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Self-organized Agglomerations and Transport Costs*

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Abstract

This paper investigates the number and structure of spatial equilibria in a continuous space for a general class of transport cost functions. The economic space is represented by a circumference on which firms and workers-consumers are perfectly mobile. We derive the conditions to be imposed on the transport cost functions under which the distributions of workers and firms are stable equilibria. We also derive the conditions under which discrete distributions of workers over equidistant points (cities) are stable equilibria for large and small number of points (cities).

Keywords: agglomeration, continuous distribution, asymptotic stability, Fourier series.

J.E.L. Classification: C62, F12, R12.

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1 Introduction

The question about the number and structure of cities that emerge from market interactions has attracted the attention of spatial economists and economic geographers. Yet, ever since Krugman (1991), many people have studied two-location models to explain the discrepancies in economic development. However, other spatial configurations are possible as an economic activity can be distributed over a continuous space or over a finite number of location points. Because such spatial configurations are richer and geographically more relevant than the two-location case, existing models in the new economic geography literature may be subject to strong qualifications. Therefore, it is important to devote additional research on the theoretical foundations of dispersion and agglomeration of economic agents in richer space structures.

At the same time, the literature seems to have overlooked the role of the shape of transport cost function on the properties of spatial equilibria. The relevance of the shape of transport costs can be presented from three points of views. First, partial equilibrium models of spatial competition show that firms' incentives to agglomerate or to separate are drastically altered by the shape of transport costs (d'Aspremont *et al.* 1979; Economides 1986). Second, in general equilibrium models, the choice of transport cost functions is usually made for analytical convenience: iceberg cost with CES utility function and linear cost with quadratic utility function. Finally, empirical evidence about the shape of transport costs is mitigated. Literature on transport logisitics reports transport cost functions with constant, increasing or decreasing returns according to the industry producing the good; freight costs are rationales for increasing returns to scale; optimization between many transport technologies gives support to concave transport costs; in pre-industrialized history, important land and sea hazards suggest increasing and convex transport costs (Boyer 1998; Gómez-Ibáñez and Tye 1999). As little seems to be known about the impact of the shape of transport cost on the emergence of economic agglomeration, its impact on the economic landscape deserves a dedicated investigation.

The present paper attempts to fill the theoretical gap in the economic geography literature by *investigating the number and structure of spatial equilibria in a continuous space, for a general class of transport cost functions.* Extending Salop's (1979) model, we assume that the economic space is described by a circumference on which firms and workers-consumers are perfectly mobile. The economic space is symmetric in that there exists no "first nature" locational advantage. We consider both continuous and discrete distributions of worker-consumers in a unified framework based on Ottaviano, Tabuchi and Thisse's (2002) model. Using the quadratic property of this model, we are able to extract several analytical results without making strong assumptions on the shape of transport costs and without having recourse to numerical simulations. In particular, we investigate a wide class of transport cost functions, including linear, exponential, step, sinusoidal, and so on. The originality of this paper lies in the use of Fourier decomposition of both spatial distributions of firms and workers *and* of transport cost functions. The equilibrium and stability properties are translated in the space of spatial frequencies in which the results are obtained.

We first study the existence and the stability of seamless spatial distributions of workers, namely spatial distributions that are piece-wise differentiable functions of distance and with no empty location. We identify a class of equilibria in which agents incur the same cost of access to the varieties that are produced over the space. According to the shape of transportation costs, uniform and non-uniform distributions may emerge as equilibria. Still, *uniform distribution – flat earth – is the unique such equilibrium compatible with all shapes of transport cost functions.* Furthermore, we find that such equilibria are usually not stable in the sense that there always exists a spatial perturbation of the distribution of firms and workers that does not converge back to the initial equilibrium. This is the case when the transport cost function has a kink at zero distance. Hence, in a continuous space, we show that *a seamless distribution of economic activity is not stable* *if the shape of transport costs does not share similar smoothness properties.*

We then investigate the equilibrium conditions of configurations in which economic activity is concentrated in a finite number of equidistant points, which stand for atomic cities. We show that the equilibrium and stability conditions are very similar to those obtained for continuous distributions. Again, configurations with both equal and unequal city sizes may be equilibria according to the shape of transportation costs. In addition, as expected, we show that equilibrium and stability conditions with an infinitely large number of cities converge to the conditions holding for a seamless distribution. Therefore, for the same reason as above, stable equilibria are not likely to include many atomic cities.

A further characterization of spatial equilibria requires more specific transport cost functions. This is why we study in more detail the spatial configurations associated with the following transport cost functions often encountered in the literature: linear and exponential functions. Linear transport cost functions lead to more cumbersome analysis, in particular for an odd number of cities. Still, examples suggest that stable equilibria are likely to exist only for small number of cities. In other words, agglomeration seems to be the likely outcome when economic agents behave independently.

The remainder of the paper is organized as follows. The model is presented in Section 2. Existence and stability of continuous equilibria are analyzed in Section 3 and 4, while existence and stability of discrete equilibria with atomic cities are discussed in Section 5 and 6. Sections 7 presents remarkable results for configurations with few cities and Section 8 discusses the particular cases of sinusoidal and linear transport costs. Section 9 concludes. All proofs are relegated to the appendices.

Related literature

In a seminal paper, Papageorgiou and Smith (1983) study the properties of a spatial equilibrium when agents are endowed with exogenous spatial externality functions and are located on a number of points that tends to infinity. They offer the first theoretical proof of stability of flat-earth in a setting that abstracts from the micro-economic foundations of spatial externalities.

The properties of the spatial equilibria are unfortunately more difficult to obtain in models that derive spatial externalities from *micro-economic interactions*. In most cases, researchers study the equilibrium properties through numerical simulation exercises or under strong and restrictive assumptions. For instance, Krugman (1993) develops a race-track economic model in which firms and workers locate around a circumference and in which goods are shipped along this circumference. Using numerical simulations and the CES Cobb-Douglas framework developed in Krugman (1991), he observes that a configuration with 12 symmetric cities of equal size is often unstable against small perturbations. His work also supports the view that the agglomeration process leads to a few cities. Brakman, Garretsen and Marrewijk (2001) provide further numerical simulations supporting the same kind of result: 2-city agglomeration is more likely to emerge than configuration with a large number of cities.

Yet few authors have contributed to the theoretical foundation of economic agglomeration and dispersion in a continuous space framework. Assuming infinitely extensive circumference, Fujita, Krugman and Venables (1999, chapters 6 and 17) provide an analytical proof of the instability of flat earth under the assumption of infinite dimension of circunference. Mossay (2003) theoretically qualifies this result in the case of workers' heterogenous preferences for location. Mossay (2006) presents a similar theoretical analysis in the simpler case of agents who can migrate and who trade a continuously renewed endowment.However, none of these works assess the impact on their results of their modeling choice about transport cost functions.

Some readers may wonder about the level of generality that the present contribution offers. Such a reservation is grounded on the criticism about Ottaviano *et al.*'s (2002) model –on which this research is based and– which assumes consumer preferences that linear in income and quadratic in consumption. Such a model has a 'partial equilibrium flavour' because it abstracts from general equilibrium income effects and therefore may appear to lack generality compared to Krugman's (1993) economic geography models based on CES-Cobb-Douglas preferences. Because both modelling schools have pros and cons, we do not intend to enter in a hot debate but we prefer to underline our motivation for the choice of the Ottaviano *et al.*'s (2002) model on three grounds. First, the latter model is a microeconomically founded model which analytical possibilities largely outweigh those of the CES-Cobb-Douglas models (Fujita *et al.* 1999, Mossay 2003,...). The latter models restrict authors to characterize only symmetric equilibria and often require the use of debatable assumptions (e.g. infinite earth perimeter, exponential iceberg costs...). Second, it is shown that traditional properties of international trade theory are confirmed under quasi-linear preferences (Dinopoulos *et al.* 2006). Such preferences thus present no pathology with respect to trade patterns. In addition, it has been suggested that quasi-linear preferences may be better suited to understand the observed 'missing trade' that cannot be explained under homothetic preferences. Finally, the strand of contributions following Ottaviano *et al.* (2002) and a series of papers based on CES preferences with no income effects have clearly shown that income effects are *not* necessary to explain the emergence of agglomeration or core-periphery patterns. Therefore, to our opinion, Ottaviano *et al.*'s (2002) model presents some limitations that are no more problematic than those of CES-Cobb-Douglas models. The former model seems better suited for the purpose of analyzing the impact of the shape of transport costs on spatial agglomeration.

It is also interesting to contrast our general equilibrium model to partial equilibrium models of spatial competition on a circular space. Our model suggests agglomeration in a few cities and allows for agglomeration in a single location when some workers-consumers are eager to choose to locate close to firms. Partial equilibrium models do not give support to this unless some geographical constraints are considered. In particular, firms disperse equidistantly around a circumference when they choose their location before their prices (Anderson, de Palma and Thisse 1992) and similar configurations are obtained when firms choose their location before their output levels (Pal 1998; Matsushima 2001; Shimizu and Matsumura 2003). While equilibria in these models are confined to atomic distribution, those in our general equilibrium model allow for distributions that can be nonuniform and seamless.

2 The Model

We assume that immobile farmers and perfectly mobile workers are located on a circumference with perimeter equal to 1. Farmers are uniformly distributed around the circumference with a distribution density equal to A. Workers are located according to the density $\lambda(y)L$ with $\int_0^1 \lambda(y)dy = 1$. There is a continuum of firms, each of which produces a single variety $i \in [0, M]$ and requires ϕ workers to operate its plant. There are thus $\lambda(y)M$ varieties produced at location y. Equilibrium in the whole labor market implies that $L = \phi M$.

Let y and $x \in [0, 1]$ be the coordinates of a producer and a consumer on the circumference. Because all varieties produced at location y are symmetric, they are consumed in equal quantities.[1] Therefore, the consumer demand of a variety produced at location y and consumed at location x is given by the function $q(y, x)$.

As in Ottaviano *et al.* (2002), consumers' preferences are identical across individuals

[1]As in Ottaviano *et al.* (2002), we assume for the sake of exposition that at the equilibrium, all firms located at the same location y set equal prices. This is in fact not an assumption. To see it, it suffices to denote quantities and prices by $q(y, z, x)$ and $p(y, z, x)$ where y is the production location and $z \in [0, \lambda(y)]$ is a variety 'secondary' address. It is straightforward to check that the maximization of consumer's utility and firm's profit yields: $q(y, z, x) = q(y, z', x)$ and $p(y, z, x) = p(y, z', x)$, $\forall z, z' \in [0, \lambda(y)]$.

and are described by the following quasi-linear utility with quadratic sub-utility functions:

$$U(q_0, q(\cdot, x)) = \alpha \int_0^1 q(y,x)\lambda(y) M \ dy \ - \frac{\beta}{2}\int_0^1 [q(y,x)]^2 \lambda(y) M \ dy$$
$$- \frac{\gamma}{2}\left[\int_0^1 q(y,x)\lambda(y) M \ dy\right]^2 + q_0$$

where q_0 is the numéraire, $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.

The budget constraint of a consumer located at x is equal to

$$\int_0^1 p(y,x)q(y,x)\lambda(y) M \ dy + q_0 \leq w(x) + \bar{q}_0$$

where $p(y,x)$ is the price of a variety produced at location y and sold at location x, $w(x)$ is his/her income residing at location x, and $\bar{q}_0$ is the consumer's initial endowment. Whereas mobile workers' income depends on their location, the immobile farmers' income does not depend on location. Indeed, it is assumed that immobile farmers produce the same constant-returns-to-scale good that can be transported at zero cost. Therefore, the market for this agricultural good clears at the same price in every location and yields the same income to farmers. We can normalize this income to 1 without loss of generality. Each consumer maximizes his/her utility, which leads to the following demand:

$$q(y,x) = \frac{\alpha}{\beta + \gamma M} - \frac{1}{\beta}p(y,x) + \frac{\gamma}{\beta(\beta + \gamma M)}P(x)$$

where $P(x) = \int_0^1 p(y,x)\lambda(y) M dy$ is the price index at location x.

Each price-discriminating firm is negligible in the sense that its action has no impact on the market. Each firm settling at location y maximizes its profit

$$\Pi(y) = \int_0^1 [p(y,x) - \tau(y,x)]\, q(y,x)\, [\lambda(x) L + A]\, dx - \phi w(y)$$

where $\tau(y,x)$ is the unit transport costs from locations y to x incurred by the firm, and $w(y)$ is the wage paid to workers employed by the firm at location y. The first-order condition yields the optimal price of variety produced and consumed at the same location x:

$$p^*(x,x) = \frac{2\alpha\beta + \gamma M \int_0^1 \tau(x,y)\lambda(y)dy}{2(2\beta + \gamma M)}$$

and the optimal price of variety produced at y and consumed at x:

$$p^*(y,x) = p^*(x,x) + \frac{1}{2}\tau(y,x)$$

In the long run, entry occurs until firms earn zero profit. Plugging the optimal prices into the free-entry condition $\Pi(x) = 0$, we obtain the workers' wage at location x:

$$w^*(x) = \frac{1}{\beta\phi}\int_0^1 [p^*(x,y) - \tau(x,y)]^2 [\lambda(y) L + A] \, dy$$

As in Ottaviano *et al.* (2002), the consumer surplus of an individual located at x is given by

$$\begin{aligned} S^*(x) &= \frac{\alpha^2 M}{2(\beta+\gamma M)} - \frac{\alpha M}{\beta+\gamma M}\int_0^1 p^*(y,x)\lambda(y)\,dy \\ &- \frac{\gamma M^2}{2\beta(\beta+\gamma M)}\left[\int_0^1 p^*(y,x)\lambda(y)\,dy\right]^2 + \frac{M}{2\beta}\int_0^1 [p^*(y,x)]^2 \lambda(y)\,dy \end{aligned}$$

The worker's indirect utility is therefore given by $V(x) = S^*(x) + w^*(x)$.

Before turning to the study of the equilibria for several types of spatial distributions, we need to be more explicit about the transport cost function. We decompose unit transport costs from locations x to y as

$$\tau(x,y) \equiv \tau T(x-y)$$

where τ is the *amplitude* of transportation costs and where $T(x)$ captures the *shape* of transportation costs. The function $T(x) : \mathbb{R} \to [0,1]$ is a periodic function such that $T(x) = T(l+x) = T(l-x)\ \forall l \in \mathcal{N}$, where $\mathcal{N}$ is the set of natural numbers, and such that $T(0) = 0$, $T(1/2) = 1$, and $T'(x) \geq 0\ \forall x \in [0,1/2]$. This function shares many similarities with the cosine function; indeed $T(x) = (1-\cos 2\pi x)/2$ fulfills these conditions. Also the crenellated periodic function $T(x) = T_1(x) \equiv 2x$ for all $x \in [0,1/2]$ fulfills these conditions; it captures linear transport cost. Figure 1 shows examples of shapes of transport cost functions.

INSERT FIGURE 1 HERE

To avoid corner solutions, we impose that trade is feasible between any pair of locations and for any distribution of firms and workers. This means that the firms' prices net of transport costs on a variety produced in x and sold in y are always positive. That is, we require

$$p^*(x,y) - \tau(x,y) = p^*(y,y) - \frac{1}{2}\tau(x,y) > 0$$

for any x, y and $\lambda(x)$, which is equivalent to

$$\tau < \frac{2\alpha\beta}{2\beta + \gamma M} \tag{1}$$

Finally, we will show in the sequel that the spatial frequency content of shape of transport costs drives the results about the existence and stability of equilibria. For this purpose, we define the Fourier decomposition of $T(x)$ and its square as

$$T(x) = \sum_{m=-\infty}^{\infty} a_m \exp(2\pi I m x) \qquad \text{and} \qquad [T(x)]^2 = \sum_{m=-\infty}^{\infty} b_m \exp(2\pi I m x)$$

where $I^2 \equiv -1$ and $\exp 2\pi I k x \equiv \cos 2\pi k x + I \sin 2\pi k x$. We readily have

$$a_m = \int_0^1 T(x) \exp(-2\pi I m x)\, dx \qquad \text{and} \qquad b_m = \int_0^1 [T(x)]^2 \exp(-2\pi I m x)\, dx$$

Because the transport costs are even functions and return real values, the Fourier coefficients are real and symmetric with respect to m: $a_m = a_{-m} \in \mathbb{R}$ and $b_m = b_{-m} \in \mathbb{R}$. In other words, transport cost functions are approached by Fourier series with cosine components only. Moreover, one easily gets the relationships: $b_0 > 0$ and

$$b_m = \sum_{k=-\infty}^{\infty} a_k a_{m-k} \tag{2}$$

3 Seamless Equilibria

In this section, we consider distributions of firms and workers with active manufacturing in all location: $\lambda(x) > 0, \forall x$ and with the following smoothness property:

Definition 1 *A spatial distribution $\lambda(x)$ is said to be seamless if $\lambda(x)$ is continuous and $\lambda'(x)$ is piecewise continuous $\forall x$.*

In the sequel, we seek the conditions under which an equilibrium exists for seamless distributions. Collecting the results of section 2, the worker's indirect utility can be rewritten as a function of $\lambda(\cdot)$ and x (see computations in Appendix 1):

$$\begin{aligned} V(x) &= W_0 - W_1 f_1(x) + W_2 f_2(x) - W_3 [f_1(x)]^2 \\ &\quad - W_4 \int_0^1 T(x-y) f_1(y) dy - W_5 \int_0^1 T(x-y) \lambda(y) f_1(y) dy \end{aligned} \tag{3}$$

where f_1 and f_2 are "accessibility measures" defined as

$$f_1(x) \equiv \int_0^1 T(x-z) \lambda(z) \, dz \qquad f_2(x) \equiv \int_0^1 [T(x-z)]^2 \lambda(z) \, dz$$

and where W_0 is a constant and

$$W_1 = \frac{\tau \alpha M (3\beta + 2\gamma M)}{(2\beta + \gamma M)^2} \qquad W_2 = \frac{3\tau^2 M}{8\beta} \qquad W_3 = \frac{\tau^2 \gamma^2 M^3}{8\beta (2\beta + \gamma M)^2}$$
$$W_4 = \frac{\tau^2 \gamma A M}{2\beta\phi (2\beta + \gamma M)} \qquad W_5 = \frac{\tau^2 \gamma M^2}{2\beta (2\beta + \gamma M)}$$

All constants W_j's are all positive and 'generically' different from zero in the sense that $W_j > 0$ for any non-zero measure of parameters $(\alpha, \beta, \gamma, \phi, \tau, L, M)$.[2]

It is known that Fourier series of seamless functions converge (Iorio and de Magalhães Iorio 2002, p.102). Hence, a seamless spatial distribution can be decomposed by its Fourier

[2]The reader will note that our micro-economically founded indirect utility function is more complicated than the utility assumed in Papageorgiou and Smith (1983). In the present continuous space model, the latter would simply be written as $V(x) = \int_0^1 E(x-z) \lambda(z) \, dz$ where the function $E(x)$ is an exogenous spatial externality that an agent has with the agents located at other locations.

series:

$$\lambda(x) \equiv \sum_{k=-\infty}^{\infty} \lambda_k \exp(2\pi I k x)$$

where $\lambda_k \in \mathbb{C}$, the set of complex numbers. Because $\lambda(x)$ returns real values and because $\int_0^1 \lambda(x)\, dx = 1$, we have that $\lambda_{-k} = \overline{\lambda_k}$, where $\overline{\lambda_k}$ is the complex conjugate of λ_k and $\lambda_0 = 1$.

The Fourier series of $\lambda(x)$ and $T(x)$ make explicit the spatial frequency contents of those functions. So, $V(x) - V(0) = \sum_{k=-\infty}^{\infty} V_k \exp(2\pi I k x)$, where

$$V_k = \lambda_k \left[-W_1 a_k + W_2 b_k - W_4 (a_k)^2\right] - \sum_{m=-\infty}^{\infty} \lambda_m a_m \lambda_{k-m} \left(W_3 a_{k-m} + W_5 a_k\right) \quad (4)$$

Equilibrium distribution with $\lambda(x) > 0$ is attained when $V(x)$ is constant for all x, which is equivalent to $V_k = 0 \ \forall k \neq 0$.

In this paper, we will focus on a particular type of equilibria: the *constant-access equilibria*. Because constant-access provides additional symmetry to the locational problem, the nature and the stability of constant-access equilibria can be determined in a general way. Non-constant access equilibria exist but their stability is difficult to assess.[3]

[3] In some cases, constant-access equilibria exist at the same time as non-constant-access equilibria. For example, when $T(x) = \sum_{k=-K}^{K} a_k \exp(2\pi I k x)$, it can be shown that non-constant-access distributions $\lambda^*(x)$ with the coefficients:

$$\begin{aligned}
&\lambda_k = 0 \qquad \text{for all } k \notin \{0, \pm K, \pm 2K\} \\
&\|\lambda_K\|^2 = \tfrac{W_2 b_{2K}}{W_3 W_5 (a_K)^4} \left[-W_1 a_K + W_2 b_K - (W_4 + W_5)(a_K)^2\right] \\
&\lambda_{2K} = -(\lambda_K)^2 \tfrac{W_3}{W_1} \tfrac{(a_K)^2}{b_{2K}}
\end{aligned}$$

are equilibria. Proof can be obtained upon request. In contrast to constant-access equilibrium distributions, this distribution changes as the economic parameters $(\alpha, \beta, \gamma, ...)$ change.Because the nature and stability of non-constant-access equilibria are too complicated to characterize in general, we will focus on constant-access equilibria in the sequel.

3.1 Constant-Access Equilibria

Constant-access distributions of workers are spatial distributions for which the access measures $f_1(x)$ and $f_2(x)$ are independent of any location x. The access measure $f_1(x)$ corresponds to the additional transport costs that workers located at x incur when they rise their consumption by one unit of each variety. The access measure $f_2(x)$ corresponds to a higher moment of this additional transport costs. Constant-access brings additional symmetry: workers have the same access to varieties wherever they locate. Conversely, firms have the same access to consumers.

Decomposing these measures in Fourier series as

$$f_1(x) = \sum_{k=-\infty}^{\infty} a_k \lambda_k \exp(2\pi I k x) \quad \text{and} \quad f_2(x) = \sum_{k=-\infty}^{\infty} b_k \lambda_k \exp(2\pi I k x)$$

and we define constant-access distributions as follows:

Definition 2 *A distribution $\lambda(x)$ is said to be constant-access if the convolutions $f_1(x)$ and $f_2(x)$ are constants.*

Using the relationship (2), this means that

$$a_k \lambda_k = a_m a_{m-k} \lambda_k = 0 \qquad \forall k \in \mathcal{N}^0, m \in \mathcal{N} \tag{5}$$

where $\mathcal{N}^0$ is the set of integers different from zero, and $\mathcal{N}$ is the set of integers.

By definition of equilibrium, a distribution with $\lambda(x) > 0$ yields an equilibrium if the worker's utility is the same everywhere. It is easily checked that expression (3) becomes a constant when the access measures $f_1(x)$ and $f_2(x)$ are replaced by constants.[4] Equivalently, it is readily shown that $V_k = 0 \ \forall k \neq 0$ under (5). This yields the following lemma.

Lemma 1 *Any constant-access, seamless distribution of workers is an equilibrium.*

[4]It should be noted that the constant-access is endogenously determined here, while it is often exogenously specified in the literature, such as Tabuchi *et al.* (2005).

The lemma has an important implication about the shape of transport cost on constant-access equilibrium distributions. Broadly speaking, the richness of the set of equilibrium distributions is complementary to the complexity of the transport cost function in term of its spatial frequency content. More formally, let $Y = Y_a \cup Y_b$, where Y_a is a set of frequencies $m \neq 0$ such that $a_m \neq 0$ and where Y_b is a set of frequencies $m - j$ such that $m, j \in Y_a$. Let also Z be a set of frequencies k such that $\lambda_k \neq 0$. The set Z includes the frequency $k = 0$. Then, any distribution with spatial frequencies in Z is an equilibrium if $Z \cap Y = \varnothing$.

It is easy to infer that flat earth $\lambda^*(x) = 1$ (i.e., $Z = \{0\}$ because $\lambda_k = 0 \; \forall k \neq 1$) is a seamless equilibrium distribution of workers that is compatible with all shapes of transport costs (i.e. for any a_m). From equality (5), it can be seen that *the set of seamless equilibria is smaller, the richer the spatial frequency content of transport cost functions.* To illustrate this, it is worth looking at specific examples.

Linear and exponential transport costs: Assume that a transport cost is given by

$$T(x) = \frac{1 - e^{-\rho x}}{1 - e^{-\rho/2}} \qquad \text{for } x \in [0, 1/2]$$

which is concave when $\rho > 0$ and convex when $\rho < 0$. Because $\lim_{\rho \to +0} T(x) = 2x$, this reduces to the linear transport cost function when ρ tends to 0. One can compute the Fourier coefficients of exponential cost functions ($\rho \neq 0$) as

$$a_m = \begin{cases} \frac{2 - 2e^{\rho/2} + \rho e^{\rho/2}}{\rho\left(e^{\rho/2} - 1\right)} & \text{for } m = 0 \\ \frac{2\rho\left[(-1)^m - e^{\rho/2}\right]}{(4\pi^2 m^2 + \rho^2)\left(e^{\rho/2} - 1\right)} & \text{for } m \neq 0 \end{cases}$$

and those of linear cost function ($\rho = 0$) as

$$a_m = \begin{cases} \frac{1}{2} & \text{for } m = 0 \\ \frac{(-1)^m - 1}{(\pi m)^2} & \text{for } m \neq 0 \end{cases}$$

Since $a_m < 0$ for odd m, we readily get $Y = \mathcal{N}^0$. Hence, flat earth ($Z = \{0\}$) is the unique seamless equilibrium distribution for linear and exponential transport costs.

Step transport costs: Assume that transport is costless within some distance ξ from production location and is constant above this distance: $T(x) = 0$ for $0 \leq x < \xi$ and $T(x) = 1$ if $\xi < x \leq 1/2$, where $0 < \xi < 1/2$. Then, $a_0 = 1 - 2\xi$ and $a_m = -\sin 2\xi m\pi/(m\pi)$for $m \geq 1$. If ξ is not a rational number so that $2\xi m \neq k$ with $m, k \in \mathcal{N}^0$, then $a_m \neq 0$ for all $m \neq 0$, and hence $Y = \mathcal{N}^0$. It must be that flat earth ($Z = \{0\}$) is the unique seamless equilibrium distribution.

Nevertheless, there exists shapes of transport costs that yield equilibria with other spatial distributions than flat earth.

Sinusoidal transport costs: Assume that $T(x) = (1 - \cos 2\pi x)/2$. Then, $a_0 = 1/2$, $a_1 = a_{-1} = -1/4$ and $a_m = 0$ otherwise. Hence, $Y = \{-2, -1, 1, 2\}$. Any spatial distribution with the shape $\lambda^*(x) = 1 + \sum_{|m| \geq 3} \lambda_m \exp(2\pi I m x) > 0$ is an equilibrium. For instance, the three-peaked distribution $\lambda^*(x) = 1 + 0.5 \cos 6\pi x$ as well as flat earth is an equilibrium.

To sum up, the shape of transport cost is an important determinant of the dimension of the set of spatial equilibria. We now turn to the issue of stability of those equilibria.

4 Stability of Seamless Equilibria

In this section, we extend Krugman's (1993) racetrack economic approach for any (non-flat) equilibrium and we study its stability. To this aim we analyze the stability against small perturbations on the seamless equilibrium distribution $\lambda^*(x)$. We first present the dynamics of workers' migration, derive the equilibrium conditions of workers' distribution around the space, and finally study whether those perturbations attenuate or amplify due to (infinitesimally) small perturbations.

4.1 Dynamic Behavior of Workers

Introducing the time variable t, the variables $\lambda(x,t)$ and $V(x,t)$ are now time dependent. We assume myopic workers in the time and space dimensions: in their migration decisions, workers consider only the current period and the utility differential with respect to their neighboring locations. More specifically, we assume that the number of migrating workers is proportional to the difference of their instantaneous utility between their current location x and their neighboring location $x \pm \varepsilon$ where $\varepsilon > 0$ is small enough. If $V(x-\varepsilon,t) > V(x,t)$, then we assume that $\nu_0 L\left[V(x-\varepsilon,t) - V(x,t)\right]$ workers move from locations x to $x-\varepsilon$, otherwise $\nu_0 L\left[V(x,t) - V(x-\varepsilon,t)\right]$ workers move from locations $x-\varepsilon$ to x, where coefficient $\nu_0 > 0$ measures the speed of adjustment. Similarly, if $V(x+\varepsilon,t) > V(x,t)$, then $\nu_0 L\left[V(x+\varepsilon,t) - V(x,t)\right]$ workers move from location x to location $x+\varepsilon$, otherwise $\nu_0 L\left[V(x,t) - V(x+\varepsilon,t)\right]$ workers move from location $x+\varepsilon$ to location x. The resulting flows yield the following local motion equation:

$$\frac{\partial}{\partial t}\lambda(x,t) = \nu_0\left[2V(x,t) - V(x-\varepsilon,t) - V(x+\varepsilon,t)\right] \tag{6}$$

This motion process respects the law of conservation of the total mass of workers. Indeed, the total number of workers remains fixed since $\int_0^1 \partial\lambda(x,t)/\partial t \; dx = 0$. Still, the motion process will be well defined provided that the number of workers at each location remains positive. That is, we must choose sufficiently small ν_0 and ε such that the RHS of (6) is less than $\lambda(x,t)$ for all x and t.

For sufficiently small ε, the above expression can be approximated to the following motion equation:

$$\frac{\partial}{\partial t}\lambda(x,t) = -\nu\frac{\partial^2}{\partial x^2}V(x,t) \tag{7}$$

where $\nu = \nu_0\varepsilon^2/2$, which is set equal to 1 without loss of generality.

We finally note that as in Mossay (2003), stability results do not depend on this local motion process where workers consider neighboring locations only. In Appendix 2, we prove the validity of our results for a global motion process where workers consider all locations in their migration choice.

4.2 Perturbed Equilibrium Distributions

Let us consider a seamless equilibrium distribution

$$\lambda^* (x) = \sum_{m \in Z} \lambda_m \exp(2\pi I m x), \tag{8}$$

where $Z \cap Y = \varnothing$ and $\lambda_0 = 1$. In order to check stability, we now allow for (infinitesimally) small temporal variations of the distribution of workers. Small perturbations are defined as $\widetilde{\lambda} (x,t) \equiv \lambda(x,t) - \lambda^*(x)$ and $\widetilde{V} (x,t) \equiv V (x,t) - V^*$, where a tilde refers to the perturbed values of these variables, and where $\lambda^*(x)$ and V^* are the equilibrium values of these variables. Also, using the definitions $\widetilde{f_1}(x,t) \equiv f_1(x,t) - f_1^* = \int_0^1 T (x-y) \widetilde{\lambda} (y,t)\, dy$ and $\widetilde{f_2}(x,t) \equiv f_2(x,t) - f_2^* = \int_0^1 [T (x-y)]^2 \widetilde{\lambda} (y,t)\, dy$, the utility function and the motion equation (7) can be linearized by dropping terms with perturbations of order strictly higher than one. This yields

$$\begin{aligned} \widetilde{V} (x,t) = {} & -W_1 \widetilde{f_1}(x,t) + W_2 \widetilde{f_2}(x,t) - W_4 \int_0^1 T (x-y) \widetilde{f_1}(y,t) dy \\ & - W_5 \int_0^1 T (x-y) \lambda^* (y) \widetilde{f_1}(y,t) dy + \text{constant} \end{aligned} \tag{9}$$

where we used the obvious facts that $\int_0^1 \widetilde{\lambda} (y,t)\, dy = 0$ and $\int_0^1 \lambda^* (y)\, dy = 1$.

The motion equation (7) can be linearized as

$$\frac{\partial \widetilde{\lambda} (x,t)}{\partial t} = -\frac{\partial^2 \widetilde{V} (x,t)}{\partial x^2} \tag{10}$$

Note that equations (9) and (10) constitute a homogenous system of linear partial differential equations. It can be studied by its 'normal modes' below.

4.3 Normal Modes and Instability

Let the system be perturbed by initial normal mode functions of the form $\exp(2\pi I k x)$, where k is the normal mode frequency. Like distributions of workers and firms, perturbations are assumed to be seamless functions. Then, the solution of the system is given by $\exp(2\pi Ikx)\exp(s_k t)$, where s_k's are amplifying parameters. Since the system of equations is linear, any linear combination of normal mode solutions is also a solution of the system. Because we focus on seamless spatial distributions, any initial perturbation can be decomposed by its Fourier series $\widetilde{\lambda}(x,0) = \sum_{k=-\infty}^{\infty} \widetilde{\lambda}_k \exp(2\pi Ikx)$, where $\widetilde{\lambda}_k$ are normal mode amplitudes. Since the perturbation does not alter the total size of the workers' population, it must be that $\int_0^1 \widetilde{\lambda}(x,0)\,dx = 0$ and thus $\widetilde{\lambda}_0 = 0$. As a result, the response of the system to such an initial perturbation is equal to

$$\widetilde{\lambda}(x,t) = \sum_{k=-\infty}^{\infty} \widetilde{\lambda}_k \exp(2\pi Ikx + s_k t)$$

Stability is related to the normal modes by the following definition:

Definition 3 *A seamless equilibrium $\lambda^*(x)$ is asymptotically stable if any sufficiently small and seamless change in the distribution results in a movement back toward the equilibrium.*

Therefore, an equilibrium is unstable if there exists a (infinitesimally small) perturbation of the equilibrium distribution that does not attenuate. That is, there exists a normal mode $k(\neq 0)$ that does not vanish: $s_k \geq 0$. We here provide the condition for stability of any constant-access equilibrium distribution $\lambda^*(x)$ given by (8). Plugging this distribution into (9) and (10) yields the following lemma:

Lemma 2 *A constant-access, seamless equilibrium distribution is unstable if and only if there exists one k such that $s_k > 0$ where*

$$s_k/(2\pi k)^2 = -\left[W_1 + (2W_3 + W_5)\,a_0\right] a_k + W_2 b_k - (W_4 + W_5)(a_k)^2$$

For general transport cost functions, a constant-access equilibrium will be unstable if, for some k, the above expression is positive. Inspecting the values of W_k, one readily checks that small transport costs τ and large manufacturing demand α decrease the likelihood of flat earth stability, whereas large farming population A always increases the likelihood of flat earth stability.

Because a_k and b_k are decreasing series, the dispersion effect of the farming population vanishes for high frequencies (see the term in $W_4 (a_k)^2$). In other words, the effect of the farming population is important only against low frequency perturbations (low k). The intuition for this goes as it follows. When workers are forced to locate more proportionally in the North of the circumference (i.e. a perturbation with frequency $k = 1$), they have incentives to relocate to the South because competition is weaker there and because they have a better access to Southern farmers. By contrast, if workers are asked to locate in many close and repeated areas (i.e. a perturbation with high frequency k), they do not have incentives to relocate far away because access and competition conditions are rather similar everywhere on the circumference. As a result, the stability properties associated to high frequencies are mostly determined by the properties of transport cost functions.

In practice, it is reasonable to argue that transport costs suddenly go up for sufficiently small distance because loading and unloading costs are not negligible. More generally, if we consider the weaker restriction of lumpiness $\lim_{x\to+0} T'(x) > 0$, the transport cost function has a kink at $x = 0$ and it is likely to generate high frequencies yielding positive s_k. This intuition is corroborated by the following proposition.

Proposition 1 *Assume that* $T(x)$ *is three-times differentiable for on the interval* $(0, 1/2)$ *and that it strictly increases at* $x = 0$ $(\lim_{x\to+0} T'(x) > 0)$. *Then, any constant-access, seamless equilibrium distribution* $\lambda^* (x)$ *is unstable.*

Fujita *et al.* (1999) and Mossay (2003) provide theoretical support to the view that flat earth is unstable in the case of exponential iceberg costs and infinitely large circumference. The present goes beyond those authors' result and suggests that *flat-earth instability is a property for a much larger class of transport cost functions and for a space with a finite dimension.* Instability of any constant-access, seamless equilibrium distribution holds for 'acceptable' transport cost functions.[5] This includes for instance linear and exponential transport cost functions. Models including fixed freight costs should also be lead to unstable equilibria to the extent that one models freight cost as transport cost functions with $T'(0) = \infty$.

The intuition of Proposition 1 lies in the fact that the consumption surplus of a variety is a decreasing and convex function of the consumption price and thus of the transport cost between the consumption and production places. If the shape of transport costs displays a kink at distance zero, the consumer surplus for distant varieties is also likely to display a kink with decreasing and convex slopes around the location of the consumer. Consumers then benefit from relocating to the peaks of any perturbation of firms'/workers' distribution. Because of this convexity of consumer surplus to distance, consumers benefit even more when the peaks get higher and narrower; that is, when the perturbation has a higher frequency k.

As it will become clear in Section 6, flat earth can be considered as the limit case of symmetric atomic cities when the number of cities goes to infinity. The proposition then tells that an equilibrium with infinitely many cities is always unstable. Therefore, the racetrack economic approach developed by Fujita *et al.* (1999) turns out to contain no stable constant-access equilibrium. Its intuition is that *workers and firms always have an incentive to move and form agglomerations in which some transport costs can be saved.*

To sum up, many shapes of transport costs yield to multiplicity of equilibria with seamless distributions of workers. However, flat earth is the unique constant-access equilibrium for all shapes of transport costs. Still, flat earth is unstable under acceptable conditions on the shape of transport costs. Because of this negative result on seamless distributions, it is worth studying alternative spatial distributions of workers. In the next section, we explore equilibria with atomic cities.

[5] See Picard and Tabuchi (2003) for a more complete characterization of stability conditions and transport costs as well as for a class of examples of transport cost functions that yield stable equilibria.

5 Equilibria with Atomic Cities

Suppose that there are n atomic cities located equidistantly on the circumference with perimeter equal to 1. Whereas the density of farmers is uniform across the circumference and equal to A, workers are now distributed over n atomic cities and located at $x_j \equiv j/n$, $j = 0, 1, \dots, n-1$. We focus on equilibria, where workers and firms locate only in atomic cities. The spatial distribution of workers is

$$\lambda(x) = \begin{cases} \frac{1}{n}\sum_{k=-\infty}^{\infty} \lambda_k \exp(2\pi I k x) & \text{if } x = x_j,\ j = 0, 1, \dots, n-1 \\ 0 & \text{otherwise} \end{cases}$$

with $\lambda_0 = 1$, $\lambda_{-k} = \overline{\lambda_k}$ and λ_k $(k \geq 1)$ are small enough to respect $\lambda(x) > 0$ for all x. A spatial distribution is called *symmetric* if $\lambda(x_j) = 1/n$ for all j.

Before studying discrete equilibrium distributions, it is natural to introduce the discrete Fourier series associated to transport cost functions as

$$T(x_j) = \sum_{m=0}^{n-1} a_m^n \exp(2\pi I m x_j) \qquad \text{and} \qquad [T(x_j)]^2 = \sum_{m=0}^{n-1} b_m^n \exp(2\pi I m x_j)$$

where the coefficients a_m^n and b_m^n are defined as

$$a_m^n \equiv \frac{1}{n}\sum_{j=0}^{n-1} T(x_j) \exp(-2\pi I m x_j) \qquad \text{and} \qquad b_m^n \equiv \frac{1}{n}\sum_{j=0}^{n-1} [T(x_j)]^2 \exp(-2\pi I m x_j)$$

These coefficients are the discrete counterpart of the Fourier coefficients in the racetrack economic model where workers' location choice is continuous. More specifically, using the identities

$$\frac{1}{n}\sum_{j=0}^{n-1} \exp(2\pi I k x_j) = \begin{cases} 1 & \text{if } k = nl,\ l \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases}$$

we get the following relationships:

$$a_m^n = a_{n-m}^n = \sum_{l=-\infty}^{\infty} a_{m-nl} \qquad \text{and} \qquad b_m^n = b_{n-m}^n = \sum_{l=-\infty}^{\infty} b_{m-nl}$$

for all $n \geq m$. Note that these discrete Fourier coefficients converge to the continuous Fourier coefficients when n becomes very large: $\lim_{n\to\infty} a_m^n = a_m$ and $\lim_{n\to\infty} b_m^n = b_m$.

The workers indirect utility with n cities becomes

$$\begin{aligned} V(x) &= W_0 - W_1 g_1(x) + W_2 g_2(x) - W_3 [g_1(x)]^2 \\ &\quad - W_4 \int_0^1 T(x-y)\, g_1(y) dy - W_5 \sum_{j=0}^{n-1} T(x - x_j)\, \lambda(x_j) g_1(x_j) \end{aligned}$$

where the discrete accessibility measures are

$$g_1(x) \equiv \sum_{i=0}^{n-1} T(x - x_i)\, \lambda(x_i) \qquad \text{and} \qquad g_2(x) \equiv \sum_{i=0}^{n-1} [T(x - x_i)]^2\, \lambda(x_i)$$

In a spatial equilibrium, workers have no incentives to move to other cities or to the hinterland between cities. According to Ginsburgh, Papageorgiou and Thisse (1985), spatial equilibrium is defined as follows.

Definition 4 *A spatial equilibrium is a distribution* $\lambda(x)$ *in the space* $[0,1]$ *such that either* $V(x) = \overline{V}$ *for* $\lambda(x) > 0$*, or* $V(x) \leq \overline{V}$ *for* $\lambda(x) = 0$.

The analysis in Section 3 and 4 can be applied here if we restrict relocations of workers to cities only, which is the case in most models with atomic cities.[6] When workers settle within $n(\geq 2)$ cities located at $x = x_j$, $j = 0, 1, \ldots, n-1$, it must be that $V(x)$ is the maximum at each city location x_j. Noting that $a_k^n = a_{k+n}^n$ and $b_k^n = b_{k+n}^n$, we get

$$\begin{aligned} g_1(x_j) &\equiv \sum_{i=0}^{n-1} T(x_j - x_i)\, \lambda(x_i) = \sum_{k=0}^{n-1} a_k^n \lambda_k^n \exp(2\pi I k x_j) \\ g_2(x_j) &\equiv \sum_{i=0}^{n-1} [T(x_j - x_i)]^2\, \lambda(x_i) = \sum_{k=0}^{n-1} b_k^n \lambda_k^n \exp(2\pi I k x_j) \end{aligned}$$

[6]To our knowledge, the literature provides no analytical, general study of hinterland conditions. Krugman (1993), Tabuchi *et al.* (2005), Anas (2004), and Behrens, Lamorgese, Ottaviano and Tabuchi (2004) model cities with no hinterland at all. Fujita and Kruman (1995) model city creation in the 'outer land' of an existing city using the specific exponential transport costs. Still, the analytical complexity obliges these authors to focus on the discussion of illustrative numerical exercises.

where

$$\lambda_k^n \equiv \sum_{l=-\infty}^{\infty} \lambda_{k-nl}$$

These two functions are constant for all $j = 0, 1, \ldots, n-1$ at an equilibrium with constant-access, whose definition is given by the following.

Definition 5 *A discrete distribution $\lambda(x_j)$ is said to be constant-access if $g_1(x_j)$ and $g_2(x_j)$ are constants.*

Using the above expressions, this means that

$$a_k^n \lambda_k^n = a_m^n a_{m-k}^n \lambda_k^n = 0 \qquad \forall k = 1, 2, \ldots, n, \ m = 0, 1, \ldots, n \tag{11}$$

The constant-access equilibrium conditions (11) are the discrete counterpart of (5). Similar to the seamless equilibria, we can therefore say the following.

Lemma 3 *When workers locate only within atomic cities, any constant-access discrete distribution of $n(\geq 2)$ atomic cities located at $\{x_0, x_1, ..., x_{n-1}\}$ is a spatial equilibrium. When workers can locate within the hinterland, the previous statement is true if, in addition, $V(x) \leq V(x_j)$ for all x belonging to the hinterland.*

The first part of this lemma is the discrete counterpart of Lemma 1. Still, when workers have access to hinterlands, the additional condition $V(x) \leq V(x_j)$ must be checked. In particular, when there are few cities, farmers in the hinterland are badly served by firms because the firms charge high prices there. Firms and workers can then find it profitable to relocate in the hinterland. As a result, symmetric distribution $\lambda^*(x_j) = 1/n$, $\forall j$ is not necessarily a spatial equilibrium in the case of atomic cities. Section 8 gives an illustration of this situation for linear transport costs and odd number of atomic cities. To our knowledge, there is no general condition that guarantees the absence of deviation in the hinterland without having recourse to the specific transport cost functions.

For a specific shape of transport costs, many non-uniform spatial distributions are likely to exist provided that transport costs include few spatial frequencies. Let $Y^n = Y_a^n \cup Y_b^n$, where Y_a^n is a set of integers $m \neq 0$ such that $a_m^n \neq 0$ and Y_b^n is a set of integers $m - j$ such that $m, j \in Y_a^n$. Let also Z^n be a set of positive integers k such that $\lambda_k^n \neq 0$. Then, any distribution with spatial frequencies in Z^n is a spatial equilibrium if $Z^n \cap Y^n = \varnothing$.

6 Stability of Atomic City Equilibria

A spatial equilibrium refers to possible deviations of an individual worker/firm to other location on the circumference. At the spatial equilibrium involving atomic cities, residing in the hinterlands between cities necessarily yields a smaller utility than in the cities. On the other hand, asymptotic stability refers to the dynamics of workers' relocation after some infinitely small perturbations of the spatial equilibrium. In the context of atomic cities, infinitely small perturbations are not able to alter the lack of attractiveness of hinterlands. Therefore the study of asymptotic stability can be restricted to perturbations of the populations located within cities.[7]

For expositional purposes, let $\Lambda \equiv (\lambda(x_0), \lambda(x_1), ..., \lambda(x_{n-1}))$ and $\Lambda^* \equiv (\lambda^*(x_0), \lambda^*(x_1), ..., \lambda^*(x_{n-1}))$. As defined before, an equilibrium Λ^* is asymptotically stable if any sufficiently small change in the distribution results in a movement back toward the equilibrium.

Analogous to the case of seamless distributions, we assume that workers compare utilities of neighboring cities. Dynamics of n cities is therefore depicted by

$$\frac{d\lambda^*(x_j)}{dt} = 2V(x_j, \Lambda) - V(x_{j-1}, \Lambda) - V(x_{j+1}, \Lambda) \qquad j = 0, 1, \dots, n-1 \tag{12}$$

where we normalize the speed of adjustment ν to 1 as before. Note that dynamics (12)

[7]Note that we do not consider the particular configurations of parameters such that workers' utility is the same within cities as at some locations outside cities. In this case, the stability criteria should be defined with respect to the hinterland too, which is beyond the scope of this paper.

satisfies the principle of a constant total mass since $\sum_{j=0}^{n-1} d\lambda^*(x_j)/dt = 0$. Asymptotic stability is then studied by linearizing the system of equations (12) around the constant-access equilibrium ($\Lambda = \Lambda^*$) and by studying the eigenvalues of this system of linear differential equations. The system is asymptotically stable when all eigenvalues are negative. Before characterizing stability, we need to define

$$c_k^n \equiv \sum_{l=-\infty}^{\infty} (a_{k-nl})^2 \geq 0$$

Note that $\lim_{n\to\infty} c_k^n = (a_k)^2$. In the Appendix 5, we obtain the following lemma:

Lemma 4 *A constant-access equilibrium with atomic cities is unstable if and only if there exists $k \in \{1, 2, \ldots, n-1\}$ such that $s_k^n > 0$ where*

$$s_k^n/n = -\left[W_1 + (2W_3 + W_5)\, a_0^n\right] a_k^n + W_2 b_k^n - \left[W_4 c_k^n + W_5 \left(a_k^n\right)^2\right] \tag{13}$$

Moreover,

$$\lim_{n\to\infty} \frac{s_k^n}{n} = \frac{s_k}{(2\pi k)^2}$$

This lemma is very similar to Lemma 2. One can readily check that it is equivalent to the latter for infinitely many atomic cities. Hence, stability of uniform distributions in atomic cities share similar properties. As in Proposition 1, *equilibrium distributions with atomic cities are unstable when n is large enough.* Inspection of expression (13) also allows us to establish the properties of stable equilibria in relation to the transport costs and the farming and manufacturing sectors. To our knowledge, the following Proposition provides the first analytical proof to Krugman's (1993) numerical exercises on the impact of transport costs and farming sector on the existence of agglomeration.

Proposition 2 *Assume that there exists j such that $T(x_j) > 0$. Constant-access equilibria with atomic cities are stable if there are sufficiently many immobile farmers (large A). They are unstable if the manufacturing demand is high (large α), if the transport costs are low (small τ) and if goods are very bad substitutes (small γ).*

It is widely known that when transport costs are small enough, full agglomeration is a stable equilibrium in the case of two-region setting, such as Krugman (1991) or Ottaviano *et al.* (2002). A natural question comes as to whether full agglomeration a single region or city can be a stable equilibrium when regions or cities lie on a continuous space like the circumference considered here. The answer is positive. When τ is small enough, the stability condition (13) can be approximated by $s_k^n = -W_1 a_k^n < 0$.Because transport costs are increasing with distance, one can show that $a_1^n < 0 \; \forall n \geq 2$,[8] and hence, $s_1^n > 0$. As a result, an equilibrium distribution with n atomic cities does not return to its initial value after a pertubation with frequency $k = 1$. In other words, if a small set of firms and workers relocate from the South to the North of the circumference, the rest of the firms and workers find it profitable to relocate to the North and any constant-access equilibrium with $n \geq 2$ atomic cities breaks down.So, constant-access equilibria in equidistant atomic cities are never stable for sufficiently small transport costs. Yet, this result generalizes for any type of spatial distribution, for any distance between atomic cities and for any equilibrium with hinterland relocation.

Proposition 3 *Assume that $T(x)$ strictly increases at $x = 0$ ($\lim_{x\to+0} T'(x) > 0$).For sufficiently small amplitude of transport costs ($\tau \to 0$), agglomeration in a single city is the unique stable equilibrium.*

The intuition is the same as in the two-region model. When transport costs fall to zero, the agglomeration force created by demand linkages fall less rapidly than the dispersion forces created by the farmers' dispersion and by product market competition. Note that in contrast to seamless distributions, this result about agglomeration in a single atomic city does not require any smoothness property on the shape of transport costs. This is because workers are not able relocate to infinitely close locations.

[8]The intuition is that increasing transport cost functions ($T'(x) = -T'(-x) > 0$, $x \in (0, 1/2)$) have a Fourier decomposition that includes the function of the form $a_1 \cos 2\pi x$ where $a_1 < 0$. This is the only Fourier basic function that increases on $x \in (0, 1/2)$. The proof is left to the reader.

7 Stable Equilibria with Specific Transport Costs

The previous sections are informative about equilibria with very small and very large number of atomic cities. We can nevertheless obtain clearer results about equilibria with intermediate numbers of cities for specific shapes of transport costs functions. In this section, we study the nature of stable equilibria of atomic cities given the following exponential shape of transport cost function:

$$T(x) = \frac{1 - e^{-\rho x}}{1 - e^{-\rho/2}} \qquad \text{for } x \in [0, 1/2]$$

This function yields respectively concave, linear or convex transport costs when ρ is respectively positive, nil ($\lim_{\rho \to 0} T(x) = |2x|$) or negative.

In the sequel, we show that more concave transport cost functions increases the equilibrium number of cities. The intuition simply stems from the fact that in the presence of transport costs, consumers purchase larger quantities of those varieties that are produced close to them. As transport costs become more concave, consumers less easily access close varieties and firms lose more revenue from close consumers than from remote ones. This entices firms and unskilled workers to spread in more numerous cities.

(i) Concave transport costs $\rho > 0$: Many models in the new economic geography literature (Baldwin *et al.* 2003) are based on the Samuelson's iceberg cost and exponential transport cost. Although such transport cost functions are generally chosen for their analytical properties in CES models, they are considered to be fair approximations of actual transport costs where distance-related shipping costs are low and distance-unrelated costs (insurance, loading and unloading) are high. Our above specification share similar properties when $\rho > 0$.

Computing the equilibrium and stability conditions for $n = 1, 2, 3, ...$ symmetric cities, we are able to determine the break and sustain points between which symmetric cities are stable equilibria. The left panel of Figure 2 shows an example of stable equilibria with symmetric cities when $\rho = 1$.[9] The solid bars represent the amplitude of average transport costs τ for which $n = 1, 2, 3, ...$ are stable equilibria. Note that the upper bound of τ is given by the feasible trade condition (1), which is 0.385. Observe that odd and even numbers of cities imply distinct properties for the ranges between break points (lowest τ) and sustain points (highest τ). Yet, for odd numbers of cities, both symmetric break points and sustain points are monotone increasing in the number of cities. The same property is true for even numbers of cities.

INSERT FIGURE 2

(ii) Linear transport cost $\rho \to 0$: Linear transport costs are intensively used in the traditional literature of spatial economics (e.g. Greenhut, Norman and Hung 1987). More recently, Ottaviano *et al.*'s (2002) have introduced the possibility of linear transport costs in the new economic geography paradigm. We here provide the characterization of symmetric cities equilibria under linear transport costs. The middle panel of Figure 2 depicts an example with linear transport cost and with the same economic parameters as in the case of concave transport costs.

Using Lemma 4, it can be shown that even numbers with $n \geq 4$ of symmetric cities are unstable.[10] Hence, as shown in the middle panel of Figure 2, there is no interval of transport costs that supports an equilibrium with even number $n \geq 4$ of atomic cities. We see that both symmetry break and sustain points are monotone increasing in the odd number of cities.

[9] Other parameters are set to $\alpha = 10, \beta = 2, \gamma = 1, \phi = 1, L = 100$ and $A = 50000$.

[10] $T(x) = 2x$ is the composition of sinusoidal cost functions like $1/2 + \sum_m a_{2m-1} \cos(2m-1)\pi x$. One can check that the associated Fourier coefficients a_k is zero for even k, which leads to $a_2^n = c_2^n = 0$ and $b_2^n > 0$, and hence $s_2^n > 0$ holds for all even $n (\geq 4)$. Thus, atomic city equilibria with even numbers $(n \geq 4)$ are always unstable.

Under linear transport costs, equilibrium conditions differ according to even or odd numbers of cities. For an even number of cities, the mill price $p^*(x,x)$ is shown to be constant across locations for even number of cities, which simplifies the periodic,indirect utility[11]

$$V(x) = \text{constant} + W_2 \left[-\frac{n-1}{n^2} |2nx| + \frac{1}{n^2} (2nx)^2 \right]$$

$V(x) = V(x + k/n)$, $k \in \mathcal{N}, x \in (-1/2n, 1/2n)$. This expression attains a maximum at city locations $x_j = 0, 1/n, \ldots, (n-1)/n$ because $W_2 > 0$. This implies that *once a symmetric equilibrium with even number of cities is reached, workers do not move to the hinterland, and hence new cities never emerge for any marginal changes in parameter values.* One can check that the utility level at the symmetric equilibrium decreases with even n and converges to the level obtained under flat earth as n goes to infinity.

For an odd number of cities, the mill price $p^*(x,x)$ is not constant, and the periodic,indirect utility is written as

$$\begin{aligned} V(x) = \text{constant} &- \frac{1}{n^2} \left[n^2 W_1 + \left(n^2 - 1\right) W_3 \right] |2nx| \\ &+ \frac{1}{2n^4} \left(2n^2 W_2 - 2W_3 + W_4\right) (2nx)^2 - \frac{1}{3n^4} W_4 |2nx|^3 \end{aligned}$$

$V(x) = V(x + k/n)$, $k \in \mathcal{N}, x \in (-1/2n, 1/2n)$. The indirect utility is a periodic function with n periods and has the following properties in the interval of $[0, 1/n]$: it is symmetric about $x = 1/2n$, each symmetric section is a cubic function of x such that $\lim_{x \to +0} V'(x) < 0$ and $\lim_{x \to 1/2n-0} V'(x) < 0$. Hence, when new cities emerges due to, say, an increase in farming population, their locations are not at midpoints of existing cities. One thus needs to check case by case whether a new city may emerge in the interval $(0, 1/2n)$. Finally, one can check that the utility level at the symmetric equilibrium is shown to be decreasing in odd n and equal to the flat earth utility for $n \to \infty$. An example of the transport costs τ for which cities are stable equilibria is shown in the middle panel of Figure 2. One again

[11]The proofs of the following results are in Picard and Tabuchi (2003).

observes that both break points (lowest τ) and sustain points (highest τ) are monotone increasing in the (odd!) number of cities.

(iii) Convex transport costs $\rho < 0$: Convex transport costs are seldom used in the spatial economics literature because of the transshipping problem: a direct shipping to a destination costs more than an indirect shipping using an intermediate destination ($T(x+y) > T(x)+T(y)$). Firms are then enticed to partition the shipping across distance. Nevertheless, it is instructive to discuss this type of transport cost in our study of the impact of cost convexity on location equilibria. In the right hand panel of Figure 2, we conduct similar computations as before for $\rho = -1$, we find that only few cities are stable equilibria ($n = 1, 3, 5$). For more convex transport costs, the number of cities at the equilibrium is even smaller. This implies that a symmetric equilibrium is less likely to exist for a sufficiently convex transport cost (although asymmetric equilibria would exist).

To sum up, the present analysis suggests that the equilibrium number of symmetric cities strongly depends on the degree of symmetry in their configuration (even n versus odd n) and that it heavily depends on the shape of transport costs. In particular, this analysis suggests that more concave transport costs yield more numerous cities at the equilibrium and that it generate a higher equilibrium indeterminacy.[12]

8 Conclusion

We have considered the racetrack economic approach, where manufacturing activities are distributed continuously and discretely around a circumference of a circle, and where

[12] One might think that an increase in the equilibrium number of cities could have been attributed to a rise in the transport cost that a more concave transport cost function would have implied. However, this is needless worry because the results in Figure 2 show a clear contrast: many stable equilibria for concave transport costs ($\rho > 0$) and few stable equilibria for convex transport costs ($\rho < 0$) regardless of the values of τ.

the economic interactions were identical to those in Ottaviano *et al.*'s (2002) model. Investigating the nature and stability of equilibria, we have shown that constant-access, seamless equilibrium distributions are unstable for 'acceptable' transport cost functions, whereas agglomeration in 1 or 2 atomic cities is stable for any economic parameters given some symmetry properties of the transport cost functions.

Our finding vindicates that the assumption of two regions in most of the literature is neither noxious nor a mathematical convenience. When compared with Hotelling or Cournot spatial competition, agglomeration is a distinct property in economic geography accruing from the existence of home market effects which results in forming a limited number of atomic cities instead of dispersed and continuous urban configurations.

Our analysis also sheds some light on the role of the shape of transport cost in the properties of spatial equilibria. The richness of the class of spatial equilibrium distributions crucially depends on the shape of the transport cost function, and more specifically on its spatial frequency content. Also, the stability of constant-access, seamless equilibria highly depends on the slope of this function. When the transport cost function is strictly increasing only at the origin ($T'(0) > 0$), these equilibria (including flat earth) are unstable. Convex transport costs drastically eliminate candidate distributions for stable equilibrium with atomic cities. This indicates that the modeling choice of transport cost function is not innocuous.

To our opinion, this piece of work provides the first unified framework that allows economists to study spatial equilibria of continuous and discrete distributions within the new economic geography paradigm. As the reader will note, the analyses of the continuous and discrete cases share so close similarities that it seems inefficient to present the two cases separately. Moreover, this work offers analytical light about issues that have been dealt with numerical exercises or under strong assumptions. The work does not only confirms and qualifies the existing results, but it also adds new results about the role of the shape of transport costs in the determination of spatial equilibria. Still, as most contributions in this topic, this work has encountered many technical limitations. In particular, the paper limits its focus on constant-access, seamless equilibria and atomic cities equilibria, which is a subset of the equilibrium set. Other types of equilibria are difficult to study analytically and may need to have recourse to numerical tools. These issues are thus left for further research.

Appendix 1: The weights w_i in (3)

We have that $p^*(y,x) = \alpha_1 + \alpha_2 f_1(x) + \frac{\tau}{2} T(x-y)$, where

$$\alpha_1 \equiv \frac{\alpha\beta}{2\beta+\gamma M} \quad \text{and} \quad \alpha_2 \equiv \frac{\tau\gamma M}{2(2\beta+\gamma M)}$$

We can compute

$$\int_0^1 p^*(y,x)\lambda(y)\,dy = \alpha_1 + \left(\alpha_2 + \frac{\tau}{2}\right) f_1(x)$$

$$\int_0^1 [p^*(y,x)]^2 \lambda(y)\,dy = (\alpha_1)^2 + (\alpha_2)^2 [f_1(x)]^2 + \left(\frac{\tau}{2}\right)^2 f_2(x) + 2\alpha_1\alpha_2 f_1(x) + \frac{\tau}{2} 2\alpha_1 f_1(x) + \frac{\tau}{2} 2\alpha_2 [f_1(x)]^2$$

and thus

$$\begin{aligned} S^*(x) = {} & \tfrac{\alpha^2 M}{2(\beta+\gamma M)} - \tfrac{\alpha M}{\beta+\gamma M}\alpha_1 - \tfrac{\gamma M^2}{2\beta(\beta+\gamma M)}(\alpha_1)^2 + \tfrac{M}{2\beta}(\alpha_1)^2 \\ & + \left[\left(-\tfrac{\alpha M}{\beta+\gamma M} - \tfrac{2\alpha_1\gamma M^2}{2\beta(\beta+\gamma M)}\right)\left(\alpha_2 + \tfrac{\tau}{2}\right) + \tfrac{M}{2\beta}\alpha_1(2\alpha_2+\tau)\right] f_1(x) \\ & + \left[-\tfrac{\gamma M^2}{2\beta(\beta+\gamma M)}\left(\alpha_2+\tfrac{\tau}{2}\right)^2 + \tfrac{M}{2\beta}\alpha_2(\alpha_2+\tau)\right][f_1(x)]^2 \\ & + \tfrac{M}{2\beta}\left(\tfrac{\tau}{2}\right)^2 f_2(x) \end{aligned}$$

$$\begin{aligned} & \textstyle\int_0^1 [p^*(x,y) - \tau(x,y)]^2 A\,dy \\ & \quad = (\alpha_1)^2 A + (\alpha_2)^2 A \textstyle\int_0^1 [f_1(y)]^2 dy + \left(\tfrac{\tau}{2}\right)^2 A \int_0^1 [T(x-y)]^2 dy \\ & \qquad + 2\alpha_1\alpha_2 A \textstyle\int_0^1 f_1(y)dy - 2\alpha_1 \tfrac{\tau}{2} A \int_0^1 T(x-y)\,dy \\ & \qquad - 2\alpha_2 \tfrac{\tau}{2} A \textstyle\int_0^1 T(x-y) f_1(y) dy \end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left[p^*(x,y) - \tau(x,y)\right]^2 L\lambda(y)\,dy \\
&\quad = (\alpha_1)^2 L + (\alpha_2)^2 L \int_0^1 [f_1(y)]^2 \lambda(y)\,dy + \left(\tfrac{\tau}{2}\right)^2 L f_2(x) \\
&\qquad + 2\alpha_1\alpha_2 L \int_0^1 f_1(y)\lambda(y)\,dy - 2\alpha_1 \tfrac{\tau}{2} L f_1(x) \\
&\qquad - 2\alpha_2 \tfrac{\tau}{2} L \int_0^1 T(x-y)\, f_1(y)\lambda(y)\,dy
\end{aligned}$$

Note that $\int_0^1 T(x-y)\,dy = a_0$ and $\int_0^1 [T(x-y)]^2\,dy = b_0$. Thus, grouping terms and substituting for α_1, α_2 and α_3, we get

$$\begin{aligned}
V(x) &= W_0 - W_1 f_1(x) + W_2 f_2(x) - W_3 [f_1(x)]^2 \\
&\quad - W_4 \int_0^1 T(x-y)\, f_1(y)dy - W_5 \int_0^1 T(x-y)\, f_1(y)\lambda(y)\,dy
\end{aligned}$$

where W_0 is a constant and

$$W_1 = \frac{\tau\alpha M(3\beta + 2\gamma M)}{(2\beta+\gamma M)^2} \qquad W_2 = \frac{3\tau^2 M}{8\beta} \qquad W_3 = \frac{\tau^2\gamma^2 M^3}{8\beta(2\beta+\gamma M)^2}$$

$$W_4 = \frac{\tau^2\gamma A M}{2\beta\phi(2\beta+\gamma M)} \qquad W_5 = \frac{\tau^2\gamma M^2}{2\beta(2\beta+\gamma M)}$$

all of which are positive.

Finally, substituting the Fourier series of $\lambda(x)$ and $T(x)$, the terms in (3) are computed as

$$\begin{aligned}
f_1(x) &= \sum_{k=-\infty}^{\infty} \lambda_k a_k \exp(2\pi I k x) \\
f_2(x) &= \sum_{k=-\infty}^{\infty} \lambda_k b_k \exp(2\pi I k x) \\
[f_1(x)]^2 &= \sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} (\lambda_m a_m \lambda_{k-m} a_{k-m})\right] \exp(2\pi I k x) \\
\int_0^1 T(x-y)\, f_1(y)dy &= \sum_{k=-\infty}^{\infty} \lambda_k (a_k)^2 \exp(2\pi I k x) \\
\int_0^1 T(x-y)\,\lambda(y)\, f_1(y)dy &= \sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} (\lambda_m a_m \lambda_{k-m} a_k)\right] \exp(2\pi I k x)
\end{aligned}$$

Appendix 2: Global motion process

This paper studies stability under a *local motion* process. We here show that stability under the local motion process is equivalent to stability under a global motion process.

For seamless distributions, a *global motion* process may be depicted by

$$\frac{d\lambda(x,t)}{dt} = V(x,t) - \int_0^1 V(z,t)\, dz$$

by which workers moves to any location that has a higher utility than the average utility on the circumference. Obviously, the total number of workers remains fixed: $\int_0^1 \frac{d\lambda(x,t)}{dt} dx = 0$ $\forall t$. For small perturbations, we get

$$\frac{d\widetilde{\lambda}(x,t)}{dt} = \widetilde{V}(x,t)$$

and for normal modes with spatial frequency k, we get

$$s_k \widetilde{\lambda}_k = \widetilde{V}_k$$

This must be compared with the local motion and its resulting normal mode equality: $s_k \widetilde{\lambda}_k = (2\pi k)^2 \widetilde{V}_k$. Obviously, the sign of s_k does not depend on whether motion is local or global.

For equilibria with atomic cities, we propose a similar global motion equation:

$$\frac{d\lambda_k}{dt} = V(x_k, \Lambda) - \frac{1}{n} \sum_{j=0}^{n-1} V(x_j, \Lambda) \qquad k = 0, 1, \dots, n-1$$

Net migration flow in a city x_k is proportional to the difference between worker's utility in a city and the average utility. Differentiating the RHS of this equation by λ_l and evaluating it at Λ^*, we get the Jacobian matrix J, whose elements are given by

$$J_{m,l} = \frac{\partial}{\partial \lambda_l} V(x_m, \Lambda^*) - \frac{1}{n} \sum_{i=0}^{n-1} \frac{\partial}{\partial \lambda_l} V(x_i, \Lambda^*)$$

Taking advantage of the symmetry property, we note that partial derivatives are equal to

$$v_j \equiv \frac{\partial}{\partial \lambda_0} V(x_j, \Lambda_0) = \frac{\partial}{\partial \lambda_l} V(x_m, \Lambda^*).$$

for any l and m such that $j = \mathrm{mod}\,|l-m|$. This also yields a circulant Jacobian matrix $J_{n,l} = v_j - \overline{v}$ where $j = |l-n|$ and where $\overline{v} = \frac{1}{k}\sum_{j=0}^{k-1} v_j$. The n eigenvalues of this matrix are

$$s_k^n = \sum_{j=0}^{n-1} (v_j - \overline{v}) \exp(2\pi I k x_j) = \sum_{j=0}^{n-1} v_j \exp(2\pi I k x_j) \qquad k = 0, 1, \ldots, n-1$$

Each eigenvalue has the same sign as that the eigenvalues (14) derived in Appendix 5 and reported in Lemma 4 for the local motion process since $1 - \cos\frac{2\pi k}{n} > 0$ holds for $1 \leq k < n$ in (14).

Appendix 3: Proof of Lemma 2

At a constant-access equilibrium $\lambda(x) = \lambda^*(x)$, we must have

$$f_1(x) = a_0 \qquad \text{and} \qquad f_2(x) = b_0$$

In the dynamic setting, small perturbations are defined as $\widetilde{\lambda}(x,t) \equiv \lambda(x,t) - \lambda^*(x)$, $\widetilde{f}_1(x,t) \equiv f_1(x,t) - a_0$ and $\widetilde{f}_2(x,t) \equiv f_2(x,t) - b_0$. Dropping terms in perturbations with order higher than one, we can write the perturbation in the worker's utility as

$$\widetilde{V}(x) = -W_1 \widetilde{f}_1(x,t) + W_2 \widetilde{f}_2(x,t) - 2W_3 \widetilde{f}_1(x,t) f_1^* - W_4 \int_0^1 T(x-y)\, \widetilde{f}_1(y,t) dy$$
$$- W_5 \int_0^1 T(x-y)\, \widetilde{\lambda}(y)\, f_1^* dy - W_5 \int_0^1 T(x-y)\, \lambda^*(y)\, \widetilde{f}_1(y,t) dy$$

Since $\widetilde{\lambda}(x,t) = \sum_{k=-\infty}^{\infty} \widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$ and since $\int_0^1 \exp(2\pi I (k-m) y)\, dy$ is equal to 1 if $k = m$ and to 0 otherwise, we have

$$\widetilde{f_1}(x,t) = \int_0^1 T(x-y)\,\widetilde{\lambda}(y)\,dy = \sum_{k=-\infty}^{\infty} a_k \widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$$

$$\widetilde{f_2}(x,t) = \sum_{k=-\infty}^{\infty} b_k \widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$$

$$\int_0^1 T(x-y)\,\widetilde{f_1}(y,t)dy = \sum_{k=-\infty}^{\infty} (a_k)^2\,\widetilde{\lambda}_k \exp(2\pi I m x + s_k t)$$

$$\int_0^1 T(x-y)\,\widetilde{\lambda}(y)\,f_1^* dy = \widetilde{f_1}(x,t) f_1^* = \sum_{k=-\infty}^{\infty} a_0 a_k \widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$$

$$\begin{aligned}
&\int_0^1 T(x-y)\,\lambda^*(y)\,\widetilde{f_1}(y,t)dy = \\
&\quad = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_l \lambda_m^* a_k \widetilde{\lambda}_k \int_0^1 \exp(2\pi I(ky + my + lx - ly) + s_k t)\,dy \\
&\quad = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_m a_k \lambda_{m-k}^* \widetilde{\lambda}_k \exp(2\pi I m x + s_k t) \\
&\quad = \sum_{k=-\infty}^{\infty} (a_k)^2\,\widetilde{\lambda}_k \exp(2\pi I k x + s_k t)
\end{aligned}$$

where the last equation is due to the condition of constant-access equilibrium: $a_m a_k \lambda_{m-k}^*$ for all $k \neq m$ and $\lambda_0^* = 1$.

Using $\widetilde{V}(x) = \sum_{k=1}^{\infty} \widetilde{V}_k \widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$, we obtain

$$\widetilde{V}_k = -W_1 a_k + W_2 b_k - (W_4 + W_5)(a_k)^2 - (2W_3 + W_5)\,a_0 a_k$$

Appendix 4: Proof of Proposition 1

We focus on sufficiently high frequencies k. Integrating by part, the Fourier coefficient a_k can successively be rewritten as

$$\begin{aligned}
a_k &= 2\int_0^{1/2} T(x)\cos 2\pi k x\ dx = -2\int_0^{1/2} \frac{T'(x)\sin 2\pi k x}{2\pi k}dx \\
&= 2\frac{(-1)^k T'(1/2) - T'(0)}{(2\pi k)^2} + 2\int_0^{1/2} \frac{T'''(x)\sin 2\pi k x}{(2\pi k)^3}dx
\end{aligned}$$

Because $T(x)$ is three-times differentiable, $T'''(x)$ is bounded and the second term in the last equality can be neglected because it has higher order in k than the first term.

Similarly, using $D(x) \equiv [T(x)]^2$, we have

$$b_k \approx 2\frac{(-1)^k D'(1/2) - D'(0)}{(2\pi k)^2} = 2\frac{(-1)^k D'(1/2)}{(2\pi k)^2}$$

where we use the equality $D'(0) = 2T'(0)T(0) = 0$.

Since $(a_k)^2 \ll a_k$ holds for sufficiently large k, the sign of s_k is the same as the sign of $-[W_1 + (2W_3 + W_5)\, a_0]\, a_k + W_2 b_k$. Inserting the above results in this expression, the sign of s_k is related to the sign of

$$[W_1 + (2W_3 + W_5)\, a_0]\, T'(0) \; - (-1)^k \{[W_1 + (2W_3 + W_5)\, a_0]\, T'(1/2) + W_2 D'(1/2)\}$$

Because $a_0 > 0$, we conclude that the first term is positive if $T'(0) > 0$ and that the second term is positive for either even or odd k. Hence, for sufficiently large $\overline{k}$, there exist $k > \overline{k}$ such that $s_k > 0$.

Appendix 5: Proof of Lemma 4

In this Appendix we study the condition for asymptotic stability of equidistant atomic cities ($x_j = j/n$, $j = 0, ..., n-1$) for which workers' distribution $\Lambda = (\lambda(x_0), ..., \lambda(x_{n-1}))$ follows the workers' motion equations (12). Linearizing the system of equations (12) around the constant-access equilibrium ($\Lambda = \Lambda^*$), we find a system of linear differential equations with the Jacobian matrix, whose element is

$$J_{j,i} = \frac{\partial}{\partial \lambda(x_i)} \left[2V(x_j, \Lambda^*) - V(x_{j-1}, \Lambda^*) - V(x_{j+1}, \Lambda^*)\right]$$

for $i = 0, ..., n-1$ and $j = 0, ...n-1$. Eigenvalues are obtained form this Jacobian matrix and determine the asymptotic stability of the equilibrium. We proceed in three steps.

In the first step, we derive the elements of the Jacobian matrix associated to the system of equations (12). The constant-access equilibrium condition is now written as $g_1(x_j, \Lambda^*)) = g_1^*$ and $g_2(y, \Lambda^*)) = g_2^*$ where g_1^* and g_2^* are constants. Using these properties and the fact that $\sum_{j=1}^{n-1} \lambda^*(x_j) = 1$, we get

$$
\begin{aligned}
\frac{\partial V(x_m, \Lambda^*)}{\partial \lambda^*(x_i)} &= -W_1 T\left(x_i - x_m\right) + W_2 \left[T\left(x_i - x_m\right)\right]^2 \\
&- 2W_3 g_1^* T\left(x_i - x_m\right) - W_4 \int_0^1 T\left(x_i - y\right) T\left(y - x_m\right) dy \\
&- W_5 \sum_{j=0}^{n-1} \left[T\left(x_i - x_m\right) + T\left(x_j - x_m\right)\right] T\left(x_i - x_j\right) \lambda^*(x_j)
\end{aligned}
$$

For any equilibrium distribution $\lambda^*(x_j)$ satisfying conditions (11), this expression has the following symmetry property: $\partial V(x_0)/\partial\lambda^*(x_i) = \partial V(x_k)/\partial\lambda^*(x_{i+k})$ for all $k \in \mathcal{N}$. Let us define

$$
v_i \equiv \frac{\partial V(x_0)}{\partial \lambda^*(x_j)} \qquad \text{for } j = \operatorname{mod}|i|
$$

where mod $|i|$ is the modulo n function: $\operatorname{mod}|i| = i - nl$ if $nl \leq i < n(l+1)$, $l \in \mathcal{N}$. Because of the symmetry, we note that $v_i = \partial V(x_k)/\partial\lambda^*(x_{i+k})$ for all $k = 0, 1, \ldots, n-1$. This yields a circulant Jacobian matrix $J_{j,i} = J_{|i-j|,0}$ where $J_{k,0} = 2v_k - v_{k-1} - v_{k+1}$.

In the second step, we relate the eigenvalues of the linearized system (12) to the elements v_i of the Jacobian matrix. According to Bellman (1970, pp.242-243) and Papageorgiou and Smith (1983), the n eigenvalues of this matrix are known as

$$
s_k^n = \sum_{j=0}^{n-1} J_{j,0} \exp\left(-2\pi I k x_j\right) \qquad \text{for } k = 0, 1, \ldots, n-1
$$

which is rewritten as

$$
\begin{aligned}
s_k^n &= \sum_{j=0}^{n-1} \left(2v_j - v_{j-1} - v_{j+1}\right) \exp\left(-2\pi I k x_j\right) \\
&= \sum_{j=0}^{n-1} v_j \left[2 \exp \frac{-2\pi I k j}{n} - \exp \frac{-2\pi I k \left(j+1\right)}{n} - \exp \frac{-2\pi I k \left(j-1\right)}{n}\right] \\
&= 2\left(1 - \cos \frac{2\pi k}{n}\right) \sum_{j=0}^{n-1} v_j \exp\left(-2\pi I k x_j\right) \qquad (14)
\end{aligned}
$$

Since $1 - \cos\frac{2\pi k}{n} > 0$, the eigenvalue s_k^n is proportional to the discrete Fourier transform of v_j. Because $s_0^n = 0$, we pay attention to s_k^n for $k = 1, \ldots, n-1$.

In the final step, we evaluate the eigenvalues s_k^n as the functions $\sum_{j=0}^{n-1} v_j \exp\left(-2\pi I k x_j\right)$ for $0 \leq k < n$, where

$$\begin{aligned} v_j = \frac{\partial V(0,\Lambda^*)}{\partial \lambda^*(x_j)} &= -W_1 T\left(x_j\right) + W_2 \left[T\left(x_j\right)\right]^2 - 2W_3 g_1^* T\left(x_j\right) \\ &- W_4 \int_0^1 T\left(x_j - y\right) T\left(y\right) dy - W_5 \sum_{i=0}^{n-1} \left[T\left(x_j\right) + T\left(x_i\right)\right] T\left(x_j - x_i\right) \lambda^*(x_i) \end{aligned}$$

We compute each line separately. The first line obviously yields

$$n\left(-W_1 a_k^n + W_2 b_k^n - 2W_3 g_1^* a_k^n\right)$$

whereas the first term on the second line gives

$$\begin{aligned} &- W_4 \sum_{j=0}^{n-1} \int_0^1 T\left(x_j - y\right) T\left(y\right) dy \exp\left(-2\pi I k x_j\right) \\ &= -W_4 \sum_{j=0}^{n-1} \sum_{l=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} a_l a_i \int_0^1 \exp\left(2\pi I \left(l x_j - l y + i y - k x_j\right)\right) dy \\ &= -nW_4 \sum_{l=-\infty}^{\infty} \left(a_{nl+k}\right)^2 = -nW_4 c_k^n \end{aligned}$$

To compute the last term on the second line, we firstly note that for $i = \{0, 1, ..., n-1\}$,

$$\begin{aligned} \lambda^*(x_i) &= \sum_{q=-\infty}^{\infty} \lambda_q^* \exp(2\pi I q i/n) = \sum_{r=0}^{n-1} \sum_{l=-\infty}^{\infty} \lambda_{r+nl}^* \exp(2\pi I i \left(r + nl\right)/n) \\ &= \sum_{r=0}^{n-1} \lambda_r^n \exp(2\pi I r x_i) \end{aligned}$$

One can then show that $\sum_{r=0}^{n-1} \lambda^*(x_i) = n\lambda_0^n = 1$, and thus $\lambda_0^n = 1/n$. Note also that

$$\sum_{i=0}^{n-1} \sum_{t=0}^{n-1} \sum_{r=0}^{n-1} a_t^n \lambda_r^n \exp\left(2\pi I \left(-t x_i + r x_i\right)\right) = n \sum_{r=0}^{n-1} a_r^n \lambda_r^n$$

$$\sum_{j=0}^{n-1} \sum_{s=0}^{n-1} \sum_{r=0}^{n-1} a_s^n a_r^n \lambda_r^n \exp\left(2\pi I \left(s x_j + r x_j - k x_j\right)\right) = n a_k^n a_0^n \lambda_0^n + 2n \sum_{r=0}^{n-1} a_{k-r}^n a_r^n \lambda_r^n$$

As a consequence, the first term in the last line can be computed as

$$\sum_{j=0}^{n-1}\left[-W_5\sum_{i=0}^{n-1}T\left(x_j\right)T\left(x_j-x_i\right)\lambda^*(x_i)\right]\exp\left(-2\pi Ikx_j\right)$$
$$=-W_5\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\sum_{s=0}^{n-1}\sum_{t=0}^{n-1}\sum_{r=0}^{n-1}a_s^na_t^n\lambda_r^n\exp\left(2\pi I\left(sx_j+tx_j-tx_i+rx_i-kx_j\right)\right)$$
$$=-nW_5a_k^na_0^n$$

where the last equality is due to constant-access condition ($a_r^n\lambda_r^n=0$ for all $r\neq 0$).

Finally, the second term of the last line is equal to

$$\sum_{j=0}^{n-1}\left[-W_5\sum_{i=0}^{n-1}T\left(x_i\right)T\left(x_j-x_i\right)\lambda^*(x_i)\right]\exp\left(-2\pi Ikx_j\right)$$
$$=-W_5\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\sum_{s=0}^{n-1}\sum_{t=0}^{n-1}\sum_{r=0}^{n-1}a_s^na_t^n\lambda_r^n\exp\left(2\pi I\left(sx_i+tx_j-tx_i+rx_i-kx_j\right)\right)$$
$$=-nW_5\left(a_k^n\right)^2$$

where the last equality is due to constant-access condition ($a_{r-k}^na_k^n\lambda_r^n$ for all $r\neq 0$).

Thus, using $g_1^*=na_0^n\lambda_0^n=a_0^n$ under constant-access condition, we get

$$\lim_{n\to\infty}s_k^n/n=\left[-W_1a_k+W_2b_k-\left(W_4c_k^n+W_5\left(a_k\right)^2\right)-\left(2W_3+W_5\right)a_0a_k\right]=\frac{s_k}{8\pi^2k^2}$$

Note that eigenvalues are real numbers.

Appendix 6: Proof of Proposition 2

For large A and thus large W_4, we obviously get $s_k^n<0$ for all k. Since a_k^n multiplies W_1 and thus α in (13), large enough α yields instability because there exists a negative a_k^n with $k\in\{1,2,\ldots,n-1\}$. Indeed, on the one hand, one can check that $T(0)=\sum_{k=0}^{n-1}a_k^n=0$, $a_0^n=(1/n)\sum_{k=0}^{n-1}T(x_k)>0$, and thus, $\sum_{k=1}^{n-1}a_k^n<0$. Hence, there always exists a strictly negative a_k^n with $k\neq 0$. The same argument applies to τ, using $[T(0)]^2=\sum_{k=0}^{n-1}b_k^n=0$, $b_0^n=(1/n)\sum_{k=0}^{n-1}[T(x_k)]^2>0$. Finally, when $\gamma\to 0$, we get $s_k^n/n=-W_1a_k^n+W_2b_k^n$. In particular,

$$\lim_{\gamma\to 0}\frac{s_0^n}{n} = -W_1 a_0^n + W_2 b_0^n = \frac{3\tau M}{8\beta n}\left[-2\alpha\sum_{k=0}^{n-1}T(x_k) + \tau\sum_{k=0}^{n-1}[T(x_k)]^2\right]$$
$$\leq \frac{3\tau M}{8\beta n}(-2\alpha+\tau)\sum_{k=0}^{n-1}T(x_k) < 0$$

where the first inequality is from $T(x) \geq [T(x)]^2$ and the second from (1) with $\gamma \to 0$. Since $\sum_{k=0}^{n-1} s_k^n/n = -W_1\sum_{k=0}^{n-1} a_k^n + W_2\sum_{k=0}^{n-1} b_k^n = 0$, there exists at least one k such that $s_k^n > 0$.

Appendix 7: Proof of Proposition 3

We know from Proposition 1 that there is no seamless equilibrium distribution that is stable. We therefore consider equilibrium distribution of atomic cities, which are not necessarily located equidistantly (i.e. $x_j \neq j/n$).

For sufficiently small τ, the indirect utility function approximates as $V(x_j) \approx$constant $-W_1 g_1(x_j)$. Differentiating the RHS of the dynamics (12) by $\lambda(x_i)$ and evaluating it at $\lambda(x_i) = \lambda^*(x_i)$, we get the Jacobian matrix J whose diagonal elements are computed as

$$J_{i,i} = \frac{\partial}{\partial\lambda(x_i)}\left[2V(x_i,\Lambda^*) - V(x_{i-1},\Lambda^*) - V(x_{i+1},\Lambda^*)\right]$$
$$= W_1\left[T(x_{i-1}-x_i) + T(x_{i+1}-x_i)\right]$$

Since $\lim_{x\to+0} T'(x) > 0$, this expression is positive and the trace of the Jacobian J is also positive. However, it is known that trace of J is the sum of all eigenvalues. That is, there exists at least one eigenvalue having a positive real part, implying that any equilibrium with atomic cities is unstable.

Finally, when there is one city with $\lambda(0) = 1$, the indirect utility is given by $V(x) \approx$constant $-W_1 T(x)$. Hence, $V(0) > V(x)$ strictly holds for all $x \in (0,1)$, which means stability of the full agglomeration.

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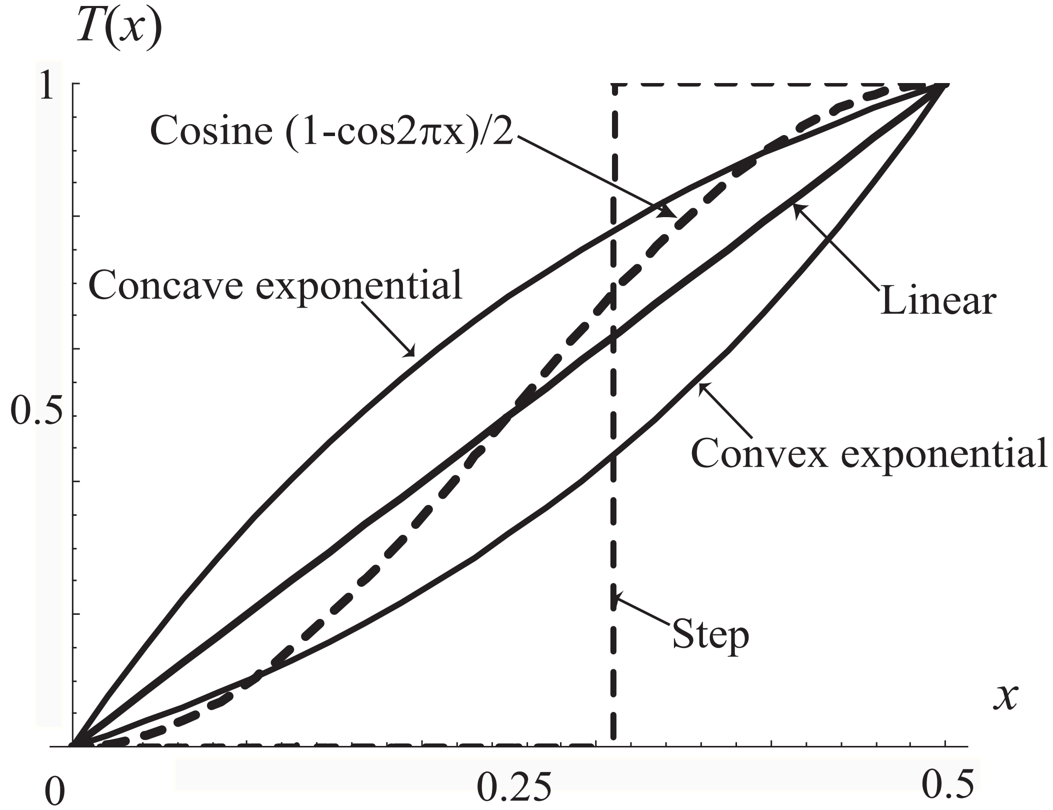

Figure 1: Shapes of transportation cost

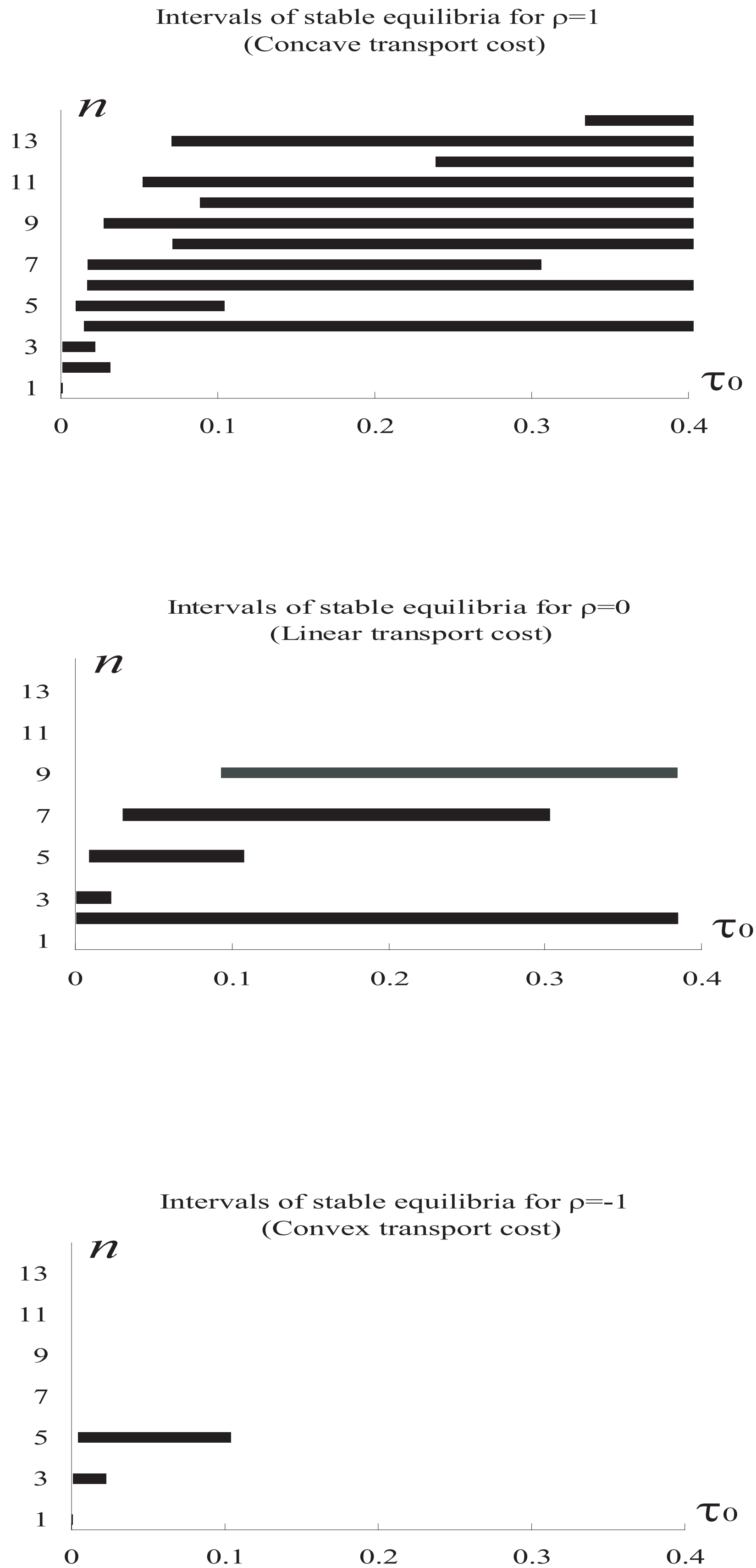

Figure 2: Intervals of stable equilibria