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Running Title: Market Games and the Core

Abstract: In view of the well known core equivalence results in atomless economies, coincidence of market game equilibrium allocations with competitive allocations is tantamount to a three way equivalence between market game mechanisms, competitive equilibria and the core. Based on this idea we study equilibrium refinements of market games, which allow us to use the core equivalence machinery in order to provide an exact market game characterization of competitive equilibria.

Keywords: market games, core, competition.

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1 Introduction

Perfectly competitive markets are viewed as mass markets with small participants. In such markets individual activities are inconsequential in the determination of prices and allocation of commodities. For this reason it is reasonable to expect that individuals exhibit a price taking behavior in such environments. However, the central position of this 'price taking' hypothesis, calls for some formal proof of this last point. The theories of the core and of strategic market games, which are modern formalizations of the ideas of Edgeworth and Cournot respectively, have been both employed to this end. By and large the theory of competition based on the core is thought of as 'cooperative', while the one based on market games is thought of as 'non cooperative' foundation of competition, though this terminology has been (directly or indirectly) contested in numerous cases.

One obvious implication of the equivalence theorems via strategic market games or the core is that in large economies the set of core outcomes are Nash equilibrium outcomes of a strategic market game. This conclusion suggests that 'cooperation' is a matter of interpretation rather than an accurate distinction of the two approaches. In fact, this three way association between the set of Walrasian, Nash equilibrium and core allocations is the starting point in this paper. Our objective here is to fuse the two approaches together and produce equilibrium concepts with both Cournotian and Edgeworthean features. We are motivated in this by the fact that those two approaches to competition are so intimately related in terms of their outcomes in large economies.

There are several conceptual reasons for pursuing this matter. We will focus on one that we consider very important. The core has often been criticized for completely lacking a description of the trading process. By contrast, strategic market games feature quite explicit descriptions of the distribution mechanism. On the other hand the core features coordination among individuals, which is entirely absent from market games. The appearance of Nash equilibria where some (or all) markets are inactive can be viewed as a consequence of this lack of coordination. Hence, both worlds could benefit from a marriage: the core could acquire a more descriptive nature and the market game could be more coordinated to avoid 'absurd' outcomes. Our results show that a successful marriage of the two approaches is possible and provides quite sensible results.

In this paper we introduce a hybrid equilibrium notion that blends together Edgeworthean and Cournotian elements. Briefly, we define the core of an economy where trades among any group of individuals are conducted via the Shapley-Shubik mechanism (see [12]). In other words we endow the economy with a specific institutional arrangement (notably the strategic market game structure) via which commodities can be traded and define a kind of 'constrained' core notion, where only allocations of commodities attainable through the institutional norms can be considered. It turns out that in an atomless economy the allocations resulting from such an equilibrium notion are precisely the core allocations and therefore competitive. The intuition behind this result is clear: in view of the well known core equivalence results in atomless exchange economies [1], the equivalence of market game mechanisms and the core is *necessary* in order to obtain exact equivalence with the competitive mechanism. In other words, equilibrium allocations of the market game mechanism, which are bounded away from the core, cannot be competitive equilibria.

The equivalence between the core and the notion we outlined above suggests that the allocations which are blocked via trades through the market game mechanism, are identical to those which are blocked when arbitrary trades are allowed. In view of this fact the important message of our results is that no more trade flexibility than that allowed by the strategic market game is necessary, in order to characterize the core or competitive allocations.

To the best of our knowledge, equilibrium notions based on coalitions in the context of market games have never been studied before. Coalitional structures in market games have appeared in [3] but that paper focuses on the structure of trading groups and does not address properties of the resulting allocations. Another related paper in the literature is [11], where the authors develop the 'Walras-core', which is also a 'constrained' core notion where all trades are restricted to those attainable via some Walrasian price system. In the sequel we will relate the Walras core to our notion.

In the section that follows we develop the model and several equilibrium concepts.

[1] See [2] for a comprehensive review of core equivalence results.

Next we proceed to prove some equivalence results. An interpretation of the model and some discussion as well as possible extensions are discussed in section four.

2 The Economy

Let $(A, \mathcal{A}, \mu)$ be a measure space of agents, where A is a complete separable metric space and μ is a Borel regular measure on A. There are L commodity types in the economy and the consumption set of each agent is identified with $\Re^L_+$. An individual is characterized by a preference relation, which is representable by a utility function $u_a : \Re^L_+ \rightarrow \Re$, and an initial endowment $e(a) \in \Re^L_+$. In order to be able to use some standard results we impose the following assumptions:

Assumption 1 *$e(a) >> 0$ ae.*

Assumption 2 *Preferences are continuous, strictly monotonic and indifference surfaces passing through the endowment do not intersect the axis.*

Let $\mathcal{P}$ denote the set of utility functions satisfying the above assumption endowed with the appropriate topology (see [7]). An *economy* is a mapping $\mathcal{E} : A \rightarrow \mathcal{P} \times \Re^L_+$ which is measurable. The standard definitions of a competitive equilibrium and the core for such an economy are as follows:

Definition 1 *A competitive equilibrium is a price system $p \in \Re^L_+$ and a measurable assignment $x : A \rightarrow \Re^L_+$ such that:*

(i) $\int_A x\,(a) \leq \int_A e\,(a)$

(ii) $x\,(a) \in \; argmax\,\{u_a\,(y) : p \cdot y \leq p \cdot e(a)\}$ *ae in* A

Definition 2 *The core is the set of all measurable assignments $x : A \rightarrow \Re^L_+$ such that:*

(i) $\int_A x\,(a) \leq \int_A e\,(a)$

(ii) $\nexists T \in \mathcal{A}$ *where* $\mu\,(T) > 0$ *and* $y : T \rightarrow \Re^L_+$ *such that*

$\int_T y(a) = \int_T e(a)$ *and* $u_a(y(a)) > u_a(x(a))$ *ae in* T.

Let $\mathcal{W}\,(A)$ and $\mathcal{C}\,(A)$, denote respectively the set of competitive equilibria and the core assignments for this economy. A celebrated result, which originally appeared in [1], asserts that under suitable conditions the above two sets coincide. Since the conditions for that result are satisfied in the model presented here we state this here as a fact for future reference.

Fact 1 $\mathcal{W}(A) = \mathcal{C}(A)$.

We now turn to develop an alternative description of the trading process based on market games.

2.1 The Market Game

Trade is organized via systems of trading posts where individuals place orders for sales and purchases of commodities. A possible scenario for the rules of exchange is presented below. The interested reader should consult [9] or [10] for a detailed account of this formulation.

In this setup purchasing orders are placed in terms of a unit of account and sale orders in terms of physical commodities. The action sets of agents are described by a measurable correspondence $S : A \longrightarrow 2^{\Re^L_+ \times \Re^L_+}$, where

$$S(a) = \left\{(b, q) \in \Re^L_+ \times \Re^L_+ : q^i \le e^i(a),\ i = 1, 2, \dots, L\right\}.$$

A *strategy profile* is a pair of measurable mappings $b : A \longrightarrow \Re^L_+$, $q : A \longrightarrow \Re^L_+$ such that $(b(a), q(a)) \in S(a)$ ae in A, i.e., a strategy profile is a measurable selection from the graph of the correspondence S, which we denote by $Gr(S)$. It is easily seen that $S : A \longrightarrow 2^{\Re^{2L}_+}$ has a measurable graph so such measurable mappings exist by Aumann's measurable selection theorem.

For a given strategy profile $(b, q) \in Gr(S)$, let $B^i = \int_{a\in A} b^i(a)$, where it is understood that $B^i = \infty$ whenever the integral is not defined, and $Q^i = \int_{a\in A} q^i(a)$.

Consumption assignments, for $i = 1, 2, \dots, L$, are determined as follows:

$$x^i_a(b(a), q(a), B, Q) = \begin{cases} e^i(a) - q^i(a) + b^i(a)\frac{Q^i}{B^i} & \text{if } \sum_{i=1}^L \frac{B^i}{Q^i} q^i(a) \ge \sum_{i=1}^L b^i(a) \\ e^i(a) - q^i(a) & \text{otherwise} \end{cases}$$

where divisions over zero are taken to equal zero. The interpretation of this allocation rule is that commodities are distributed to non bankrupt individuals in proportion to their bids, while the purchases of bankrupt individuals are confiscated. Notice that when $B^iQ^i \neq 0$, the vector defined as $\pi(b,q) = \left(\frac{B^i}{Q^i}\right)_{i=1}^{L}$ has a natural interpretation as a 'price vector'. Furthermore, since we can always normalize individual bids, in what follows we may assume without loss of generality that $\pi(b,q) \in \Delta^L$.

Given a profile $(b,q) \in Gr(S)$ consumers are viewed as solving the following problem:

$$\max_{(\hat{b},\hat{q})\in S(a)} u_a(x_a(\hat{b},\hat{q},B,Q)) \tag{1}$$

In this way we have a game in normal form that describes trade in this economy. We define below the standard pure strategy Nash equilibrium notion for this game.

DEFINITION 3 *A strategy profile $(b,q) \in Gr(S)$ is a Nash equilibrium of the market game, iff:*

$u_a(x_a(b(a),q(a),B,Q)) \geq u_a(x_a(\hat{b},\hat{q},B,Q))$, $\forall(\hat{b},\hat{q}) \in S(a)$ *ae in* A.

Let $\mathbf{N}(A)$ denote the set of Nash equilibrium strategy profiles and $\mathcal{N}(A)$ the set of allocations which correspond to elements of $\mathbf{N}(A)$.

Notice that, due to the bankruptcy rule above, at a Nash equilibrium with positive bids and offers individuals can be viewed as solving the following problem:

$$\begin{aligned}&\max_{(\hat{b},\hat{q})\in S(a)} u_a(x_a(\hat{b},\hat{q},B,Q))\\ &s.t.\ \pi(b,q)\cdot\hat{q} \geq \textstyle\sum_{i=1}^{L}\hat{b}^i\end{aligned} \tag{2}$$

We now abandon the non cooperative framework and allow individuals to coordinate their activities in a market game. In this way we can add an Edgeworthean flavor in the Cournotian setup of the market game. Our purpose is to capture the idea that any group of agents can apply the mechanism described above to exchange among themselves, if they find it profitable to do so. In other words the distribution mechanism allows for exclusion. This is the idea behind the following definition, which states that a given profile is an equilibrium if no group of agents could derive a potential benefit from using the market game mechanism exclusively for its members.

Definition 4 *A strategy profile $(b,q) \in Gr(S)$ is in the Shapley-Shubik core ($SS-core$) of the market game with market exclusion, iff:*

$\nexists\ T \in \mathcal{A}$, $\mu(T) > 0$ *and* $(\hat{b},\hat{q}) \in Gr(S)$, *where* $(\hat{b}(a),\hat{q}(a)) = (0,0)$ *ae in* $A \backslash T$, *s.t.* $u_a(x_a(\hat{b}(a),\hat{q}(a),\hat{B},\hat{Q})) > u_a\left(x_a(b(a),q(a),B,Q)\right)$ *ae in* T.

Let $\mathbf{C}_e^{ss}(A)$ denote the set of $SS-core$ strategies and $\mathcal{C}_e^{ss}(A)$ the set of allocations which correspond to the elements of $\mathbf{C}_e^{ss}(A)$.

The key premise of the above definition is that any group of individuals can set up trading posts and trade among themselves. In other words we replace the hypothesis of a single trading post where all bids and offers are placed, with the hypothesis that any group of traders can replicate the mechanism by which prices and allocations are calculated.[2] In that case the definition of equilibrium must require that the *structure* of trading posts be in equilibrium in the sense that no group of individuals should have an incentive to set up further trading posts.

A well known property of the Shapley-Shubik market game mechanism is that individual strategies can be altered in a way so that prices, budgets and allocations remain the same. The following fact records this property.

Fact 2 *Given any $(b,q) \in Gr(S)$, all strategy profiles $(\hat{b},\hat{q}) \in Gr(S)$, which satisfy $\hat{b}(a) = \left(b^i(a) + \pi^i(b,q)\left(\hat{q}^i(a) - q^i(a)\right)\right)_{i=1}^{L}$ ae in A, give rise to the same prices, budgets and allocations for each $a \in A$.*

In view of the above fact one can fix the offers of individuals at the endowment level and describe $SS-core$ strategy profiles via a corresponding profile of bids. Note that in the $SS-core$, strategy profiles are relevant only to the effect that they give rise to certain allocations. Therefore, the altered strategy profile according to the fact above, is in the $SS-core$ if and only if the original strategy profile is in the $SS-core$.[3] In conclusion every allocation $x \in \mathcal{C}_e^{ss}(A)$ has a 'standard' representation $(b,e) \in \mathbf{C}_e^{ss}(A)$.

[2] In [6] it is demonstrated that in a finite economy the structure of trading posts is not inconsequential for the set of Nash equilibria. However, the structure of trading posts is exogenous there and is not associated with groups of individuals.

[3] Recall that this is not true in general for Nash equilibria.

We now proceed to relate the $SS - core$ to the core, the competitive and Nash equilibria of the atomless economy.

3 Results

We begin with some results relating the Core and the $SS - core$ of the economy.

PROPOSITION 1 $\mathcal{C}(A) \subset \mathcal{C}_e^{ss}(A)$.

Proof: Let $x \in \mathcal{C}(A)$. By fact (1) we have that $x \in \mathcal{W}(A)$, so there is $p \in \Re^L_{++}$ so that $p \cdot x(a) \leq p \cdot e(a)$ ae in A. Define the strategy profile $(b, q) : A \longrightarrow \Re^{2L}_{+}$ as follows: $(b(a), q(a)) = \left((p^i x^i(a), e^i(a))_{i=1}^L\right)$. Clearly, b and q are measurable and by construction $(b(a), q(a)) \in S(a)$, ae in A so $(b, q) \in Gr(S)$. Notice that $\pi(b, q) = p$.

For this strategy profile we have that ae in A:

$$\pi(b, q) \cdot q(a) = p \cdot e(a) = p \cdot x(a) = \sum_{i=1}^{L} b^i(a)$$

Therefore, from the allocation rule we deduce that:

$$x_a(b(a), q(a), B, Q) = \left(\frac{b^i(a)}{p^i}\right)_{i=1}^{L} = x(a)$$

ae in A, i.e., the strategy profile (b, q) implements via the market game, the core assignment x.

Suppose that $x \notin \mathcal{C}_e^{ss}(A)$. Then $\exists T \in \mathcal{A}$, $\mu(T) > 0$ and $(\hat{b}, \hat{q}) \in Gr(S)$, where $(\hat{b}, \hat{q}) = (0, 0)$ for all $a \notin T$, so that the corresponding assignment is such that:

$$\begin{aligned} u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q})) &> u_a\left(x_a(b(a), q(a), B, Q)\right) \\ &= u_a\left(x(a)\right) \ ae \text{ in } T. \end{aligned} \tag{3}$$

From the definition of the allocation rule it follows that:

$$\int_T x_a(\hat{b}, \hat{q}, \hat{B}, \hat{Q})) = \left(\int_T e^i(a) + \frac{\hat{Q}^i}{\hat{B}^i} \int_T \hat{b}^i(a) - \int_T \hat{q}^i(a)\right)_{i=1}^{L}$$

$$= \int_T e(a) \tag{4}$$

But then (3) and (4) imply that $x \notin \mathcal{C}(A)$, which is a contradiction. Therefore, it must be $x \in \mathcal{C}_e^{ss}(A)$. $\square$

The next result shows that the reverse inclusion is also true.

PROPOSITION 2 $\mathcal{C}_e^{ss}(A) \subset \mathcal{C}(A)$.

Proof: Let $x : A \to \Re_+^L$ be a measurable assignment such that $x \notin \mathcal{C}(A)$. We will show that $x \notin \mathcal{C}_e^{ss}(A)$.

If $\int_A x(a) \nleq \int_A e(a)$ then clearly $x \notin \mathcal{C}_e^{ss}(A)$.

If $\int_A x(a) \leq \int_A e(a)$ then $x \notin \mathcal{C}(A)$ implies that: $\exists\, T \in \mathcal{A}$, $\mu(T) > 0$ and $y : T \to \Re_+^L$ s.t. $\int_T y(a) = \int_T e(a)$ and $u_a(y(a)) > u_a(x(a))$ *ae* in T.

For each $p \in \Delta^L$ define $(b_p, q) : A \to \Re_+^{2L}$ as: $(b_p(a), q(a)) = \left((p^i e^i(a), e^i(a))_{i=1}^L\right)$. Let $B_p = \int_A b_p(a)$, $Q = \int_A q(a)$ and note that $\pi(b_p, q) = p$.

For each $a \in T$ define now the correspondence:

$$b(a;p) = argmax\left\{u_a(x_a(\hat{b}, q(a), B_p, Q)) : \pi(b_p, q)\, q(a) \geq \sum_{i=1}^{L} \hat{b}^i\right\}$$

This correspondence assigns to each $a \in T$ the 'best response' bids to (b_p, q), when offers are fixed at the endowment level. In view of assumptions (1), (2), $b(a;\cdot)$ is nonempty, *uhc* and $\hat{b} > 0$ for all $\hat{b} \in b(a;\cdot)$ *ae* in T. Moreover, $b(\cdot;p)$ has a measurable graph (see [13]).

Let

$$T(p) = \left\{a \in T : u_a(x_a(\hat{b}, q(a), B_p, Q)) > u_a(y(a)) \text{ for some } \hat{b} \in b(a;p)\right\}$$

We distinguish the following two cases:

Case I: $\exists \bar{p} \in \Delta^L$ so that $\mu(T(\bar{p})) = 0$

i.e., $\exists \hat{b}(a) \in b(a;\bar{p})$ so that $u_a(y(a)) \geq u_a(x_a(\hat{b}(a), q(a), B_{\bar{p}}, Q))$ *ae* in T.

In this case consider the strategy profile defined as follows:

$$\left(\bar{b}(a), q(a)\right) = \begin{cases} \left((\bar{p}^i y^i(a), e^i(a))_{i=1}^L\right) & \text{if } a \in T \\ (0,0) & \text{otherwise} \end{cases} \tag{5}$$

Notice that $\pi\left(\bar{b}, q\right) = \bar{p} = \pi\left(b_{\bar{p}}, q\right)$.

Suppose that for some $a \in T$, $\sum_{i=1}^L \bar{b}^i(a) < \pi\left(b_{\bar{p}}, q\right) q(a)$.

Then since $x_a(\bar{b}(a), q(a), B_{\bar{p}}, Q) = y(a)$ by the hypothesis of this case we would have $u_a(x_a(\bar{b}(a), q(a), B_{\bar{p}}, Q)) = u_a\left(y(a)\right) \geq u_a(x_a(\hat{b}(a), q(a), B_{\bar{p}}, Q))$, which contradicts $\hat{b}(a) \in b(a;\bar{p})$.

So it must be $\sum_{i=1}^L \bar{b}^i(a) \geq \pi\left(b_{\bar{p}}, q\right) q(a)$ *ae* in T. But if for $a \in T' \subset T$ with $\mu(T') > 0$, we had $\sum_{i=1}^L \bar{b}^i(a) > \pi\left(b_{\bar{p}}, q\right) q(a)$ then

$$\sum_{i=1}^L \int_T \bar{b}^i(a) > \pi\left(b_{\bar{p}}, q\right) \int_T q(a) = \bar{p} \int_T e(a) = \bar{p} \int_T y(a) = \sum_{i=1}^L \int_T \bar{b}^i(a)$$

which is a contradiction. It follows that

$$\begin{aligned} \sum_{i=1}^L \bar{b}^i(a) &= \pi\left(b_{\bar{p}}, q\right) q(a) \\ &= \pi\left(\bar{b}, q\right) q(a) \end{aligned}$$

ae in T. Therefore, the coalition T along with the strategy profile defined by (5) implies $x \notin \mathcal{C}_e^{ss}(A)$.

Case II: $\forall p \in \Delta^L \; \mu(T(p)) > 0$.

For each $a \in T$ consider the correspondence defined as follows:

$$\bar{b}(a;p) = \begin{cases} b(a;p) & \text{if } u_a\left(x_a(b(a;p), q(a), B_p, Q)\right) > u_a(y(a)) \\ b(a;p) \cup \{b_p(a)\} & \text{if } u_a\left(x_a(b(a;p), q(a), B_p, Q)\right) = u_a(y(a)) \\ \{b_p(a)\} & \text{if } u_a\left(x_a(b(a;p), q(a), B_p, Q)\right) < u_a(y(a)) \end{cases} \tag{6}$$

This correspondence is nonempty and *uhc*.

Consider now the correspondence $\bar{B}(p) = \int_T \bar{b}(a;p)$. Since this correspondence is also nonempty, *uhc* and by Lyapunov's theorem convex valued, so is the correspondence

$\varphi : \Delta^L \longrightarrow 2^{\Delta^L}$ defined as $\varphi(p) = \pi(\bar{b}(p), q)$.

It follows that φ has a fixed point: $\bar{p} \in \pi(\bar{b}(\bar{p}), q)$, i.e., there is a measurable function $\bar{b} : T \longrightarrow \Re^L_+$ where $\bar{b}(a) \in \bar{b}(a; \bar{p})$ *ae* in T so that $\bar{p} = \pi(\bar{b}, q)$. Consider now the set

$$U = \Big\{a \in T : u_a(x_a(\bar{b}(a), q(a), B_{\bar{p}}, Q)) \geq u_a(y(a)) \text{ and } \bar{b}(a; \bar{p}) = b(a; \bar{p})\Big\}$$

Since $T(\bar{p}) \subset U$ we have $\mu(U) > 0$. Define now the following strategy profile:

$$(b(a), q(a)) = \begin{cases} \left(\bar{b}(a), q(a)\right) & \text{if } a \in U \\ (0, 0) & \text{otherwise} \end{cases} \tag{7}$$

The coalition U along with the strategy profile $(b(a), q(a))$ defined above will do for our purpose. The key observation is that for each $i = 1, 2, \ldots, L$ we have:

$$\begin{aligned} \bar{p}^i &= \pi^i(\bar{b}, q) \\ &= \frac{\int_T \bar{b}^i(a)}{\int_T q^i(a)} \\ &= \frac{\int_U \bar{b}^i(a) + \int_{T\setminus U} \bar{b}^i(a)}{\int_U q^i(a) + \int_{T\setminus U} q^i(a)} \\ &= \frac{\int_U \bar{b}^i(a) + \bar{p}^i \int_{T\setminus U} e^i(a)}{\int_U e^i(a) + \int_{T\setminus U} e^i(a)} \end{aligned}$$

Therefore,

$$\bar{p}^i = \frac{\int_U \bar{b}^i(a)}{\int_U e^i(a)} = \frac{\int_U b^i(a)}{\int_U e^i(a)} = \pi^i(b, q)$$

Furthermore, for each $a \in U$:

$$\pi\left(b, q\right) q(a) = \bar{p} q(a) = \pi\left(b_{\bar{p}}, q\right) q(a) \geq \sum_{i=1}^{L} \bar{b}^i(a) = \sum_{i=1}^{L} b^i(a)$$

So for each $a \in U$

$$x_a(b(a), q(a), B, Q) = x_a(b(a), q(a), B_{\bar{p}}, Q) = x_a(\bar{b}^i(a), q(a), B_{\bar{p}}, Q)$$

Therefore, *ae* in U

$$
\begin{aligned}
u_a\left(x_a(b(a),q(a),B,Q)\right) &= u_a(x_a(\bar{b}^i(a),q(a),B_p,Q)) \\
&\geq u_a(y(a)) \\
&> u_a(x(a))
\end{aligned}
$$

In conclusion, the coalition U can block using the strategy (b,q) defined in (7).

Therefore, $x \notin \mathcal{C}_e^{ss}(A)$ as desired. □

Remark 1 In fact, the restriction of the strategy profile (b,q) *to* U*, denoted as* $(b,q)\mid_U$ *turns out to be a Nash equilibrium profile for the market game among individuals in* U*, i.e.,* $(b,q)\mid_U \in \mathbf{N}(U)$*. This means that any assignment that is not in the core can be blocked by a Nash equilibrium of a market game played within a coalition* U*. In other words the market game can serve as a* non cooperative *mechanism in order to enforce blocking allocations among the members of a coalition.*

The following theorem, is a consequence of the preceding two propositions along with fact (1).

Theorem 1 $\mathcal{C}_e^{ss} = \mathcal{C}(A) = \mathcal{W}(A)$.

A few comments about the above results are in order. The implication of the above results is that the trade opportunities that the strategic market game rules allow, are sufficient to characterize the core or competitive equilibria. In other words, opportunities for additional arbitrary trades among groups are redundant. However, a word of caution is necessary: it would be a fallacy to infer from this fact that the opportunities allowed by the strategic market game exhaust the blocking opportunities of a given coalition! A carefull reading of our proofs makes evident that it is conceivable that a given coalition can find a feasible allocation(s) preferable to its members, but no such allocation can be implemented via the market game mechanism.

In order to highlight the fact that the Edgeworthean equilibrium refinement proposed here avoids 'trivial' equilibria of the market game mechanism we report the following result:

Theorem 2 $(0,0) \in \mathbf{C}_e^{ss}(A)$ *iff the endowment assignment is Pareto optimal.*

Proof: - $(0,0) \in \mathbf{C}_e^{ss}(A) \Rightarrow$ Endowments are Pareto optimal.

Notice that the allocation resulting from $(0,0)$ is precisely the endowment allocation, which by the previous theorem is a competitive allocation so by the first welfare theorem it is Pareto optimal.

- Endowments are Pareto optimal $\Rightarrow (0,0) \in \mathbf{C}_e^{ss}(A)$.

Since the endowment assignment is Pareto optimal, by the second welfare theorem it can be supported as a competitive assignment for some price vector $p \in \Re_+$, i.e., $e \in \mathcal{W}(A)$ and hence by fact (1) $e \in \mathcal{C}(A)$.

Suppose that $(0,0) \notin \mathbf{C}_e^{ss}(A)$. Then there must be some $T \in \mathcal{A}$, $\mu(T) > 0$, and $(\hat{b}, \hat{q}) \in Gr(S)$, where $(\hat{b}, \hat{q}) = (0,0)$ for all $a \notin T$, so that

$$u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q})) > u_a(e(a))$$

ae in T. But since $\int_T x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q}) = \int_T e(a)$, it follows that $e \notin \mathcal{C}(A)$, which contradicts our previous statement. $\square$

We can now relate the $SS-core$ to the set of Nash equilibria of the market game.

Proposition 3 $\mathcal{C}_e^{ss}(A) \subset \mathcal{N}(A)$.

Proof: Let $x : A \to \Re_+^L$ be an assignment such that $\int_A x(a) = \int_A e(a)$ and suppose that $x \notin \mathcal{N}(A)$. We will show that $x \notin \mathcal{C}_e^{ss}(A)$.

For each $p \in \Delta^L$ define $(b_p, q) : A \to \Re_+^{2L}$ as follows:

$$(b_p(a), q(a)) = \left((p^i e^i(a), e^i(a))_{i=1}^L\right)$$

Let $B_p = \int_A b_p(a)$, $Q = \int_A q(a)$ and note that $\pi(b_p, q) = p$.

For each $a \in A$ define now the correspondence:

$$b(a;p) = argmax\left\{u_a(x_a(\hat{b}, q(a), B_p, Q)) : \pi(b_p, q)\, q(a) \geq \sum_{i=1}^{L} \hat{b}^i\right\}$$

In view of assumptions (1), (2), $b(a;\cdot)$ is nonempty, uhc and $\hat{b} > 0$ for all $\hat{b} \in b(a;\cdot)$ ae in A. Moreover, $b(\cdot;p)$ has a measurable graph.

Define

$$T(p) = \left\{ a \in A : u_a(x_a(\hat{b}, q(a), B_p, Q)) > u_a(x(a)) \text{ for some } \hat{b} \in b(a; p) \right\}$$

Lemma 1 $\forall p \in \Delta^L \ \mu(T(p)) > 0$.

Proof Suppose $\exists \bar{p} \in \Delta^L$ so that $\mu(T(\bar{p})) = 0$, i.e., $\exists \hat{b}(a) \in b(a; \bar{p})$ so that *ae* in A

$$u_a(x(a)) \geq u_a(x_a(\hat{b}(a), q(a), B_{\bar{p}}, Q))$$

Consider $\left(\bar{b}(a), q(a)\right) = \left((\bar{p}^i x^i(a), e^i(a))_{i=1}^L\right)$ and notice that

$$\pi\left(\bar{b}, q\right) = \bar{p} = \pi\left(b_{\bar{p}}, q\right)$$

Fix one $a \in A$ and suppose that $\sum_{i=1}^L \bar{b}^i(a) < \pi\left(b_{\bar{p}}, q\right) q(a)$.

Then since $x_a(\bar{b}(a), q(a), B_{\bar{p}}, Q) = x(a)$ we would have

$$u_a\left(x_a(\bar{b}(a), q(a), B_{\bar{p}}, Q)\right) = u_a\left(x(a)\right) \geq u_a(x_a(\hat{b}(a), q(a), B_{\bar{p}}, Q))$$

which contradicts $\hat{b}(a) \in b(a; \bar{p})$.

So it must be $\sum_{i=1}^L \bar{b}^i(a) \geq \pi\left(b_{\bar{p}}, q\right) q(a)$ *ae* in A.

Let $T = \left\{ a \in A : \sum_{i=1}^L \bar{b}^i(a) > \pi\left(b_{\bar{p}}, q\right) q(a) \right\}$. If $\mu(T) > 0$ then

$$\sum_{i=1}^L \int_T \bar{b}^i(a) > \pi\left(b_{\bar{p}}, q\right) \int_T q(a) = \bar{p} \int_T e(a) = \bar{p} \int_T x(a) = \sum_{i=1}^L \int_T \bar{b}^i(a)$$

which is a contradiction. So, it must be $\mu(T) = 0$. Thus, it follows that *ae* in A:

$$\begin{aligned} \sum_{i=1}^L \bar{b}^i(a) &= \pi\left(b_{\bar{p}}, q\right) q(a) \\ &= \pi\left(\bar{b}, q\right) q(a) \end{aligned} \tag{8}$$

so from the allocation rule we conclude that $x_a(\bar{b}(a), q(a), \bar{B}, Q) = x(a)$ *ae* in A.

Let us consider any $a \in A$ and let $(\tilde{b}, \tilde{q}) \in S(a)$ be any strategy which satisfies $\sum_{i=1}^L \tilde{b}^i \leq \pi\left(\bar{b}, q\right) \tilde{q}$. By fact (2) we may assume that $\tilde{q} = q(a)$.

Since $\pi\left(\tilde{b}, q\right) = \pi\left(b_{\bar{p}}, q\right)$ we have that $\sum_{i=1}^{L} \tilde{b}^{i} \leq \pi\left(b_{\bar{p}}, q\right)\tilde{q} = \pi\left(b_{\bar{p}}, q\right)q(a)$ as well. Furthermore,

$$x_a(\tilde{b}, q(a), B_{\bar{p}}, Q) = x_a(\tilde{b}, q(a), \bar{B}, Q) = x_a(\tilde{b}, \tilde{q}, \bar{B}, Q)$$

By definition of $b(a;\bar{p})$ it follows that:

$$\begin{aligned} u_a(x_a(\tilde{b}, \tilde{q}, \bar{B}, Q)) &= u_a(x_a(\tilde{b}, q(a), B_{\bar{p}}, Q)) \\ &\leq u_a\left(x_a(\hat{b}(a), q(a), B_{\bar{p}}, Q)\right) \\ &\leq u_a\left(x(a)\right) \\ &= u_a\left(x_a(\bar{b}(a), q(a), \bar{B}, Q)\right) \end{aligned} \tag{9}$$

ae in A. But (8) along with (9) imply that $x \in \mathcal{N}(A)$ which is a contradiction to the original hypothesis. $\square$

For each $a \in A$ consider the correspondence defined as follows:

$$\bar{b}(a;p) = \begin{cases} b(a;p) & \text{if } u_a\left(x_a(b(a;p), q(a), B_p, Q)\right) > u_a(y(a)) \\ b(a;p) \cup \{b_p(a)\} & \text{if } u_a\left(x_a(b(a;p), q(a), B_p, Q)\right) = u_a(y(a)) \\ \{b_p(a)\} & \text{if } u_a\left(x_a(b(a;p), q(a), B_p, Q)\right) < u_a(y(a)) \end{cases} \tag{10}$$

Since the correspondence $\bar{B}(p) = \int_T \bar{b}(a;p)$ is nonempty, *uhc* and by Lyapunov's theorem convex valued, so is the correspondence $\varphi : \Delta^L \to 2^{\Delta^L}$ which is defined as $\varphi(p) = \pi(\bar{b}(p), q)$. It follows that φ has a fixed point, i.e., there is $\bar{p} = \pi(\bar{b}(\bar{p}), q)$. Therefore, there is a measurable function $\bar{b} : A \to \Re_+^L$ where $\bar{b}(a) \in \bar{b}(a;\bar{p})$ *ae* in A so that $\bar{p} = \pi(\bar{b}, q)$.

Consider now the set

$$U = \left\{a \in A : u_a\left(x_a(\bar{b}(a), q(a), B_{\bar{p}}, Q)\right) \geq u_a(x(a)) \text{ and } \bar{b}(a;\bar{p}) = b(a;\bar{p})\right\}$$

Since $T(\bar{p}) \subset U$ we have $\mu(U) > 0$. Define now the following strategy profile:

$$(b(a), q(a)) = \begin{cases} \left(\bar{b}(a), q(a)\right) & \text{if } a \in U \\ (0, 0) & \text{otherwise} \end{cases} \tag{11}$$

We now focus on the coalition U along with the strategy profile $(b(a), q(a))$. First note that for each $i = 1, 2, \ldots, L$ we have:

$$\begin{aligned}
\bar{p}^i &= \pi^i(\bar{b}, q) \\
&= \frac{\int_A \bar{b}^i(a)}{\int_A q^i(a)} \\
&= \frac{\int_U \bar{b}^i(a) + \int_{A\setminus U} \bar{b}^i(a)}{\int_U q^i(a) + \int_{A\setminus U} q^i(a)} \\
&= \frac{\int_U \bar{b}^i(a) + \bar{p}^i \int_{A\setminus U} e^i(a)}{\int_U e^i(a) + \int_{A\setminus U} e^i(a)}
\end{aligned}$$

which implies

$$\bar{p}^i = \frac{\int_U \bar{b}^i(a)}{\int_U e^i(a)} = \frac{\int_U b^i(a)}{\int_U e^i(a)} = \pi^i(b, q)$$

Furthermore, for each $a \in U$:

$$\pi\,(b, q)\, q(a) = \bar{p}q(a) = \pi\,(b_{\bar{p}}, q)\, q(a) \geq \sum_{i=1}^{L} \bar{b}^i(a) = \sum_{i=1}^{L} b^i(a)$$

It follows that for each $a \in U$

$$x_a(b(a), q(a), B, Q) = x_a(b(a), q(a), B_{\bar{p}}, Q) = x_a(\bar{b}(a), q(a), B_{\bar{p}}, Q)$$

Therefore, *ae* in U

$$u_a\,(x_a(b(a), q(a), B, Q)) = u_a\left(x_a(\bar{b}(a), q(a), B_p, Q)\right) \geq u_a(x(a))$$

where the last inequality is strict for $a \in T(\bar{p}) \subset U$ (recall that $\mu(T(\bar{p})) > 0$). Since preferences are monotonic and continuous, by a standard argument we can find an assignment $y : U \to \Re^L_+$ such that $\int_U y(a) = \int_U x_a(b(a), q(a), B, Q) = \int_U e(a)$ and $u_a(y(a)) > u_a(x(a))$ *ae* in U. It follows that $x \notin \mathcal{C}\,(A)$ so by theorem (1) $x \notin \mathcal{C}^{ss}_e\,(A)$. $\square$

This proposition is akin to the results of [7] relating the core to competitive allocations. It has a very interesting implication from a conceptual point of view, regarding the core of an atomless economy:

Corollary 1 $\mathcal{C}\,(A) \subset \mathcal{N}\,(A)$

The meaning of this corollary, which is complementary to that of remark (1), is that core allocations can be *non cooperatively* implemented via the market game.

Another interesting point of this corollary is that it provides a formal argument that non cooperative equilibria with (some) inactive markets are due to lack of coordination. Notice that the reverse inclusion does not generally hold, as there exist Nash equilibria with any subset of trading posts inactive, which are blocked (see theorem (1) above). The literature has attributed the presence of such equilibria to coordination failure. Corollary (1) formally confirms this point.

3.1 A further comparison

This section features a comparison of the $SS-core$ with the Walras-core ($W-core$) developed in [11]. The $W-core$, denoted $\mathcal{C}^w(A)$, requires blocking assignments to be feasible and for some price vector also budget feasible for all agents. This notion can be criticized on two counts. First, it lacks a description of price formation. Second, it is not very usefull as a foundation (characterization) of competition: using a concept which is built on price taking, in order to provide a foundation of price taking behavior is self defeating.

The following lemma can be used as a basis to address both of the above criticisms.

Lemma 2 $\mathcal{C}^w(A) = \mathcal{C}_e^{ss}(A)$

Proof: $\mathcal{C}_e^{ss}(A) \subset \mathcal{C}^w(A)$

Suppose $x \in \mathcal{C}_e^{ss}(A)$, i.e., for some $(b,e) \in \mathbf{C}_e^{ss}(A)$, $x(a) = \left(\frac{b^i(a)}{\pi^i(b,e)}\right)_{i=1}^L$, ae in A. By individual rationality of x, it follows that $\sum_{i=1}^L b^i(a) \leq \pi(b,e)e(a)$, ae in A. Therefore, $\pi(b,e)x(a) \leq \pi(b,e)e(a)$, ae in A.

Furthermore, $\int_A x(a) = \left(\int_A \frac{b^i(a)}{\pi^i(b,e)}\right)_{i=1}^L = \int_A e(a)$.

It follows that $x \notin \mathcal{C}^w(A) \Rightarrow \exists T$ and $y : T \to \Re_+^L$ $s.t.$ $\int_T y(a) = \int_T e(a)$, $py(a) = pe(a)$ ae in T for some $p \in \Delta^L$ and $u_a(y(a)) > u_a(x(a))$ ae in T.

Define the profile:

$$\left(\hat{b}^i(a), \hat{q}^i(a)\right)_{i=1}^L = \begin{cases} (p^i y^i(a), e^i(a))_{i=1}^L & \text{if } a \in T \\ (0,0) & \text{if } a \notin T \end{cases}$$

We have that $\sum_{i=1}^L \hat{b}^i(a) = py(a) = pe(a) = \pi(b,e)e(a)$, ae in A.

Therefore, $u_a(x_a(\hat{b}, \hat{q}, \hat{B}, \hat{Q})) = u_a(y(a)) > u_a(x(a))$ ae in T.

It follows that $x \notin \mathcal{C}_e^{ss}(A)$, which contradicts our initial statement.

Finally, a similar argument establishes that $\mathcal{C}^w(A) \subset \mathcal{C}_e^{ss}(A)$. □

Thus, the $SS-core$ serves well as an interpretation of the $W-core$ and provides a game theoretic foundation for it. The following corollary, which follows directly from the lemma above along with proposition (3), crystallizes this idea.

Corollary 2 $\mathcal{C}^w(A) \subset \mathcal{N}(A)$

4 Conclusion

In this note we have established the equivalence between the core (and consequently the competitive mechanism) and non cooperative trade within groups of agents. This equivalence has been obtained via equality of the equilibrium allocations corresponding to the two concepts.

The central message of our result is that even within the institutional constraint imposed by the strategic market game, the economy can attain outcomes which are precisely the same as in the core. In other words, any additional trade opportunities available to groups of individuals are redundant. Another conclusion that can be drawn from our results regards the interpretation of the core. We showed that allocations which do not belong to the core can be blocked by Nash equilibria of a market game played among the members of a coalition. In other words blocking allocations can be *non cooperatively* implemented. This discussion casts some doubts on the traditional view of the core as a purely 'cooperative' concept. It would be false however to interpret our results as asserting that no coordination at all is necessary in the interpretation of the core. The set of Nash equilibria of the market game played among a group of individuals may not be singleton. The selection of the 'right' Nash equilibrium requires some level of coordination. Nevertheless, our results (see remark (1) above) do imply that once individuals coordinate on the 'right' profile of strategies, the execution of those strategies is in a way 'self enforcing': once everyone else in the group executes the agreed strategy doing likewise is, from an individual point of view, a best response.

The foregoing discussion can be used to shed some light on the informational requirements of the core. The core has been criticized for the severe informational requirements that it imposes on individuals. On the face of it, in the standard definition of the core the formation of a blocking coalition would require knowledge of individual characteristics, which is an arduous requirement compared with the competitive price mechanism, where knowledge of the price vector suffices. [4]

The $SS-core$ and its equality with the core that we prove here suggests that this comparison may be unfair. In the $SS-core$ the formation of a coalition can be thought of as coming about as follows: prices are announced and a blocking coalition forms by all individuals who possess strategies which, at those prices, result in trades preferable than in other price vectors. In other words, a coalition is formed by all individuals who express a wish to join in trading, at some prices, by submitting bids and offers in the corresponding trading posts. An equilibrium occurs following the announcement of a price vector, such that all individuals are willing to submit bids and offers at those prices. Notice that in this interpretation the informational requirement imposed on individuals is the knowledge of price vector(s), which is very similar to the competitive price mechanism.

It should be noted that the descriptive nature of the market game mechanism, which the $SS-core$ is endowed with, may prove usefull in modelling formally a process such as the one described above. However, this issue is beyond the scope of this paper. Finally, we believe that the modeling of these ideas can be improved by taking advantage of the equivalence between competitive equilibria and the f-core (see [5]) in economies with anonymous externalities. The market game structure makes the f-core an appealing concept to extend our results to a context without exclusion.

[4] See [8] for a discussion of this issue.

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