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Formulating and computing a continuous time overlapping generations policy model

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Welfare Analysis of HIV/AIDS: Formulating and Computing a Continuous Time Overlapping Generations Policy Model*

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Abstract

We introduce a continuous time overlapping generations demographic model, in which a social planner seeks to generate an optimal policy for influencing the demographic change of the underlying population in a neoclassical growth model. The model has the notable feature that the underlying state space is a continuum, leading to a Hamilton-Jacobi-Bellman PDE system which is defined over a Hilbert space generated by the ages of the population cohorts. In this technical report the dynamic programming problem is presented and the numerical approximation using a finite difference approximation is derived. This analysis is part of a larger research program on welfare analysis and policy development for the HIV/AIDS global pandemic.

Keywords: optimal control, continuous time overlapping generations, Hamilton-Jacobi-Bellman PDE, finite difference approximation, HIV/AIDS, demographic modeling

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1 Introduction

This is a technical report on formulating a continuous time overlapping generations (OLG) model with an underlying time-varying demographic population, whose growth (positive

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or negative) is influenced by policy variables selected by a social planner.

The report is part of a larger research program designed to help uncover the optimal policy decisions and welfare implications of the HIV/AIDS pandemic. There is an enormous amount of work on the economic effects of the disease in the literature, much of it focusing on modeling, the measurement of the economic and welfare costs, and the allocation of resources to treatment vs. prevention investment. Perhaps not surprisingly, there is at present no consensus on how best to model the economic effects of the disease, and different sub-disciplines of economic and socio-economic analysis have generated different standards. For example, the International Monetary Fund's *The Macroeconomics of HIV/AIDS* [Haacker 2004] contains a variety of neoclassical models which assess both the economic and welfare consequences of HIV/AIDS, concluding that the disease has dire consequences for growth, income, life expectancy, and the development of human capital (see also Nattrass [2006] for a review of Haacker [2004], where this negative conclusion is criticized). The use of growth models to study the effects of HIV/AIDS is also adopted in Arndt and Lewis [2000] (a Computable General Equilibrium model and simulation), Robalino, Jenkins and Maroufi [2002] (a calibrated neoclassical growth model), and McDonald and Roberts [2006], Dixon, McDonald and Roberts [2001] (a linearized augmented Solow growth model). Generally, all of these modeling environments measure large negative effects on growth and income from the disease, even if the prevalence of HIV/AIDS in the population is relatively small.

The effects of the pandemic upon the development of capital are treated in Bell and Gersbach [2006], who use a discrete time OLG framework to argue for greater targeting of susceptible subgroups of the population to combat the disease. By contrast, Bonnel [2000] examines the interplay between physical, human and social capital, and concludes that HIV/AIDS can destroy the social fabric of an economy, in particular its institutions, as well as both the physical and human capital stock of the economy.

Both Arrow [2003] and Kirman, Luchini and Moatti [2003] advocate the use of welfare analysis to understand the costs of HIV/AIDS, with Kirman et al. [2003] concentrating upon the possible role of greater treatment of those already infected with HIV/AIDS (through e.g. anti-retroviral treatment), as this provides a higher quality of life and/or increase longevity for those with the disease. This welfare role is also taken up in Deininger, Crommelynck and Kempaka [2005] and in Crafts and Haacker [2003], who include in agent preferences a desire for longer life expectancy to assess the welfare costs of increased disease mortality. They find, again, that the welfare losses due to HIV/AIDS can be substantial.

On the 'prevention vs. treatment' debate there is no clear-cut conclusion, as the economic costing which accompanies discussions of prevention are often balanced by the purely welfare gains from treatment. In an interesting policy prescription, Oster [2005] highlights the possibility of reducing the impact of HIV/AIDS through prevention, by targeting the spread of other sexually transmitted diseases (which may then prevent the spread of the disease). In order to assess whether or not treatment or prevention should receive most attention, Canning [2006] advocates a greater reliance upon cost-effectiveness as a strategy to assess the impact of alternative investments into prevention or treatment of the disease (or of alternative investments between different diseases or health crises), and Gaffeo [2003] supports its use while also examining the effects of market failure and HIV/AIDS' long incubation period on the ability to form a cost-effectiveness strategy. By contrast, Kumaranayake and Watts [2001] state that data availability (at least for Sub-Saharan Africa) is too poor to allow cost-effectiveness a role in policy formation, and Dike [2002] examines many of the challenges facing researchers in collecting both quantitative and qualitative data (particularly in developing nations).

Nevertheless, some form of cost-effectiveness has been adopted for regional and sectoral studies, e.g. on the demand for food [Agbola, Damoense and Saini 2004], the impact on the agricultural sector [Ambert 2002], or on the construction sector [Jayne, Villarreal, Pingali and Hemrich 2005]. In the health care sector, DeSimone and Schumacher [2004] estimate the wage differential which results from the added risk of health care practitioners (particularly nurses) being exposed to HIV/AIDS.

Finally, substantial resources have been devoted to combating HIV/AIDS by non-governmental organizations such as the International Monetary Fund, the Organisation for Economic Cooperation and Development, and the World Bank. In their assessment of aid effectiveness (see e.g. World Bank Operations Evaluation Department [2005], OECD Development Assistance Committee and UNAIDS [2004], Over et al. [2004]), it is generally concluded that effectiveness of aid can be increased by improving the institutions of the developing nation that is combating the disease, while at the same time improving both monitoring and evaluation of the aid package. This emphasis on institutional development is also reinforced by Godinho et al. [2005].

Overall, there are a plethora of modeling environments to choose from, but what appears to be lacking is a single framework which can make many of questions outlined above (e.g. cost-effectiveness vs. welfare, prevention vs. treatment) outcomes of the model rather than modeling choices of the researcher. In addition, few of the above models use results provided by the mathematical biology and epidemiology literature on the spread and diffusion of HIV/AIDS–these literatures themselves often compute simple, sometimes *ad hoc* estimates of both the economic cost of the disease and its welfare effects. It would be beneficial, then, to propose a unifying framework which could combine the relative merits of both approaches, and this research project aims to provide a meaningful contribution to fill this gap.

The mechanism by which the effects of HIV/AIDS, or indeed of any infectious disease, is modeled in the current study will be well understood by most (if not all) economists, but the modeling from epidemiology and its demographic effects may not be. In addition, the particular welfare measurement environment adopted here, the continuous time OLG framework, is relatively novel in the literature. The overall goal of this framework is to allow the following questions to be addressed:

1. What resources, and in what form, should be dedicated to the fight against the HIV/AIDS pandemic, if such resources are to come from the productive output of an economy (an economy defined here as a region, province, nation, or group of

nations)?

2. Because resources are scarce, is there a way to measure the trade-off which must take place when such resources are devoted to combating HIV/AIDS, that takes into account not only the effects on the here-and-now, but on future generations to come? In particular, what is the trade-off between investment in treatment, investment in prevention, and investment in other alternative resource allocation projects?

3. Can we meaningfully measure the cost (in terms of welfare, production, life expectancy, per-capita income, health, etc.) of an additional incremental increase in the funding of policies designed to combat HIV/AIDS?

4. Can these measurements be *modularized*, i.e. can new innovations in our understanding of both the optimal policies and of the disease itself be incorporated into a modeling framework with a minimum of 'retooling' required?

5. Can this model be computed in a time frame which allows policy makers to assess alternative policy tools, so that in addition to its *descriptive* capacity the model also provides *predictive* power through simulation?

This report will not travel far along the road towards a complete characterization of the answers to the above questions. But it is hoped that the reader will see how such a (relatively) simple framework as that described below can be extended to incorporate more realistic features, and at the same time provide insights on how to eventually use this framework in actual policy-making decisions of such crucial importance as combating HIV/AIDS.

The structure of this paper is the following: Section 2 provides a characterization of the full demographic model, a continuous time OLG framework with population growth affected by the HIV/AIDS disease. Already here several simplifications are made, e.g. capital is absent and there is no subjective discounting performed by the social planner (so that future generations are valued equally highly as current generations),[1] and the social planner has a finite planning horizon with a known terminal condition.

After introducing the full model, which is analyzed and treated elsewhere, Section 3 develops a much-simplified version of the model in which a single population of agents does not suffer from disease, but simply from mortality, and the social planner can implement policy to affect the *rate* of mortality loss. In a sense this is a super-aggregated version of the full model, as we could (for example) just think of this single population as 'the' population of the economy, and the mortality rate as 'the' mortality rate for this overall population. Policy impacts would thus take place at this super-aggregated level–this might model, say, the overall impact of an anti-infectious disease campaign on the entire population's mortality rate.

Although radically simplified from the full model given in section 2, this single population model shares the feature from the full model that the welfare valuation of the social planner depends upon the entire distribution of individuals according to their age, which is (in this context) a continuum. The dynamics of this model are most properly studied at the theoretical level as a Hamilton-Jacobi-Bellman system over a Hilbert space. We defer this analysis to a later paper, noting in passing that such systems have in fact been studied in Economics in the past (e.g. Faggian [2005]). What we show in Section 4, and what is the main result, is that in spite of this complication it is possible to numerically compute this Hamilton-Jacobi-Bellman system on a computer using finite-difference methods for solving first-order, nonlinear partial differential equations.

The final section, which is still work in progress, will be a demonstration of the scale at which the model can be computed using high performance computing resources at Manchester Computing (the University of Manchester, UK) and the National Center for Supercomputing Applications (the University of Illinois at Urbana-Champaign). Currently available desktop solutions (e.g. Linux or other *nix systems running at approx. 2Ghz clock speed on dual-core CPUs, with 2GB RAM) indicate that both computation

[1]Note, however, that there *is* discounting in the model, though, as the population suffers from a mortality rate which cannot be extinguished.

of the solution and even storage of the results is a serious consideration when solving the relatively simple problem derived in Section 3. When the high performance computing results are complete they will be included in an expanded version of this document, which is always available at `http://www.ihs.ac.at/~shorish`.

2 A Welfare Model of HIV/AIDS

The following model is a synthesis of what might be termed 'conventional demographic growth models' on the one hand, and 'models of demographic epidemiology' on the other, applied to the HIV/AIDS pandemic. Economic growth models are usually analyzed within the context of 'welfare', which is a way of measuring the preferences of a population of inhabitants in a well-defined, axiomatic fashion. While there is (and will no doubt always be) a certain degree of debate over whether or not human welfare can be measured in this way, it is clear that as long as humans have some structure over how they rank alternatives (be they alternative decisions, consumption goods, careers, etc.), this structure should not be ignored if at all possible.

For the current model we adopt a weaker contention than that humans 'actually' rank alternatives using a welfare measure such as utility, and instead *define* a social planner's measurement of others' welfare by such a measure. In the present context, we think of a social planner as a government or non-governmental agency which has at its disposal a technology allowing it to affect large sections of the population of agents at a time, through *economic policy.* The social planner funds this policy activity by collecting resources made available in the economy, so that the policy is constrained both by the sum total of resources available, and also by the alternative uses (e.g. alternative policies or activities) that those resources could be allocated.

Underlying this social planning problem (as it is known) is the assumption that a proportion of the population is suffering from HIV/AIDS. We adopt a standard framework from epidemiology and mathematical biology and suppose that there exist three categories of people, those *susceptible* to infection, those *infected* with HIV but not expressing AIDS, and those who have *acquired* infection and are now expressing AIDS.

Transitions between these populations are the main mechanism by which the disease progresses, and are the target of any economic policy designed to combat this progression. For example, HIV carries with it a *force of infection*, which indicates how (over a small time interval) susceptible population members become infected with HIV. In this transition is usually included the transmission of HIV through sexual contact (heterosexual and homosexual), intravenous drug use (needle sharing), and mother-to-child transmission of the virus during pregnancy. Economic policies which are targeted to one or more of these transmission mechanisms, then, are directly targeting the force of infection.

In addition, there is a *force of expression* of HIV into AIDS, which indicates how many HIV-infected population members express AIDS. Here again there is scope for economic policy, because of the use of e.g. anti-retroviral drug therapy can delay the onset of AIDS many years after an HIV infection, leading to a higher quality of life through increased longevity.

Finally, there is a latent mortality rate for all three categories which humans can do nothing about, or which are affected by other factors not represented by economic policy prescriptions against HIV/AIDS. These terms are present in the demographic evolution of all three populations, and are dependent not only upon the population type but also upon their age.

2.1 A Mathematical Formulation

The demographic model of infectious disease adopted here is based upon a class of models introduced in 1927 by Kermack and McKendrick [1991] which are a standard of the epidemiology literature. This model is extended to allow age-dependent disease incidence and progression, as in Inaba [2006] (see also Bacaër and Ye [2006], Garnett and Anderson [1993], Anderson [1991], Inaba [1990, 1989], Tudor [1985] and Hoppensteadt [1974] for an overview of the progression of similar models in the epidemiology literature). The population is classified according to the stage of the infection–in the literature on HIV/AIDS there are commonly three categories, or ‘compartments’ which any individual will be classified, and individuals move between categories according to certain transitions (or transition probabilities). This analysis focuses upon three categories:

1. Susceptibles: these population members are currently uninfected by the disease, but may catch the disease in the future. They are given by a density $s(t, x)$ of the population at time t and age x, with $x \leq t$ always.

2. Infecteds: these members are infected with the HIV virus, but do not yet exhibit acquired infections, i.e. they do not exhibit the symptoms of AIDS. There is a density $i(t, \tau, x)$ of infecteds, which indicates at time t the proportion of population members who were infected with HIV at age x, τ periods earlier. Obviously it must be that $x + \tau \leq t$, since here the age of the individual is $x + \tau$.

3. Acquireds: these members are infected with HIV and moreover exhibit the symptoms of AIDS–their quality of life and mortality are sufficiently different from HIV-infected-only agents that they have a category of their own. The density of acquireds $a(t, \tau, x)$ defines the fraction of the population at time t who began exhibiting AIDS at age x, τ periods ago. As before, $x + \tau \leq t$.

As the population is measured in terms of densities it must be true that

$$\int_0^t \left(s(t, x) + \int_0^x i(t, \tau, x - \tau) d\tau + \int_0^x a(t, \tau, x - \tau) d\tau \right) dx = 1 \, \forall t, \tag{2.1}$$

since $i(t, \tau, x - \tau)$ and $a(t, \tau, x - \tau)$ represent the density of individuals at t of infecteds and aquireds, respectively, who were infected $x - \tau$ periods before, i.e. who are of age x at t.

Each of these densities (s, i and a) evolves according to the transition from one state to the next ($s \to i \to a$) and also to the absorbing state “death”, represented by mortality rate transitions. We take as given the following full specification of the transitions between states, where certain transitions may be influenced by a vector of *policy variables* denoted $p(t)$:

$$\frac{d}{dt}s(t,x) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) s(t,x) = -\left[\mu(x) + \lambda(x, p(t))\right] s(t,x), \tag{2.2}$$

$$\frac{d}{dt}i(t,\tau,x) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right) i(t,\tau,x) = -\left[\mu(x+\tau) + \gamma(\tau, x, p(t))\right] i(t,\tau,x), \tag{2.3}$$

$$\frac{d}{dt}a(t,\tau,x) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right) a(t,\tau,x) = -\left[\mu(x+\tau) + \delta(\tau, x, p(t))\right] a(t,\tau,x), \tag{2.4}$$

$$s(0,x) = S_0(x),\ s(t,0) = B(t). \tag{2.5}$$

In the above system the 'instantaneous mortality rate' $\mu(x)$ denotes that part of the population's mortality rate which is independent of policy and is uninfluenced by disease. In (2.2) $\mu(x)$ depends upon the age of the population x, while in (2.3) and (2.4) $\mu(x+\tau)$ denotes those population members who were of age x when infected τ periods, before, i.e. those who are of age $x+\tau$ now.

There is also an additional mortality rate term $\delta(\tau, x, p(t))$ in (2.4), which is introduced because those population members who exhibit AIDS are more likely to die (given their age) than either HIV infected or uninfected individuals. This term depends upon the time AIDS has been exhibited (τ), the age at which the individual first exhibited AIDS (x), and the policy variable $p(t)$–the latter dependence is due to the possibility of e.g. anti-retroviral treatment extending the lifespan of those with AIDS.

The terms $\lambda(x, p(t))$ and $\gamma(\tau, x, p(t))$ are forces of infection and acquisition, indicating the likelihood of a susceptible becoming infected with HIV and the likelihood of an HIV-infected person exhibiting AIDS, respectively. It is in these two terms, along with $\delta(\tau, x, p(t))$, that policy devoted to combating HIV/AIDS has its greatest impact–it is also in these terms that modern epidemiology, sociology, and economics have the most to say. Indeed, entire research agendas (not just individual research articles) are devoted to understanding these rates, and no full explanation of the ability of a social planner to influence the progression of such a disease can be considered realistic without this understanding. We shall unfortunately not treat this question in the current analysis, as our focus is on the general welfare analysis methodology once $\lambda(x, p(t))$, $\gamma(\tau, x, p(t))$, and

$\delta(\tau, x, p(t))$ are completely specified. But these are absolutely vital for our understanding of how to combat the disease.

Finally, $S_0(x)$ and $B(t)$ indicate the initial distribution of susceptibles at time $t = 0$ and the reproduction rate for every period t, respectively.

The innovation we have introduced into the age-weighted demographic model is to include the policy variable $p(t)$, which may be a vector of policy tools, explicitly into the forces of infection and acquisition, and into the mortality rate adjustment for those exhibiting AIDS.[2] We next turn to how a social planner selects $p(t)$ in an optimal fashion.

The continuous time nature of the infectious disease lends support to the adoption of a continuous time OLG growth model framework, with an endogenous labor supply (see Brito and Dilao [2006] for an example of a continuous time OLG model of demographic population growth for an exchange economy). This allows the social planner to use standard tools in optimal control theory to solve the following resource allocation problem: given a specification for the economy, what is the optimal resource investment towards combating HIV/AIDS? Notice that the natural response, 'as much as possible', is uninformative on its own, but is exactly what we wish to know–how much is 'as much as possible', exactly?

The social planner arrives at an answer by measuring the welfare of the population, and constructing trade-offs between benefits that the population would receive directly from an economy's production (by consuming goods), and benefits received by forgoing consumption and battling the disease instead (these benefits are measured both for prevention, by lowering the risk of infection of future generations, and also for treatment, by improving the current quality of life of those already infected).

[2]Lloyd [1991] derives optimal policy instruments in a simpler model where HIV infected agents respond to policy decisions on the part of a social planner. While we feel this policy response is an important determinant of the overall impact of policy decisions upon the evolution of the disease, we incorporate this effect into the force of infection and acquisition terms given above–one might term these the 'net policy effects' after agent responses are incorporated.

This welfare measure is assumed to be time separable and additive:

$$U = \int_t^T \int_0^v \Big[u_s(c_s(v,x))s(v,x) + u_i(c_i(v,x)) \int_0^x i(v,\tau,x-\tau)d\tau \\ + u_a(c_a(v,x)) \int_0^x a(v,\tau,x-\tau)d\tau \Big] dxdv. \tag{2.6}$$

The social planner measures an individual's welfare with the utility functions u_s, u_i and u_a, for susceptibles, infecteds and acquireds, respectively, which take as respective arguments the per capita consumption levels c_s, c_i and c_a. The above welfare measure U is simply a weighted average of the welfare of everyone in the economy over all three types, for all overlapping generations which exist between time t and the finite horizon of the social planner, T. As usual, we assume that the utility functions are at least C^1, strictly increasing and concave in consumption.

The economic trade-offs for the social planner's decision problem are captured in the economy's *resource constraint*. This is a neoclassical aggregate resource constraint, and states that in any time period, the sum of current population consumption and the funding devoted to reducing the mortality rate of the population cannot exceed the available resources of the economy:

$$\int_0^t \Big[c_s(t,x)s(t,x)dx + c_i(t,x) \int_0^x i(v,\tau,x-\tau)d\tau + \\ c_a(t,x) \int_0^x a(v,\tau,x-\tau)d\tau \Big] dx + C(p(t)) \leq F\left(\int_{\underline{t}}^{\bar{t}} s(t,x)dx \right), \tag{2.7}$$

where $C(p(t))$ is the resource cost of selecting the policy vector $p(t)$, $F(\cdot)$ is the production function of the economy (assumed to be either constant or decreasing returns to scale), and $[\underline{t}, \bar{t}]$ is the interval between which agents are of working age (and thus contribute to production).

Finally, we can state the full optimal control problem of the social planner. The planner selects a set of consumption levels for every individual in the economy at every time period, as well as a vector of policy variables (also at every time period) to maximize the welfare of the population. This is expressed as an optimal control problem, where the solution, the optimal *value function*, is given by:

$$V := \max_{\{c_s, c_i, c_a, p\}} \int_t^T \int_0^v \Big[u_s(c_s(v,x))s(v,x) + u_i(c_i(v,x)) \int_0^x i(v,\tau,x-\tau)d\tau \\ + u_a(c_a(v,x)) \int_0^x a(v,\tau,x-\tau)d\tau \Big] dxdv, \tag{2.8}$$

such that (2.2) - (2.5) and (2.7) hold, and it is understood that by e.g. 'c_s' we mean the entire family of values $c_s(v,x)$ for all times $v \in [t,T]$ and all ages $x \in [0,v]$.

The formulation of this problem is similar to previous studies on the optimal control (treatment) of infectious disease age-structured models studied in, for example, Feichtinger, Tragler and Veliov [2003] and Faggian and Gozzi [2004], with the additional complexity of an overlapping generations framework and a resource allocation problem. The model is complicated by the dependence of the value function upon the current state of the population $s(t,x)$, which makes the resulting dynamic programming problem (see below) infinite dimensional. Both Feichtinger et al. [2003] and Faggian and Gozzi [2004] tackle the necessary conditions for similar problems, while Faggian [2005] outlines regular solutions for a general class of convex boundary control problems over a Hilbert space (this builds upon earlier research on the Hamilton-Jacobi system in infinite dimensional spaces from Crandall and Lions [1985]). It is not known at present if these results can be extended in a straightforward fashion to demonstrate the existence of regular solutions to the above system.

3 The Simplified Model

For this report we shall make several further assumptions that will help simplify the analysis, and (it is hoped) significantly improve expositional clarity. Strictly speaking, these simplifications will recast the demographic problem away from infectious disease, as there will be only a single population of uninfected agents $s(t,x)$. This population will, nevertheless, have its mortality rate influenced by economic policy, so that the economic tradeoff between greater current living conditions for the present or a lower mortality rate in the future is still present. Moreover, passing from this simplified model to the full model just described preserves both the numerical approximation methodology and the computational implementation strategy–this is, of course another way of saying that even this simplified model has as much complexity as the full model.

We make the following assumptions. First, we assume that there is only one type of population, the uninfected (no longer 'susceptible' since there is no threat of infection). This population density, which will be denoted (as before) by $s(t,x)$, has an associated mortality rate $\mu(x,p(t))$ which is influenced by policy. There is no birth rate and the mortality rate is always positive, so in this model the initial distribution of agents $S_0(x)$ always declines over time. The social planner may mitigate this decline by an optimal choice of $p(t)$, as described below.

To keep policy analysis simple we suppose that $p(t)$ is a scalar and $C(p(t)) = p(t)$. Moreover, we let $F(x) = x \; \forall x$, $\underline{t} = 0$ and $\bar{t} = t$, so that the resource constraint may be simplified to

$$\int_0^t c(t,x)s(t,x)dx + p(t) \leq \int_0^t s(t,x)dx = 1. \tag{3.1}$$

The policy variable $p(t)$ influences the demography of the economy only by impacting the mortality rate of the population. In a full compartment model of infectious disease this may be a vector of policy variables (as defined earlier), and different stages of a disease will be impacted by one or more of these variables. Here we assume that there is only one 'compartment', which is a population of generic individuals who age over time. We model the demographic change of the population as:

$$\frac{d}{dt}s(t,x) = \frac{\partial s}{\partial t}dt + \frac{\partial s}{\partial x}dx = -\mu(t,x)s(t,x)dt,$$

which (since $dx = dt$) can be written as the partial differential equation (PDE) system

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) s(t,x) = -\mu(x, p(t)) s(t,x). \tag{3.2}$$

The term $\mu(x, p(t))$ is the instantaneous mortality rate of the population–given a policy $p(t)$ at time t, the fraction of the population of age x will be adjusted downward at the rate $-\mu(x, p(t))$. One form of the mortality rate will be specified in the modeling section below, but for now we note that it is the impact of the policy variable upon the transition rate of the population, in this case the mortality rate, which is the demographic impact of economic policy. Although in a more complete model the impact of e.g. high consumption on mortality or on reproduction could be considered, we abstract away from this in what follows.

Finally, we suppose that there is in the current model no new births in the population, i.e. the population of the economy is at its maximum at $t = 0$, with an initial population distribution $s(0, x) := S_0(x)$. This assumption is for exposition only, and a full model would include a birth rate that could be influenced by economic policy.

3.1 The Optimal Control Problem

With these assumptions in hand, for the remainder of this analysis we consider the following optimal control problem:

$$v(t, s) := \max_{c,p} \int_t^T \int_0^v u(c(v,x)) s(v,x) dx dv \tag{3.3}$$

such that

$$\int_0^t c(t,x)s(t,x)dx + p(t) \leq 1, \tag{3.4}$$

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \frac{\partial s}{\partial x} = -\mu(x, p(t))s(t,x), \tag{3.5}$$

$$v(T,\cdot) = V_T(x), \quad s(0,x) = S_0(x) \tag{3.6}$$

$$\int_0^t s(t,x)dx = 1 \quad \forall t. \tag{3.7}$$

This class of optimal control problems represents the decision problem facing a social planner with a finite planning horizon and a continuum of agents indexed by age. The instantaneous utility function $u(c)$ is assumed to be increasing and strictly concave–in the numerical computations below the functional form is given the standard $u(c) := \ln(c)$ specification. This utility function is assumed to be the same for every age cohort, an assumption made for exposition only and which can be safely dispensed with in more realistic models.

The resource constraint (3.4) has been likewise simplified–we assume a closed economy, so that the policy maker only has at their disposal the output of a domestic economy. There is no capital, so only the available labor supply influences production. To keep our attention focused upon the problem of computation, the production function is assumed to be linear in aggregate labor and the linear multiplier is set equal to one without loss of generality. It should also be emphasized that although $s(t,x)$ is a density function (i.e., the number of agents in a small age-interval $(x, x+dx)$ at time t is equal to $s(t,x)dx$), the consumption levels $c(t,x)$ are aggregate per-capita levels and not densities. This means, for instance, that the resource constraint has the equivalent form

$$\mathbb{E}_{s(t,\cdot)}c(t,\cdot) = 1 - p(t). \tag{3.8}$$

The problem is non-standard because the value function $v(t,s)$ depends upon every value of the density of susceptibles $s(t,x)$ for each x–the state space is thus a continuum. Naturally in any computation using a finite state machine the state space will have to be discretized, and the nature of the discretization will be addressed in the upcoming section. The convergence properties of the numerical approximation to the true value function is a subject of future research, however, and caution should be used in mapping numerical computational results back to the theoretical problem at hand.

To render the problem easier to treat numerically we can recast the problem as a Hamilton-Jacobi-Bellman first order nonlinear partial differential equation. We arrive at:

$$-\frac{\partial v}{\partial t} = \max_{c,p} \left\{ \int_0^t u(c(t,x)s(t,x)dx + \int_0^t \frac{\partial v}{\partial s} \cdot (-\mu(x, p(t)))s(t,x)dx \right\} \tag{3.9}$$

such that constraints (3.4), (3.6) and (3.7) are satisfied. We have slightly abused notation here to highlight that there is a continuum of partial derivatives of v with respect to s in the last term of (3.9), so the second integral may be considered as an inner product on an Hilbert space between the gradient of the value function defined over the space of all probability densities over x, and the representative draw from this space labeled $s(t,x)$ (multiplied by $\mu(x, p(t))$. The proper definition of this derivative will be presented below, when passing to the discretization for numerical computation.

3.2 Deriving $s(t,x)$ when the Policy Function is Known

If the trajectory of $p(t)$ is known for all t (or is irrelevant for the time evolution of $s(t,x)$) then solving for the density function of susceptibles $s(t,x)$ is straightforward using the method of characteristics to convert the partial differential equation (3.5) into a set of ordinary differential equations. This separates the time evolution of the demographic component from the policy decisions, as is often performed in standard economic models with (say) an exogenously defined rate of population growth. We next briefly outline how the population of susceptibles would evolve in the absence of policy–for further references on the method of characteristics see e.g. John [2005].

The characteristic curve defines a mapping from a parameter $z \in \mathbb{R}$ to $(x(z), t(z))$, values for which the unknown density S is unchanging over time. Hence,

$$\frac{ds}{dz} = \frac{\partial s}{\partial t}\frac{dt}{dz} + \frac{\partial s}{\partial x}\frac{dx}{dz}. \tag{3.10}$$

But from (3.5) we can match coefficients, yielding

$$\frac{dt}{dz} = 1, \tag{3.11}$$

$$\frac{dx}{dz} = 1, \tag{3.12}$$

$$\frac{ds}{dz} = -\mu(x(z), p(t(z)))s(x(z), t(z)). \tag{3.13}$$

The first two ordinary differential equations may be integrated to yield

$$t(z) = z + t_0, \tag{3.14}$$

$$x(z) = z + x_0. \tag{3.15}$$

We choose the constant of integration $t_0 = 0$, which allows us to write

$$x(z) = x(t, x_0) = t + x_0. \tag{3.16}$$

Meanwhile, the last ODE can also be integrated, choosing as the limits of integration $(0, t)$:

$$\ln(s) = -\int_0^t \mu(x(z), p(t(z)))dz + S_0, \tag{3.17}$$

with S_0 a constant.

To determine the constant of integration we use (3.16) and the initial condition for $s(0, x)$ from (3.6):

$$s(0, x(0, x_0)) = S_0(x_0). \tag{3.18}$$

Finally, we know that $x = t + x_0 \rightarrow x_0 = x - t$, so that

$$s(t,x) = \exp\left[-\int_0^t \mu(z+x_0, p(z))dz\right] + S_0(x_0),$$

or

$$s(t,x) = \exp\left[-\int_0^t \mu(z+x-t, p(z))dz\right] + S_0(x-t). \tag{3.19}$$

The susceptible population at t is thus determined when the policy function $p(t)$ is known for every time period up to t. Unfortunately, the value of the policy function is itself determined by the time evolution of the susceptible population, and for the actual optimal control problem the standard technique outlined here cannot be used.

3.3 First Order Conditions

The first order necessary conditions to solve system (3.3) - (3.7) rely upon slightly reformulating the HJB system (3.9) using a Lagrange multiplier to incorporate the resource constraint. The Lagrangian is defined as:

$$\begin{aligned}\mathcal{L}(c,p,s,\lambda) := \frac{\partial v}{\partial t} + \max_{c,p}\Bigg\{\int_0^t u(c(t,x))s(t,x)dx + \int_0^t \frac{\partial v}{\partial s}\cdot(-\mu(x,p(t)))s(t,x)dx \\ + \lambda\left[1 - \int_0^t c(t,x)s(t,x)dx - p(t)\right]\Bigg\},\end{aligned} \tag{3.20}$$

such that (3.6), (3.7) hold. Note that at the optimum, the Lagrangian is equal to zero.

Provided that the problem carries an interior solution for points off the boundary, the first order necessary conditions associated with this problem may be written as:

$$\begin{aligned}\left.\frac{\partial u}{\partial c}\right|_{(t,x)} &= \left.\frac{\partial v}{\partial s}\left(-\frac{\partial \mu}{\partial p}\right)\right|_{(t,x)}, \\ \int_0^t c(t,x)s(t,x)dx + p(t) &= 1.\end{aligned} \tag{3.21}$$

If we assume that $u(c) := \ln(c)$, the conditions may be rewritten in terms of the optimal policies $c^*(t,x)$ and $p^*(t)$ as

$$c^*(t,x) = \left[\frac{\partial v}{\partial s} \left(-\frac{\partial \mu}{\partial p} \right) \Bigg|_{(t,x)} \right]^{-1},$$

$$p^*(t) = 1 - \int_0^t \left[\frac{\partial v}{\partial s} \left(-\frac{\partial \mu}{\partial p} \right) \Bigg|_{(t,x)} \right]^{-1} s(t,x)dx. \tag{3.22}$$

4 Numerical Dynamic Programming

4.1 Fixing the Mortality Rate μ

The analytical treatment of the first order conditions and the existence of a solution are treated in a companion paper. The focus of this report is to determine how to compute the solutions of (3.21) on a computer, given the functional forms for production and utility assumed above. In order to embark upon computation an additional functional form assumption must be made for $\mu(x,p)$. In a sense this expression is the 'heart' of the economics, for all policy effects to change the progression of the demographic population variable $s(t,x)$ are given by this term.[3]

To keep the exposition as clear as possible we assume that $\mu(x,p)$ has a simple form which represents the following stylized qualities:

1. the mortality rate is decreasing and convex in the policy variable p, i.e.

$$\frac{\partial \mu}{\partial p} \leq 0, \; \frac{\partial^2 \mu}{\partial p^2} \geq 0,$$

2. the mortality rate is maximized when $p = 0$, i.e.

$$\mu(x,0) > \mu(x,p) \; \forall p,$$

[3]As mentioned earlier, in the complete model of the epidemiology of infectious disease, economic policy will have an impact upon non-disease-related mortality, disease-related mortality, and the 'force of infection', i.e. the rate at which those susceptible to disease are actually infected.

3. the mortality rate tends to a lower bound as p goes to infinity, i.e.

$$\lim_{p \to \infty} \mu(x, p) =: \bar{\mu}(x) < \mu(x, p) \; \forall p.$$

These stylized qualities are designed to capture the effects of a policy with clearly-defined effects on the mortality rate, without externalities or distortions. Naturally in the real world policy effects are less clear-cut, and some ambiguity will obtain regarding not only the optimal policy level with respect to aggregate consumption (as here) but also with respect to deleterious (or unanticipated) effects that the policy might have upon mortality. Without such externalities, the qualities given above state that increased policy spending decreases mortality, but does so at a declining rate. For every age cohort, the mortality rate is maximized when no policy spending is implemented, while if policy spending were totally unconstrained (so that an infinite amount could be allocated) then the mortality rate would decline to a fixed (age dependent) minimum. The latter assumption prevents the 'eternal youth' problem of allowing the mortality rate to decline to zero with unbounded policy spending, which retains the common-sense intuition that regardless of what we do, there are uninsurable risks which cannot be eliminated by policy spending.

For computation, we further restrict $\mu(x, p)$ to an explicit functional form that respects the stylized qualities defined above:

Assumption 4.1. *The instantaneous mortality rate $\mu(x, p)$ is defined as:*

$$\mu(x, p) := (1 + x^2)(1 + e^{-p}). \tag{4.1}$$

Note that in addition to satisfying the qualities required for the policy function, this functional form also states that mortality is increasing and convex in age (x). Although plausible, it is worth noting here that this functional form assumption is made for ease of computation only (that is, it facilitates the calculation of an exact gradient of μ). Indeed, recall that *all of the important demographic effects are captured by* μ, and so the functional form of μ with respect to x is most properly a data-driven specification rather than a modeling specification. It is assumed that when the model is implemented as a full-scale policy engine, the researcher will identify precisely that $\mu(x,p)$ which best fits the data on age-weighted and policy-influenced mortality. In the final analysis, discovering the true specification of $\mu(x,p)$ is an *econometric* question which can only be given by the data.

4.2 Computing the Value Function

As with most finite horizon dynamic programming problems we begin at the end and work backwards. Starting from the terminal condition $v(T,\cdot) = V_T(x)$ we solve

$$
\begin{aligned}
c^*(T,x) &= \left[\frac{\partial V_T}{\partial s} \left(-\frac{\partial \mu}{\partial p} \right) \bigg|_{(T,x)} \right]^{-1}, \\
p^*(T) &= 1 - \int_0^T \left[\frac{\partial V_T}{\partial s} \left(-\frac{\partial \mu}{\partial p} \right) \bigg|_{(T,x)} \right]^{-1} s(T,x)dx,
\end{aligned} \tag{4.2}
$$

for every possible $s(T,x)$ such that $\int_0^T s(T,x)dx = 1$. Of course, here we run immediately into the problem that no matter how many $s(T,x)$ families we compute, there are always a continuum more waiting in the wings. On a finite state computing machine, unfortunately, we must approximate the entire space of possible populations by a finite grid, and 'only' compute values on the grid.

To this end, we first define the range of the population density function $s(t,x)$ by the interval $[0,\bar{s}]$, and define a set P_s of n_s grid points from this set. Note that the upper limit $\bar{s}$ can prevent sharply-peaked probability density functions from being considered– but in demographic models with birth and death processes there are normally fairly widely dispersed distributions. For notational exposition we shall denote a typical value from P_s by $s_j \in P_s$, for $j = 1,\ldots,n_s$.

Thus, each possible value of $s(t,x)$ is constrained to lie on the grid of dimension n_s over $[0,\bar{s}]$. But there is a second complication which is potentially more serious than constraining the values of $s(t,x)$, and that is the problem that the dimension of the state space *as required for* $v(t, s)$ is given by the mapping between the underlying age x and $s(\cdot, x)$. The age space must also be partitioned–since in a finite horizon problem the maximum age for the economy is T, we assume that from $X := [0, T]$ a set P_x of n_x grid points is selected. Again, denote for exposition a typical element of P_x by $x_k \in P_x$, $k = 1, \ldots n_x$.

With these partitions, a given state s for $v(t, s)$ is actually an n_x-dimensional vector

$$s = (s_1, s_2, \ldots, s_{n_x}) \in (P_s)^{n_x}.$$

To compute the solutions (4.2) requires that $s(T, x)$ be specified for all $(n_s)^{n_x}$ possible grid points. This allows the integrals in the first order conditions to be computed with respect to $s(T, x^k)$ at each point $x^k \in P_x$, which allows in turn the computation of $c^*(T, x^k)$ and $p^*(T)$.

We shall return to the now obvious 'curse of dimensionality' problem just stated in a moment, and will assume that the optimal controls c^* and p^* have been found. The next step in the dynamic programming problem is to solve (3.9), i.e.

$$-\frac{\partial v}{\partial t} = \int_0^T u(c^*(T, x)s(T, x)dx + \int_0^T \frac{\partial v}{\partial s(T, x)} \cdot (-\mu(x, p^*(T))s(T, x)dx \tag{4.3}$$

for the value function $v(t, s)$. This can be performed in a variety of ways–one may approximate the PDE itself using finite difference methods (as we do here), or use the integral form of the expression for finite volume methods, or approximate the solution of the PDE instead and use finite element methods. We choose to use finite difference methods here simply because they are straightforward to implement, and preliminary simulation results indicated that they are less prone to spurious oscillations (as occurred with finite volume methods).

The simplest finite difference approximation of the above system approximates the derivatives appearing in the expression by small changes along the predefined grid points for s, x and t. We partition the time interval $[0, T]$ into a grid, and use the Hamilton-

Jacobi-Bellman equation to approximate a small time step backward. To this end, define the set of grid points over $[0, T]$ by P_t, with n_t points, and a typical element of P_t by $t^n \in P_t$, $n = 1, \ldots, n_t$.

For added simplicity, we presume that all grid partitions are equidistant, so that we can use the standard notation for expression finite differences on grid points. For example, the partition of $[0, T]$ is into $n_t - 1$ equal subintervals of width $\Delta t = \frac{T}{n_t - 1}$, and we can express the nth interval by $n\Delta t$, with $n = 1, \ldots, n_t - 1$. A function value defined on a time grid point n will carry a superscript, so that if e.g. y is a function of time, then y^n is the value of the function at grid point n. We assume that both grid partitions for s and x are also equidistant, and that the policy function $p(t)$ is partitioned over P_t.

Note again a feature not common for standard finite difference models–as described before, $s(t, x)$ is defined for all x, so that there are $(n_s)^{n_x}$ grid points which represent the discrete approximation of the continuous state space. Thus, a function y which is defined over (t, s) will carry the indices $y^n_{k_1,\ldots,k_{n_x}}$, where each of the k_i takes represents a grid point from P_s. This means that *as the grid over x becomes finer, the number of dimensions of the problem increases.* This is different from a standard finite difference model, where an increasingly fine grid means that more computations must be performed per dimension, but not that the number of dimensions itself increases.

Armed with this notation we begin with a backward difference approximation to the time derivative of the value function, i.e.

$$\frac{\partial v}{\partial t} \simeq \frac{v^n_{\mathbf{k}} - v^{n-1}_{\mathbf{k}}}{\Delta t}, \ n = T, T-1, \ldots, 2,$$

where $\mathbf{k}$ represents an n_x-dimensional vector of indices for the point $s(\cdot, x_j)$, $j = 1, \ldots, n_x$. Notice that because we are working in a dynamic programming environment, the discretization of $s(t, x)$ is independent of t—we are forming the closed-loop policy functions $c^*(t, x)$ and $p^*(t)$, so we must compute the value function for every admissible point in the state space (regardless of whether or not actual state trajectories achieve these points).

Discretizing the rest of the system is more involved, and requires a re-examination of the first order conditions (4.2). First, we note that in the expression for $c^*(T, x)$, there is a partial derivative of the value function V_T and a partial derivative of the mortality rate μ. The latter partial derivative is straightforward, but the former partial derivative is a functional derivative with respect to $s(t, x)$. In other words, considering any perturbation $h(t, x)$ of $s(t, x)$ such that

$$\|h(t,\cdot)\| := \int_0^t h(t,x)dx = 1\,\forall t,$$

we define the (Gâteaux) derivative of $v(t, s)$ with respect to s by

$$\frac{\partial v}{\partial s} := \frac{d}{d\alpha}v(t, s+\alpha h)\bigg|_{\alpha=0},$$

where $\alpha \in \mathbb{R}$ is a scalar. The reason we denote this derivative by $\frac{\partial v}{\partial s}$ instead of $\frac{dv}{ds}$, as is mathematically correct, is that in order to approximate this derivative on a discretized grid, *we must take the derivative for every point in* P_x, i.e. for each of n_x different points there exists a different derivative. We could have specified this derivative as e.g. $\frac{dv}{ds_j}$, but the partial derivative notation is more suggestive–it represents the fact that in reality the dependence of one function upon another, which in turn depends upon an underlying continuous space, is actually a dependence of one function upon a continuum of functions ('indexed', if you will, by the points in the underlying space). To compute a meaningful variation in the value function v for variations in s on a computer, then, requires that the discretization keep track of how many points in the underlying space (here X) are used for its approximation (here P_x).

Armed with this intuition we can proceed to the actual discretization of the partial derivative $\frac{\partial v}{\partial s}$. The discretized version is:

$$\frac{\partial v}{\partial s}\bigg|_{(t,x)} \simeq \Delta_s v^n_{\mathbf{k}j} := \frac{v^n_{\mathbf{k}} - v^n_{\mathbf{k}_{-1j}}}{\Delta_j s}, \tag{4.4}$$

where the notation $\mathbf{k}_{-1j}$ means

$$\mathbf{k}_{-1j} := (k_1, \ldots, k_{j-1}, k_j - 1, \ldots, k_{n_x}),\ j = 1, \ldots, n_x,$$

and $\Delta_j S$ represents

$$\Delta_j S := S_{k_j} - S_{k_j - 1}.$$

(The notation for the finite difference approximation $\Delta_s v_{\mathbf{k}j}^n$ of $\frac{\partial v}{\partial s}$ will be used below.)

Approximation (4.4) is a backward partial difference operation, where the value function v at time t is evaluated first at a grid point $(S_{k_1}, \ldots, S_{k_{n_x}})$, and then at a neighboring point $(S_{k_1}, \ldots, S_{k_{j-1}}, S_{k_j - 1}, S_{k_{j+1}}, \ldots, S_{k_{n_x}})$. These values are subtracted and then divided by $\Delta_j S$, so that the total represents the jth partial derivative of $v(t, s)$ at the underlying grid point x_j.

The backward difference approximation for $\frac{\partial \mu}{\partial p}$ is more straightforward:

$$\frac{\partial \mu}{\partial p} \simeq \Delta_p \mu_j^n := \frac{\mu_j^n - \mu_j^{n-1}}{\Delta p}, \tag{4.5}$$

where Δp is the interval from partitioning the grid for values of the policy function p. As above, $\Delta_p \mu_j^n$ represents this finite difference approximation to $\frac{\partial \mu}{\partial p}$.

The functional form of μ given by Assumption 4.1 admits a closed-form solution for the exact derivative, and this is what is used in the numerical computations here. But when the functional form of μ is an outcome of an underlying model, which may not admit a closed-form solution, the use of a numerical derivative such as that in (4.5) will be necessary.

The finite difference approximation for $c^*(t, x)$ for any period t can now be specified:

$$c^*(t, x) \simeq c_{\mathbf{k}j}^n = \left[-\Delta_s v_{\mathbf{k}j}^n \Delta_p \mu_j^n\right]^{-1},\ n = n_t, \ldots, 1. \tag{4.6}$$

In similar fashion, the policy function $p^*(t)$ may be approximated, using the solution (4.2)–we express the numerical integral which must be computed in a generic format below, but any numerical integration method (quadrature, rhomberg, trapezoidal, etc.) may be used. The approximation is:

$$p^*(t) \simeq p^n_{\mathbf{k}} = 1 - \sum_{j=1}^{n_x} c^n_{\mathbf{k}j} s_{\mathbf{k}j} \Delta x_j,\ n = n_t, \ldots, 1. \tag{4.7}$$

Once the optimal policies have been discretized the Hamilton-Jacobi-Bellman equation can also be approximated. Recall that for the left hand side of (4.3) we have

$$\frac{\partial v}{\partial t} \simeq \frac{v^n_{\mathbf{k}} - v^{n-1}_{\mathbf{k}}}{\Delta t}. \tag{4.8}$$

The right hand side of (4.3) is another numerical integral (again, this is a generic template for the integral):

$$\int_0^t \left[u(c^*(t,x)) - \frac{\partial v}{\partial s} \cdot (\mu(x, p^*(t)) \right] s(t,x) dx \simeq \\ \sum_{j=1}^{n_x} \left[\ln\left(c^n_{\mathbf{k}j}\right) - \Delta_s v^n_{\mathbf{k}j} \mu(x_j, p^n_{\mathbf{k}}) \right] s_{\mathbf{k}j} \Delta x_j. \tag{4.9}$$

Finally, we can combine (4.8) and (4.9) to yield

$$v^{n-1}_{\mathbf{k}} = v^n_{\mathbf{k}} + \sum_{j=1}^{n_x} \left[\ln\left(c^n_{\mathbf{k}j}\right) - \Delta_s v^n_{\mathbf{k}j} \mu(x_j, p^n_{\mathbf{k}}) \right] s_{\mathbf{k}j} \Delta x_j \Delta t, \tag{4.10}$$

with $c^n_{\mathbf{k}j}$ and $p^n_{\mathbf{k}}$ defined as in (4.6) and (4.7), respectively.

4.3 The algorithm

The algorithm proceeds in standard fashion, starting from the terminal period T:

1. the grid sizes n_t, n_s and n_x are chosen, and the grid partitions are formed. For x, the age space $[0, T]$ is used at the beginning, defining $\Delta x = \frac{T}{n_x - 1}$. Let an index counter $i = 1$, let $n = n_t$, and $v^n_{\mathbf{k}j} := V_T(x_j)$.

2. Compute optimal policies $c^n_{\mathbf{k}j}$ and $p^n_{\mathbf{k}}$ for every state $s_{\mathbf{k}}$ and (in the case of consump-

tion) underlying index grid point j. This involves the computation of $\Delta_s v^n_{\mathbf{k}j}$ and $\Delta_p \mu^n_j$.

3. The next iteration $v^{n-1}_{\mathbf{k}}$ is computed from (4.10).

4. The age space is redefined to $[0, T - i\Delta t]$ and new grid points for x are chosen, with $\Delta x = \frac{T - i\Delta t}{n_x - 1}$.[4] Let $n = n_t - i$. Finally, let $i = i + 1$.

5. Repeat algorithm from 2 until $i = n_t$.

5 Algorithm Convergence, Viscosity Solutions

Algorithm convergence is an open question and will be addressed in future research, as will the relative merits of the 'naive' finite differencing outlined above vs. a more sophisticated solution method using the viscosity approach (see. e.g. Crandall and Lions [1984, 1983] for a comprehensive treatment of this method for solving HJB systems).

[4]This can be automated in the computation of the numerical integrals, where this partition is used.

References

Agbola, F. W., Damoense, M. Y. and Saini, Y. K.: 2004, South africa: Impact of hiv/aids on food demand, *International Journal of Social Economics* **31**(7-8), 721–731.

Ambert, C.: 2002, The economic impact of hiv/aids on the construction sector: Supply side implications for housing policy, *South African Journal of Economics* **70**(7), 1235–1261.

Anderson, R. M.: 1991, Discussion: the kermack-mckendrick epidemic threshold theorem, *Bulletin of Mathematical Biology* **53**, 3–32.

Arndt, C. and Lewis, J. D.: 2000, The macro implications of hiv/aids in south africa: A preliminary assessment, *South African Journal of Economics* **68**(5), 380–392.

Arrow, K. J.: 2003, The locus of decisions about aids/hiv, malaria treatment: What does welfare economics say? a question, *Economie Publique* pp. 9–11.

Bacaër, Nicolas, X. A. and Ye, J.: 2006, Modeling the hiv/aids epidemic among injecting drug users and sex workers in kunming, china, *Bulletin of Mathematical Biology* **68**, 525–550.

Bell, C. and Gersbach, H.: 2006, *The Macroeconomics of Targeting: The Case of an Enduring Epidemic*, C.E.P.R. Discussion Papers: 5714.

Bonnel, R.: 2000, Hiv/aids and economic growth: A global perspective, *South African Journal of Economics* **68**(5), 820–855.

Brito, P. and Dilao, R.: 2006, Equilibrium price dynamics in an overlapping-generations exchange economy, *Technical report*, Munich Personal RePEc Archive Paper No. 699.

Canning, D.: 2006, The economics of hiv/aids in low-income countries: The case for prevention, *Journal of Economic Perspectives* **20**(3), 121–142.

Crafts, N. F. R. and Haacker, M.: 2003, *Welfare Implications of HIV/AIDS*, International Monetary Fund, IMF Working Papers: 03/118.

Crandall, M. G. and Lions, P.-L.: 1983, Viscosity solutions of hamilton-jacobi equations, *Transactions of the American Mathematical Society* **277**, 1–42.

Crandall, M. G. and Lions, P.-L.: 1984, Some properties of viscosity solutions of hamilton-jacobi equations, *Transactions of the American Mathematical Society* **282**, 487–502.

Crandall, M. G. and Lions, P.-L.: 1985, Hamilton-jacobi equations in infinite dimensions I. uniqueness of viscosity solutions, *Journal of Functional Analysis* **62**, 379–396.

Deininger, K., Crommelynck, A. and Kempaka, G.: 2005, Impact of aids on family composition, welfare, and investment: Evidence from uganda, *Review of Development Economics* **9**(3), 303–324.

DeSimone, J. and Schumacher, E. J.: 2004, *Compensating Wage Differentials and AIDS Risk*, National Bureau of Economic Research, Inc, NBER Working Papers: 10861.

Dike, S.: 2002, Research on the economic impact of hiv/aids in south africa: Review of methodology and lessons learned, *South African Journal of Economics* **70**(7), 1283–1291.

Dixon, S., McDonald, S. and Roberts, J.: 2001, Hiv/aids and development in africa, *Journal of International Development* **13**(4), 381–389.

Faggian, S.: 2005, Regular solutions of first-order hamilton-jacobi equations for boundary control problems and applications to economics, *Applied Mathematics and Optimization* **51**, 123–162.

Faggian, S. and Gozzi, F.: 2004, On the dynamic programming approach for optimal control problems of pdes with age structure, *Mathematical Population Studies* **11**, 233–270.

Feichtinger, G., Tragler, G. and Veliov, V. M.: 2003, Optimality conditions for age-structured control systems, *Journal of Mathematical Analysis and Applications* **288**, 47–68.

Gaffeo, E.: 2003, The economics of hiv/aids: A survey, *Development Policy Review* **21**(1), 27–49.

Garnett, G. P. and Anderson, R. M.: 1993, Factors controlling the spread of hiv in heterosexual communities in developing countries: Patterns of mixing between different age and sexual activity classes, *Philosophical Transactions: Biological Sciences* **342**, 137–159.

Godinho, J. et al.: 2005, *Reversing the Tide: Priorities for HIV/AIDS Prevention in Central Asia*, World Bank Working Paper, no. 54; Washington, D.C.:; World Bank.

Haacker, M. (ed.): 2004, *The macroeconomics of HIV/AIDS*, Washington, D.C.:; International Monetary Fund.

Hoppensteadt, F.: 1974, An age dependent epidemic model, *Journal of the Franklin Institute* **297**, 325–333.

Inaba, H.: 1989, Weak ergodicity of population evoluation processes, *Mathematical Biosciences* **96**, 195–219.

Inaba, H.: 1990, Theshold and stability results for an age-structured epidemic model, *Journal of Mathematical Biology* **28**, 411–434.

Inaba, H.: 2006, Endemic threhold results in an age-duration-structured population model for hiv infection, *Mathematical Biosciences* **201**, 15–47.

Jayne, T. S., Villarreal, M., Pingali, P. and Hemrich, G.: 2005, Hiv/aids and the agricultural sector: Implications for policy in eastern and southern africa, *eJADE: Electronic Journal of Agricultural and Development Economics* **2**(2), 158–181.

John, F.: 2005, *Partial Differential Equations*, 4th edn, Springer.

Kermack, W. O. and McKendrick, A. G.: 1991, Contributions to the mathematical theory of epidemics - i, *Bulletin of Mathematical Biology* **53**, 33–55. Reprinted from *Proceedings of the Royal Society*, V. 115A, pp. 700-721, 1927.

Kirman, A. P., Luchini, S. and Moatti, J.-P.: 2003, Can economic analysis contribute to the current debate on access to hiv/aids care in developing countries?, *Economie Publique* **12**, 131–151.

Kumaranayake, L. and Watts, C.: 2001, Resource allocation and priority setting of hiv/aids interventions: Addressing the generalized epidemic in sub-saharan africa, *Journal of International Development* **13**(4), 451–466.

Lloyd, P. J.: 1991, Aids: An economic approach to the choice of policies, *Economic Record* **67**(197), 126–138.

McDonald, S. and Roberts, J.: 2006, Aids and economic growth: A human capital approach, *Journal of Development Economics* **80**(1), 228–250.

Nattrass, N.: 2006, Review: The macroeconomics of hiv/aids, *Journal of Economic Literature* **44**(1), 161–163.

OECD Development Assistance Committee and UNAIDS: 2004, Analysis of aid in support of hiv/aids control: 2000-2002, OECD.

Oster, E.: 2005, Sexually transmitted infections, sexual behavior, and the hiv/aids epidemic, *Quarterly Journal of Economics* **120**(2), 467–515.

Over, M. et al.: 2004, *HIV/AIDS treatment and prevention in India: Modeling the costs and consequences*, Human Development Network Health, Nutrition and Population Series; Washington, D.C.: World Bank.

Robalino, D. A., Jenkins, C. and Maroufi, K. E.: 2002, *The Risks and Macroeconomic Impact of HIV/AIDS in the Middle East and North Africa: Why Waiting to Intervene Can Be Costly*, The World Bank, Policy Research Working Paper Series: 2874.

Tudor, D.: 1985, An age-dependent epidemic model with application to measles, *Mathematical Biosciences* **73**, 131–147.

World Bank Operations Evaluation Department: 2005, *Committing to Results: Improving the Effectiveness of HIV/AIDS Assistance–An OED Evaluation of the World Bank's Assistance for HIV/AIDS Control*, Washington, D.C.: World Bank.