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Heterogeneity and misspecifications in learning

By Michele Berardi

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Economics
School of Social Sciences
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom

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Michele Berardi*†
School of Social Sciences
University of Manchester

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Abstract

In this paper we consider a linear, stochastic, univariate, forward looking model with one lag under adaptive heterogeneous learning. The system is populated by two different types of agents who learn through recursive least squares techniques the parameter values in their forecasting models. The two groups are constrained to have different information sets, one being always a subset of the other. We analyze convergence of these two interacting learning processes under different specifications of the forecasting model, and in one case we find that an Heterogenous Expectations Equilibrium emerges.

Key words: Adaptive learning, expectations formation, heterogenous expectations, misspecifications.

JEL classification: C62, D83, E37.

*Contact information: School of Social Sciences, Economics, University of Manchester, Dover Street Building, Manchester M13 9PL. E-mail: Michele.Berardi@manchester.ac.uk; Tel.: +44 (0)161 2754834.

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1. Introduction

Literature on learning in macroeconomics often assumes that agents have a correctly specified perceived law of motion, i.e., that they include in their learning model all and only those variables that actually play a role in determining the dynamics of the economy. But as agents are considered as econometricians, they face the same problem of choosing the most appropriate specification for their model. Guided by their idiosyncratic beliefs, they will start by using a particular model and will then change it only if rejected by data, but nothing assures that this model will include all and only those variables that actually contribute to the dynamics of the economy. If agents use an overparameterized model and include more variables than necessary, they can learn over time to discard the irrelevant ones. But if instead they use an underparameterized model, one that neglects one or more variables that are actually relevant for the process to be forecast, they might end up in what is called a restricted perception equilibrium (RPE), where data do not reject their misspecified model because the forecast errors are orthogonal to the variables included in their information set.

In this paper we use this idea to introduce heterogeneity in the learning process implemented by two different groups of agents, with one group using only a subset of the variables used by the other. The group with the smaller information set might be underparameterizing its model with respect to the actual process describing the evolution of the system. The reason why agents would use an undeparameterized model in their learning activity is that, acting as econometricians, they face the same problems of computational limits and degree of freedom limitations that arise in econometrics (Branch and Evans, 2006). In this perspective, the fact that different agents use different models can be interpreted as them having different computational capabilities.

Agents that make use of an underparameterized perceived law of motion (PLM) can not possibly learn the true process describing the evolution of the endogenous variable and, as a consequence, the system can not converge towards a rational expectations equilibrium (REE). Nevertheless, we show that it can still converge to an equilibrium in which all beliefs are confirmed by data, in the sense that the errors agents are making in their forecasts are uncorrelated with the variables included in their own information set, though agents in the two groups have different information sets. The equilibrium is named *Heterogeneous Expectations Equilibrium* (HEE), and we provide conditions for its learnability.

It has been shown by Evans and McGough (2005) that the learnability of the REE in an AR(1) model depends on its representation. We will thus consider two different representations of the equilibrium we analyze: the *general form* representation and the *common factor* representation. Since both representations correspond to the equilibrium solution, agents could use either of them in their learning process, and in the absence of coordination this could give rise to heterogeneity. When one group of agents uses a general form (GF) representation and the other uses a common factor (CF) representation, a situation similar to that in which part of the agents use a misspecified model arises, i.e., one group is underparameterizing its PLM. In fact, while under rational expectations the equilibrium can be represented in both ways, this is not true on the path towards equilibrium: under learning dynamics, when some agents include in their learning scheme the extra variables required by the GF representation these variables become part of the system and are needed to fully characterize its dynamics. Agents that neglect those variables, by using a CF representation, are therefore using an underparameterized PLM.

It is worth pointing out that while in this work both groups of agents use the same type of learning algorithm, namely recursive least squares (RLS), the analysis could be easily generalized to allow for the learning algorithms to differ between the two groups of agents. In addition, more than two groups of agents could be considered, each using a different PLM. We leave these exercises to future work.

The paper is organized as follows: section 2 outlines some background theory for restricted perceptions and for heterogeneity in expectations; section 3 presents the model and solves it under RE; section 4 introduces heterogeneous learning into the model, derives the HEE and considers alternative learning dynamics; section 5 concludes.

2. Some background theory

2.1. Restricted perceptions equilibria and other related concepts

The concept of rational expectations (RE) has come to play a central role in macroeconomics over the last thirty years, as it imposes rigor on an otherwise free component of modern macroeconomic models. Recent works have tried to maintain the same rigor while relaxing some of the most stringent requirements that RE impose on agents in terms of information and computational abilities. We present here a brief exposition of the different equilibrium concepts that can be obtained by departing from full rationality while preserving the original idea of Muth (1961) that agents fully exploit the information available to them.

In the learning literature, the REE is interpreted as the limit point of the learning process undertaken by agents. But for convergence to obtain, the model used by agents, their PLM, must be correctly specified or possibly overparameterized, i.e., it must include at least all the variables that play a role in the actual law of motion (ALM). If instead the learning activity is implemented by means of an underparameterized PLM, one that neglects some variable(s) relevant for the dynamics of the system, then the learning process can not possibly converge to the REE but it may nevertheless converge to a different equilibrium, which is called *Restricted Perceptions Equilibrium* (RPE) by Evans and Honkapohja (2001). This equilibrium is the equivalent of an REE for models in which the PLM is restricted to be of a particular (misspecified) form, as it imposes the same orthogonality condition between the forecasting errors and the variables included into the information set of the agents. The fact that this information set is constrained to exclude one or more variables means that expectations can not be rational in the traditional sense, but they are nevertheless optimal given the restrictions imposed on the PLM: agents do the best they can given the model they are endowed with.

In Sargent (1999) a similar kind of equilibrium is called *Self-Confirming Equilibrium.* The difference is that while in an RPE agents could in principle recognize the non rationality of their expectations if they were to allow the neglected variables to play a role in their PLM, in Sargent's self-confirming equilibrium agents can realize that they are using a wrong model only along off equilibrium paths. In equilibrium, both models are equally good.

An extension of the RPE concept is the *Misspecification Equilibrium* proposed by Branch and Evans (2006), where the misspecification is endogenized: agents choose the model that performs best in the set of all underparameterized models. For cognitive and computational constraints, as well as for degrees of freedom limitations, agents use a misspecified model, but they choose among the possible functional forms the one that allows them to achieve higher profits.

A particular type of RPE is the *Stochastic Consistent Expectations Equilibrium*, developed by Hommes and Sorger (1998): in this case the true process describing the dynamics of the economy is non-linear, but agents use a linear model to form their forecasts. The equilibrium is reached when the mean and the correlation coefficients in the actual data, generated by a non-linear stochastic process, coincide with those predicted by the linear model employed by agents (see also Branch and McGough, 2005).

The main feature of an RPE is the orthogonality condition between the forecast errors and the variables included in the information set of the agents. Calling z_t the vector of variables used in forming expectations, x_t the actual process for the variable to be forecast and $\hat{x}_t$ the value obtained using the PLM, in an RPE the orthogonality condition

$$Ez_t(x_t - \hat{x}_t) = 0 \tag{1}$$

gives the equilibrium least squares estimate for the parameters in the PLM. If z_t were to comprise all the variables determining x_t, then the equilibrium obtained would be an REE. When instead some relevant variables are left out from z_t, if an equilibrium obtains this is an RPE. The values of the coefficients in the PLM will depend in this case also on the covariances between the variables included in z_t and those excluded, as the ALM must be projected on the restricted PLM in order to find the estimated values for the parameters. The fixed point of the system formed by the ALM and the expectation formation algorithms represents the equilibrium. If lagged endogenous variables play a role in determining the dynamics of the system and are left out from the PLM, then the covariance terms will not be exogenous but will depend on the value of the estimated parameters in the PLM, as they are used to form expectations which then determine the actual value of the endogenous variables. This endogeneity of the covariances makes the problem more complicated than, for example, the cases presented in Evans and Honkapohja (2001), ch. 13, where only exogenous variables affect the economy. We will need to consider also this additional feature of the system in order to find the equilibrium fixed point of our model.

2.2. Heterogenous expectations

The RE concept with full information requires homogeneity of expectations, as subjective probabilities must coincide with the objective ones, and thus with each other. There still can be structural heterogeneity in the economy, in the sense that expectations of different groups of agents can enter the model differently and affect the economy in different ways, but those expectations must be the same. If instead we allow for some of the agents to depart from full rationality, or to have limited information, then heterogeneity in expectations can arise even in equilibrium. In this case not only expectations of different agents can affect the economy differently, but those expectations can be different from each other. Structural heterogeneity arises for example in the New Keynesian model when closed with an expectations based policy rule in the case in which the central bank (CB) responds to its own expectations. In this case, in fact, private expectations affect the economy through the structural parameters in the IS equation and the Phillips curve, while CB expectations impact on the economy through monetary policy. If CB and private sector have the same expectations, the model features structural heterogeneity but homogeneity in expectations, while if the two parts are allowed to have different beliefs then the model features also heterogeneity of expectations. Honkapohja and Mitra (2006) and (2005) consider both types of heterogeneity, but the heterogeneity in expectations vanishes as agents learn and their beliefs converge to rationality, and to each other: the limit equilibrium point is the usual REE.

Models have been studied in which there is structural homogeneity but expectations are heterogenous: different groups of agents have different expectations, but those expectations affect the economy through the same mechanism. In this case the expectations that enter the structural equations are usually a weighted average of the expectations held by different agents. See for example the models presented in Branch and McGough (2004) and Giannitsarou (2001).

Branch and McGough (2004) define the concept of *Heterogeneous Expectations Equilibrium* (HEE) in a way that is in some sense similar to the HEE that we are going to define, but differs from it in a fundamental way. Apart from featuring structural homogeneity, in their HEE one group of agents is rational, while the other has adaptive (static) expectations with fixed parameters: there is no learning activity going on, and agents having adaptive expectations would see their forecasts persistently falsified by data. This HEE could not represent the limit point of a learning process, as our HEE instead is. In this sense, our definition of equilibrium is a more rigorous refinement of their HEE, and it captures the spirit of rational expectations: all agents use the information available to them efficiently, and their beliefs are not falsified by experience.

3. The model

The structural model we are going to use in this study is a simple forward looking, stochastic, univariate model in which the endogenous variable depends on current expectations of its next period's value, on its lagged value and on an exogenous stationary AR(1) process. There are two different groups of agents populating the model, each forming its own expectations independently. The analysis could be carried out in a multivariate model as well, but though the algebra would be much more tedious, little would be gained in terms of insights. We thus choose to keep things easy and go for the univariate case.

The structural equations are

$$x_t = aE_t^1 x_{t+1} + bE_t^2 x_{t+1} + cx_{t-1} + dw_t \tag{2}$$

$$w_t = \rho w_{t-1} + v_t, \tag{3}$$

where x_t is an endogenous variable, w_t is exogenous and observable at time t and v_t is an i.i.d. random disturbance. E^i indicates expectations, not necessarily rational. ρ is assumed to be between 0 and 1.

Note that structural heterogeneity arises for $a \neq b$. a and b can be decomposed in two terms, one referring to the way in which expectations feed through on the endogenous variable and one representing a weight on expectations that depends on the proportion of agents sharing the same beliefs. Thus we could write

$$a = \alpha_1 \mu \tag{4}$$

$$b = \alpha_2 (1 - \mu) \tag{5}$$

where μ is a measure of the relative size of the group of agents holding type 1 expectations. In particular, given a continuum of agents over the unit interval, μ of them are of type 1, and the remaining $(1 - \mu)$ are of type 2. When there is structural homogeneity, $\alpha_1 = \alpha_2 = \alpha$, and if also expectational homogeneity holds (i.e., $E^1 = E^2 = E^C$), then $(aE^1 + bE^2) = \alpha E^C$, E^C where stands for common expectation.

Following Honkapohja and Mitra (2005) we can also define the concept of *average expectations:*

$$x_t = mE_t^{AV} x_{t+1} + cx_{t-1} + dw_t \tag{6}$$

where average expectations are

$$E_t^{AV} x_{t+1} = m^{-1}(aE_t^1 x_{t+1} + bE_t^2 x_{t+1}) \tag{7}$$

and

$$m = a + b \tag{8}$$

is the *aggregate characteristics* or *average economy.*

3.1. REE and its representations

To find the REE of this model, we impose $E^1 = E^2 = E$, where E is the rational expectations operator. Moreover, we impose structural homogeneity. The model becomes

$$x_t = \alpha E_t x_{t+1} + c x_{t-1} + d w_t \tag{9}$$

$$w_t = \rho w_{t-1} + v_t, \tag{10}$$

In finding a solution, we must distinguish between the regular and the irregular case.

We can rewrite (9) as

$$x_{t+1} = \frac{1}{\alpha} x_t - \frac{c}{\alpha} x_{t-1} - \frac{d}{\alpha} w_t + \varepsilon_{t+1} \tag{11}$$

where

$$\varepsilon_{t+1} = x_{t+1} - E_t x_{t+1} \tag{12}$$

is a martingale difference sequence (mds).

This is called by Evans and McGough (2005) *general form* representation: every equilibrium satisfying (9) must also satisfy this equation. Rewrite the system in first order form

$$\begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha} & -\frac{c}{\alpha} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} \frac{d}{\alpha} \\ 0 \end{pmatrix} w_t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varepsilon_t, \tag{13}$$

define

$$A = \begin{pmatrix} \frac{1}{\alpha} & -\frac{c}{\alpha} \\ 1 & 0 \end{pmatrix} \tag{14}$$

and call e_1, e_2 the two eigenvalues of A. We have the regular case when $|e_1| < 1 < |e_2|$, the irregular case when $|e_1| < |e_2| < 1$ and the system is said to be explosive if $1 < |e_1| < |e_2|$. In this last case there are no stable solutions to the system, while in the irregular case there are infinitely many of them.

We consider in this study only the regular case, where there is only one solution to (9)-(10) that is stationary, and this is the minimum state variable (MSV) solution that depends only on the fundamentals of the economy.[1] This solution nevertheless can be represented in different ways. The standard way, the one called by Evans and McGough (2005) *common factor* representation, is to represent the current endogenous variable as solely function of its lagged value and the exogenous forcing variable

$$x_t = e_1 x_{t-1} + \frac{d}{1 - \alpha(e_1 + \rho)} w_t, \tag{15}$$

where

$$e_1 = \frac{1 - \sqrt{1 - 4\alpha c}}{2\alpha} \tag{16}$$

is the stable root of A.

But if we multiply both sides of (15) by $(1 - \eta L)$, $\forall\ \eta \neq 0$, we obtain an

[1] The relationship between a determinate solution and the MSV solution has been analysed by McCallum (2004). He shows that in the univariate case the two solutions coincide. For a more general discussion of the MSV solution, see McCallum (1999).

ARMA(2,1) representation of the same REE:[2]

$$x_t = (e_1 + \eta)x_{t-1} - (e_1\eta)x_{t-2} + \frac{d}{1-\alpha(e_1+\rho)}w_t - \eta\frac{d}{1-\alpha(e_1+\rho)}w_{t-1}. \quad (17)$$

Note that the general form representation (11) can be obtained from this ARMA(2,1) representation by imposing $\eta = e_2$ and defining $\varepsilon_t = \frac{d}{1-\alpha(e_1+\rho)}v_t$. This shows that the equilibrium is determinate, as the expectational error ε_t is uniquely pinned down by the fundamentals of the economy.

4. Introducing learning

We now step back from the rational expectations assumption and suppose agents learn by extracting information from data through a RLS procedure. Moreover, we allow for two different groups of agents that form expectations using different models.

In evaluating learnability we will make use where possible of the E-stability principle that establishes a direct link between learnability in real time through techniques such as RLS and the stability of a differential equation derived by mapping the PLM into the ALM.[3]

Agents, in their learning activity, use a reduced form model, one that is meant to capture the form of the ALM for the endogenous variables of the system. But as we have said, there is not a unique representation of the equilibrium process. Therefore agents, when deciding what model to estimate, might choose either the CF representation or the GF representation, as both correctly capture the structure of the economy in equilibrium. Then, for any of the two cases, we consider what happens when some agents underparameterize that model. Moreover, it could also happen that at the same time part of the agents use the CF rep-

[2] We consider only the case of a doubly infinite REE. For an initialized process, appropriate initial conditions need to be specified.

[3] For an extensive treatment of this and other related concepts, see Evans and Honkapohja (2001).

resentation and some the GF representation: this can be regarded as a lack of coordination among agents. We address this case in section 4.3.

4.1. Learning through Common factor representation

4.1.1. Homogeneous case

We start by analyzing the standard case of homogeneity of expectations and structural homogeneity. All agents use the same model, the CF representation, in order to form their expectations:

$$\text{PLM}^1 = \text{PLM}^2 : x_t = \phi_1 x_{t-1} + \phi_2 w_t. \tag{18}$$

In this case the PLM can be mapped into the ALM in the standard way,[4] obtaining an ODE whose asymptotic behavior is the same as that of the original RLS algorithm. Learnability of the REE can thus be assessed by analyzing local stability of this ODE at the equilibrium point $(\bar{\phi}_1, \bar{\phi}_2)$.

The E-stability condition reduces to

$$\alpha\bar{\phi}_1 + \alpha\rho < 1 \tag{19}$$

which using

$$\bar{\phi}_1 = \frac{1 - \sqrt{1 - 4\alpha c}}{2\alpha} \tag{20}$$

becomes

$$\alpha\rho < \frac{1 + \sqrt{1 - 4\alpha c}}{2}. \tag{21}$$

Proposition 1. *The fundamental REE of the model (9)-(10), with all agents using the CF representation (18) in their learning process, is E-stable when condition (21) is satisfied.*

[4] See Evans and Honkapohja (2001) for a complete exposition of the techniques used here, and in particular for the applicability of the E-stability principle.

4.1.2. Heterogeneous case

We turn now to analyze the learning properties of the equilibrium with the AR(1) CF representation in case of heterogeneity of expectations. In order to introduce heterogeneity, we constrain agents of group 2 to neglect the lagged endogenous component x_{t-1} in their PLM.

Agents of group 1 have a correctly specified PLM, i.e., one that takes the same form as the reduced form for the actual process:

$$x_t = \phi_1 x_{t-1} + \phi_2 w_t, \tag{22}$$

while agents of group 2 use the underparameterized PLM

$$x_t = \theta w_t. \tag{23}$$

All agents recurrently estimate the parameters in their model through recursive least squares techniques. If we stack together the two recursive algorithms, we get

$$\xi_t = \xi_{t-1} + t^{-1} \begin{pmatrix} P_t^{-1} z_t \left(x_{t-1} - \hat{x}_{t-1}^1 \right) \\ R_t^{-1} w_{t-1} \left(x_{t-1} - \hat{x}_{t-1}^2 \right) \end{pmatrix} \tag{24}$$

where

$$\xi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \theta \end{pmatrix} \tag{25}$$

$$z_t = \begin{pmatrix} x_{t-2} \\ w_{t-1} \end{pmatrix} \tag{26}$$

$$P_t = P_{t-1} + t^{-1}(z_t z_t^{'} - P_{t-1}) \tag{27}$$

$$R_t = R_{t-1} + t^{-1}(w_{t-1} w_{t-1} - R_{t-1}) \tag{28}$$

$$
\begin{aligned}
\left(x_{t-1}-\hat{x}_{t-1}^{1}\right) &= (a\phi_1^2+c-\phi_1)x_{t-2}-(a\phi_1\phi_2+a\rho\phi_2+b\theta\rho+d-\phi_2)w_{t-1} \quad (29)\\
\left(x_{t-1}-\hat{x}_{t-1}^{2}\right) &= (a\phi_1^2+c)x_{t-2}-(a\phi_1\phi_2+a\rho\phi_2+b\theta\rho+d-\theta)w_{t-1}. \quad (30)
\end{aligned}
$$

This shows how the updating processes implemented by the two groups of agents will jointly determine the equilibrium outcome for the economy. Once agents have estimated their own model, they use it to make forecasts. Note that agents, when updating the parameters in their PLM at time t and making forecasts for time $t+1$, do not know the current value x_t. This assumption is common in the literature, and is justified by the fact that x_t depends on the estimates for time $t+1$, so when these are made the current endogenous variable can not be known.

The ensuing expectations for the two groups are

$$
\begin{aligned}
E_t^1 x_{t+1} &= \phi_1^2 x_{t-1}+(\phi_1\phi_2+\rho\phi_2)w_t \quad (31)\\
E_t^2 x_{t+1} &= \theta\rho w_t, \quad (32)
\end{aligned}
$$

where ϕ_1, ϕ_2, θ are the most recent estimates of those parameters.[5] Inserting these expectations into (2) leads to the temporary equilibrium or ALM

$$
x_t=(a\phi_1^2+c)x_{t-1}+(a\phi_1\phi_2+a\rho\phi_2+b\theta\rho+d)w_t. \quad (33)
$$

It is easy to find the map from the PLM to the ALM for agents in group 1:

$$
\begin{aligned}
\phi_1 &\rightarrow a\phi_1^2+c \quad (34)\\
\phi_2 &\rightarrow a\phi_1\phi_2+a\rho\phi_2+b\theta\rho+d, \quad (35)
\end{aligned}
$$

while things are more complicated for agents in group 2, as the ALM must be projected on their restricted PLM. The relevant stochastic recursive algorithm

[5] For simplicity of exposition, we have dropped the subscript t from the parameters of the PLMs.

(SRA) in this case is

$$\theta_t = \theta_{t-1} + t^{-1}R_t^{-1}w_{t-1}(x_{t-1} - \hat{x}_{t-1}) \tag{36}$$

$$R_t = R_{t-1} + t^{-1}(w_{t-1}w_{t-1} - R_{t-1}) \tag{37}$$

where $\hat{x}_{t-1} = \theta_{t-1}w_{t-1}$. In order to write this SRA in the standard form, we need to perform the transformation $R_t = S_{t-1}$, as on the r.h.s. we can only have lagged values of the parameters. The associated ordinary differential equation (ODE) that governs stability of this SRA is

$$\frac{d\Phi}{d\tau} = h(\Phi) = \lim_{t\to\infty} EQ(t, \Phi, z_t) \tag{38}$$

where $\Phi = (\theta,\ S)'$, $z_t = (x_{t-2,}w_{t-1})$ and E denotes expectations of $Q(t, \Phi, z_t)$ taken over the invariant distribution of z_t, for fixed Φ. $Q(t, \Phi, z_t)$ comes from the SRA and it is defined as

$$Q(t, \Phi, z_t) = \begin{pmatrix} S^{-1}w_{t-1}(x_{t-1} - \hat{x}_{t-1}) \\ (\frac{t}{t+1})(w_{t-1}w_{t-1} - S) \end{pmatrix}. \tag{39}$$

Here we can see that the orthogonality condition (1) holds at the resting point of the ODE for θ. By substituting in the first line the expressions for x_{t-1} and $\hat{x}_{t-1}$ we get:

$$Q_\theta(t, \Phi, z_t) = S^{-1}w_{t-1}((a\phi_1^2 + c)x_{t-2} + (a\phi_1\phi_2 + a\rho\phi_2 + b\theta\rho + d - \theta)w_{t-1}). \tag{40}$$

It thus follows that

$$h_\theta(\Phi) = \lim_{t\to\infty} ES^{-1}w_{t-1}((a\phi_1^2 + c)x_{t-2} + (a\phi_1\phi_2 + a\rho\phi_2 + b\theta\rho + d - \theta)w_{t-1})$$

and

$$h_S(\Phi) = \lim_{t\to\infty}(\frac{t}{t+1})E(w_{t-1}w_{t-1} - S) = M - S, \tag{41}$$

where $M = Ew_{t-1}w_{t-1} = Ew_t w_t$. From (41) we have that $S \to M$, which leads to

$$h_\theta(\Phi) = a\phi_1\phi_2 + a\rho\phi_2 + b\theta\rho + d + (a\phi_1^2 + c)Ew_{t-1}x_{t-2}M^{-1} - \theta. \tag{42}$$

M is here just the variance of the stochastic process w that for clarity we call now $\sigma_w^2 = Ew_t^2$ and is given exogenously. Instead $Ew_{t-1}x_{t-2}$ is determined endogenously and depends on the parameters in the PLMs:

$$Ew_{t-1}x_{t-2} = E(\rho w_{t-2} + v_{t-1})((a\phi_1^2 + c)x_{t-3} + (a\phi_1\phi_2 + a\rho\phi_2 + b\theta\rho + d)w_{t-2}). \tag{43}$$

Given that the variables are taken to be stationary, we have the asymptotic result $Ew_{t-1}x_{t-2} = Ew_{t-2}x_{t-3} = \sigma_{wx}^2$. Moreover, given that v_{t-1} is an i.i.d. process, it is uncorrelated with all the other variables. It then follows:

$$\sigma_{wx}^2 = \rho(a\phi_1\phi_2 + a\rho\phi_2 + b\theta\rho + d)(1 - \rho(a\phi_1^2 + c))^{-1}\sigma_w^2 \tag{44}$$

which substituted in (42) gives

$$h(\theta) = (a\phi_1\phi_2 + a\rho\phi_2 + b\theta\rho + d)\left(1 + \frac{(a\phi_1^2 + c)\rho}{1 - \rho(a\phi_1^2 + c)}\right) - \theta. \tag{45}$$

We can see that σ_w^2 gets canceled out, as it enters also in the expression for σ_{wx}^2: this happens because the variable relevant for the process but neglected in the forecasts is the lag of the endogenous variable. If instead this variable was exogenous, then both the variance of the included variable and the covariance between the two exogenous variables would be part of the solution value for the estimated parameter.

The ODE (45), together with (34) and (35) give the system

$$\begin{aligned}
\dot{\phi}_1 &= a\phi_1^2 + c - \phi_1 && (46)\\
\dot{\phi}_2 &= a\phi_1\phi_2 + a\rho\phi_2 + b\theta\rho + d - \phi_2 && (47)\\
\dot{\theta} &= \frac{a\phi_1\phi_2 + a\rho\phi_2 + b\theta\rho + d}{1-\rho(a\phi_1^2 + c)} - \theta. && (48)
\end{aligned}$$

Any fixed point $(\bar{\phi}_1, \bar{\phi}_2, \bar{\theta})$ of this system represents an equilibrium. Equation (46) is clearly independent from the other two's, so it can be solved autonomously. It is a quadratic equation, that has the two solutions

$$\phi_1^{+,-} = \frac{1 \pm \sqrt{1-4ac}}{2a}. \tag{49}$$

The minimum state variable (MSV) solution, in the sense of McCallum (1983), is given by ϕ_1^- as it is the one that gives a coefficient equal to zero when c is zero. This solution is real for $ac < \frac{1}{4}$, which is therefore taken to be a constraint for the model to be sensible. Given $\bar{\phi}_1 = \phi_1^-$, the values for $\bar{\phi}_2$ and $\bar{\theta}$ are determined uniquely by (47) and (48).

Equations (47) and (48) form a system, and must be considered simultaneously. Using matrix form

$$\begin{pmatrix} \dot{\phi}_2 \\ \dot{\theta} \end{pmatrix} = \Gamma \begin{pmatrix} \phi_2 \\ \theta \end{pmatrix} + \Psi, \tag{50}$$

where

$$\Gamma = \begin{pmatrix} a\bar{\phi}_1 + a\rho - 1 & b\rho \\ \frac{a\bar{\phi}_1 + a\rho}{1-\rho(a\bar{\phi}_1^2 + c)} & \frac{b\rho}{1-\rho(a\bar{\phi}_1^2 + c)} - 1 \end{pmatrix} \tag{51}$$

and

$$\Psi = \begin{pmatrix} d \\ \frac{d}{1-\rho(a\bar{\phi}_1^2 + c)} \end{pmatrix}. \tag{52}$$

Convergence[6] to the fix point $(\bar{\phi}_1, \bar{\phi}_2, \bar{\theta})$ requires that $2a\bar{\phi}_1 - 1 < 0$ and that all the eigenvalues of the matrix Γ have negative real part. The first of the two conditions is satisfied by any real solution for ϕ_1^- from (49). The second, by applying the Routh Theorem (see Chiang, 1984, p. 546), requires $\Gamma_{11} + \Gamma_{22} < 0$ and $\Gamma_{11}\Gamma_{22} - \Gamma_{12}\Gamma_{21} > 0$, which reduces to

$$a\bar{\phi}_1 + a\rho + \frac{b\rho}{1 - \rho(a\bar{\phi}_1^2 + c)} < 1. \tag{53}$$

The equilibrium values for $(\bar{\phi}_2, \bar{\theta})$ are given by

$$\begin{pmatrix} \bar{\phi}_2 \\ \bar{\theta} \end{pmatrix} = -\Gamma^{-1}\Psi, \tag{54}$$

with Γ taken to be invertible.

At the fixed point $(\bar{\phi}_1, \bar{\phi}_2, \bar{\theta})$ expectations of agents in group 1 are rational, while those of group two are restricted to be misspecified, but with optimally tuned parameters. We thus call it heterogenous expectations equilibrium.

Definition 2. *A heterogeneous expectations equilibrium (HEE) is a stationary stochastic process for x_t which solves the expectational difference equations (9) and (10) given that (i.) $E^1 = E$, the rational expectations operator, and (ii.) E^2 is formed using a misspecified model such as (23), with θ satisfying the equilibrium condition (1).*

We have just shown that the economy can actually converge locally towards this equilibrium, provided a restriction on the parameters is satisfied:

Proposition 3. *Provided that the $\bar{\phi}_1$ obtained as MSV solution from (49) is real, the economy represented by (2), (3), (31) and (32) will converge locally to the heterogeneous expectations equilibrium $(\bar{\phi}_1, \bar{\phi}_2, \bar{\theta})$ iff condition (53) is satisfied.*

[6] Here convergence is to be considered local, as there is more than one equilibrium point.

We can consider the case of structural homogeneity ($\alpha_1 = \alpha_2 = \alpha$), so as to disentangle the effects on stability due to heterogeneity of expectations alone. Rewrite condition (53) as

$$\alpha\mu\bar{\phi}_1 + \alpha\mu\rho + \frac{\alpha(1-\mu)\rho}{1-\rho(\alpha\mu\bar{\phi}_1^2 + c)} < 1 \tag{55}$$

and remember that

$$\bar{\phi}_1 = \frac{1-\sqrt{1-4\alpha\mu c}}{2\alpha\mu}. \tag{56}$$

We want to compare here the E-stability condition (19), derived for the homogeneous economy, with the condition (55), derived for the economy with heterogeneity of expectations.

Note first that

$$\lim_{\mu\to 1}(55) = (19), \tag{57}$$

that is, as the proportion of agents using the correct model increases, the E-stability condition for the heterogeneous case resembles more and more that for the homogeneous case, and in the limit the two coincide. Moreover

$$\lim_{\mu\to 0}(55) = \left[\frac{\alpha\rho}{1-\rho c} < 1\right], \tag{58}$$

which is the stability condition when all agents are of type 2, as can be seen from (48). Note that in the limit case when $\mu = 0$, the HEE becomes what is known as RPE.

The way in which μ enters into the stability condition (55) is too complicated to be analyzed analytically: we thus fix values for the structural parameters and use numerical calculations in order to understand how heterogeneity affects the E-stability of equilibrium. We plot a lattice on the space (α, c) for different values of μ, fixing ρ at .5 and $d = 0$ (note that this last parameter doesn't affect the equilibrium solution and its properties in terms of stability). Figure 1 shows the case for $\mu = 0$ (all agents use the restricted model), Figure 2 is drawn for $\mu = .5$

(heterogeneity) and Figure 3 for $\mu = 1$ (all agents use the correct model). In each figure, the HEE with the specified μ is compared with the REE: each point on the lattice is marked with a symbol indicating the properties of the two associated equilibria: "x" means that both REE and HEE are E-stable, "o" means that none is E-stable, "+" that only REE is E-stable and "*" means that only HEE is E-stable.

When $\mu = 1$, the two equilibria coincide, so either both are stable or both unstable. In fact Figure 3 shows that both equilibria are E-stable in the lower and left part of the graph, while unstable in the higher right part. As μ decreases below 1 and the HEE becomes different from the REE, it arises a region in which only the HEE is E-stable, and a region where only the REE is E-stable, with both regions increasing in size as μ decreases towards zero and the two equilibria become more and more different from each other. It is interesting to note the region marked with "*": here heterogeneity helps agents' learning and they can converge towards an equilibrium.

These graphs show that the proportion of agents using each model affects the possibility for the HEE to be learned. This equilibrium is E-stable in the regions marked with "x" and "*": as μ goes from 1 to 0 a region that was previously unstable (approximately, α greater than 1 and c positive) becomes progressively more stable, while the region that was previously stable becomes partly unstable.

4.2. Learning through General form representation

4.2.1. Homogeneous case

We start again with the case of homogeneity, both structural and expectational, so that the relevant structural model is given by (9)-(10). All agents employ now a GF representation as their econometric model to be estimated, and use it to make one-step-ahead forecasts. There is a technical problem in this case: if they included the past value of the exogenous process w_{t-1} as forcing variable in their PLM, as it should be in theory, then the current value w_t would show up in the

ALM, and the PLM would turn out to be misspecified (underparameterized). As we want them to have a correctly specified model for now, we allow them to use the current value w_t, so that the PLM and the ALM are consistent with each other;[7] we consider then also the case in which agents include both w_t and w_{t-1}, in which case the coefficient attached to w_{t-1} will turn out to be undetermined.

When agents use only w_t as forcing variable, the PLM is

$$x_t = \gamma_1 x_{t-1} + \gamma_2 x_{t-2} + \gamma_3 w_t + \varepsilon_t. \tag{59}$$

The RE values $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are as in (11) while $\bar{\gamma}_3$ is $-\frac{d}{\alpha\rho}$, i.e., the value in (11) corrected to take into account the fact that the current value w_t is used instead of the lagged value.[8] Constructing the map from the PLM to the ensuing ALM, the relevant Jacobian for E-stability can be found to be

$$\begin{pmatrix} 2\alpha\gamma_1 - 1 & \alpha & 0 \\ \alpha\gamma_2 & \alpha\gamma_1 - 1 & 0 \\ \alpha\gamma_3 & 0 & \alpha\gamma_1 + \alpha\rho - 1 \end{pmatrix} \tag{60}$$

whose eigenvalues are

$$\lambda_1 = \alpha\gamma_1 + \alpha\rho - 1 \tag{61}$$

and

$$\lambda_2, \lambda_3 = \frac{1}{2}\left[(3\alpha\gamma_1 - 2) \pm \sqrt{(3\alpha\gamma_1 - 2)^2 - 4(2\alpha^2\gamma_1^2 - 3\alpha\gamma_1 - \alpha^2\gamma_2 + 1)}\right], \tag{62}$$

to be evaluated at the RE equilibrium point $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$. This gives $\lambda_1 = \alpha\rho$, $\lambda_{2,3} = \frac{1 \pm \sqrt{1-4\alpha c}}{2}$. E-stability requires all three eigenvalues to have negative real part. The first would be negative for $\alpha < 0$ (given the restriction $0 < \rho < 1$),

[7] This is due to the fact that at time t, when agents form expectations for time $t+1$, w_t is assumed to be observable, while w_{t+1} is not.

[8] Of course, there is another fixed point for the system of differential equations, and this is the MSV/REE equilibrium, where γ_2 is zero and γ_1 and γ_3 are the values in (15).

but the second (the one obtained with plus sign) can not have negative real part. Therefore the REE equilibrium is never E-stable under the GF representation.

If instead agents use both w_t and w_{t-1} in their learning scheme, the equilibrium values are $\bar{\gamma}_1 = \frac{1}{\alpha}$, $\bar{\gamma}_2 = -\frac{c}{\alpha}$, $\bar{\gamma}_3 = -\frac{d}{\rho\alpha}$, with $\bar{\gamma}_4$ undetermined, where $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4$ correspond respectively to $x_{t-1}, x_{t-2}, w_t, w_{t-1}$.[9]

The Jacobian relevant for E-stability in this case is

$$\begin{pmatrix} 2\alpha\gamma_1 - 1 & \alpha & 0 & 0 \\ \alpha\gamma_2 & \alpha\gamma_1 - 1 & 0 & 0 \\ \alpha\gamma_3 & 0 & \alpha\gamma_1 + \alpha\rho - 1 & \alpha \\ \alpha\gamma_4 & 0 & 0 & \alpha\gamma_1 - 1 \end{pmatrix} \tag{63}$$

to be evaluated at the point $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4)$. Eigenvalues are $0, \alpha\rho, 1 \pm \sqrt{1 - 4\alpha c}$, and thus the equilibrium again is not E-stable.

Proposition 4. *The fundamental REE of the model (9)-(10) with GF representation is never E-stable.*

Simulations confirm that the GF/REE solution values are not locally stable under learning: even when agents start off with beliefs very close to the equilibrium values, the learning algorithms do not converge towards them.

Moreover, it can be shown that the CF/MSV equilibrium values are *strongly* E-stable in a subregion of the parameter space, i.e., they are locally stable even when agents overparameterize their model using a GF representation. Computing eigenvalues of (63) for the CF/MSV solution, it turns out that this solution is learnable using the GF representation provided that

$$\alpha c < 1 \text{ and} \tag{64}$$

$$\alpha\rho < \frac{1 + \sqrt{1 - 4\alpha c}}{2}. \tag{65}$$

[9] Here, again, the MSV/REE values are also a fixed point of the differential equations, with $\gamma_2 = \gamma_4 = 0$.

Note that the first condition is always satisfied for real solutions, and the second is the same as condition (21).

4.2.2. Heterogeneous case

We now consider the case where part of the agents use a GF representation and the rest underparameterize their model with respect to this representation. Remember that the GF representation for the REE takes the form

$$x_t = \frac{1}{a+b}x_{t-1} - \frac{c}{a+b}x_{t-2} - \frac{d}{a+b}w_{t-1} + \varepsilon_t. \tag{66}$$

Agents with the correctly specified law of motion correctly recognize the need to use both x_{t-1} and x_{t-2}. They also recognize the need to include the exogenous forcing process w. But again, if they were to include only w_{t-1} in their PLM, the ALM would turn out to include also w_t, so the PLM would be misspecified. As we want to have only one group with misspecified PLM, we include also w_t in the PLMs: this ensures that part of the agents can in principle converge towards the REE values. The rest of the agents neglect the twice lagged endogenous component, but correctly recognize the importance of the exogenous variables.

The two PLMs are thus

$$\mathrm{PLM}^1 : x_t = \gamma_1 x_{t-1} + \gamma_2 x_{t-2} + \gamma_3 w_t + \gamma_4 w_{t-1} + \varepsilon_t \tag{67}$$

and

$$\mathrm{PLM}^2 : x_t = \zeta_1 x_{t-1} + \zeta_3 w_t + \zeta_4 w_{t-1} + \varepsilon_t. \tag{68}$$

Deriving the ensuing expectations and plugging them into the structural model we get the ALM. Direct maps can be derived for coefficients from PLM^1 to ALM, which give the differential equations:

$$
\begin{aligned}
\dot{\gamma}_1 &= a(\gamma_1^2 + \gamma_2) + b\zeta_1^2 + c - \gamma_1 && (69) \\
\dot{\gamma}_2 &= a\gamma_1\gamma_2 - \gamma_2 && (70) \\
\dot{\gamma}_3 &= a(\gamma_1\gamma_3 + \rho\gamma_3 + \gamma_4) + b(\zeta_1\zeta_3 + \rho\zeta_3 + \zeta_4) + d - \gamma_3 && (71) \\
\dot{\gamma}_4 &= a\gamma_1\gamma_4 + b\zeta_1\zeta_4 - \gamma_4. && (72)
\end{aligned}
$$

Note that these ODEs include also the parameters of the PLM^2, so the solution values for the two groups must be found together. But a direct map from PLM^2 to ALM can not be derived: instead, we must project the ALM onto the space of the PLM^2. By projecting we get the following differential equations governing the limiting dynamics for the parameters in the PLM^2:

$$
\begin{aligned}
\dot{\zeta}_1 &= k_1 + k_2[(R^{-1})^{11}\delta\sigma_w^2 + (R^{-1})^{12}\rho^2\psi\sigma_w^2 + (R^{-1})^{13}\rho\psi\sigma_w^2] - \zeta_1 && (73) \\
\dot{\zeta}_3 &= k_3 + k_2[(R^{-1})^{21}\delta\sigma_w^2 + (R^{-1})^{22}\rho^2\psi\sigma_w^2 + (R^{-1})^{23}\rho\psi\sigma_w^2] - \zeta_3 && (74) \\
\dot{\zeta}_4 &= k_4 + k_2[(R^{-1})^{31}\delta\sigma_w^2 + (R^{-1})^{32}\rho^2\psi\sigma_w^2 + (R^{-1})^{33}\rho\psi\sigma_w^2] - \zeta_4. && (75)
\end{aligned}
$$

The Appendix gives the expressions for the new parameters introduced here. Note that these ODEs are the same as (69), (71), (72) plus a correction term for the missing variable x_{t-2}. This component prevents us from finding analytical solution to the above system of seven ODEs, as the terms $k's$, δ and ψ and the matrix R all depend in a highly non linear way on the parameters being learned.

Since we can not solve this system analytically, we simulate the RLS algorithms for the two groups of agents together. We find that the heterogeneous learning processes have a locally stable equilibrium at the CF/MSV solution values. Both groups of agents are using a model that is overparameterized with respect to the CF representation but over time, as they learn from data, the parameters in their models converge to those in the CF/MSV solution, and they discard the extra variables in their model. This result extends the strong E-stability property found in the homogeneous case to an heterogeneous setting.

We also investigate through simulations the range of parameter configurations $(a,\ b)$ for which this convergence obtains, and results are shown in Figure 4:[10] stars indicate stability, circles instability.[11] We can see that for small values of a and b the equilibrium is learnable, but as these values increase, an unstable region emerges. Remember that in the homogeneous case, while the GF representation of the equilibrium was never learnable, the CF solution was learnable when condition (65) was satisfied: Figure 5 shows the stability/instability regions for this condition in the (a, b) space.[12] If we compare this to Figure 4, we can see that when we introduce heterogeneity the stability region shrinks: in particular, for $a > 0.8$ there is no value of b that can ensure convergence of the learning schemes.

4.3. General form and Common factor representations together

The last case we consider is slightly different from the others: heterogeneity now does not arise because of some of the agents neglecting some of the relevant variables but because of a lack of coordination among agents. In fact, when more than one model can represent well the equilibrium solution *in* equilibrium, as we have seen to be the case for the two representations of the REE, heterogeneity can arise because agents do not agree about which model to use in forming their expectations. Without coordination, it is possible that part of the agents will use one model and part the other. In this case, those agents that are using the model that includes a larger set of variables make the other model be underparameterized. We thus use the analysis of the previous section to study the situation in which one group of agents uses the GF representation (group 1) while the other group uses the CF representation (group 2). With respect to the situation presented in the previous section, the only difference is that ζ_4 in (68) is now zero.

For agents using the GF representation, the ODEs (69)-(70) must be modified

[10] The other structural parameter values are fixed at $c = .3$, $d = .5$, $\rho = .7$

[11] In the unstable regions we have included also the regions where the REE parameters are complex numbers.

[12] Here what really matters is the *sum* of a and b, which determines α. The rest of the parameters are set as before: $c = .3$, $d = .5$, $\rho = .7$.

to account for the fact that $\zeta_4 \equiv 0$. For the other group instead, we have to project the ALM onto the smaller space of the PLM^2. The matrix R is now the 2 by 2 matrix

$$R = E(x_{t-1}\ w_t)'(x_{t-1}\ w_t) = \begin{pmatrix} Ex_t x_{t-1} & \sigma_{wx_{-1}} \\ \sigma_{wx_{-1}} & \sigma_w^2 \end{pmatrix}, \tag{76}$$

where all the components have to be modified with respect to the previous case by setting $\zeta_4 \equiv 0$.

The differential equations for ζ_1 and ζ_3 become

$$\dot{\zeta}_1 = k_1 + k_2(R^{-1}\Pi)^{11} + k_4(R^{-1}\Pi)^{12} - \zeta_1 \tag{77}$$

$$\dot{\zeta}_3 = k_3 + k_2(R^{-1}\Pi)^{21} + k_4(R^{-1}\Pi)^{22} - \zeta_3, \tag{78}$$

where

$$\Pi = \sigma_w^2 = \begin{pmatrix} \delta & \psi \\ \rho^2\psi & \rho\psi \end{pmatrix}. \tag{79}$$

These ODEs are similar to those found before in (73)-(75), except that now we have two correction terms, one for each of the two variables (x_{t-2} and w_{t-1}) appearing in the GF but not in the CF representation. Again, we can not find an analytical solution, and we use simulations to investigate convergence of these learning schemes.

Figures 6 and 7 show the evolution of the estimates for the parameters in the GF and in the CF representation, respectively. Figures are drawn for the parameter configuration: $a = .2$, $b = .3$, $c = .3$, $d = .5$, $\rho = .7$ (with structural homogeneity this choice for a and b corresponds to $\alpha = .5$, $\mu = .4$). The estimates tend to stabilize at fixed values, which are the CF/MSV solution values for the homogeneous case. Parameters in PLM^1 and PLM^2 converge to each other and to those in ALM: the economy reaches an equilibrium. As before, we then investigate the parameter space for which convergence obtains: results are shown in Figure 8, which can be seen to be similar to Figure 4.

It is interesting to note that while a CF representation is correctly specified if all agents use that representation, or even if some of the agents underparameterize their PLM by using a smaller set of variables, the same CF representation becomes underparameterized when some of the agents use a GF representation, which includes a larger set of variables. But once the equilibrium has been reached, the CF representation is again correctly specified. This can be seen by computing the mean square errors for forecasts using the two models: on average, errors are smaller with the PLM^1 on the transition path, while they become the same once convergence has been reached. This result shows that the fact that some agents are overparameterizing their learning model can worsen the predictions of the others, by making extra variables relevant for the dynamics of the economy.

5. Conclusions

In this paper we have considered an univariate forward looking stochastic model under adaptive heterogeneous learning. Heterogeneity has been introduced by forcing agents to use different forecasting models, an assumption that could be justified by the fact that agents, when depicted as adaptive learners, face the problem of selecting which set of variables to use in their model. We did not analyze how this choice is made, but we considered what happens *given* the choice made by agents. In particular, we wanted to see whether this choice, though possibly not optimal, could lead to an equilibrium situation in which agents do not change their model; in other words, we wanted to see whether the heterogeneous learning algorithms can converge to a fixed point.

The specific form of the learning algorithm used is RLS, and where possible we have applied the E-stability principle that establishes a connection between learnability through such an algorithm and the E-stability property. When not possible, we have resorted to stochastic simulations of the learning algorithms.

Heterogeneity has been introduced by underparameterizing the model used by some agents, and we have considered two different representations of the equilibrium, the CF and the GF representation.

The two homogeneous cases could be tackled analytically and the E-stability conditions were found; the same was then done for the simpler heterogeneous model, the one derived with the CF representation. In the other two heterogeneous cases we could not derive an analytical solution and stochastic simulations were employed to gain some insights about the properties of the learning processes.

In the first case, where all agents use a CF representation, a restriction on the structural parameters defines the E-stability region: for parameter values inside this region, agents can learn over time and their estimates converge towards the CF/MSV solution values. Once heterogeneity is introduced in this system, the equilibrium values for the parameters in the forecasting models change, but still an equilibrium can be reached as estimates converge to a fixed point in which the forecasting errors of each group are uncorrelated with the variables in their own information set. We have called this equilibrium Heterogeneous Expectations Equilibrium, in which part of the agents are rational, in the sense that variables and parameter values in their PLM are the same as those in the ALM, and the rest of the agents are using an underparameterized model, but with parameters that are optimally tuned to those of the ALM. We have provided conditions for this HEE to be learnable by least squares techniques, and in particular we have shown the impact on learnability of the proportion of agents using each model.

In the second homogeneous case, where all agents use a GF representation, the system has a locally stable equilibrium (the CF/MSV solution values) and a locally unstable one (the GF solution values): agents can not learn the GF representation of the equilibrium, but if their starting beliefs are not too far off and the strong E-stability restriction on the structural parameters holds then they can learn the CF/MSV solution. When heterogeneity is introduced in this setting, a locally stable equilibrium is still present, and this corresponds to the CF/MSV solution of the homogeneous case. This happens because the CF representation is embedded in both models used by agents, and through learning agents are able to identify the unnecessary variables and disregard them. Of course, through simulations we could not check for the existence of locally unstable equilibria.[13]

The last case is the one in which both CF and GF representations are used by different groups of agents. As the REE can be represented equally well in both ways, agents can use either model in their learning processes, but though these models are equally good *in* equilibrium, they are not equivalent out of equilibrium, when agents are learning. Agents using the model with a smaller information set are actually underparameterizing their model on the learning path. For a subregion of the parameter space considered, the learning process of both groups of agents can converge: again this locally stable solution is the CF/MSV solution.

To summarize, four messages seem to emerge from this work: first, the MSV/CF solution is quite robust under learning, even when heterogeneity is introduced, as it remains the unique locally stable point for the learning algorithms when agents overparameterize; second, the GF solution representation is not learnable, and heterogeneity does not change the result; third, when agents underparameterize their model with respect to the CF representation a different equilibrium emerges, the HEE, and this equilibrium can be learned by agents; and last, if some agents overparameterize their model, they can make an otherwise correctly specified model be misspecified over the learning path.

6. Appendix

In order to project the ALM onto the PLM[2] we need the variance-covariance matrix for the regressors:

$$R = E(x_{t-1}\ w_t\ w_{t-1})'(x_{t-1}\ w_t\ w_{t-1}) = \begin{pmatrix} Ex_t x_{t-1} & \sigma_{wx_{-1}} & \sigma_{wx} \\ \sigma_{wx_{-1}} & \sigma_w^2 & \rho\sigma_w^2 \\ \sigma_{wx} & \rho\sigma_w^2 & \sigma_w^2 \end{pmatrix} \tag{80}$$

[13] By looking at equations (69)-(72) and (73)-(75) we can see that more than one solution, possibly non-real, should exist. These would be unstable equilibrium points of the learning algorithms.

where

$$
\begin{aligned}
Ex_t x_{t-1} &= \delta \sigma_w^2 && (81)\\
\sigma_{wx} &= \psi \sigma_w^2 && (82)\\
\sigma_{wx_{-1}} &= \rho \psi \sigma_w^2 && (83)\\
\sigma_{wx-2} &= \rho^2 \psi \sigma_w^2 && (84)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi &= \frac{k_3 + \rho k_4}{1 - \rho k_1 - \rho^2 k_2} && (85)\\
\delta &= f(k_1, k_2, k_3, k_4, \rho), && (86)
\end{aligned}
$$

with f a non linear function of its variables, and

$$
\begin{aligned}
k_1 &= a(\gamma_1^2 + \gamma_2) + b\zeta_1^2 + c && (87)\\
k_2 &= a\gamma_1\gamma_2 && (88)\\
k_3 &= a(\gamma_1\gamma_3 + \rho\gamma_3 + \gamma_4) + b(\zeta_1\zeta_3 + \rho\zeta_3 + \zeta_4) + d && (89)\\
k_4 &= a\gamma_1\gamma_4 + b\zeta_1\zeta_4. && (90)
\end{aligned}
$$

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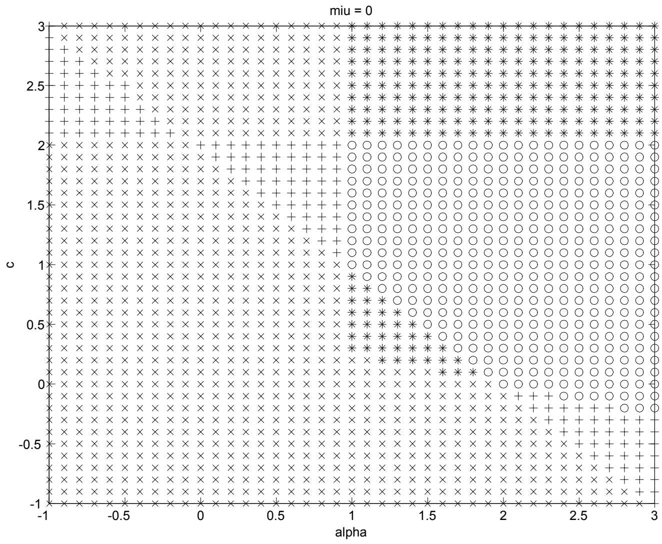

Figure 1: E-stability properties of HEE and REE; CF representation; $\mu = 0$.

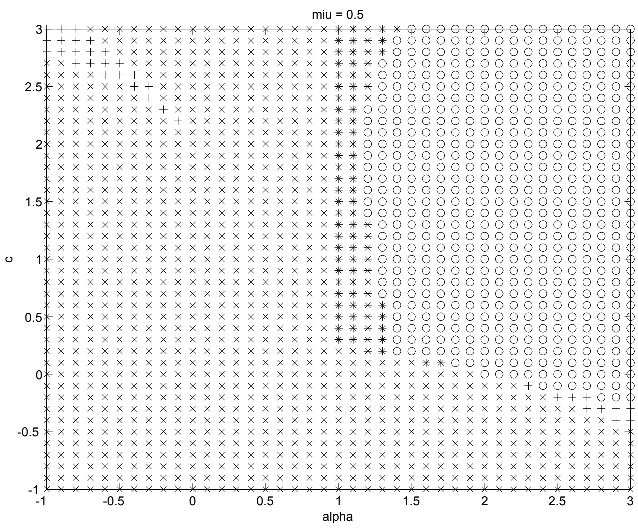

Figure 2: E-stability properties of HEE and REE; CF representation; $\mu = .5$.

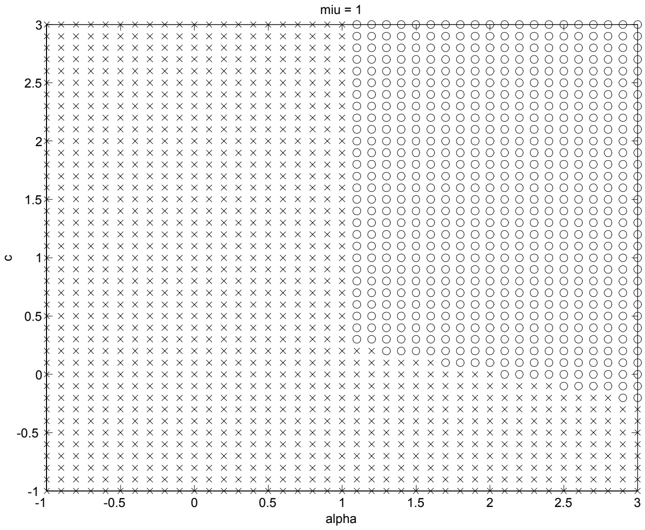

Figure 3: E-stability properties of HEE and REE; CF representation; $\mu = 1$.

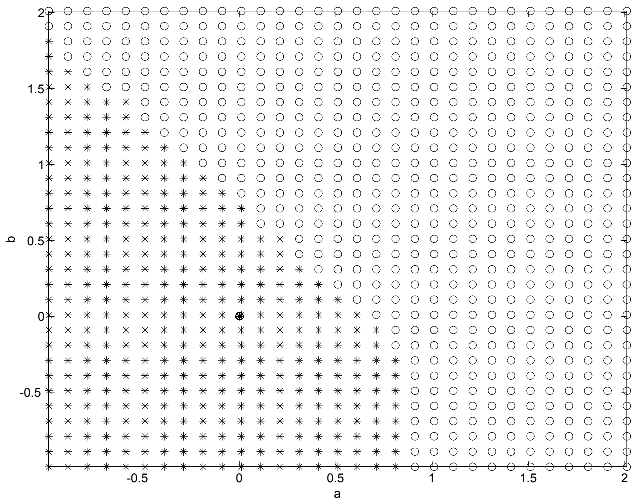

Figure 4: Stability region (*) for the CF solution when agents use a GF representation with misspecifications.

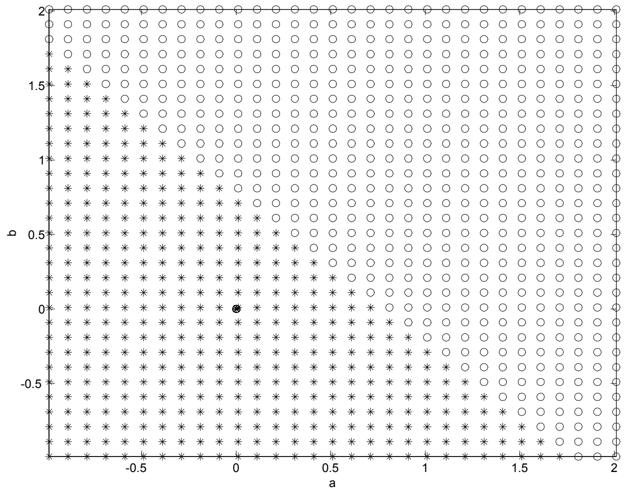

Figure 5: Strong E-stability condition for the CF/MSV solution. Stars represent stable region, circles unstable region.

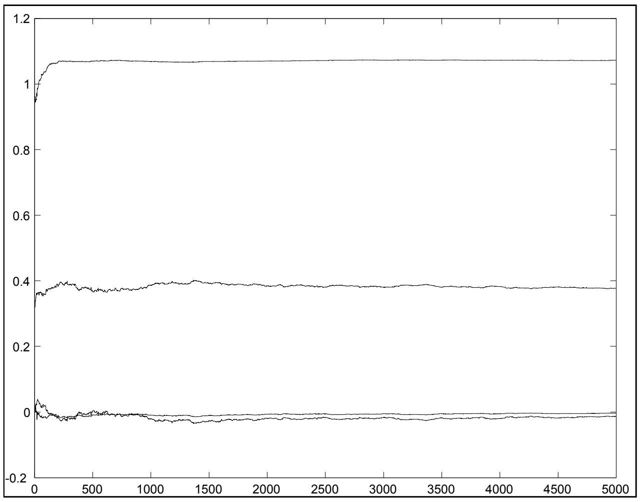

Figure 6: HEE: convergence of parameters in GF representation. From top to bottom, lines represent the evolution of estimates for $\gamma_{3,}\gamma_1, \gamma_2, \gamma_4$.

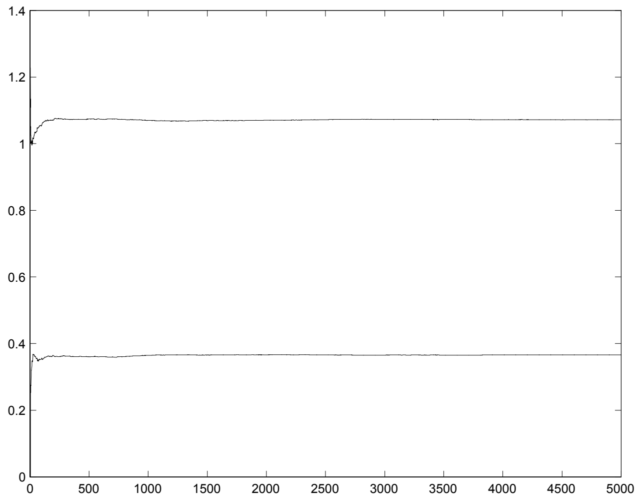

Figure 7: HEE convergence of parameters in CF representation. From top to bottom, lines represent the evolution of estimates for ζ_3 and $\zeta_{1.}$

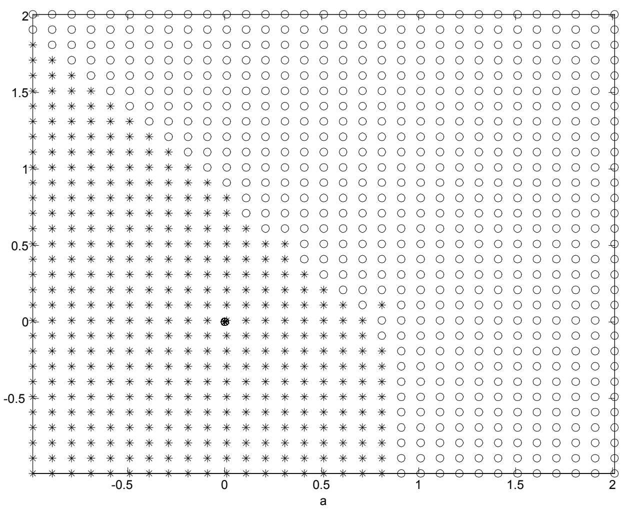

Figure 8: Stable (stars) and unstable (circles) regions in the (a,b) space for the HEE with CF and GF representations.