

First, the model with unconstrained locations [9] illustrates that the duopolists may, in fact, choose to differentiate beyond the market's boundaries when permitted to do so. In particular, the unique pure strategy Subgame-Perfect NE has the firms located a quarter of the market's length outside the opposite boundaries.

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On the other hand, in the model with discriminatory pricing [5] the firms deliver the goods to the consumers with the discretion to set a different price at each location. Here, the unique SPNE is socially optimal with the firms now located inside the market, at its first and third quartile.

However, one assumption that could be criticised as somewhat unrealistic, is that the firms are certain of what the distribution of consumer tastes is, even before making the first-stage location decisions. In fact, a number of papers introduced some form of demand uncertainty into the Hotelling setting. Balvers and Szerb [2] study the effect of random shocks to the products' desirability under fixed prices. Harter [6] examines the uncertainty in the form of a uniformly distributed random shift of the (uniform) customer distribution, where the firms locate sequentially. Other papers, such as [1], concentrate on the strategic effect of acquiring information about the demand through price-experimentation.

But there are two approaches to the problem which are of a particular importance to this study. Unlike the papers listed above, they preserve the structure of the conventional "certainty" models (despite, of course, adding uncertain demand), so that these models can be used as benchmarks when assessing the effect of uncertain demand. In particular, Izaga [8] adapts the same form of uncertainty as Harter, but the duopolists locate simultaneously. The second study, conducted by Meagher and Zauner ([10], [11]), adapts the same setting, but also parametrizes the support of the random variable that shifts the customer distribution, which allows for comparative statics.

Those two approaches, discussed in more detail in the next section, rely on the idea of a uniformly distributed additive shock as a representation of the demand uncertainty. First introducing the shock and then increasing the spread of its distribution is said to represent the growing uncertainty about consumer tastes. Consequently, any resulting changes in the outcome of the game are attributed to the fact that the demand is becoming more uncertain, ignoring the potential effects of changes in the *average* demand. As the spread of the shock increases, some people in some states of nature begin to display preferences that were not seen before, while other sorts of tastes are becoming less common. It is therefore not obvious that the associated changes in equilibrium characteristics are due to the increase of uncertainty and not to the fact that, on average terms, the customer distribution is changing its shape.

For instance, consider a market for sweets in which the only two firms present are certain that the consumers prefer reduced sugar products and this is what they choose to provide. Suppose now that the duopolists come to believe that it is equally likely that people will start choosing ordinary sweets in favour of the reduced-sugar ones. As a result, each firm decides to specialise in a different type of product, i.e. product differentiation increases. But is this an effect of the increased demand uncertainty or simply of the fact that, on average across all states of nature, the spread of consumer tastes has become wider?

The above seems to suggest a way of re-examining the comparative statics of the Hotelling game with respect to demand uncertainty. Maintaining the general structure of Meagher and Zauner's analysis, it might be interesting to adjust it so that it would allow to study the effects of changing the uncertainty of the demand, while leaving it on average the same. In other words, the question is what happens when the consumer mass in any area of interest to the firms still has the same expected value, but is more variable than before. This will be addressed by modelling a change in demand uncertainty in the following way.

Take an arbitrary point on the line representing consumer tastes and consider a set of all points located on one side (i.e. to the left or to the right) of the point selected (in what follows, any such set will be called a "half-market"). That is to say, take any subset of the space of consumer tastes that a duopolist might capture for some combination of locations and prices. With any such half-market we can associate a random variable representing the total mass of consumers located there. This paper investigates what happens when, for every half-market, the distribution of the associated random variable undergoes a mean-preserving spread, i.e. when the associated demand allocation becomes more risky, but remains on average the same.

This could allow to disentangle the effect of changing the demand uncertainty from that of changes in its average level. In a similar way, in order to determine an individual's attitude towards risk, it is more helpful to present this person with a choice of two lotteries with equal expected values but different levels of risk, rather than with a pair of lotteries that differ with respect to both of these factors. For those reasons, a model that allows increasing the uncertainty of the demand while leaving it on average the same could be a valuable addition to the ones present in the existing literature.

The paper is structured as follows. Section Two defines the general form of the problem to be discussed, and provides a more detailed assessment of the existing research. Sections Three and Four describe the model and derive its SPNE respectively. Section Five examines the implications of the obtained equilibrium characteristics, conducting comparative statics of product differentiation, profits and social welfare. Section Six summarises the results.

2 Theoretical Background

2.1 Location-then-price Game with Demand Uncertainty

Suppose that, as in the standard Hotelling framework, there are two firms competing in a linear market with uniformly distributed consumers and quadratic transportation costs.

In the first stage of the game, firms simultaneously choose locations, x_1 and x_2, where without loss of generality $x_1 \leq x_2$. In the second stage, with locations fixed, the firms simultaneously set prices, p_1 and p_2, knowing that a consumer located at x will purchase one unit of the good from firm 1 if:

$$p_1 + t(x - x_1)^2 < p_2 + t(x - x_2)^2$$

or from firm 2 if the opposite strict inequality holds ($t > 0$ represents transportation costs and we assume without loss of generality that the production cost is zero).

The difference from the 'standard' model is that at the time of choosing x_1 and x_2, the firms are uncertain of the exact location of the demand. All they know is that in each state of the world the distribution of customers will be uniform, specifically $U(Z, Z + c)$, where $Z \sim U(a, b)$ is itself random and a, b, c are (known) constants.

Crucially, the firms learn the realisation of Z *after* they choose locations but *before* setting prices. Assuming risk-neutrality, it is then possible to define payoffs in the reduced single stage game as the firms' expectations of the second-stage equilibrium profits with respect to the distribution of Z. One can then find the SPNE locations as the equilibrium strategies of the reduced game.

2.2 Existing research

Two studies of the demand uncertainty problem, as outlined above, can be found in the literature:

1. Izaga [8] compares the standard certainty model with unconstrained locations (as in [9]) with a particular instance of the above problem: $a = 0$, $b = c = 1$. He finds that in equilibrium $x_2^* - x_1^* = 1\frac{5}{9} > 1\frac{1}{2}$(the certainty case) and concludes that demand uncertainty increases product differentiation.

2. Meagher and Zauner ([10] and [11]) extend this analysis to allow for comparative statics of welfare. They introduce a parameter $L > 0$ intended to represent the degree of uncertainty and consider a continuum of games where $a = \frac{-L-1}{2}$, $b = \frac{L-1}{2}$ and $c = 1$. Having derived the SPNE locations *in terms of* L, they conclude that demand uncertainty increases product differentiation and can have a positive *or* negative welfare effect, depending on the value of L.

Both studies rely on spreading the distribution of Z (i.e. increasing $b - a$) while holding c fixed as a representation of an increase in the demand uncertainty. The next subsection argues that this could lead to certain difficulties with interpreting the resulting changes in equilibrium characteristics.

2.3 The average customer distribution

As indicated before, in this paper we are interested in comparing situations different with respect to the degree of demand uncertainty but with other factors held constant. In particular, the customer distribution, despite becoming more or less uncertain, should remain *on average the same*. Before proceeding, the latter notion needs to be clarified.

Definition 1 *the* ***average customer distribution*** *(over all states of nature) is given by the following density function:*

$$h(x) = \int_{-\infty}^{+\infty} f(x,z) \times g(z)\, dz \tag{1}$$

where $f(x,z)$ is the customer density at location x when the state-of-nature determining variable Z takes a value z, and g is the PDF of Z

This implies that the expected value of the customer mass located between any two points x_0 and x_1 is equal to $\int_{x_0}^{x_1} h(x)dx$. In particular, for the model described in subsection 2.1, we have $Z \sim U(a,b)$ and $f(x,z)$ is the PDF of $U(z, z+c)$, i.e.:

$$g(z) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq z \leq b \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f(x,z) = \begin{cases} \frac{1}{c} & \text{for } z \leq x \leq z+c \\ 0 & \text{otherwise} \end{cases}$$

Hence, (1) takes the following form, illustrated in Fig. 1:

$$h(x) = \frac{1}{c\,(b-a)} \times \begin{cases} x-a & a \leq x < \min\{b, a+c\} \\ \min\{c, b-a\} & \min\{b, a+c\} \leq x < \max\{b, a+c\} \\ b-x+c & \max\{b, a+c\} \leq x \leq b+c \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

It follows from (2) that the "uncertainty" case in Izaga's model ($a = 0$, $b = c = 1$) is associated with the following average customer distribution:

$$h(x) = \begin{cases} x & 0 \leq x < 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

This is a symmetric triangular distribution with support on $[0, 2]$, clearly different from $U(0, 1)$.

Similarly, in Meagher and Zauner's model the spread of the average distribution is (see Fig. 1): $(b + c) - a = L + 1$, increasing in L. Moreover, for $L < 1$ the average distribution becomes more centralised as L increases, for $L = 1$ it is symmetric triangular and for $L > 1$ it decentralises again.

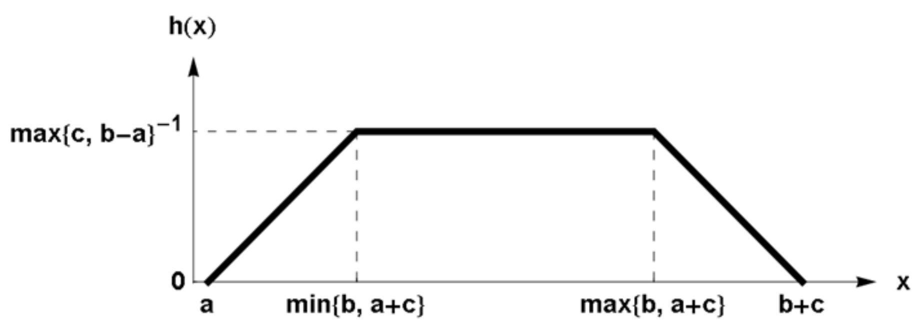

Fig. 1 The average customer distribution over all states of nature

Thus, in both cases, one may suspect that the observed changes in equilibrium characteristics are brought about by, loosely speaking, a convolution of three factors:

1. a change in the spread of the average customer distribution: any increase of $(b + c) - a$ is likely to also increase the equilibrium level of product differentiation, just as happens under certainty.[3] Expected profit maximizers are motivated to move away from one another into more distant areas when more consumers (on average) are located there. In fact, this could help understand the unbounded increase of both product differentiation and profits in the model by Meagher and Zauner when $L \to \infty$, as this is exactly what happens in the certainty case when the support of the customer distribution expands (but is difficult to interpret as a result of simply increasing the uncertainty)[4]

2. a change in the shape of the average distribution (rectangular vs. trapezoidal / triangular): the firms have a motivation to move closer to one another, i.e. reduce product differentiation, when on average there are more customers in the centre of the market. See [13] for a study of this effect in the certainty case.[5]

3. a change in the demand uncertainty, understood as a change of the riskiness of the customer mass distribution for at least some half-markets.

Consequently, it is difficult to assess how much of the observed variation in the equilibrium locations is due to (3.) and how much due to the accompanying changes (1.) and (2.). Hence, in this paper, we attempt to disentangle the effect of (3.) from those of (2.) and (1.). The following section presents a model in which it is possible to manipulate the spread of the additive shock while holding the support of the average distribution fixed, thereby eliminating any potential effect of (1.). Within this framework, a way is then proposed of making pairwise comparisons such that any variation in (2.) is also removed and the effect of (3.) is unambiguous, in the sense that the customer mass distribution becomes more risky for *every* half-market.

After the SPNE of the model are worked out, this leads to a number of statements as to the *ceteris paribus* effect of demand uncertainty, represented by a mean-preserving spread of the customer mass distribution for every half-market.

[3] it follows directly from [9] that $x_2^* - x_1^* = \frac{3}{2}c$, where c is the length of the market and also the spread of the average distribution under certainty (i.e. when $a = b = 0$)

[4] This also means that the consumers' reservation price, even if high enough to be ignored in the certainty case, is exceeded for L big enough and the validity of the model could then be questioned.

[5] For a symmetric triangular distribution with support on $[0, 1]$, there are two assymmetric SPNE, each with a differentiation of $\frac{7}{3\sqrt{6}} \approx 0.95 < \frac{3}{2}$ (the $[0, 1]$ uniform distribution case)

3 The Model

Suppose the setting is as in subsection 2.1 with $a = 0, b = 1 - m, c = m$, where $m \in [0, 1]$ is a parameter determining the size of the market in each state of nature, as well as the support of the shock Z to the market's position. In particular:

- $m = 1$ implies $a = b = 0$, i.e. the shock is always zero and the size of the market is $c = m = 1$, which means the customer distribution is always $U(0, 1)$: the standard certainty case
- as m decreases, so does the size of the market in a particular state. At the same time, the spread of the distribution of Z increases by the same amount. Hence, the ratio of the size of the market in a particular state to the size of the support of Z decreases, just as happens in the model by Meagher and Zauner when L is getting bigger.
- finally, when $m = 0$, we have $c = 0$, which we interpret as everyone being located at the same point, the distribution of which is $U(0, 1)$. Interestingly, the following statement is true.

Proposition 2 *The Hotelling mill-pricing Model with demand uncertainty and $a = c = 0, b = 1$ is equivalent to the certainty model with price discrimination and customer distribution $U(0, 1)$.*

Proof. Consider the second stage subgame of the game with uncertainty where all customers are located at z. Because of that, the firm that is cheaper to buy from for consumers at z will attract the total (unit) demand and the two firms will engage in a Bertrand-style competition. In equilibrium, the firm closer to z (an equivalent of a cost advantage) takes the entire market by setting a price that makes everyone indifferent between its offer and that of the other firm given it sets its price to zero. Hence, the profit of firm i in state z is:

$$\pi_i(x_1, x_2, z) = \max\{t \left[(x_{-i} - z)^2 - (x_i - z)^2 \right], 0\}$$

and is the same as the profit of firm i in the subgame associated with location z in the certainty model with price discrimination. Profits in the reduced single-stage game are expectations of the above with respect to the distribution of Z, i.e.:

$$\Pi_i(x_1, x_2) = \int_0^1 \pi_i(x_1, x_2, z)\ dz$$

which is the same as the payoff function of the reduced game in the certainty model (see [5]). ■

We may take Fig.1 and substitute for a, b, c to see how changing the value of m affects the average customer distribution.

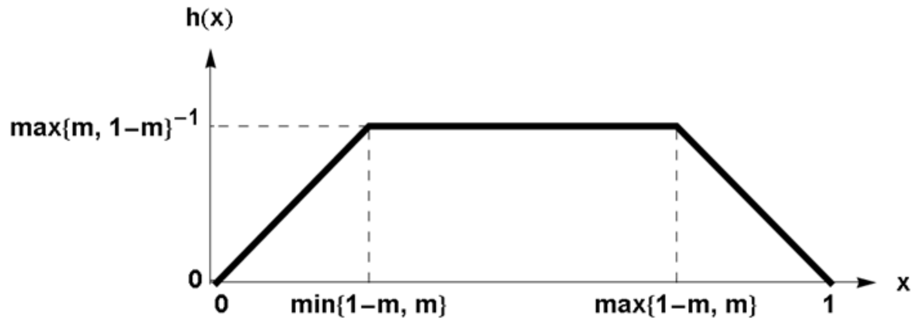

Fig. 2 The average customer distribution for a given value of m

First of all, the support of h is $[0,1]$, regardless of m. That is not to say that the shape of the distribution is maintained. For $m = 0$ the distribution is $U(0,1)$, but as uncertainty decreases, it becomes trapezoidal and the "uniform" segment $[\min\{1-m,m\}, \max\{1-m,m\}]$ initially decreases, until at $m = \frac{1}{2}$ the distribution becomes triangular and the process is reversed, until it is again $U(0,1)$ at $m = 1$. Thus, if we compare equilibria in this model associated with different values of m, we need to be aware of the fact that, despite holding the support of the average distribution fixed, some variation in firm locations may still be an effect of the average distribution becoming more or less centralised.

The problem may be avoided by considering pairs of situations where $m = \frac{1}{2} \pm v$, $v \in \left(0; \frac{1}{2}\right]$ (i.e. two values of m symmetric with respect to $\frac{1}{2}$). In the case of $m = \frac{1}{2} - v$ we have $a = 0, b = \frac{1}{2} + v, c = \frac{1}{2} - v$. Hence, (2) becomes:

$$h(x) = \frac{1}{\left(\frac{1}{2} - v\right)\left(\frac{1}{2} + v\right)} \times \begin{cases} x & x \in \left[0, \frac{1}{2} - v\right) \\ \frac{1}{2} - v & x \in \left[\frac{1}{2} - v, \frac{1}{2} + v\right) \\ 1 - x & x \in \left[\frac{1}{2} + v, 1\right] \\ 0 & \text{otherwise} \end{cases}$$

which is also the case for $m = \frac{1}{2} + v = 1 - \left(\frac{1}{2} - v\right) \Longrightarrow a = 0, b = \frac{1}{2} - v, c = \frac{1}{2} + v$. Let D_x^m denote the random variable representing the customer mass located to the left of x, given a particular value of m. It follows that:

$$\forall x, m : E\left(D_x^m\right) = E\left(D_x^{1-m}\right)$$

Moreover, the following is true.

Proposition 3 *For every $m \in (0, 1/2)$ and every $x \in (0,1)$, the distribution of D_x^m is a mean-preserving spread of the distribution of D_x^{1-m}.*

Proof. Let $F_x^m(s) = \Pr\left(D_x^m \le s\right)$. The demand located to the left of x is non-increasing in the value of the shock (z). We have:

$$(x - z)\frac{1}{m} = s \Leftrightarrow z = x - ms$$

Hence, for $s \in [0,1]$, $Z \sim U(0, 1-m)$ and $Z_0 \sim U(0,1)$:

$$F_x^m(s) = \Pr(Z \geq x - ms) = \Pr(Z_0 \geq (x - ms)/(1-m))$$

so that:

$$\text{for } \frac{x - ms}{1-m} \leq \frac{x - (1-m)s}{1-(1-m)} \Leftrightarrow s \leq x \text{ we have } F_x^m(s) \geq F_x^{1-m}(s)$$

Conversely, for $s \geq x$ we have $F_x^m(s) \leq F_x^{1-m}(s)$. Together with the equality of means ($\int sdF_x^m(s) = \int sdF_x^{1-m}(s)$) this implies:

$$\forall y : \int_0^y F_x^m(s)\ ds \geq \int_0^y F_x^{1-m}(s)\ ds$$

i.e. $F_x^m(s)$ is a mean-preserving spread of $F_x^{1-m}(s)$. ■

By symmetry, Proposition 3 is also valid when we think of D_x^m as the consumer mass located *to the right* of x. This means that the customer mass distribution associated with any half-market has exactly the same expected value, but is more risky for the smaller of any two symmetric values of m. We will refer to this situation as an *unambiguous mean-preserving increase of uncertainty.*

4 SPNE analysis

Proposition 4 *The location-then-price Hotelling game with uncertainty and customer distribution $U(Z, Z+m)$, where $Z \sim U(0, 1-m)$, has a unique pure strategy SPNE with the equilibrium locations as follows:*

$$\begin{cases} x_1^* = 1 - x_2^* = \frac{8m^2-1}{8m-4} & \text{if } 0 \leq m < \frac{1}{4} \\ x_1^* = 1 - x_2^* = \frac{20m-28m^2-1}{36m} & \text{if } \frac{1}{4} \leq m \leq 1 \end{cases}$$

Proof.[6] To begin with, consider the 2nd stage subgame of the above game associated with locations x_1, x_2 and state of nature z, i.e. a price-competition

[6] As an alternative, one could make an assumption of scale invariance and indirectly infer the solution from the one given by Meagher and Zauner:

$$\begin{cases} -x_1^* = x_2^* = \left(27 - L^2\right)/36 & \text{if } 0 < L \leq 3 \\ -x_1^* = x_2^* = \left(5 + L^2 - 2L\right)/4(L-1) & \text{if } L > 3 \end{cases}$$

Shifting the distribution of Z by $(L+1)/2$ and re-scaling it, as well as the size of the market, by the same factor of $1/(L+1)$ should result in the equilibrium undergoing the same transformation, namely:

$$\begin{cases} \widehat{x}_1^* = 1 - \widehat{x}_2^* = \left(\frac{L^2-27}{36} + \frac{L+1}{2}\right)\frac{1}{L+1} = \frac{18L-L^2-9}{36(1+L)} & \text{if } 0 < L \leq 3 \\ \widehat{x}_1^* = 1 - \widehat{x}_2^* = \left(\frac{5+L^2-2L}{4-4L} + \frac{L+1}{2}\right)\frac{1}{L+1} = \frac{2L+L^2-7}{4(L^2-1)} & \text{if } L > 3 \end{cases}$$

to be the SPNE locations when $a = 0, b = L/(L+1), c = 1/(L+1)$. Setting $L = (1-m)/m$ gives $a = 0, b = 1-m, c = m$ and the above solution reduces to the one stated in the proposition. Despite validating this conjecture, the (longer) proof given here also has the advantage of providing explicit formulae for the payoff and best-response functions, which is useful when interpreting the results in Section 5.

with given firm locations and customer distribution $U(z, z+m)$. It follows (as a simple generalisation) from [9] that the firms' equilibrium profits satisfy one of the three cases:

1. if $z < \overline{x} - 2m$, where $\overline{x} = \frac{x_1+x_2}{2}$, then the customer distribution is close enough to the 1st firm and sufficiently far away from the 2nd for the first firm to capture the entire market in equilibrium. Profits are then:

$$\begin{cases} t(2(z+m) - x_1 - x_2)(x_1 - x_2) & \text{player 1} \\ 0 & \text{player 2} \end{cases} \tag{4}$$

2. if $\overline{x} - 2m \leq z < \overline{x} + m$, then no firm can capture the entire market and equilibrium profits are:

$$\begin{cases} \frac{t(x_2-x_1)(x_1+x_2-2z+2m)^2}{18m} & \text{player 1} \\ \frac{t(x_2-x_1)(x_1+x_2-2z-4m)^2}{18m} & \text{player 2} \end{cases} \tag{5}$$

3. if $\overline{x} + m \leq z$, then case 1 is reversed and firm 2 captures the entire market, resulting in profits of:

$$\begin{cases} 0 & \text{player 1} \\ t(x_1 - x_2)(x_1 + x_2 - 2z) & \text{player 2} \end{cases} \tag{6}$$

The players' payoffs in the game are expectations of the above profits with respect to the distribution of z, i.e. $U(0, 1-m)$. Taking into account that one only needs to consider the values of z inside $[0, 1-m]$, it is possible to define $z_1 = \max\{\min\{\overline{x} - 2m, 1-m\}, 0\}$, $z_2 = \max\{\min\{\overline{x} + m, 1-m\}, 0\}$ and write the payoff of player 1 as:

$$\overbrace{\int_0^{z_1} \frac{t(2(z+m) - x_1 - x_2)(x_1 - x_2)}{1-m}\, dz}^{\text{CASE 1.}} + \overbrace{\int_{z_1}^{z_2} \frac{t(x_2 - x_1)(x_1 + x_2 - 2z + 2m)^2}{18m\,(1-m)}\, dz}^{\text{CASE 2.}}$$

By considering different cases with respect to z_1, z_2 and integrating accordingly, the above expression may be written as a piecewise function, the precise form of which depends on the value of m.

As for the second player, consider a pair of locations $x_1' = 1 - x_2$, $x_2' = 1 - x_1$ and $z' = 1 - m - z$. Observe that according to (4)-(6) the NE profit of player 2 given locations (x_1', x_2') and customer distribution $U(z', z'+m)$ is the same as the NE profit of player 1 given locations (x_1, x_2) and customer distribution $U(z, z+m)$. In other words, given a pair of locations symmetric with respect to $\frac{1}{2}$ to the original ones, for every s.o.n. z there is a (symmetric) $z' \in [0, 1-m]$ such that the NE profit of the player located at x_1' in state z' is equal to the NE profit of the player located at x_2 in state z. Since z is uniformly distributed, the expected profit of the x_2 player must be equal to that of the x_1' player, i.e. we may write:

$$\mathrm{E}\Pi_2\left(x_1, x_2\right) = \mathrm{E}\Pi_1\left(1 - x_2, 1 - x_1\right) \tag{7}$$

where $\mathrm{E}\Pi_i\left(\cdot\right)$ is the expected profit of player i. Thus, in what follows we focus on the analysis of player 1, as the payoff and best-response functions of the second player can then be easily obtained. The analysis is split into two parts.

4.1 Case $0 < m < \frac{1}{4}$ ("A")

In this situation it is impossible for the firms to share the market in all states of nature, i.e. $z_1 = 0 \Leftrightarrow \overline{x} - 2m \leq 0$ implies $\overline{x} + m < 1 - m \Leftrightarrow z_2 < 1 - m$ and $z_2 = 1 - m$ implies $z_1 > 0$. Consequently, the payoff of player 1, $\mathrm{E}\Pi_1^{\mathrm{A}}\left(x_1, x_2\right)$, becomes:

$$\begin{cases} 0 & x_1 \leq -2m - x_2 \\ \frac{2t(x_2-x_1)(m+\overline{x})^3}{27(1-m)m} & -2m - x_2 < x_1 \leq 4m - x_2 \\ \frac{t(x_2-x_1)\left(2m(m-\overline{x})+\overline{x}^2\right)}{1-m} & 4m - x_2 < x_1 \leq 2 - 4m - x_2 \\ \frac{38m^3-2(\overline{x}-1)^3+3m(2+\overline{x})(5\overline{x}-2)-6m^2(13\overline{x}-4)}{27(1-m)m[t(x_2-x_1)]^{-1}} & 2 - 4m - x_2 < x_1 \leq 2 + 2m - x_2 \\ t(x_2 - x_1)(2\overline{x} - 1 - m) & x_1 > 2 + 2m - x_2 \end{cases}$$

Notation 5 *Let $\varphi[[j.a]]$ denote the functional form constituting the j-th segment of the piecewise function φ and let $\varphi[[j.b]]$ denote the associated inequality condition.*

For instance, $\mathrm{E}\Pi_1^{\mathrm{A}}[[2.a]] = \frac{2t(x_2-x_1)(m+\overline{x})^3}{27(1-m)m}$ and $\mathrm{E}\Pi_1^{\mathrm{A}}[[1.b]]$ represents $x_1 \leq -2m - x_2$.

The corresponding best-response function $\mathrm{BR}_1^{\mathrm{A}}\left(x_2\right)$ is:

$$\begin{cases} x_2 & x_2 \leq -m \\ \frac{1}{2}(x_2 - m) & -m < x_2 \leq 3m \\ \frac{1}{3}\left(4m - x_2 + 2\sqrt{x_2^2 - 2m^2 - 2mx_2}\right) & 3m < x_2 \leq \frac{16m-22m^2-3}{-2+6m} \\ \text{2nd root (with respect to } x_1\text{) of: } \frac{\partial \mathrm{E}\Pi_1^{\mathrm{A}}[[4.a]]}{\partial x_1} & \frac{16m-22m^2-3}{-2+6m} < x_2 \leq \frac{3+3m}{2} \\ \frac{1+m}{2} & x_2 > \frac{3+3m}{2} \end{cases}$$

and the unique SPNE locations are:

$$x_1^* = 1 - x_2^* = \frac{8m^2 - 1}{8m - 4} \tag{8}$$

(see the Appendix for details of the proof).

4.2 Case $\frac{1}{4} \leq m < 1$ ("B")

In this case it is possible for the firms to share the market in all states of nature, i.e. we may have $z_1 = 0 \iff \overline{x} - 2m \leq 0$ and $z_2 = 1 - m \Leftrightarrow \overline{x} + m \geq 1 - m$, since $1 - 2m \leq 2m \Leftrightarrow m \geq \frac{1}{4}$. Moreover, $z_1 > 0$ implies $z_2 = 1 - m$ and $z_2 < 1 - m$ implies $z_1 = 0$, i.e. at most one firm can capture the whole market in some states of nature. The payoff of player 1, $\mathrm{E}\Pi_1^{\mathrm{B}}(x_1, x_2)$, becomes:

$$\begin{cases} 0 & \mathrm{E}\Pi_1^{\mathrm{A}}[[1.b]] \\ \mathrm{E}\Pi_1^{\mathrm{A}}[[2.a]] & -2m - x_2 < x_1 < 2 - 4m - x_2 \\ \frac{2t(x_2-x_1)\left((\overline{x}+m)^3-(\overline{x}+2m-1)^3\right)}{27(1-m)m} & 2 - 4m - x_2 \leq x_1 \leq 4m - x_2 \\ \mathrm{E}\Pi_1^{\mathrm{A}}[[4.a]] & 4m - x_2 < x_1 < 2 + 2m - x_2 \\ \mathrm{E}\Pi_1^{\mathrm{A}}[[5.a]] & \mathrm{E}\Pi_1^{\mathrm{A}}[[5.b]] \end{cases}$$

leading to a best response function $\mathrm{BR}_1^{\mathrm{B}}(x_2)$:

$$\begin{cases} x_2 & x_2 \leq -m \\ \mathrm{BR}_1^{\mathrm{A}}[[2.a]] & -m < x_2 \leq \frac{1}{3}(4-7m) \\ \frac{2-6m-x_2+2\left(2m^2-m-x_2+3mx_2+x_2{}^2\right)^{1/2}}{3} & \frac{1}{3}(4-7m) < x_2 \leq \varphi_0(m) \\ \mathrm{BR}_1^{\mathrm{A}}[[4.a]] & \varphi_0(m) < x_2 \leq \varphi_1(m) \\ \mathrm{BR}_1^{\mathrm{A}}[[5.a]] & x_2 > \varphi_1(m) \end{cases}$$

where $\varphi_0(m)$ and $\varphi_1(m)$ are certain increasing functions of m. $\mathrm{BR}_1^{\mathrm{B}}(x_2)$ is discontinuous for $m > \frac{2}{3}$, but it does not affect the equilibrium. The SPNE locations are still unique, specifically:

$$x_1^* = 1 - x_2^* = \left(20m - 28m^2 - 1\right)/36m \tag{9}$$

(again, see the Appendix for details).

5 Comparative Statics

5.1 Product Differentiation

It follows from (9) that for $m \to 1$ the SPNE locations approach the ones associated with the unconstrained "certainty" Hotelling Model (as in [9]). This is also the case as $L \to 0$ in Meagher & Zauner's model.

The new feature here is that the other limiting case, for $m \to 0$, is the "certainty" Hotelling Model with price discrimination, in which the SPNE locations are $x_1^* = \frac{1}{4}$ and $x_2^* = \frac{3}{4}$.

The piecewise function giving the SPNE locations for a given m (depicted in Fig. 3 below) has a "kink" at $m = \frac{1}{4}$, i.e. at the point of "switching" from case "A" and the solution (8) to case "B" and the solution (9). The locations are closest to one another at $m^* = \frac{1}{4}\left(2 - \sqrt{2}\right) \approx 0.15$, with equilibrium product differentiation decreasing in m for $m \in [0, m^*]$ and increasing otherwise. Most importantly, when we look at the pairs of symmetric values of m, i.e. consider $m = \frac{1}{2} \pm v$, $v \in \left(0; \frac{1}{2}\right]$, it is obvious that the lower value of m is always associated with less product differentiation.

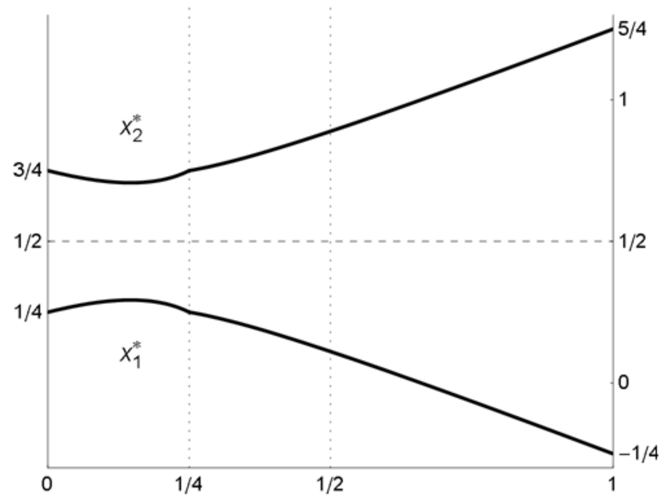

Fig. 3 The SPNE locations for different values of m

Corollary 6 *In a location-then-price Hotelling game with uncertainty and customer distribution $U(Z, Z+m)$, where $Z \sim U(0, 1-m)$, an unambiguous mean-preserving increase of uncertainty decreases the SPNE product differentiation.*

What could explain the effect associated with $m \in [0, m^*]$ is the fact that for $m < \frac{1}{2}$ any increase of m results in the average customer distribution becoming more centralised, despite its support being held constant. A larger customer base in the centre of the market is what convinces the firms to move even more towards it. However, the same increase of m also creates opportunities for sharing the market in more and more states of nature. Thus, it is becoming more and more beneficial to "accommodate", rather than "fight" and there is no better way in this framework to weaken the price competition than by spatial differentiation. Eventually, that second tendency "overpowers" the first one and the equilibrium level of differentiation begins to increase.

This can be seen by looking at the best-response curves depicted in Fig. 4 below. The dashed downwards sloping line corresponds to $x_1 = 1 - x_2$ and hence its intersection with the BR function gives the symmetric SPNE locations. In general, as m increases, the BR curve shifts downwards, approaching the dotted bottom BR curve of the "certainty" mill-pricing model. Consequently, the duopolists locate further away from each other.

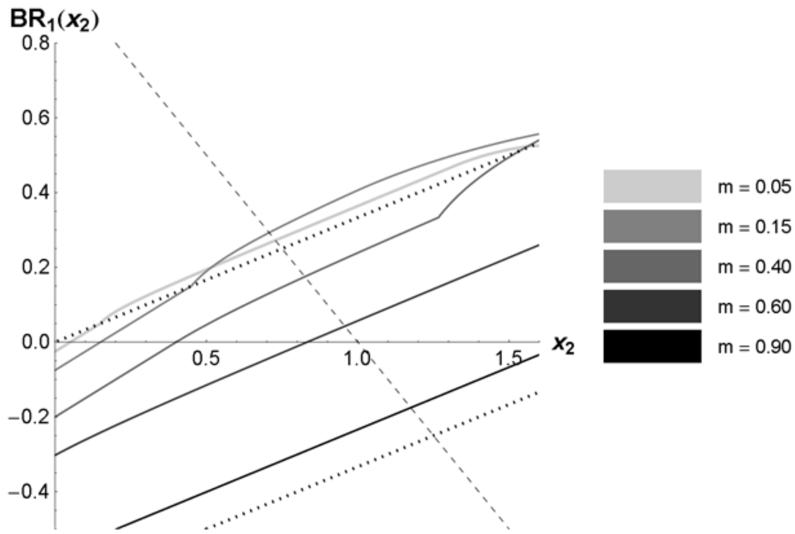

Fig. 4 The best-response function of player 1 for different values of m

This is, however, not true for $m < 0.15$ and the relatively "more concave" section of the BR function, corresponding to Player 1 securing a "hinterland" of states of nature in which she takes the entire market. Initially, this section lies above the other dotted line, representing the BR in the "certainty" model with price discrimination, and bends upwards as m increases. This is because moving further to the right allows to add some of the now more densely populated central areas of the average distribution to the "monopolised" section of the probability space. However, an increase of m also pushes the concave segment to the right, as Player 2 now needs to be located further away from the centre for Player 1 to be able ever to take the entire market. Otherwise, moving rightwards no longer increases one's "monopoly area", instead making price competition stronger when the market is shared.

5.2 Profits

Substituting the SPNE locations, as stated in Proposition 4 into the profit function (specifically, into $\mathrm{E}\Pi_1^{\mathrm{A}}[[3.a]]$ and $\mathrm{E}\Pi_1^{\mathrm{B}}[[3.a]]$) derived in Section 4, one obtains the SPNE profit:

$$\mathrm{E}\Pi_1\left(x_1^*, x_2^*\right) = \begin{cases} \frac{\left(1-4m+8m^2\right)^2 t}{8+8m(2m-3)} & 0 < m < \frac{1}{4} \\ \frac{\left(1-2m+28m^2\right)^2 t}{972m^2} & \frac{1}{4} \leq m < 1 \end{cases} \tag{10}$$

which is the same for both players, since:

$$\mathrm{E}\Pi_2\left(x_1^*, x_2^*\right) = \mathrm{E}\Pi_1\left(1-x_2^*, 1-x_1^*\right) = \mathrm{E}\Pi_1\left(x_1^*, x_2^*\right)$$

It is interesting to compare (10) with the profits the firms would earn were they certain of the exact location of the demand at the time of choosing locations. By a simple generalisation of [9] (to a market of size m rather than 1), the SPNE locations are:

$$x_1^* = -\frac{m}{4},\ x_2^* = \frac{5m}{4}$$

giving a profit of $\frac{3m^2t}{4}$ to both players. This can be shown to be less than (10) for $0 \leq m \leq 1$. It is also clear from Fig. 5 that this profit loss due to more information is larger for $m = \frac{1}{2} - v$ than for $m = \frac{1}{2} + v$, where $v \in (0, \frac{1}{2}]$.

Corollary 7 *In a location-then-price Hotelling game with uncertainty and customer distribution $U(Z, Z+m)$, where $Z \sim U(0, 1-m)$, an unambiguous mean-preserving increase of uncertainty decreases the SPNE profits and increases the profit loss due to certainty.*

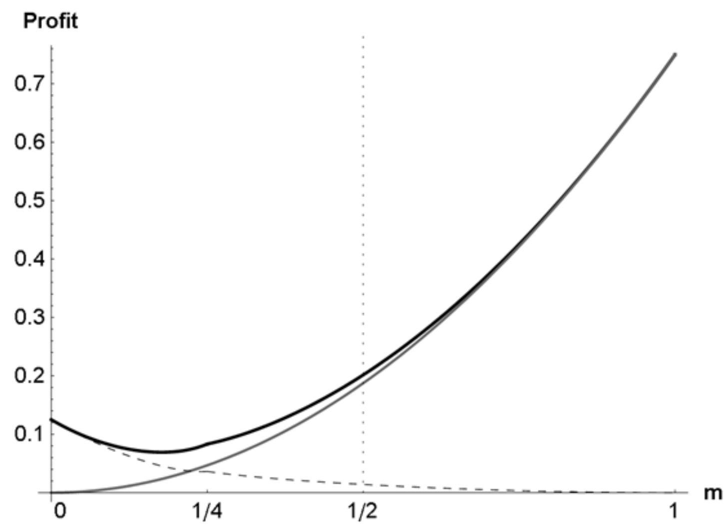

Fig. 5 the SPNE profit associated with uncertainty (black), certainty (grey) and the difference between the two (dashed) for different values of m (and $t = 1$)

5.3 Social Welfare

What is, perhaps, more important, is the effect of uncertainty on social welfare. Since the demand is inelastic, the social welfare in a particular state of nature can be measured by (minus) the total transportation cost incurred by all consumers in this state. But since the state is uncertain, we consider the expectation, or average of this cost over all states. This is equal to:

$$\mathrm{ETC}\,(m) = \begin{cases} \frac{7m-32m^2+120m^3-288m^4+224m^5-1}{48(1-2m)^2(m-1)/t} & \text{for } 0 < m < \frac{1}{4} \\ \frac{28m+120m^2-296m^3+1208m^4-7}{3888m^2/t} & \text{for } \frac{1}{4} \leq m < 1 \end{cases}$$

(see the Appendix for derivation)

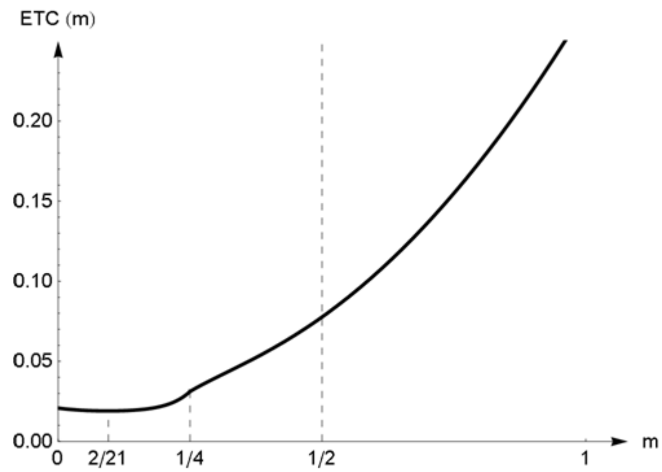

Fig. 6 Expected Total Cost in equilibrium for different values of m and $t = 1$

The Expected Cost is minimized at $m \approx 2/21$ and slightly decreasing in m below this value. This could be explained by the fact that, with locations close to $\frac{1}{2}$, the influx of customers (in average terms) to the centre of the market will decrease the average distance one has to travel in order to make a purchase.

Still, it is obvious from Fig. 6 that for any pair of symmetric values of m, the lower value will result in a lower ETC. Hence:

Corollary 8 *In a location-then-price Hotelling game with uncertainty and customer distribution $U(Z, Z+m)$, where $Z \sim U(0, 1-m)$, an unambiguous mean-preserving increase of uncertainty leads to a social-welfare improvement.*

We may also want to compare the equilibrium ETC with the Planner's first-best optimal solution, where:

- 2nd stage prices are set equal to one another, so that each consumer buys from the closer firm
- the (symmetric) locations $x_1^{so} = 1 - x_2^{so}$ are chosen so as to minimize the resulting Expected Total Cost

The corresponding socially-optimal ETC is:

$$\mathrm{ETC}^{\mathrm{so}}(m) = \begin{cases} \frac{3-12m+24m^2-24m^3+8m^4}{144(m-1)^2/t} & \text{for } m < \frac{1}{2} \\ \frac{4m-8m^3+8m^4-1}{144m^2/t} & \text{for } m \geq \frac{1}{2} \end{cases} \tag{11}$$

(again, see the Appendix for derivation).

Not surprisingly, this is symmetric, so that the symmetric pairs of values of m are associated not only with the same average customer distribution but also with the same minimum level of expected total transportation costs. In fact, the difference between the equilibrium and optimal expected costs is always increasing in m (see Fig. 7 below).

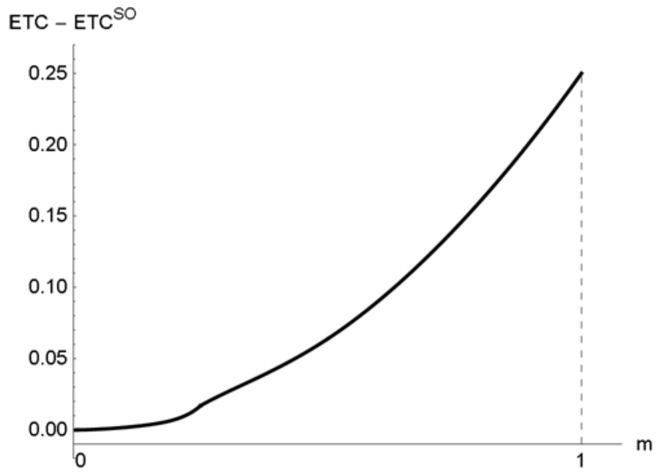

Fig. 7 The difference between the equilibrium and the socially optimal Expected Total Cost

As $m \to 0$, the model approaches the limiting case of the price-discrimination model with the equilibrium being socially optimal and $\mathrm{ETC} = \mathrm{ETC}^{\mathrm{SO}}$.

5.4 Extensions

It is interesting to observe how the above results are affected by two sorts of changes:

- **a change in timing.** Suppose the value of Z is revealed to the players only after they have chosen their prices. The (expected) payoff of player i in a second-stage (price-setting) subgame associated with a pair of locations x_1, x_2 is now:

$$\pi_i(x_1, x_2, p_1, p_2) = p_i \int_0^{1-m} D_i(x_1, x_2, p_1, p_2, z) \frac{1}{1-m}\, dz \tag{12}$$

  where $D_i(\cdot)$ is the demand in state z given locations and prices, i.e.:

$$D_1(x_1, x_2, p_1, p_2, z) = \int_{-\infty}^{\widetilde{x}} f(x, z)\, dx = 1 - D_2(x_1, x_2, p_1, p_2, z)$$

  where $\widetilde{x}$ (the "indifferent consumer") solves $p_1 + t(\widetilde{x} - x_1)^2 = p_2 + t(\widetilde{x} - x_2)^2$. Noting that $f(\cdot)$ is jointly measurable and non-negative, we may reverse the order of integration and, using (12), write the profit of pl.1 $\pi_1(\cdot)$ as:

$$p_1 \int_0^{1-m} \left[\int_{-\infty}^{\widetilde{x}} f(x, z)\, dx \right] \frac{1}{1-m}\, dz = p_1 \int_0^{1-m} \left[\int_{-\infty}^{\widetilde{x}} \frac{f(x, z)}{1-m}\, dx \right] dz$$

$$= p_1 \int_{-\infty}^{\widetilde{x}} \left[\int_0^{1-m} \frac{f(x, z)}{1-m}\, dz \right] dx = p_1 \int_{-\infty}^{\widetilde{x}} h(x)\, dx$$

i.e. the payoff function is the same as in the certainty game with the customer distribution equal to the average customer distribution of the game with uncertainty (recall Definition 1). As the same is true for player 2, the two games are equivalent. Consequently, demand uncertainty itself will have no effect on the outcome of the game when disentangled from the effect of changes in the average demand. In particular, $m = \frac{1}{2} - v$ now represents exactly the same game as $m = \frac{1}{2} + v$, because the average distribution is unchanged and so are the payoff functions.

- **a change in the pricing rules.** Specifically, suppose that mill pricing is replaced by price discrimination, as in the model described by [5]. This means a Bertrand-style competition at every location in every state of nature. In particular, the equilibrium profit of firm i at location x is:

$$\pi_i(x_1, x_2, x) = \max\{t\left[(x_{-i} - x)^2 - (x_i - x)^2\right], 0\}$$

This is integrated with respect to the state-dependant customer distribution function to give the total profit (from all locations) in state z:

$$\int_{-\infty}^{+\infty} \pi_i(x_1, x_2, x) \times f(x, z)\ dx$$

Consequently, the total expected profit (the payoff function of the reduced game) is:

$$\Pi_i(x_1, x_2) = \int_0^{1-m} \left[\int_{-\infty}^{+\infty} \pi_i(x_1, x_2, x) f(x, z)\ dx\right] \times \frac{1}{1-m}\ dz =$$
$$\int_{-\infty}^{+\infty} \left[\int_0^{1-m} \frac{\pi_i(x_1, x_2, x) f(x, z)}{1-m} dz\right] dx = \int_{-\infty}^{+\infty} \pi_i(x_1, x_2, x) h(x)\, dx$$

and is the same as the payoff in the *price discrimination* game under certainty, with the customer distribution equal to the average customer distribution in the game with uncertainty[7]. Hence, an unambiguous mean-preserving increase of uncertainty has again no effect on the outcome.

This suggests that the role of demand uncertainty lies in the extra discretion or flexibility of the firms to differentiate their prices across states of nature. As the market becomes smaller relative to the support of the shock affecting its placement, this differentiation becomes more and more similar to perfect price discrimination with respect to physical locations. But when the said discretion is eliminated by a change of timing, or when it is already embedded in the pricing

[7]Note that the joint measurability of $\pi_i(x_1, x_2, x) \times f(x, z)$ was used to reverse the order of integration.

rule, introducing demand uncertainty or increasing it has, *ceteris paribus,* no effect at all. The question then simply reduces to investigating the effect of re-shaping the customer distribution in a certainty model and has already been considered in the literature (see, for example [13] or [12]).

6 Concluding Remarks

In this paper, the problem of a Hotelling duopoly facing an uncertain demand is re-examined in a slightly different framework to the one present in the literature. The simplifying assumption of uniformly distributed additive shock to the (uniform) customer distribution is preserved. However, in this modified setting demand uncertainty is modelled as a mean-preserving spread of the customer mass distribution in every half-market.

Once the SPNE locations are obtained, the above feature makes it possible to see what happens when customer tastes become more uncertain while remaining on average the same. This allows re-examination of the comparative statics with respect to demand uncertainty, with the effect of changes in the average distribution removed from the analysis.

The results are strikingly different from the existing studies of this problem. Most importantly, the effect of demand uncertainty seems to be similar to that of price discrimination in a certainty model. When allowed to set different prices for different realisations of customer demand, the firms are effectively price-discriminating between states of nature. As the size of the market in a particular state decreases, the flexibility of price differentiation increases, approaching the discretion of setting prices independently at each physical location. Furthermore, the second-stage price-competition becomes more and more similar to the Bertrand Model with asymmetric costs. The firm closer to the customer distribution (equivalent of a cost advantage) takes the entire market, earning a profit that depends on how much the transportation costs of getting to the "losing" firm exceed those of reaching the winner. As a result, the individual (expected) profits become more and more related to the total (expected) transportation costs incurred by the customers. Consequently, the firms implicitly begin to pursue a socially desirable objective. Whenever the uncertainty increases for a given average demand, product differentiation falls, profits decline and social welfare gets closer to the optimal level. As the size of the market in a particular state goes to zero, the model approaches the limiting case of the standard Hotelling Model with price discrimination and a socially optimal outcome.

On the other hand, when differentiating the prices is ruled out by a change of timing, or when it is already present under certainty, then making the demand more uncertain, while holding it on average the same, ceases to have any effect on the actions of expected-profit maximizing firms.

APPENDIX

Best-Response and Equilibrium derivation for subs. 4.1

We proceed to derive the best response function of firm 1 to the opponent's location, i.e. the maximum of $\mathrm{E}\Pi_1^{\mathrm{A}}$ over $x_1 \in (-\infty, x_2]$ for a given value of x_2.

The first thing to do is to find all local maximum points of the payoff function, by taking each of its segments in turn, finding any local maximum points of the associated function and the values of x_2 such that those local maximum values of x_1 will satisfy the associated condition on x_1. We have:

1. $\frac{\partial \mathrm{E}\Pi_1^{\mathrm{A}}[[2.a]]}{\partial x_1} = \frac{t(m+2x_1-x_2)(2m+x_1+x_2)^2}{54(-1+m)m}$. This has two roots, one of which, $x_1 = \frac{1}{2}(x_2 - m)$, is a local maximum.

   We have $-2m - x_2 < \frac{1}{2}(x_2 - m) \leq 4m - x_2 \Leftrightarrow -m < x_2 \leq 3m$: for those values of x_2 this is the local maximum of the payoff function.

2. $\frac{\partial \mathrm{E}\Pi_1^{\mathrm{A}}[[3.a]]}{\partial x_1} = \frac{t\left(8m^2-8mx_1+(3x_1-x_2)(x_1+x_2)\right)}{4(-1+m)}$ has two roots, where the larger one, $x_1 = \frac{1}{3}\left(4m - x_2 + 2\sqrt{x_2^2 - 2m^2 - 2mx_2}\right)$ is a local maximum and satisfies $\mathrm{E}\Pi_1^{\mathrm{A}}[[3.b]]$ iff $3m < x_2 \leq \frac{16m-22m^2-3}{6m-2}$

3. $\frac{\partial \mathrm{E}\Pi_1^{\mathrm{A}}[[4.a]]}{\partial x_1} =$
   $$= \frac{38m^3+6m^2(4-13x_1)-2(2x_1-x_2-1)(\overline{x}-1)^2+\frac{3m}{4}\left(32x_1+15x_1^2+10x_1x_2-5x_2-16\right)}{27(m-1)m/t}$$

   The function $\mathrm{E}\Pi_1^{\mathrm{A}}[[4.a]]$ is a 4th order polynomial in x_1 and has a limit of $+\infty$ as $x_1 \to \pm\infty$. Thus, we are interested in the 2nd (middle) root of the corresponding cubic derivative. This root is real and satisfies $\mathrm{E}\Pi_1^{\mathrm{A}}[[4.b]]$ iff $\frac{-3+16m-22m^2}{-2+6m} < x_2 \leq \frac{1}{2}(3+3m)$.

4. $\frac{\partial \mathrm{E}\Pi_1^{\mathrm{A}}[[5.a]]}{\partial x_1} = t(1+m-2x_1)$ has a unique root $x_1 = \frac{1+m}{2}$ which is a local maximum and satisfies $\mathrm{E}\Pi_1^{\mathrm{A}}[[5.b]]$ iff $x_2 > \frac{1}{2}(3+3m)$.

The above intervals on x_2 are mutually exclusive: for every value of x_2 player 1's payoff function has exactly one local maximum (except for $x_2 \leq -m$, when the payoff is always zero and we assume player 1 chooses $x_1 = x_2$ so as to reduce the payoff of player 2 to zero as well). Moreover:

- both the payoff and its derivative are continuous (to see this, it is sufficient to verify that there are no discontinuities at the points of switching from one functional form to another)
- $\lim_{x_1 \to -\infty} \mathrm{E}\Pi_1^{\mathrm{A}}(x_1, x_2) = \lim_{x_1 \to x_2^-} \mathrm{E}\Pi_1^{\mathrm{A}}(x_1, x_2) = 0$
- $\forall x_1 < x_2 : \mathrm{E}\Pi_1^{\mathrm{A}}(x_1, x_2) \geq 0$

Hence, we may conclude that any local maximum is a global maximum of the payoff function on the $(-\infty; x_2]$ interval and write the best response function $\mathrm{BR}_1^{\mathrm{A}}(x_2)$ as stated in the main body of the paper.

We proceed to find the Nash Equilibrium. It follows from symmetry of the game (and, specifically, equation (7)) that $\mathrm{BR}_2^{\mathrm{A}}(x_1) = 1 - \mathrm{BR}_1^{\mathrm{A}}(1-x_1)$. Hence, we may write the necessary condition as:

$$\begin{cases} x_1^* = \mathrm{BR}_1^{\mathrm{A}}(x_2^*) \\ x_2^* = \mathrm{BR}_2^{\mathrm{A}}(x_1^*) = 1 - \mathrm{BR}_1^{\mathrm{A}}(1-x_1^*) \iff 1 - x_2^* = \mathrm{BR}_1^{\mathrm{A}}(1-x_1^*) \end{cases} \tag{13}$$

Let $x = x_2^*$, $f = x_1^*$, $x' = 1 - x_1^*$ and $f' = 1 - x_2^*$. The two points (x, f) and (x', f') must lie on the best-response curve of player 1. But since:

$$x' - x = 1 - x_1^* - x_2^* = f' - f$$

the points must also lie on a straight line with a slope of 1. It is straightforward to verify that $\mathrm{BR}_1^{\mathrm{A}}(\cdot)$ is continuous and that $\frac{\partial \mathrm{BR}_1^{\mathrm{A}}(x_2)}{\partial x_2} < 1$ for $x_2 > -m$. When $x_2 \leq -m$ we have $\mathrm{BR}_1^{\mathrm{A}}(x_2) = x_2$, which ensures zero payoffs for both firms in all states of nature, so we can reject this as a potential equilibrium. Hence, the only possibility for (13) to be satisfied is when:

$$x' - x = 0 \iff x_1^* = 1 - x_2^*$$

i.e. when the equilibrium locations are symmetric. So, we are looking for an x_2^* such that:

$$1 - x_2^* = \mathrm{BR}_1^{\mathrm{A}}(x_2^*) < \frac{1}{2}$$

Since $\mathrm{BR}_1^{\mathrm{A}}(\cdot)$ is non-decreasing, the solution must be unique. Solving:

$$1 - x_2^* = \mathrm{BR}_1^{\mathrm{A}}[[3.a]](x_2^*) = \frac{1}{3}\left(4m - x_2^* + 2\sqrt{-2m^2 - 2mx_2 + (x_2^*)^2}\right)$$

we obtain a location $x_2^* = \frac{8m - 8m^2 - 3}{8m - 4}$ satisfying $\mathrm{BR}_1^{\mathrm{A}}[[3.b]]$, i.e.:

$$3m < x_2^* \leq \frac{16m - 22m^2 - 3}{-2 + 6m}$$

as long as $0 < m < \frac{1}{4}$.

Consequently, the pair of locations $x_1^* = 1 - x_2^* = \frac{8m^2 - 1}{8m - 4}$ is the only one such that x_1^* maximizes $\mathrm{E}\Pi_1^{\mathrm{A}}(x_1, x_2^*)$ over $x_1 \in (-\infty; x_2^*]$ and x_2^* maximizes $\mathrm{E}\Pi_2^{\mathrm{A}}(x_1^*, x_2)$ over $x_2 \in [x_1^*; \infty)$.

It remains to verify that player 1 would not wish to locate optimally to the right rather than to the left of player 2, which, by symmetry, also ensures that player 2 would not wish to locate optimally to the left rather than to the right of player 1. We need:

$$\begin{aligned} \mathrm{E}\Pi_1^{\mathrm{A}}(x_1^*, x_2^*) = \mathrm{E}\Pi_1^{\mathrm{A}}(\mathrm{BR}_1^{\mathrm{A}}(x_2^*), x_2^*) \geq \\ \mathrm{E}\Pi_2^{\mathrm{A}}(x_2^*, \mathrm{BR}_2^{\mathrm{A}}(x_2^*)) = \mathrm{E}\Pi_2^{\mathrm{A}}(x_2^*, 1 - \mathrm{BR}_1^{\mathrm{A}}(1 - x_2^*)) = \\ \mathrm{E}\Pi_1^{\mathrm{A}}(\mathrm{BR}_1^{\mathrm{A}}(1 - x_2^*), 1 - x_2^*) = \mathrm{E}\Pi_1^{\mathrm{A}}(\mathrm{BR}_1^{\mathrm{A}}(x_1^*), x_1^*) \end{aligned}$$

Intuitively, $\mathrm{E}\Pi_1^{\mathrm{A}}(\mathrm{BR}_1^{\mathrm{A}}(x_2^*), x_2^*) \geq \mathrm{E}\Pi_1^{\mathrm{A}}(\mathrm{BR}_1^{\mathrm{A}}(x_1^*), x_1^*)$ is true, i.e. it is better to locate optimally to the left of x_2^* than to the left of $x_1^* < x_2^*$. Indeed, it is easy to see from (4)-(6) that as x_1 and x_2 increase by the same amount, the profit of player 1 does not decrease in any state z and that it strictly increases for some z. Hence, the expected profit $\mathrm{E}\Pi_1^{\mathrm{A}}(x_1, x_2)$ must increase. Consequently:

$$\mathrm{E}\Pi_1^{\mathrm{A}}(\mathrm{BR}_1^{\mathrm{A}}(x_1^*), x_1^*) < \mathrm{E}\Pi_1^{\mathrm{A}}(\mathrm{BR}_1^{\mathrm{A}}(x_1^*) + (x_2^* - x_1^*), x_2^*) \leq \mathrm{E}\Pi_1^{\mathrm{A}}(\mathrm{BR}_1^{\mathrm{A}}(x_2^*), x_2^*)$$

where the last inequality follows from the definition of $\mathrm{BR}_1^{\mathrm{A}}(\cdot)$.

To conclude, (8) constitutes the unique SPNE, given that $0 < m < \frac{1}{4}$.

Best-Response and Equilibrium derivation for subs. 4.2

As in the previous case, we proceed to find all local maximum points of the payoff function $\mathrm{E}\Pi_1^{\mathrm{B}}(x_2)$:

1. We have: $\mathrm{E}\Pi_1^{\mathrm{B}}[[2.a]] = \mathrm{E}\Pi_1^{\mathrm{A}}[[2.a]]$, so that $x_1 = \frac{1}{2}(x_2 - m)$ is again the local maximum and satisfies $\mathrm{E}\Pi_1^{\mathrm{B}}[[2.b]]$ iff $-2m - x_2 < \frac{1}{2}(x_2 - m) \leq 2 - 4m - x_2 \Leftrightarrow -m < x_2 \leq \frac{1}{3}(4 - 7m)$: for those values of x_2 this is a local maximum of the payoff function.
2. $\frac{\partial \mathrm{E}\Pi_1^{\mathrm{B}}[[3.a]]}{\partial x_1} = -\frac{t\left(28m^2+(2-3x_1)^2+4m(-5+9x_1)+6x_1x_2-3x_2{}^2\right)}{54m}$ has two roots. The larger one, $x_1 = \frac{1}{3}\left(2 - 6m - x_2 + 2\sqrt{2m^2 - x_2 + 3mx_2 + x_2{}^2 - m}\right)$ is a local maximum and satisfies $\mathrm{E}\Pi_1^{\mathrm{B}}[[3.b]]$ iff $\frac{1}{3}(4 - 7m) \leq x_2 \leq \frac{1-17m+79m^2}{-3+21m}$
3. The 2nd root of $\frac{\partial \mathrm{E}\Pi_1^{\mathrm{A}}[[4.a]]}{\partial x_1}$ is real, corresponds to a local maximum and satisfies $\frac{\partial \mathrm{E}\Pi_1^{\mathrm{B}}[[4.b]]}{\partial x_1}$ iff:
   (a) $\frac{1}{4} \leq m < \frac{2}{3}$ and $\frac{1-17m+79m^2}{-3+21m} < x_2 < \frac{1}{2}(3 + 3m)$, OR:
   (b) $\frac{2}{3} \leq m < \frac{11}{13}$ and $\phi(m) \leq x_2 < \frac{1}{2}(3 + 3m)$, where $\phi(\cdot)$ is a certain function of m
4. Since $\mathrm{E}\Pi_1^{\mathrm{B}}[[5.a]] = \mathrm{E}\Pi_1^{\mathrm{A}}[[5.a]]$ and $\mathrm{E}\Pi_1^{\mathrm{B}}[[5.b]] = \mathrm{E}\Pi_1^{\mathrm{A}}[[5.b]]$, $x_1 = \frac{1+m}{2}$ is again a local maximum iff $x_2 > \frac{1}{2}(3 + 3m)$.

Unfortunately, the above intervals are mutually excludable for all values of x_2 only as long as $\frac{1}{4} \leq m < \frac{2}{3}$ (see Fig. 8 below). When $\frac{2}{3} \leq m < \frac{11}{13}$, there is a range of values of x_2 such that local maxima associated with cases (2) and (3) above will both appear (area A), while for $m > \frac{44+9\sqrt{11}}{95} \approx \frac{7}{9}$ there is a range where the same happens for the maxima associated with cases (2) and (4) (area B). As far as area "B" is concerned, it can be shown that as long as $m > \frac{4}{5}$ and $x_2 < \frac{18}{35} + \frac{30}{11}m$, the local maximum (2) is associated with a larger value of the payoff function. The same is true for the part of area "A" where $m \leq \frac{4}{5}$ and $x_2 < \frac{1}{16}(137 - 57m)m - \frac{103}{55}$.

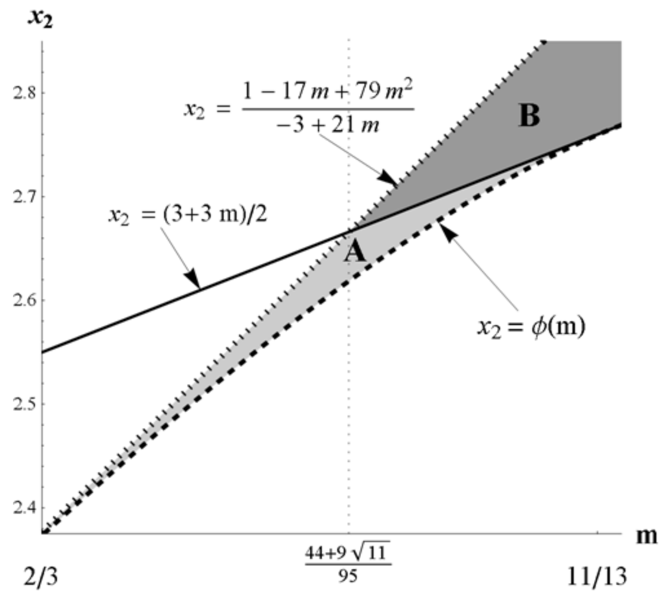

Fig. 8 Non-uniqueness of local maxima for $m > \frac{2}{3}$

We may therefore define:

$$\varphi_0(m) = \begin{cases} \frac{1-17m+79m^2}{-3+21m} & \frac{1}{4} \le m \le \frac{2}{3} \\ \frac{1}{16}(137-57m)m - \frac{103}{55} & \frac{2}{3} < m \le \frac{4}{5} \\ \frac{18}{35} + \frac{30}{11}m & \frac{4}{5} < m < 1 \end{cases}$$

$$\text{and } \varphi_1(m) = \begin{cases} \frac{1}{2}(3+3m) & \frac{1}{4} \le m \le \frac{4}{5} \\ \frac{18}{35} + \frac{30}{11}m & \frac{4}{5} < m < 1 \end{cases}$$

and write the best-response function $\mathrm{BR}_1^{\mathrm{B}}(x_2)$ as stated. For $\frac{2}{3} < m \le \frac{4}{5}$ there is a discontinuity at $\varphi_0[[2.a]](m)$: the global maximum "switches" from local maximum (2) to a distant (3). For $\frac{4}{5} < m$ there is a similar discontinuity at $\varphi_1[[2.a]](m)$, associated with switching from (2) to (4).

As in the case of $0 < m < \frac{1}{4}$, the necessary condition for a NE to occur is:

$$\begin{cases} x_1^* = \mathrm{BR}_1^{\mathrm{B}}(x_2^*) \\ 1 - x_2^* = \mathrm{BR}_1^{\mathrm{B}}(1 - x_1^*) \end{cases}$$

Unlike in the previous case, however, it is not always true that $\frac{\partial \mathrm{BR}_1^{\mathrm{B}}(x_2)}{\partial x_2} < 1$ for $x_2 > -m$ and $\mathrm{BR}_1^{\mathrm{B}}(\cdot)$ is not always continuous. Hence, one has to proceed as follows.

First of all, it must be that any equilibrium has $x_1^* < \frac{1}{2} < x_2^*$.If both firms were located on the same side of $\frac{1}{2}$, say $x_1^* < x_2^* < \frac{1}{2}$, then player 1 could instead locate optimally to the right of 2 and obtain:

$$\mathrm{E}\Pi_2^{\mathrm{B}}(x_2^*, \mathrm{BR}_2^{\mathrm{B}}(x_2^*)) \ge \mathrm{E}\Pi_2^{\mathrm{B}}(x_2^*, x_2^* + (x_2^* - x_1^*)) = \mathrm{E}\Pi_1^{\mathrm{B}}(1 - 2x_2^* + x_1^*, 1 - x_2^*) >$$
$$\mathrm{E}\Pi_1^{\mathrm{B}}(1 - 2x_2^* + x_1^* - (1 - 2x_2^*), 1 - x_2^* - (1 - 2x_2^*)) = \mathrm{E}\Pi_1^{\mathrm{B}}(x_1^*, x_2^*)$$

We can therefore rule out $x_2^* > \varphi_1(m)$, because then $x_1^* = \mathrm{BR}_1^{\mathrm{B}}(x_2^*) = (1+m)/2 > \frac{1}{2}$. Similarly we can rule out $1 - x_1^* > \varphi_1(m) \Leftrightarrow \mathrm{BR}_1^{\mathrm{B}}(1 - x_1^*) = (1+m)/2 = 1 - x_2^* > \frac{1}{2} \iff x_2^* < \frac{1}{2}$. It is also true that $x_2^* > -m$, since $x_2^* > \frac{1}{2}$ and that $1 - x_1^* > -m$, since $x_1^* < \frac{1}{2} \iff 1 - x_1^* > \frac{1}{2}$. Recalling the earlier notation:

$$x = x_2^*, f = x_1^*, x' = 1 - x_1^*, f' = 1 - x_2^*$$

we may, at this stage, conclude that the points (x, f) and (x', f') need to lie somewhere on the three "middle" segments of $\mathrm{BR}_1^{\mathrm{B}}(\cdot)$ and to the right of $\frac{1}{2}$. Moreover, since they must also lie on a line with a slope of 1 and we still have $\frac{\partial \mathrm{BR}_1^{\mathrm{B}}(x_2)}{\partial x_2} < 1$ for $x_2 \in (-m, \varphi_0(m)]$, at least the point located further to the right must lie on $\mathrm{BR}_1^{\mathrm{B}}[[4.a]](\cdot)$ if the NE is not to be symmetric. The following cases need to be considered:

1. both points lie on $\mathrm{BR}_1^{\mathrm{B}}[[4.a]](\cdot)$. This can be ruled out, since this section of the best-response function corresponds to the maximum of $\mathrm{E\Pi}_1^{\mathrm{B}}(\cdot)$ being attained on the segment $\mathrm{E\Pi}_1^{\mathrm{B}}[[4.a]](\cdot)$. This means $\mathrm{E\Pi}_1^{\mathrm{B}}[[4.b]]$ must be satisfied, i.e. $4m - x_2 < x_1 < 2 + 2m - x_2 \Leftrightarrow 0 < \overline{x} - 2m < 1 - m \Rightarrow \overline{x} + m > 1 - m$, so that $z_1 = \overline{x} - 2m$ and $z_2 = 1 - m$. Hence, the player responding according to $\mathrm{BR}_1^{\mathrm{B}}[[4.a]](\cdot)$ claims the entire market in some states and a positive fraction of it in all the other states. Clearly, this cannot be true for both players.

2. one point lies on $\mathrm{BR}_1^{\mathrm{B}}[[3.a]](\cdot)$. The player responding with $\mathrm{BR}_1^{\mathrm{B}}[[3.a]](\cdot)$ maximizes $\mathrm{E\Pi}_1^{\mathrm{B}}(\cdot)$ where it is given by $\mathrm{E\Pi}_1^{\mathrm{B}}[[3.a]](\cdot)$, which happens iff $2 - 4m - x_2 \leq x_1 \leq 4m - x_2 \Leftrightarrow (\overline{x} + m \geq 1 - m \wedge \overline{x} - 2m \leq 0)$, so that $z_1 = 0$ and $z_2 = 1 - m$. Hence, the player shares the market with the rival in every state, so the rival cannot take the entire market in any state and the other point cannot lie on $\mathrm{BR}_1^{\mathrm{B}}[[4.a]](\cdot)$.

3. finally, let $f = \mathrm{BR}_1^{\mathrm{B}}[[2.a]](x)$. Then $x' = 1 - f = 1 + \frac{m-x}{2}$ and $\mathrm{BR}_1^{\mathrm{B}}[[4.a]](x')$ should be equal to $f' = 1 - x$ whenever $\varphi_0(m) < x' \leq \varphi_1(m)$. However, substituting $1 + \frac{m-x}{2}$ for x_2 and $1 - x$ for x_1 in $\frac{\partial \mathrm{E\Pi}_1^{\mathrm{A}}[[4.a]]}{\partial x_1}$, we obtain an expression which is equal to zero iff:

   $$22m^3 + x^3 + m^2(45x - 32) + 8m(2 + (x-4)x) = 0$$

   which can be shown not to hold as long as $1/4 \leq m < 1$ and $\varphi_0(m) < 1 + \frac{m-x}{2} \leq \varphi_1(m)$. Hence, $x_1 = 1 - x$ is not a root of $\frac{\partial \mathrm{E\Pi}_1^{\mathrm{A}}[[4.a]]}{\partial x_1}$. When player 1 responds to x_2^* according to $\mathrm{BR}_1^{\mathrm{B}}[[2.a]](\cdot)$, then even if player 2 wants to respond according to $\mathrm{BR}_1^{\mathrm{B}}[[4.a]](\cdot)$, this best-response to x_1^* is different from x_2^*.

It follows that an asymmetric NE is not possible, i.e. we must have:

$$1 - x_2^* = \mathrm{BR}_1^{\mathrm{B}}(x_2^*) < \frac{1}{2}$$

where the solution, once again, must be unique. Solving:

$$1 - x_2^* = \mathrm{BR}_1^{\mathrm{B}}\left[[3.a]\right](x_2^*) = \frac{2 - 6m - x_2^* + 2\sqrt{2m^2 - m - x_2^* + 3mx_2^* + (x_2^*)^2}}{3}$$

we obtain a location $x_2^* = \frac{1+16m+28m^2}{36m}$ satisfying $\mathrm{BR}_1^{\mathrm{B}}[[3.b]]$ for $\frac{1}{4} \leq m < 1$. A pair of symmetric locations (9) is the unique SPNE, since no player will want to relocate to the other side of the competitor (as established in section 4.1).

SPNE ETC derivation for subsection 5.3

Given customer distribution $U(z, z+m)$, the following cases are possible:

1. when $z \leq \overline{x} - 2m$, everyone buys from firm $i = 1$ and the total cost is:

$$\mathrm{TTC}_i = \int_z^{z+m} \frac{1}{m} t\,(x - x_i)^2\, dx$$

   where for $z \geq \overline{x} + m$ the case is reversed with $i = 2$.

2. when $\overline{x} - 2m < z < \overline{x} + m$ the two firms share the market with the division at:

$$\widetilde{x} = \frac{1}{6}(4z + 2m + x_1 + x_2) \text{ (a simple generalisation of the result in [3])}$$

   and consequently the total cost becomes:

$$\mathrm{TTC}_{1,2} = \int_z^{\widetilde{x}} \frac{1}{m} t\,(x - x_1)^2\, dx + \int_{\widetilde{x}}^{z+m} \frac{1}{m} t\,(x - x_2)^2\, dx$$

But since the state of nature is uncertain, we are interested in the expected, or average transportation cost over all states of nature.

With symmetric locations, we have $\overline{x} = \frac{1}{2}$ and if, in addition, $m > \frac{1}{4}$, then we have $\overline{x} - 2m < z < \overline{x} + m$ for every $z \in [0, 1-m]$. Consequently, case 2 is true in every state and the expected total cost is:

$$\int_0^{1-m} \frac{1}{1-m} \mathrm{TTC}_{1,2}\; dz \tag{14}$$

substituting the SPNE locations (9) for x_1 and x_2, (14) becomes:

$$\mathrm{ETC}_2 = \frac{28m + 120m^2 - 296m^3 + 1208m^4 - 7}{3888m^2/t}$$

On the other hand, for $0 < m \le \frac{1}{4}$ case 1 is true in some states and the expected equilibrium total cost is:

$$\mathrm{ETC}_{1,2} = \frac{1}{1-m}\left[\int_{0}^{\frac{1}{2}-2m} \mathrm{TTC}_1\ dz + \int_{\frac{1}{2}-2m}^{\frac{1}{2}+m} \mathrm{TTC}_{1,2}\ dz + \int_{\frac{1}{2}+m}^{1-m} \mathrm{TTC}_2\ dz\right]$$

substituting the SPNE locations as given by (8), this becomes:

$$\mathrm{ETC}_{1,2} = \frac{7m - 32m^2 + 120m^3 - 288m^4 + 224m^5 - 1}{48(1-2m)^2(m-1)/t}$$

We can therefore define:

$$\mathrm{ETC}\,(m) = \begin{cases} \mathrm{ETC}_{1,2} & \text{for } 0 < m \le \frac{1}{4} \\ \mathrm{ETC}_2 & \text{for } \frac{1}{4} < m < 1 \end{cases}$$

Socially-Optimal ETC derivation for subsection 5.3

In each s.o.n. everyone to the left of $\frac{1}{2}$ buys from firm 1 and everyone else buys from 2.The total cost in a particular state is then:

- $\mathrm{TTC}_i = \int_{z}^{z+m} \frac{1}{m} t\left(x - x_i^{so}\right)^2 dx$, for $\frac{1}{2} \notin [z, z+m]$, where $i = 1$ for $\frac{1}{2} > z+m$ and $i = 2$ for $\frac{1}{2} < z$
- $\mathrm{TTC}_{1,2} = \int_{z}^{1/2} \frac{1}{m} t\left(x - x_1^{so}\right)^2 dx + \int_{1/2}^{z+m} \frac{1}{m} t\left(x - x_2^{so}\right)^2 dx$ for $\frac{1}{2} \in [z, z+m]$

and the Expected Total Cost is:

$$\begin{cases} \frac{1}{1-m}\left[\int_{0}^{\frac{1}{2}-m} \mathrm{TTC}_1\ dz + \int_{\frac{1}{2}-m}^{\frac{1}{2}} \mathrm{TTC}_{1,2}\ dz + \int_{\frac{1}{2}}^{1-m} \mathrm{TTC}_2\ dz\right] & \text{for } m < \frac{1}{2} \\ \int_{0}^{1-m} \frac{1}{1-m} \mathrm{TTC}_{1,2}\ dz & \text{for } m \ge \frac{1}{2} \end{cases}$$

substituting $1 - x_1^{so}$ for x_2^{so}, integrating and minimizing the above expression with respect to x_1 we obtain the optimal Expected Total Cost as stated in (11).

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