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Enrico Diecidue, Ulrich Schmidt, and Horst Zank

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Correspondence email: Horst.Zank@manchester.ac.uk

School of Social Sciences,
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom

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by Enrico Diecidue, Ulrich Schmidt, and Horst Zank*

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Abstract. This paper provides behavioral foundations for parametric weighting functions under rank-dependent utility. This is achieved by decomposing the independence axiom of expected utility into separate meaningful properties. These conditions allow us to characterize power and exponential weighting functions. Moreover, by restricting the conditions to subsets of the probability interval, parametric inverse-S shaped weighting functions are obtained.

Keywords: Expected utility, comonotonic independence, parametric weighting function, preference foundation, rank-dependent utility.

Journal of Economic Literature Classification Numbers: D81.

*Correspondence: Horst Zank, Economics, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom; Telephone: ++44 161 275 4872, Fax: ++44 161 275 4812, E-mail: horst.zank@manchester.ac.uk

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1 Introduction

Many empirical studies have shown that expected utility theory (EU), in particular its crucial independence axiom, does not provide an accurate description of people's actual choice behavior. This evidence has motivated researchers to develop alternative more flexible models. Several avenues of research have been pursued.

A very general model, proposed by Machina (1982), requires that preferences are smooth. This model dispenses completely of independence, and instead only requires that indifference curves can be approached by EU. Machina's model can accommodate many of EU's descriptive shortcomings but, as it requires EU locally, the model does not permit for reference-dependent or rank-dependent preferences.

Other theories propose specific relaxations of independence. One stream has maintained linearity of indifference curves (Fishburn 1983, Chew 1983, Dekel 1986), while another direction of research has focused on quasi-concave or quasi-convex indifference curves (Chew, Epstein and Segal 1991). Yet another stream of research has retained additive separability across events by restricting independence to hold only on comonotonic sets (Weymark 1981, Quiggin 1981, 1982, Yaari 1987, Green and Jullien 1988, Schmeidler 1989, Tversky and Kahneman 1992, Safra and Segal 1998, Chateauneuf 1999). By further requiring a separation of utility from probability weighting, these models allow for the analysis of probabilistic risk attitudes (e.g., Chew, Karni and Safra 1987, Epstein and Zin 1990, Quiggin 1991, Chateauneuf and Cohen 1994, Wakker 1994, Chateauneuf, Cohen and Meilijson 2004). This latter family of theories, which has provided the most prominent alternatives to EU, is central in this paper.

We focus on decision under risk and retain additive separability. Further, we propose other specific restrictions of independence to obtain the separation of probabilistic risk attitudes from utility. We obtain several variants of rank-dependent utility (RDU) which differ in the parametric specifications required for the probability weighting functions. There are additional features that make our preference foundations for RDU distinctive.

Most derivations of RDU require some structural richness on the set of consequences because the proposed preference conditions focus on the derivation of continuous cardinal utility. In those approaches the weighting functions are obtained as a bonus. In this paper we follow the traditional approach put forward by von Neumann and Morgenstern (1944) by focusing on the structure naturally offered by the probability interval and provide preference conditions that focus on the derivation of the probability weighting function. Typical for this approach is that cardinal utility is obtained as a bonus.

Axiomatizations of general RDU, without invoking any structural assumptions on the set of consequences, have been provided by Nakamura (1995) and more recently by Abdellaoui (2002) and Zank (2004). In these approaches the weighting function is unrestricted. Empirical evidence, however, suggests a particular pattern for probability weighting: small probabilities are overweighted while large ones are underweighted. Specific parametric forms have been proposed in the literature to accommodate these features. Some involve a single parameter (Karmarkar 1978, 1979, Röell 1987, Currim and Sarin 1989, Tversky and Kahneman 1992, Luce, Mellers and Chang 1993, Hey and Orme 1994, Safra and Segal 1998) while others use two or more parameters (Bell 1985, Goldstein and Einhorn 1987, Currim and Sarin 1989, Lattimore Baker and Witte 1992, Prelec 1998).

Despite the large interest in parametric specifications for the weighting function under RDU, little research has been invested in the axiomatic analysis of appropriate preference conditions. Further, all preference foundations we are aware of require a rich topological structure for the set of consequences (Safra and Segal 1998, Prelec 1998, Gonzalez and Wu 1999). This means that those models cannot immediately be adopted to many real world applications because the set of consequences may lack such additional structure. As a consequence, it is unclear how to extend the underlying preference foundations and, therefore, it is unclear whether these models will remain valid.

The preference conditions presented in this paper apply to general sets of consequences, which makes these models generally applicable. As mentioned before, our goal here is to characterize parametric weighting functions. Except for weak ordering and continuity, the properties that we propose are all implied by the independence axiom. For instance, we retain stochastic dominance and, in line with all rank-dependent theories, we assume comonotonic independence. These two implications ensure additive separability. But further assumptions on preferences are required to get a separation of probability weighting and utility. In fact, by focusing on specific functional forms for the weighting functions, the preference conditions that characterize these forms deliver this latter separation free of charge.

Specific implications of the independence axiom have been analyzed before and, although the focus has not been on the weighting function under RDU, there are some common aspects underlying those preference conditions and the ones proposed in this paper. Machina (1989) distinguished two properties which are termed mixture separability and replacement separability. Mixture separability demands that the preference between two lotteries is invariant to mixing them with a common degenerate lottery. Replacement separability holds if the preference between two lotteries remains unaffected when in both lotteries a common consequence with identical probability is replaced by any different consequence. We explore the implications of these separability conditions within our rank-dependent framework, where we have to restrict these conditions.

There are two sorts of restrictions that we impose. At an initial stage we focus on extreme, that is, best or worst consequences. Our variant of replacement separability permits only the substitution of common best consequences by common worst consequences. This property characterizes RDU with a linear or exponential weighting function. A first variant of mixture separability demands that the preference is invariant to mixing with worst consequences, and leads to RDU with a power weighting function. Another variant of mixture separability demands invariance of mixtures with best consequences, and characterizes RDU with a dual power weighting function. Requiring any combination of two of the latter three properties characterizes EU. This follows since the only function that is shared by any two of the parametric specifications is the linear function. Therefore, we also obtain several alternative axiomatizations of expected utility.

The secondary stage restrictions have a different motivation. The separability conditions mentioned before are descriptively problematic. For example, they are violated by the two famous paradoxes of Allais (1953). More precisely, the common ratio effect constitutes a direct violation of our version of mixture separability that generates the power weighting function, while the common consequence effect provides a violation of our version of replacement separability. More generally, because the afore mentioned weighting functions each involve a single parameter, they cannot accommodate at the same time probabilistic risk seeking and probabilistic risk aversion within the probability interval. That is, they are incompatible with the inverse-S shaped form, concave for small probabilities and convex for large probabilities, that received extensive empirical support (e.g., Camerer and Ho 1994, Wu and Gonzalez 1996, Tversky and Fox 1995, Gonzalez and Wu 1999, Abdellaoui 2000, Bleichrodt and Pinto 2000, Kilka and Weber 2001, Abdellaoui, Vossmann and Weber 2005).

To accommodate mixed probabilistic risk attitudes, we need to relax the previous preference conditions further, namely to hold only on specific subsets of the probability interval. This way, we can provide foundations for inverse-S shaped weighting functions under RDU, which are entirely based on behavioral preference conditions that do not require additional structural assumptions on the set of consequences.

Our analysis of inverse-S shaped weighting functions focuses on functional forms that may involve three parameters. One parameter describes the probabilistic risk attitudes for small probabilities while a second one describes such attitudes for large probabilities. The role of the third parameter is to separate the region of probabilistic risk aversion from the region of probabilistic risk seeking. Therefore, these parametric forms have a similar interpretation to that proposed by Tversky and Kahneman (1992) because of the relation of these parameters with the idea of modelling sensitivity to changes from impossibility and certainty, respectively.

The organization of the paper is as follows. In Section 2 general notation and preliminary results are presented. We indicate how the results of Wakker (1993) and Chateauneuf and Wakker (1993) can be used to derive additive separability, the latter being a common point of departure for all our models. In Section 3 we review expected utility, and we provide a new preference foundation for this classical theory by decomposing independence into stochastic dominance and an addition invariance property. Next, we proceed with a further separation of independence into specific variants of the separability conditions proposed by Machina (1989). In Section 4 we analyze mixture separability restricted to worst consequences, and in Section 5 we analyze replacement separability restricted to best and worst consequences. Section 6 analyses the implications of mixture separability now restricted to the best consequence. Finally, in Section 7 we provide results for parametric inverse-S shaped probability weighting functions. The majority of proofs are deferred to the Appendix, but we kept some in the main text because we think that they may clarify ideas and help understanding important step in the derivation of the theories.

2 Preliminaries

Let X denote the set of consequences. For simplicity of exposition, we assume a finite set of consequences, such that $X = \{x_0, \dots, x_n\}$ for $n \geq 3$. The results presented below can easily be extended to more general sets of consequences along the lines indicated in Abdellaoui (2002) and Zank (2004). A *lottery* is a finite probability distribution over the set X. It is represented by $P = (\tilde{p}_0, x_0; \dots; \tilde{p}_n, x_n)$ meaning that probability $\tilde{p}_j$ is assigned to consequence $x_j \in X$, for $j = 0, \dots, n$. Let L denote the set of all lotteries. The set of lotteries L is a mixture space endowed with the operation of probability mixing, i.e., for $P, Q \in L$ and $\alpha \in [0, 1]$ the mixture $\alpha P + (1 - \alpha)Q$ is also a lottery in L.

A preference relation $\succcurlyeq$ is assumed over L, and its restriction to subsets of L (e.g., all degenerate lotteries) is also denoted by $\succcurlyeq$. The symbol $\succ$ denotes strict preference, $\sim$ denotes indifference, and $\preccurlyeq$ respectively $\prec$ are the corresponding reversed preferences. We assume that no two consequences in X are indifferent, and further, that consequences are ordered from worst to best, i.e., $x_0 \prec \cdots \prec x_n$. This will simplify the subsequent presentation but also, as $n \geq 3$, the former assumption entails a non-degeneracy condition on $\succcurlyeq$ together with a mild richness assumption on X. It can be shown that our results below hold whenever there are at least four strictly ordered consequences.

In this paper we present several preference conditions which become more transparent if formulated for decumulative distributions instead of lotteries. With this in mind we can identify lotteries with their corresponding decumulative probability distribution through the mapping

$$P \mapsto (p_1, \dots, p_n),$$

where $p_j = \sum_{i=j}^{n} \tilde{p}_i$ denotes the likelihood of getting at least x_j, $j = 1, \dots, n$. As the set of consequences is fixed we have simplified the notation above by suppressing the consequences and by noting that the worst consequence x_0 always has decumulative probability equal to 1. Therefore, the set of lotteries L is identified with the set $\{(p_1, \ldots, p_n) : 1 \geq p_1 \geq \cdots \geq p_n \geq 0\}$, which consists of probability tuples that are rank-ordered from highest to lowest.

In what follows we provide preference conditions for $\succcurlyeq$ in order to *represent* the preference relation over L by a function V. That is, V is a mapping from L into the set of real numbers, $I\!R$, such that for all $P, Q \in L$,

$$P \succcurlyeq Q \Leftrightarrow V(P) \geq V(Q).$$

This necessarily implies that $\succcurlyeq$ must be a *weak order*, i.e. $\succcurlyeq$ is *complete* ($P \succcurlyeq Q$ or $P \preccurlyeq Q$ for all $P, Q \in L$) and *transitive* ($P \succcurlyeq Q$ and $Q \succcurlyeq R$ implies $P \succcurlyeq R$ for all $P, Q, R \in L$).

The preference relation satisfies *monotonicity* if $P \succ Q$ whenever $p_j \geq q_j$ for all $j = 1, \ldots, n$ and $P \neq Q$. The preference relation $\succcurlyeq$ satisfies *Jensen-continuity* on the set of lotteries L if for all lotteries $P \succ Q$ and R there exist $\rho, \mu \in (0, 1)$ such that

$$\rho P + (1 - \rho)R \succ Q \text{ and } P \succ \mu R + (1 - \mu)Q.$$

Jensen-continuity is somewhat weaker than (Euclidean) continuity on L, but in the presence of weak order and monotonicity it implies the latter (see Abdellaoui 2002, Lemma 18). The preference relation $\succcurlyeq$ satisfies (Euclidean) *continuity* if for all $P \in L$ the sets $\{Q \in L : Q \succ P\}$ and $\{Q \in L : Q \prec P\}$ are open sets in L.

Recall that L is identified with a subset of the Cartesian product space $[0, 1]^n$, which is endowed with a rich topological structure inherited naturally from the structure given on the probability interval $[0, 1]$. We can therefore invoke a classical result of Debreu (1954) to derive the following statement:

THEOREM 1 *Assume that the preference relation $\succcurlyeq$ on the set of decumulative distributions L is a Jensen-continuous monotonic weak order. Then there exists a continuous function V :*

$L \to I\!R$, strictly increasing in each decumulative probability, that represents $\succcurlyeq$. The function V is unique up to strictly increasing continuous transformations. □

A further preference condition that is used below is independence of common decumulative probabilities. To define this property we introduce some useful notation. For $i \in \{1, \ldots, n\}$, $P \in L$ and $\alpha \in [0, 1]$, we denote by $\alpha_i P$ the distribution that agrees with P except that p_i is replaced by α. Whenever this notation is used it is implicitly assumed that $p_{i-1} \geq \alpha \geq p_{i+1}$ (respectively, $\alpha \geq p_{i+1}$ if $i = 1$ and $p_{i-1} \geq \alpha$ if $i = n$) to ensure that $\alpha_i P \in L$. Similarly, for $I \subset \{1, \ldots, n\}$ we write $\alpha_I P$ for the distribution that agrees with P except that p_i is replaced by α for $i \in I$, and assume that the probabilities in $\alpha_I P$ are ranked from highest to lowest. The preference relation $\succcurlyeq$ satisfies *comonotonic independence* if $\alpha_i P \succcurlyeq \alpha_i Q \Leftrightarrow \beta_i P \succcurlyeq \beta_i Q$ for all $\alpha_i P, \alpha_i Q, \beta_i P, \beta_i Q \in L$.

Because of the implicit requirement of decumulative probabilities being ranked from highest to lowest, it is natural to use the term comonotonic independence instead of independence of common decumulative probabilities. Formulated for consequences, Wakker (1989, 1993) has called this condition coordinate independence, and obviously it belongs to the family of independence conditions with comonotonicity restrictions put forward by Schmeidler (1989).

Comonotonic independence is a weak form of replacement separability as analyzed in Machina (1989). Recall that replacement separability demands that the preference between two lotteries is invariant when common consequences with equal probability are replaced by other common consequences. The restricted variant of replacement separability used here has the interpretation that, when comparing two lotteries, common consequences can be replaced by other common consequences only if they have a common decumulative likelihood. On reflection, one observes that this restriction implies that only common consequences of adjacent rank can be replaced.

Below, in Theorem 2, we use comonotonic independence to derive an additive separable representation. As pointed out in Chateauneuf and Wakker (1993), in order to obtain this separability, the condition could have been formulated somewhat weaker in analogy to the ordinal independence of Green and Jullien (1988), also called tail independence in Zank (2001) and Wakker and Zank (2002).

Without the comonotonicity restriction on decumulative distributions in L we could adopt well-known results of Debreu (1960) to derive additive separability of the representing function in Theorem 1. Deriving additive separability on rank-ordered sets is not trivially extended from Debreu's classical result, but invokes more complex mathematical tools. The next theorem follows by using results of Wakker (1993) and Chateauneuf and Wakker (1993).

Theorem 2 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is represented by an additive function*

$$V(P) = \sum_{j=1}^{n} V_j(p_j),$$

with continuous strictly monotonic functions $V_1, \ldots, V_n : [0,1] \to I\!R$ which are bounded except maybe V_1 and V_n which could be infinite at extreme probabilities (i.e., at 0, or 1).

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence.*

The functions $V_1, \ldots, V_n$ are jointly cardinal, that is, they are unique up to location and common scale. □

Next we provide preference foundations for specific rank-dependent utility models using as common point of departure the results obtained above. We start in the next section by reconsidering expected utility and then extend that model by providing weaker and natural preference conditions implied by von Neumann and Morgenstern independence. Before proceeding we recall the general form of rank-dependent utility.

Rank-dependent utility (RDU) holds if the preference relation is represented by the function

$$V(P) = u(x_0) + \sum_{j=1}^{n} w(p_j)[u(x_j) - u(x_{j-1})], \tag{1}$$

where the utility function $u : X \to I\!R$ agrees with $\succcurlyeq$ on X, and the *weighting function* $w : [0,1] \to [0,1]$ is strictly increasing and continuous with $w(0) = 0$ and $w(1) = 1$. Under RDU utility is cardinal and the weighting function is uniquely determined. If the weighting function is linear then RDU reduces to *expected utility* (EU).

3 Expected Utility

In this section we provide an alternative derivation of expected utility using mathematical tools that have been useful for the derivation of additive separability with continuous cardinal utility. The advantage of this approach is that we provide a unifying framework in which it is transparent how relaxations of the critical von Neuman-Morgenstern independence condition lead to more general decision models.

The preference relation $\succcurlyeq$ satisfies *vNM-independence* (short for *von Neumann-Morgenstern independence*) if for all $P, Q, R \in L$ and all $\alpha \in (0,1)$ it holds that

$$P \succcurlyeq Q \Leftrightarrow \alpha P + (1-\alpha)R \succcurlyeq \alpha Q + (1-\alpha)R.$$

That is, the preference between P and Q remains unaffected if both, P and Q, are mixed with a common R. Note that in the definition of vNM-independence no restrictions apply to the choice of R. We derive two immediate implications of vNM-independence: monotonicity and additivity, defined below.

Lemma 3 *Assume $\succcurlyeq$ is a weak order on L that satisfies vNM-independence. Then, $\succcurlyeq$ satisfies monotonicity.* $\square$

The next condition, which is also an implication of vNM-independence, allows us to give an alternative interpretation for the effect of common increases (respectively decreases) in decumulative probabilities. The condition requires that such common changes do not alter original preferences. For $P \in L, R = (r_1, \ldots, r_n)$ we write $P + R$ for $(p_1 + r_1, \ldots, p_n + r_n)$; $P + R \in L$ if $1 \geqslant p_1 + r_1 \geqslant \cdots \geqslant p_n + r_n \geqslant 0$. The preference relation $\succcurlyeq$ satisfies *additivity*, if

$$P \succcurlyeq Q \Leftrightarrow P + R \succcurlyeq Q + R,$$

whenever $P, Q, P+R, Q+R \in L$. Additivity demands that the preference between two lotteries remains unaffected if the likelihood of getting some consequence is increased (or decreased) by the same probability in both lotteries. Obviously, such increments (decrements) result by simultaneously reducing (increasing) the likelihood of some other consequences. This indicates that the marginal impact of probability changes is independent of the magnitude and of the rank of those consequences, thereby suggesting a linear treatment of (decumulative) probabilities.

Lemma 4 *Assume $\succcurlyeq$ is a weak order on L that satisfies vNM-independence. Then, $\succcurlyeq$ satisfies additivity.* $\square$

Given the structure considered here, it is well-known (e.g., Herstein and Milnor 1953, Fishburn 1970) that a preference relation $\succcurlyeq$ satisfies weak ordering, Jensen-continuity and vNM-independence on L if and only if it can be represented by expected utility. To derive expected utility using additivity and monotonicity instead of vNM-independence we need to show that comonotonic independence holds. Then we can apply the results from the previous section, and exploit further the power of additivity.

LEMMA 5 *Assume $\succcurlyeq$ is a weak order on L that satisfies additivity. Then, $\succcurlyeq$ satisfies comonotonic independence.* $\square$

We can now indicate an alternative and simple way of deriving expected utility. Assume that on L the preference relation $\succcurlyeq$ is a weak order that satisfies Jensen-continuity and vNM-independence. Then, $\succcurlyeq$ also satisfies monotonicity (Lemma 3) and additivity (Lemma 4). Hence, it satisfies (Euclidean) continuity and also comonotonic independence (Lemma 5). By Theorem 1 it follows that the preference relation is represented by a continuous function V on L. From Theorem 2 it follows that the function V is additively separable, say $V = \sum_{j=1}^{n} V_j$, with functions V_j as described in the theorem. Using again additivity, it follows that the functions V_j, $j = 1, \ldots, n$, are linear. Hence, they differ only by their scale and location. (Note that this excludes, in particular, the case that V_1 and V_n are unbounded.) By fixing a common location $V_j(0) = 0$ for all j, the functions differ only by their positive slopes, say s_j, which we use to define utility iteratively as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \ldots, n$. Therefore, $V_j(p) = ps_j = p[u(x_j) - u(x_{j-1})]$ for $j = 1, \ldots, n$ with strictly monotonic utility u. From the joint cardinality of the functions V_j it follows that u is cardinal. This way, expected utility has been obtained. We summarize the result in the next theorem using the weaker implication of additivity and monotonicity instead of vNM-independence.

THEOREM 6 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is represented by expected utility*

$$V(P) = u(x_0) + \sum_{j=1}^{n} p_j[u(x_j) - u(x_{j-1})],$$

with monotonic utility function $u : X \to I\!R$.

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order satisfying additivity.*

The function u is cardinal, that is unique up to scale and location. $\square$

Note that in the above theorem the expected utility formula is written with respect to decumulative probabilities p_j. To obtain the more familiar form with likelihood of consequences $\tilde{p}_j$ instead of decumulative probabilities, recall that $\tilde{p}_0 = 1-p_1, \tilde{p}_j = p_j - p_{j+1}$ for $j = 1, \ldots, n-1$, and $\tilde{p}_n = p_n$. Substituting the latter into the formula of statement (ii) of the above theorem gives $V(P) = \sum_{j=0}^{n} \tilde{p}_j u(x_j)$, the familiar weighted average of utilities expression.

In the derivation of Theorem 6 we could also have used the tools developed in Weymark (1981). Weymark used additivity on rank-ordered nonnegative income vectors, instead of probability vectors as we do, but without explicitly using comonotonic independence. Also he replaced Jensen-continuity and monotonicity with (Euclidean) continuity and a local non-satiation property, respectively. We prefer to use comonotonic independence because of its relevance for rank-dependence, but also because in what follows we look at weakening additivity further, and this leads to natural extensions of expected utility with comonotonic independence being included in all those derivations.

4 Common Ratio Invariant Preferences

One of the difficulties of expected utility is to accommodate preferences that exhibit the common ratio effect: Allais (1953) compared the choice behavior for the following two decision problems.

In problem 1 there is the choice between the following lotteries:

$$A_1 = (1, 1M) \text{ and } B_1 = (0.2, 0M; 0.8, 5M),$$

where M denotes \$-millions. In problem 2 the choice is between

$$A_2 = (0.95, 0M; 0.05, 1M) \text{ and } B_2 = (0.96, 0M; 0.04, 5M).$$

The literature has reported (e.g., in Allais 1953, MacCrimmon and Larsson 1979, Chew and Waller 1986, Wu 1994) that a significant majority of people exhibited a preference for A_1 in the first choice problem and a preference for B_2 in the second choice problem. Substituting expected utility immediately reveals that this leads to a conflicting relationship.

Note that the ratio of probabilities of the positive consequences in the first choice problem (0.8/1) equals the ratio of probabilities of the positive consequences in the second choice problem (0.04/0.05), hence the name common ratio effect for the above EU-paradox. Obviously, by reducing in the first choice problem at a common rate the likelihood of the positive consequences, the second choice problem is generated. This reduction of likelihood for the positive consequences necessarily requires a corresponding increase in the likelihood of ending up with nothing, which is the worst consequence. Formally, using lottery notation, we observe that $A_2 = (0.05)A_1 + (1 - 0.05)(1, 0)$ and $B_2 = (0.05)B_1 + (1 - 0.05)(1, 0)$. Hence, a first preference $A_1 \succ B_1$ together with a second preference $A_2 \prec B_2$ directly violates the vNM-independence condition.

Looking at the implications of vNM-independence, as described in the previous section, we can observe that common ratio type behavior is not in conflict with monotonicity and neither with comonotonic independence. It is a different aspect of vNM-independence that is violated by such preferences, which gives rise to the following property. The preference relation $\succcurlyeq$

satisfies *common ratio invariance for decumulative distributions* if

$$(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n) \Leftrightarrow (\alpha p_1, \ldots, \alpha p_n) \sim (\alpha q_1, \ldots, \alpha q_n),$$

whenever $(p_1, \ldots, p_n), (q_1, \ldots, q_n), (\alpha p_1, \ldots, \alpha p_n), (\alpha q_1, \ldots, \alpha q_n) \in L$.

Common ratio invariance for decumulative distributions says that shifting proportionally probability mass from good consequences to the worst consequence (or doing the opposite) leaves preferences unaffected. We have formulated the condition with indifference instead of weak preferences, which makes the condition more general.

Common ratio invariance for decumulative distributions is a weak form of mixture separability (Machina 1989). Recall, that the latter demands that a preference between two lotteries is maintained if each of the lotteries is mixed with any common consequence. In contrast, common ratio invariance for decumulative distributions demands that such mixtures are only permitted if the common consequence is the worst.

The condition has also appeared in Safra and Segal (1998), called zero-independence, where it has been used in the derivation of a specific version of Yaari (1987)'s dual theory, namely RDU with linear utility and power weighting function. Here we show that the condition is powerful enough to yield RDU-preferences with power weighting without restricting the form of the utility function (see Theorem 8 below). First, we note that additivity implies common ratio invariance for decumulative distributions. More precisely, we have the following lemma:

LEMMA 7 *Assume $\succcurlyeq$ is a weak order on L that satisfies additivity and monotonicity. Then, $\succcurlyeq$ satisfies common ratio invariance for decumulative distributions.* □

Let us now assume that RDU holds and further that the weighting function is a power function, i.e., it is of the form $w(p) = p^b$ for $b > 0$. Then, for a given indifference $(p_1, \ldots, p_n) \sim$

$(q_1, \ldots, q_n)$, substituting RDU with the power weighting function, we get

$$
\begin{aligned}
(p_1, \ldots, p_n) &\sim (q_1, \ldots, q_n) \\
&\Leftrightarrow \\
u(x_0) + \sum_{j=1}^{n} p_j^b [u(x_j) - u(x_{j-1})] &= u(x_0) + \sum_{j=1}^{n} q_j^b [u(x_j) - u(x_{j-1})].
\end{aligned}
$$

Cancelling $u(x_0)$ on both sides of the equation, then multiplying both sides by α^b for any $\alpha > 0$ such that $\alpha p_1 \leq 1$ and $\alpha q_1 \leq 1$, and then adding $u(x_0)$ to both sides gives

$$
\begin{aligned}
u(x_0) + \sum_{j=1}^{n} (\alpha p_j)^b [u(x_j) - u(x_{j-1})] &= u(x_0) + \sum_{j=1}^{n} (\alpha q_j)^b [u(x_j) - u(x_{j-1})] \\
&\Leftrightarrow \\
(\alpha p_1, \ldots, \alpha p_n) &\sim (\alpha q_1, \ldots, \alpha q_n).
\end{aligned}
$$

This shows that RDU-preferences with power weighting functions imply common ratio invariance for decumulative distributions for the preference relation $\succcurlyeq$ on L. Below we show that replacing additivity with its weaker implications of comonotonic independence and common ratio invariance for decumulative distributions characterizes exactly this class of preferences.

THEOREM 8 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is represented by rank-dependent utility with a power weighting function, i.e.,*

$$V(P) = u(x_0) + \sum_{j=1}^{n} p_j^b [u(x_j) - u(x_{j-1})],$$

with $b > 0$, and monotonic utility function $u : X \to I\!R$.

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance for decumulative distributions.*

The function u is cardinal. $\square$

It has previously been documented that preferences exhibiting the paradoxical common ratio effect exclude RDU preferences with power weighting. Our result above demonstrates that it is precisely this class of RDU-preferences with power weighting, including EU-preferences, that cannot accommodate common ratio effect preferences. That the result is very general can also be inferred from the fact that, except for monotonicity, no further restrictions apply to utility.

5 Extreme Replacement Separability

We start in this section by reconsidering the common consequence paradox of Allais (1953), and relate this to a new preference condition concerning the replacement of common consequences.

The common consequence paradox originates from observing behavior among the following pairs of choice problems. In problem 3 the choice is between

$$A_3 = (1, 1M) \text{ and } B_3 = (0.01, 0M; 0.89, 1M; 0.1, 5M),$$

and in problem 4 the choice is between

$$A_4 = (0.89, 0M; 0.11, 1M) \text{ and } B_4 = (0.9, 0M; 0.1, 5M).$$

It has been observed in experiments that a significant majority of people exhibit a preference for A_3 in the former choice problem and a preference for B_4 in the latter choice problem (e.g., in Allais 1953, MacCrimmon and Larsson 1979, Chew and Waller 1986, Wu 1994, but see also related evidence in Wakker, Erev and Weber 1994, Birnbaum and Navarette 1998, Birnbaum 2004). If one writes the previous lotteries as decumulative distributions over consequences 0, $1M$, and $5M$, then one can immediately see that $A_4 = (0.11, 0)$ and $A_3 = A_4 + (0.89, 0)$, and that $B_4 = (0.1, 0.1)$ and $B_3 = B_4 + (0.89, 0)$. Clearly, exhibiting initially $A_3 \succ B_3$ together with a second preference $A_4 \prec B_4$ directly violates additivity.

In the common consequence paradox the interpretation is that people are sensitive to replacing the good common consequence of getting "1 Million with probability 0.89" with a bad common consequence of getting "0 with probability 0.89." Therefore, also replacement separability (Machina 1989) is violated. Although empirically it is yet to be verified we think that such sensitivity would also be exhibited when the best consequence is replaced by the worst consequence. For example, consider the following modification of the common consequence effect problem of Allais (1953), in which problem 3 is replaced by problem 5, where the choice is between lotteries

$$A_5 = (0.11, 1M; 0.89, 5M) \text{ and } B_5 = (0.01, 0M; 0.99, 5M).$$

Written as decumulative probability distributions we observe $A_4 = (0.11, 0)$ and $A_5 = A_4 + (0.89, 0.89)$, and that $B_4 = (0.1, 0.1)$ and $B_5 = B_4 + (0.89, 0.89)$. In other words the 0.89 likelihood of getting the best consequence of 5 Million has been replaced by an 0.89 likelihood of getting 0 in the respective lotteries, resulting in the problem 4. Clearly, exhibiting initially $A_5 \succ B_5$ together with a second preference $A_4 \prec B_4$ violates additivity.

The example above suggests that the following variant of replacement separability is critical. The preference relation $\succcurlyeq$ satisfies *extreme replacement separability* if

$$(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n) \Leftrightarrow (p_1 + \alpha, \ldots, p_n + \alpha) \sim (q_1 + \alpha, \ldots, q_n + \alpha),$$

whenever $(p_1, \ldots, p_n), (q_1, \ldots, q_n), (p_1 + \alpha, \ldots, p_n + \alpha), (q_1 + \alpha, \ldots, q_n + \alpha) \in L$.

The next lemma notes that extreme replacement separability is implied by additivity. The proof is trivial (take $R = (\alpha, \ldots, \alpha)$) and therefore omitted.

LEMMA 9 *Assume $\succcurlyeq$ is a weak order on L that satisfies additivity. Then, $\succcurlyeq$ satisfies extreme replacement separability.* □

Let us now assume that RDU holds, and further that the weighting function is an exponential function, i.e., it is of the form $w(p) = [\exp(cp) - 1]/[\exp(c) - 1]$, $c \neq 0$ (which ensures that w is strictly increasing with $w(0) = 0$, $w(1) = 1$). Then, for a given indifference $(p_1, \dots, p_n) \sim (q_1, \dots, q_n)$, by substituting RDU with the exponential weighting function, we get

$$(p_1, \dots, p_n) \sim (q_1, \dots, q_n)$$

$$\Leftrightarrow$$

$$u(x_0) + \sum_{j=1}^{n} \frac{e^{cp_j} - 1}{e^c - 1}[u(x_j) - u(x_{j-1})] = u(x_0) + \sum_{j=1}^{n} \frac{e^{cq_j} - 1}{e^c - 1}[u(x_j) - u(x_{j-1})].$$

After cancelling common terms, we obtain

$$\sum_{j=1}^{n} e^{cp_j}[u(x_j) - u(x_{j-1})] = \sum_{j=1}^{n} e^{cq_j}[u(x_j) - u(x_{j-1})].$$

A sequence of simple calculus is described next. First, both sides of the latter equality are multiplied by $e^{c\alpha}$ for any α such that $0 \leq p_n + \alpha, 0 \leq q_n + \alpha$, and $p_1 + \alpha \leq 1, q_1 + \alpha \leq 1$. Next we subtract from both sides $\sum_{j=1}^{n}[u(x_j) - u(x_{j-1})]$, then divide both sides by $e^c - 1$, and finally, we add $u(x_0)$ to both sides. This gives

$$u(x_0) + \sum_{j=1}^{n} \frac{e^{c(p_j+\alpha)} - 1}{e^c - 1}[u(x_j) - u(x_{j-1})] = u(x_0) + \sum_{j=1}^{n} \frac{e^{c(q_j+\alpha)} - 1}{e^c - 1}[u(x_j) - u(x_{j-1})]$$

$$\Leftrightarrow$$

$$(p_1 + \alpha, \dots, p_n + \alpha) \sim (q_1 + \alpha, \dots, q_n + \alpha).$$

This demonstrates that RDU with exponential weighting implies extreme replacement separability for a preference $\succcurlyeq$ on L. When c approaches 0, we observe that $w(p) = [\exp(cp) - 1]/[\exp(c) - 1]$ converges to $w(p) = p$, the case of expected utility. Indeed, expected utility also implies extreme replacement separability. The following theorem shows that for RDU-preferences there are no other weighting functions that are able to accommodate extreme replacement separability.

THEOREM 10 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is either represented by expected utility, or it is represented by rank-dependent utility with an exponential weighting function, i.e.,*

$$V(P) = u(x_0) + \sum_{j=1}^{n} \frac{e^{cp_j} - 1}{e^c - 1}[u(x_j) - u(x_{j-1})],$$

with $c \neq 0$, and monotonic utility function $u : X \to I\!R$.

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and extreme replacement separability.*

The function u is cardinal. □

Note that RDU-preferences satisfying both common ratio invariance for decumulative distributions and extreme replacement separability can only be represented by expected utility. This follows immediately by observing that the only possible weighting function that is common in Theorems 8 and 10 is the linear weighting function $w(p) = p$. We state this observation below as it provides a further alternative derivation of expected utility.

COROLLARY 11 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is represented by expected utility with monotonic utility function $u : X \to I\!R$.*

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence, common ratio invariance for decumulative distributions and extreme replacement separability.*

The function u is cardinal. $\square$

6 A Dual Analysis

The properties considered in the previous sections can easily be formulated for cumulative distributions. Jensen-continuity, monotonicity, additivity, comonotonic independence, and also extreme replacement separability have equivalent counterparts which are obtained by simply replacing the decumulative distributions by the corresponding cumulative ones. However, doing the same for the common ratio invariance leads to a different but closely related property. This can be inferred from the corresponding RDU-representation with a weighting function that is the dual of a power function (see Theorem 12 below).

Before we formulate this new property we note that if a lottery is written as a decumulative distribution $P = (p_1, \ldots, p_n)$ then writing the same lottery as a cumulative distribution results in $\tilde{P} = (1 - p_1, \ldots, 1 - p_n)$. The difference in the latter notation is that the cumulative probability $1 - p_i$ denotes the likelihood of getting at most x_{i-1}, $i = 1, \ldots, n$, whereas the decumulative probability p_i is associated with the consequences x_i, $i = 1, \ldots, n$. We denote by $\tilde{L}$ the set of cumulative distributions.

The preference relation $\succcurlyeq$ satisfies *common ratio invariance for cumulative distributions* if

$$(1 - p_1, \ldots, 1 - p_n) \sim (1 - q_1, \ldots, 1 - q_n)$$

$$\Leftrightarrow$$

$$(\alpha(1 - p_1), \ldots, \alpha(1 - p_n)) \sim (\alpha(1 - q_1), \ldots, \alpha(1 - q_n)),$$

whenever $(1-p_1,\dots,1-p_n),(1-q_1,\dots,1-q_n),(\alpha(1-p_1),\dots,\alpha(1-p_n)),(\alpha(1-q_1),\dots,\alpha(1-q_n))\in\tilde{L}$.

This variant of common ratio invariance, which says that shifting probability mass proportionally from all consequences to the best consequence leaves preferences unaffected, is also a weak form of mixture separability (Machina 1989).

We note, without proof, that similar to Lemma 7 one can demonstrate that common ratio invariance for cumulative distributions is implied by additivity and monotonicity of a weak order on L. We get the following analog result to Theorem 8.

THEOREM 12 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(i) *The preference relation $\succcurlyeq$ on L is represented by rank-dependent utility with a dual power weighting function, i.e.,*

$$V(P)=u(x_0)+\sum_{j=1}^{n}[1-(1-p_j)^d][u(x_j)-u(x_{j-1})],$$

with $d>0$, and monotonic utility function $u: X\to I\!R$.

(ii) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance for cumulative distributions.*

The function u is cardinal. □

We formally state the following implication as it also provides an alternative derivation of expected utility.

COROLLARY 13 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(i) *The preference relation $\succcurlyeq$ on L is represented by expected utility with monotonic utility function $u: X\to I\!R$.*

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence, common ratio invariance for cumulative distributions and extreme replacement separability.*

The function u is cardinal. $\square$

A further alternative derivation of expected utility is obtained if both common ratio invariance for decumulative and cumulative distributions are demanded. This result is presented next.

Corollary 14 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is represented by expected utility with monotonic utility function $u : X \to I\!R$.*

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence, common ratio invariance for decumulative distributions and common ratio invariance for cumulative distributions.*

The function u is cardinal. $\square$

7 Inverse-S shaped Weighting Functions

The parametric forms derived in the previous sections are somewhat inflexible in modeling probabilistic risk attitudes. Such risk attitudes are reflected in the shape of the probability weighting function as being concave or convex (see Chew, Karni and Safra 1987, Chateauneuf and Cohen 1994, Wakker 1994, Abdellaoui 2002, Chateauneuf, Cohen and Meilijson 2004). The afore mentioned RDU-preferences either exhibit exclusively probabilistic risk aversion or exclusively probabilistic risk seeking throughout the probability interval. That is, in Theorem 8 either the parameter $b > 1$ (w convex) or $b < 1$ (w concave); in Theorem 10 either $c > 0$ (w convex) or $c < 0$ (w concave); and in Theorem 12 either the parameter $d < 1$ (w convex) or $d > 1$ (w concave). While there is theoretical interest in overall convex/concave probability weighting, empirical findings suggest that a combination of probabilistic risk seeking for small probabilities and probabilistic risk aversion for large probabilities is an appropriate way of modeling sensitivity towards probabilities. Because the concave region for small probabilities is followed by a convex region for larger probabilities (see, e.g., Tversky and Kahneman 1992, Tversky and Fox 1995, Wu and Gonzalez 1996, Abdellaoui 2000), such weighting functions are referred to as inverse-S shaped.

A few parametric forms have been proposed for inverse-S shaped weighting functions (Karmarkar 1978, 1979, Goldstein and Einhorn 1987, Currim and Sarin 1989, Lattimore, Baker and Witte 1992, Tversky and Kahneman 1992, Prelec 1998), and their parameters have been estimated in many empirical studies (Camerer and Ho 1994, Tversky and Fox 1995, Wu and Gonzalez 1996, Gonzalez and Wu 1999, Abdellaoui 2000, Bleichrodt and Pinto 2000, Kilka and Weber 2001, Etchart-Vincent 2004, Abdellaoui, Vossmann and Weber 2005). Most of these parametric forms lack an appropriate axiomatic underpinning. This is problematic because it is unclear what kind of preference condition must be assumed to generate such weighting functions, and therefore, it is unclear what kind of behavioral properties are captured within a specific parametric family of weighting functions.

Axiomatizations have been proposed for the class of weighting functions introduced by Prelec (1998) (see also Luce 2001), and the class introduced by Goldstein and Einhorn (1987), which was axiomatized by Gonzalez and Wu (1999). In the axiomatic derivation of these families of weighting functions it is necessary to assume a rich set of consequences, and further, the representing functional must also be continuous with respect to consequences. From an empirical point of view, this dependence on consequences is a demanding restriction. A further restrictive point in these axiomatizations is that a representing functional, where the continuous utility is already separated from probability weighting, must be assumed prior to invoking the additional invariance property that generates the required parametric form. An open and from an empirical point of view important question is whether, on their own, those characterizing properties are powerful enough to induce such a separation once additive separability as given in Theorem 2 has been derived.

Recall that the results presented in the previous sections are free of restrictions on the richness of the set of consequences, and also free of additional separability conditions that ensure RDU to hold prior to invoking the invariance properties. But note at the same time that these preference conditions are too rigid to permit inverse-S shaped probability weighting functions under RDU. We would like to have both preference conditions that are independent of consequences and also axiomatizations that allow for inverse-S shaped weighting functions under RDU. In what follows we propose such preference conditions, and show that these lead to new families of parametric weighting functions.

To derive RDU with inverse-S shaped weighting functions we restrict the preference conditions presented in the Sections 4–6 to hold only on specific intervals of probabilities. This seems to be a reasonable compromise because, as we show below, these conditions are still powerful enough to separate utility from probability weighting if additive separability holds, that is, if they are added in statement (ii) of Theorem 2. The idea, in line with the empirical evidence, is to impose a first invariance condition for distributions involving small probabilities and a second invariance property for distributions involving large probabilities. This will then give sufficient flexibility in deriving the required weighting functions. However, as we indicate in the next subsection, some unwarranted features relating to the utility functions may occur.

7.1 Switch-power Weighting Functions

The results presented in this section focus on the class of weighting functions which are power functions for probabilities below some $\hat{p} \in (0,1)$, and dual power functions above $\hat{p}$, i.e.,

$$w(p) = \begin{cases} cp^a, & \text{if } p \leqslant \hat{p}, \\ 1 - d(1-p)^b, & \text{if } p > \hat{p}, \end{cases}$$

with the parameters involved as discussed below. We call these functions *switch-power weighting functions.*

We presented the function above with five parameters a, b, c, d and $\hat{p}$. However, these reduce to four because of continuity of w on $[0,1]$, and if differentiability is assumed, a reduction to three parameters on $(0,1)$ is obtained. Let us elaborate on these reductions. Continuity at 0 implies that $a > 0$, and monotonicity implies that $c > 0$. Continuity at 1 implies that $b > 0$, and monotonicity implies that $d > 0$. Continuity and differentiability at $\hat{p}$ relates a, c to b, d and $\hat{p}$ through

$$c = \frac{1}{\hat{p}^a} - \frac{d(1-\hat{p})^b}{\hat{p}^a},$$

and

$$c = \frac{db(1-\hat{p})^{b-1}}{a\hat{p}^{a-1}},$$

respectively. Combining the two gives

$$\begin{aligned} c &= \hat{p}^{-a}\left[\frac{b\hat{p}}{b\hat{p}+a(1-\hat{p})}\right], \\ d &= (1-\hat{p})^{-b}\left[\frac{a(1-\hat{p})}{b\hat{p}+a(1-\hat{p})}\right]. \end{aligned}$$

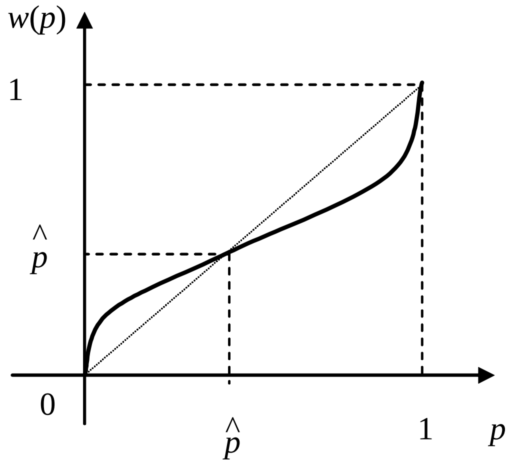
If $0 < a \leq 1$ the probability weighting function is concave on $(0,\hat{p})$, and if $0 < b \leq 1$ it is convex on $(\hat{p},1)$, hence has an inverse-S shape. For $a, b \geq 1$ we have a S-shaped probability weighting function, which is convex on $(0, \hat{p})$ and concave on $(\hat{p}, 1)$. When $\hat{p}$ approaches 1 or 0, the weighting function reduces to a power weighting function or a dual power weighting function, respectively. Moreover, substitution of $\hat{p}$ into w gives

$$\begin{aligned} w(\hat{p}) &= \frac{b\hat{p}}{b\hat{p} + a(1-\hat{p})} \\ &= 1 - \frac{a(1-\hat{p})}{b\hat{p} + a(1-\hat{p})}, \end{aligned}$$

from which one can easily derive the relationship

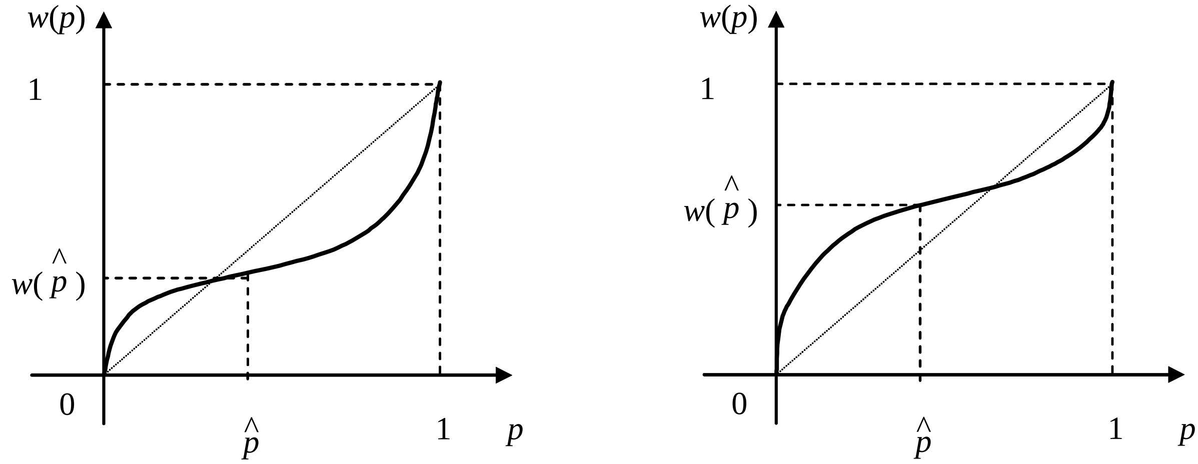
$$w(\hat{p}) \leqslant \hat{p} \Leftrightarrow b \leqslant a.$$

In particular, this shows that whenever $a = b$ the weighting function intersects the 45° line precisely at $\hat{p}$ (see Figure 1). One should also note that in this case the derivative of w at $\hat{p}$ equals a, and therefore this parameter controls for the curvature of the weighting function. The parameter $\hat{p}$, however, indicates whether the interval for overweighting of probabilities is larger than the interval for underweighting, and therefore controls for the elevation of the weighting function (see also Gonzalez and Wu (1999) for a similar interpretation of the parameters in the "linear in log-odds" weighting function of Goldstein and Einhorn (1987)).



A two parameter switch-power weighting function.

In general, when $a \neq b$, two parameters control for curvature. In that case $\hat{p}$ need not demarcate the regions of over and underweighting because it may not lie on the 45° line. Nevertheless, $\hat{p}$ will still influence elevation, however, whether there is more overweighting relative to underweighting now also depends on the relationship between the magnitudes of the parameters a and b. The following figure depicts, for the case of an inverse-S shaped weighting function, the two scenarios of underweighting ($0 < b < a < 1$), respectively, overweighting ($0 < a < b < 1$) at $\hat{p}$.



3-parameter weighting function with underweighting respectively overweighting at $\hat{p}$.

As it turns out, it is more appropriate to interpret these parameters as was initially proposed by Tversky and Kahneman (1992). All parameters may influence elevation, however, the main role of $\hat{p}$ is to demarcate the interval of probabilistic risk aversion from the interval of probabilistic risk seeking. The magnitude of the parameter a indicates diminishing (or increasing) sensitivity to changes from impossibility to possibility. This can be inferred by inspecting the derivative of w for probabilities in the range $(0, \min\{\hat{p}, 1 - \hat{p}\})$. Observe, that for $q \in (0, \min\{\hat{p}, 1-\hat{p}\})$ we get

$$\begin{aligned} w'(p)_{|p=q} &= (cap^{a-1})_{|p=q} \\ &= \hat{p}^{-a}\left[\frac{ab\hat{p}}{b\hat{p}+a(1-\hat{p})}\right] q^{a-1}. \end{aligned}$$

Therefore, sensitivity increases if $a > 1$ and decreases if $a < 1$. Note also that for $a = 1$ sensitivity is constant. Note also that the right-derivative at 0, $w'(0_-) = 0$ if $a > 1$ and is unbounded if $a < 1$, the latter indicating extreme sensitivity for changes from possible to impossible.

Similarly, as one moves away from certainty, sensitivity increases if $b > 1$ and decreases if $b < 1$, while for $b = 1$ sensitivity is constant. There is extreme sensitivity for changes from certainty to possibility if $b < 1$.

The switch-power weighting function also allows for a comparison of the sensitivity to changes from 0 relative to the sensitivity to changes from 1. Considering the ratio of derivatives at q and $1-q$ for $q \in (0, \min\{\hat{p}, 1-\hat{p}\})$ we observe

$$\frac{w'(p)_{|p=q}}{w'(p)_{|p=1-q}} = \left[\frac{(1-\hat{p})^{b-1}}{\hat{p}^{a-1}}\right] q^{a-b}.$$

Therefore, this relative sensitivity is constant when $a = b$, but otherwise there is more (less) sensitivity for changes from 0 than for changes from 1 if $a < b$ ($a > b$). As q approaches $\min\{\hat{p}, 1-\hat{p}\}$, the ratio $w'(q)/w'(1-q)$ is decreasing (increasing) towards

$$\frac{w'(\hat{p})}{w'(1-\hat{p})} = \begin{cases} [(1-\hat{p})/\hat{p}]^{b-1}, & \text{if } \hat{p} \leqslant 1/2, \\ [\hat{p}/(1-\hat{p})]^{a-1}, & \text{if } \hat{p} > 1/2. \end{cases}$$

There are some extreme cases that should be mentioned here. Taking limits when only a approaches 0 gives a weighting function that equals 0 at 0 and is constant equal to 1 on $(0, 1]$. Taking limits when only b approaches 0 we get a weighting function that equals 1 at 1 and is constant equal to 0 on $[0,1)$. These latter weighting functions do not exhibit continuity or monotonicity, and therefore fall outside the RDU-functionals considered in this paper. Similarly, this holds for the classes of weighting functions where $a=b$ and a approaches 0, or when $a \neq b$ and either a or b approach infinity.

The preference condition that is necessary for RDU with (inverse) S-shaped switch-power weighting function is defined next. *Common ratio invariance* holds if there exists a probability $\hat{p} \in (0,1)$ such that

$$(p_1,\dots,p_n) \sim (q_1,\dots,q_n) \Leftrightarrow (\alpha p_1,\dots,\alpha p_n) \sim (\alpha q_1,\dots,\alpha q_n),$$

whenever all $(p_1,\dots,p_n),(q_1,\dots,q_n),(\alpha p_1,\dots,\alpha p_n),(\alpha q_1,\dots,\alpha q_n) \in L_{\hat{p}} := \{R \in L : r_1 \leqslant \hat{p}\}$ and

$$\begin{array}{ccc} (1-p_1,\dots,1-p_n) & \sim & (1-q_1,\dots,1-q_n) \\ & \Leftrightarrow & \\ (\beta(1-p_1),\dots,\alpha(1-p_n)) & \sim & (\beta(1-q_1),\dots,\alpha(1-q_n)), \end{array}$$

whenever $(1-p_1,\dots,1-p_n)$, $(1-q_1,\dots,1-q_n)$, $(\beta(1-p_1),\dots,\beta(1-p_n))$, and $(\beta(1-q_1),\dots,\beta(1-q_n)) \in \tilde{L}_{\hat{p}} := \{R \in \tilde{L} : 1-r_n \leqslant 1-\hat{p}\}$.

Clearly common ratio invariance requires preferences to be immune to common proportional changes in decumulative probabilities whenever these are all smaller than some $\hat{p} \in (0,1)$ and it does also require immunity of preferences to common proportional changes in cumulative probabilities if these are all larger than $\hat{p}$. As the result below shows, replacing common ratio invariance for (de)cumulative distributions in (Theorem 8) Theorem 12 with the weaker common ratio invariance does not necessarily give RDU. As it turns out this property leads to a more general class of preferences represented by a RDU-like functional that combines a unique switch-power weighting function with possibly two utility functions depending on the evaluated distribution. We state this result before we analyze this aspect further.

THEOREM 15 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is represented by an additive representation as in Theorem 2 with functions V_j as follows:*

$$V_j(p) = \begin{cases} s_j[cp^a], & \text{if } p \leqslant \hat{p}, \\ \hat{s}_j[1 - d(1-p)^b], & \text{if } p > \hat{p}, \end{cases}$$

for some $\hat{p} \in (0,1)$ with $a, b, c, d > 0$, and positive $s_j, \hat{s}_j$ for all $j = 1, \ldots, n$.

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance.*

The parameters $\hat{p}, a, b, d$ are uniquely determined, and $c = 1/\hat{p}^a - d(1-\hat{p})^b/\hat{p}^a$. Further the s_j's and the $\hat{s}_j$'s can be replaced by corresponding ts_j's and $t\hat{s}_j$'s for any positive t. □

This theorem shows that, by making the sensitivity towards small probabilities independent from that for large probabilities, a more general functional than RDU is obtained. However, when we restrict to specific sets of distributions the derived representing functional still gives RDU. We elaborate on this point next.

Take $k \in \{0, \ldots, n\}$ and define $L(k) := \{P \in L : p_k \leqslant \hat{p} < p_{k+1}\}$. Then, on $L(k)$ the functions derived in Theorem 15 take the form

$$V_j(p) = \begin{cases} s_j[cp^a], & \text{if } p \leqslant \hat{p}, \\ \hat{s}_j[1 - d(1-p)^b], & \text{if } p > \hat{p}, \end{cases}$$

for some $\hat{p} \in (0,1)$ with $a, b, d > 0, c = 1/\hat{p}^a - d(1-\hat{p})^b/\hat{p}^a$ and positive $s_j, \hat{s}_j$ for all $j = 1, \ldots, n$. In this case we define $u(x_0) = 0$ and iteratively $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \ldots, k$ and

$u(x_j) = u(x_{j-1}) + \hat{s}_j$ for $j = k+1, \ldots, n$. This means that on $L(k)$ the preference relation is represented by

$$RDU_k(P) = u(x_0) + \sum_{j=1}^{n} w(p_j)[u(x_j) - u(x_{j-1})],$$

with switch-power weighting function

$$w(p) = \begin{cases} cp^a, & \text{if } p \leqslant \hat{p}, \\ 1 - d(1-p)^b, & \text{if } p > \hat{p}, \end{cases}$$

and strictly monotonic cardinal utility u. Hence, RDU has been obtained for $\succcurlyeq$ on $L(k)$.

In general, for different values of k, the RDU-functionals (or RDU-restrictions) must not agree. This shows the price that we pay for further relaxing the common ratio invariance properties of the previous sections so that they apply only on restricted sets of distributions.

An additional preference condition is now required to derive RDU for $\succcurlyeq$ on L. Such a condition has been proposed in Zank (2004). There it was shown that, in the presence of Theorem 2, the probabilistic consistency condition is necessary and sufficient to give general RDU, hence cardinal utility, without requiring any structural assumptions on the set of consequences. In this paper we present a version of that condition that is much weaker, and on its own not sufficient to give RDU, but when added to statement (ii) of Theorem 15 above, the property implies RDU with switch-power weighting function.

The preference relation $\succcurlyeq$ satisfies *consistency* if

$$p_I(\gamma, \ldots, \gamma) \sim \hat{p}_I(\delta, \ldots, \delta)$$

and

$$\hat{p}_I(\gamma, \ldots, \gamma) \sim q_I(\delta, \ldots, \delta)$$

imply

$$p_{I\setminus\{i\}}\hat{p}_i(\gamma, \ldots, \gamma) \sim \hat{p}_{I\setminus\{i\}}q_i(\delta, \ldots, \delta),$$

whenever $I = \{1,\dots,i\}$ or $I = \{i,\dots,n\}, i \in \{1,\dots,n\}$, and $q < \hat{p} < p$ are such that the above distributions are in L.

Note that, given monotonicity and continuity, the first two indifferences can always be derived locally. Consistency then requires that the measured indifferences for consequence x_i remains valid when measured for consequence x_{i-1} (respectively x_{i+1}). Under the assumptions of Theorem 15 the condition will preclude the possibility of having two utility functions that determine choice behavior. This can be inferred from the following calculus.

Suppose that $I = \{1,\dots,i\}$ for some $1 < i < n$, and for given $\hat{p}$ take $p > q$, and $\gamma < \delta$ such that $p_I(\gamma,\dots,\gamma) \sim \hat{p}_I(\delta,\dots,\delta)$, $\hat{p}_I(\gamma,\dots,\gamma) \sim q_I(\delta,\dots,\delta)$, and by consistency $p_{I\setminus\{i\}}\hat{p}_i(\gamma,\dots,\gamma) \sim \hat{p}_{I\setminus\{i\}}q_i(\delta,\dots,\delta)$. Then, taking the first and third indifference, substituting the functional form described in statement (i) of Theorem 15, and subtracting the two equations, we get

$$\hat{s}_i w(p) + s_i w(q) = \hat{s}_i w(\hat{p}) + s_i w(\hat{p}),$$

after cancelling common terms.

Similarly, taking the first and second indifference we get

$$\sum_{j=1}^{i}[\hat{s}_j w(p) + s_j w(q)] = \sum_{j=1}^{i}[\hat{s}_j w(\hat{p}) + s_j w(\hat{p})].$$

Therefore, for $i = 2$, we observe

$$\hat{s}_2 w(p) + s_2 w(q) = \hat{s}_2 w(\hat{p}) + s_2 w(\hat{p})$$

and

$$\hat{s}_1 w(p) + s_1 w(q) + \hat{s}_2 w(p) + s_2 w(q) = \hat{s}_1 w(\hat{p}) + s_1 w(\hat{p}) + \hat{s}_2 w(\hat{p}) + s_2 w(\hat{p}).$$

After substituting the first equation in the latter and cancelling common terms, we get the equivalent equations

$$\begin{aligned}\hat{s}_2 w(p) + s_2 w(q) &= \hat{s}_2 w(\hat{p}) + s_2 w(\hat{p}),\\ \hat{s}_1 w(p) + s_1 w(q) &= \hat{s}_1 w(\hat{p}) + s_1 w(\hat{p}),\end{aligned}$$

from which

$$\frac{\hat{s}_1}{s_1} = \frac{\hat{s}_2}{s_2}$$

follows. More generally it follows by induction on i that

$$\frac{\hat{s}_{i-1}}{s_{i-1}} = \frac{\hat{s}_i}{s_i}$$

holds for all $i = 2, \ldots, n$.[2] If one normalizes the positive s_i's and $\hat{s}_i$'s such that they each sum to one, which can always be done, one observes that $s_i = \hat{s}_i$ must hold. Therefore RDU with a switch-power weighting function has been obtained. We summarize this analysis in the next theorem:

THEOREM 16 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(*i*) *The preference relation $\succcurlyeq$ on L is represented by RDU with a switch-power utility,*

$$w(p) = \begin{cases} cp^a, & \text{if } p \leqslant \hat{p},\\ 1 - d(1-p)^b, & \text{if } p > \hat{p},\end{cases}$$

for some $\hat{p} \in (0,1)$ with $a, b, c, d > 0$.

(*ii*) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance.*

The parameters $\hat{p}, a, b, d$ are uniquely determined and $c = 1/\hat{p}^a - d(1-\hat{p})^b/\hat{p}^a$. Further, the utility function u is cardinal. □

[2] To get $\hat{s}_{n-1}/s_{n-1} = \hat{s}_n/s_n$ one must use consistency with $I = \{n-1, n\}$.

7.2 Reversed Switch-power Weighting Functions

Following the line of argument presented in the previous subsection, one can also provide axiomatic characterizations for RDU with an analog to the switch-power weighting function that first is a dual power weighting function followed by a power weighting function, or, alternatively, a switch-exponential weighting function made up of two different exponential functions. We present these forms below and discuss them briefly. As the characteristic preference conditions can be derived in analogy to the common ratio invariance property, we analyze these weighting functions without explicitly presenting the corresponding axiomatic foundation.

The *"reversed" switch-power weighting* function has the form:

$$w(p) = \begin{cases} c[1-(1-p)^a], & \text{if } p \leqslant \hat{p}, \\ 1-d(1-p^b), & \text{if } p > \hat{p}, \end{cases}$$

where continuity at 0 and 1 has already been exploited. Monotonicity implies that $ca > 0$ and $db > 0$. By requiring continuity and differentiability at $\hat{p}$ one can determine the parameters c and d in terms of a, b, and $\hat{p}$, giving the following, somewhat complex expressions:

$$\begin{aligned} c &= \frac{(1-\hat{p})b\hat{p}^b}{\hat{p}a(1-\hat{p})^a(1-\hat{p}^b)+(1-\hat{p})b\hat{p}^b[1-(1-\hat{p})^a]}, \\ d &= \frac{\hat{p}a(1-\hat{p})^a}{\hat{p}a(1-\hat{p})^a(1-\hat{p}^b)+(1-\hat{p})b\hat{p}^b[1-(1-\hat{p})^a]}. \end{aligned}$$

Diminishing (increasing) sensitivity at 0 occurs iff $a > 1$ ($a < 1$) while diminishing (increasing) sensitivity at 1 occurs iff $b > 1$ ($b < 1$). Again, $\hat{p}$ separates the regions of possible distinct probabilistic risk behavior. In contrast to the switch power weighting function in Theorem 16, it is not immediate to conclude from the relation between the magnitude of a and b whether there is overweighting at $\hat{p}$. The condition for overweighting at $\hat{p}$ is given as

$$w(\hat{p}) > \hat{p} \Leftrightarrow \frac{a}{b} < \frac{1-(1-\hat{p})^a}{(1-\hat{p})^{a-2}} \frac{\hat{p}^{b-2}}{1-\hat{p}^b},$$

and hence overweighting, respectively, underweighting at $\hat{p}$ does also depend on the magnitude of $\hat{p}$.

A further point of contrast to the switch-power weighting function concerns the boundedness of the slope at extreme probabilities. A steep derivative at 0 or 1 has been criticized elsewhere for leading to implausible choice behavior (e.g., Schmidt and Zank 2005, Rieger and Wang 2006).The slope of the reversed switch-power weighting function does not approach 0 or ∞ at certainty and impossibility, except when $\hat{p} \in \{0, 1\}$. The bounds on the slope are $w'(0) = ca$, respectively, $w'(1) = db$, and, using the expressions for c, d derived above, one observes that whether there is greater sensitivity at 0 compared to 1 will depend on all three parameters a, b, and $\hat{p}$. That is,

$$\frac{w'(0)}{w'(1)} = \frac{\hat{p}^{b-1}}{(1-\hat{p})^{a-1}}.$$

Note also that this latter ratio reappears when comparing the degrees of sensitivity for equal deviations from impossibility and certainty, i.e., the ratio of derivatives at q and $1 - q$ for $q \in (0, \min\{\hat{p}, 1 - \hat{p}\})$ equals

$$\frac{w'(p)_{|p=q}}{w'(p)_{|p=1-q}} = \frac{\hat{p}^{b-1}}{(1-\hat{p})^{a-1}}[1-q]^{a-b}.$$

Therefore, we can conclude that this measure of relative sensitivity is constant if $a = b$, but otherwise it increases (decreases) with q if $a < b$ ($a > b$). As q approaches $\min\{\hat{p}, 1 - \hat{p}\}$, the ratio $w'(q)/w'(1 - q)$ is increasing (decreasing) towards

$$[\hat{p}/(1-\hat{p})]^{b-1}, \text{ if } \hat{p} \leqslant 1/2, \text{ or } [(1-\hat{p})/\hat{p}]^{a-1}, \text{ if } \hat{p} > 1/2.$$

7.3 The Switch-exponential Weighting Function

Let us now consider the switch-exponential weighting function.[3] Exploiting continuity at 0 and 1, the general from of this class of weighting functions is

$$w(p) = \begin{cases} c(e^{ap} - 1), & \text{if } p \leqslant \hat{p}, \\ 1 - d(e^{b} - e^{bp}), & \text{if } p > \hat{p}, \end{cases}$$

with $ac > 0, db > 0$ by monotonicity, and due to continuity at $\hat{p}$ it holds that

$$c = \frac{1}{e^{a\hat{p}} - 1} - \frac{d(e^{b} - e^{b\hat{p}})}{e^{a\hat{p}} - 1}.$$

Requiring differentiability at $\hat{p}$ implies

$$c = \frac{db}{a}\frac{e^{b\hat{p}}}{e^{a\hat{p}}},$$

which, combined with the previous expression for c, allows us to determine both c, d in terms of a, b, and $\hat{p}$:

$$\begin{aligned} c &= \frac{be^{b\hat{p}}}{ae^{a\hat{p}}(e^{b} - e^{b\hat{p}}) + be^{b\hat{p}}(e^{a\hat{p}} - 1)}, \\ d &= \frac{ae^{a\hat{p}}}{ae^{a\hat{p}}(e^{b} - e^{b\hat{p}}) + be^{b\hat{p}}(e^{a\hat{p}} - 1)}. \end{aligned}$$

One can immediately derive the conditions for which there is diminishing (increasing) sensitivity at 0 and 1. An inverse-S shaped weighting function is obtained if $a < 0$ and $b > 0$, while an S-shaped weighting functions must have $a > 0$ and $b < 0$.

In the case of an inverse-S weighting function (i.e., $a < 0, b > 0$), the condition for overweighting at $\hat{p}$ comes down to

$$w(\hat{p}) > \hat{p} \Leftrightarrow \frac{-a}{b} < \frac{e^{-a\hat{p}} - 1}{e^{b(1-\hat{p})} - 1}\frac{1 - \hat{p}}{\hat{p}}.$$

[3] We restrict our analysis to the cases that the weighting function is exponential below some parameter $\hat{p}$ and exponential above it. As can be inferred from Theorem 10, the characterizing preference condition will allow also for linearity below or above the parameter $\hat{p}$.

Observe that the sensitivity to changes from impossibility is given by $w'(0) = ca$ and the sensitivity to changes from certainty is $w'(1) = dbe^b$, and that both expressions must exceed 1 in order to have overweighting for small probabilities and underweighting for large ones. By substituting for c and d we can determine if there is more sensitivity at 0 compared to sensitivity at 1 through

$$\frac{w'(0)}{w'(1)} = \frac{e^{(b-a)\hat{p}}}{e^b}.$$

Note that in the case of an inverse-S shaped weighting function one obtains

$$\frac{w'(0)}{w'(1)} > 1 \Leftrightarrow \hat{p} > \frac{b}{b-a},$$

hence, whether there is greater sensitivity at 0 compared to 1 will depend on all three parameters a, b, and $\hat{p}$. We compare how this relative sensitivity evolves as one moves away from the extreme probabilities. For $q \in (0, \min\{\hat{p}, 1-\hat{p}\})$ it holds that

$$\frac{w'(p)_{|p=q}}{w'(p)_{|p=1-q}} = \frac{e^{(b-a)\hat{p}}}{e^b} e^{(a+b)q},$$

hence relative sensitivity increases if $b > -a$ (decreases if $b < -a$), reaching its maximum (minimum) at $\min\{\hat{p}, 1-\hat{p}\}$ as follows:

$$e^{b(2\hat{p}-1)}, \text{ if } \hat{p} \leqslant 1/2, \text{ or } e^{a(2\hat{p}-1)}, \text{ if } \hat{p} > 1/2.$$

An analog statement can be concluded for the case of an S-shaped weighting function. Note that there is constant relative sensitivity if $b = -a$.

8 Summary

Our main objective in this paper has been to provide preference foundations for parametric weighting functions in a general RDU framework where the set of consequences is arbitrary.

Inevitably, these preference foundations have to employ conditions that exploit the mathematical structure offered by the probability interval. Initially, we have derived three classes of such RDU-forms with a single parameter for probability weighting. In all these derivations cardinal utility is obtained as a bonus in addition to the specific parametric form (power, exponential, or dual power) of the weighting functions.

The power weighting function is directly related to the common ratio pattern of preferences (Allais 1953) that has been discussed extensively in the literature in relation to violations of the vNM-independence axiom. It has also been pointed out that the exponential weighting function is directly related to the common consequence pattern of preferences (Allais 1953), a somewhat surprising connection that has not been mentioned before in the literature. The dual power weighting function has no documented EU-paradox to be liked to, but we think that a dual analog of the common ratio paradox of Allais can easily be constructed, even though we assess the benefits of obtaining a new option for criticizing expected utility as limited. However, viewed from a different perspective, the preference conditions that give rise to these weighting functions will hopefully lead to a better understanding of how demanding EU is, and in particular how demanding the vNM-independence axiom actually is.

The one-parameter classes of weighting functions have shortcomings for descriptive applications. In particular it not possible to separate sensitivity to changes in small probabilities from sensitivity to changes in large probabilities because there is a single parameter that has to govern both. Empirical studies suggest that there is extreme sensitivity to changes from certainty or impossibility to possibility, and also that this sensitivity diminishes as one approaches moderate probabilities. Taking account of this evidence, we have proposed to separate the probability interval into two exhaustive regions on which the preference conditions that implied the one-parameter weighting functions still hold. Therefore, we had to specify in advance where the boundary is that separates the intervals of distinct sensitivity to changes in probabilities, and this boundary probability appears as one of the parameters in our weighting functions. This is different to the axiomatizations offered by Prelec (1998) and Gonzalez and Wu (1999) because there the probability value that separates the regions of distinct sensitivity is implicit in the corresponding preference conditions. It should be noted, however, that those axiomatizations do not apply to our framework, in particular, because the preference conditions characterizing those weighting functions may not be well-defined here. Also, Prelec (1998) and Gonzalez and Wu (1999), in fact, model sensitivity to changes in the logarithm of probabilities instead of probabilities as we do. From a technical point of view this is an important difference as the interval of transformed probabilities is large enough to generate, with the appropriate axiom assumed, two regions in which changes in log-odds point in opposite direction.[4] We think that modeling sensitivity to probability changes is more natural under RDU, certainly this is the case if one works in the general framework that we have adopted in this paper.

By specifying exogenously the parameter separating sensitivity regions within the probability interval, we have also induced additional flexibility for the representing functions. By simply restricting some preference conditions to hold on particular subsets of the probability interval, the resulting representing functionals belong to a much larger class than that of RDU-preferences. That is, although we can obtain unique parametric weighting functions, in general there may be two cardinal utility functions that govern choice behavior. Further, the number of parameters that we get for the weighting functions —four— seems too large. To

[4] The argument used here is best exemplified for the case of, e.g., positive power functions that apply to positive numbers. Assume that the power exceeds 1. For numbers smaller than 1 applying the power function leads to decreases of the original number, while application to numbers larger than 1 results in increases. So, naturally, 1 demarcates these opposite changes in magnitude.

resolve these issues we have employed additional conditions. To retain RDU with a parametric inverse-S shaped weighting function we have introduced an axiom that explicitly requires consistency of measured preferences irrespective of consequences. This then gives a single cardinal utility, hence RDU. And to reduce the number of parameters we assume differentiability of the weighting function, which, although it seems a reasonable constraint, is enforced exogenously.

However, except for the parametrizations presented in this paper there are no other foundations of RDU in the literature that combine parametric weighting functions and general utility. The previous parametrizations either lack preference foundations or their preference foundations are meaningful only in the special case of continuous utility. Neither is satisfactory. To some extent we have been able to resolve these shortcomings. For example, we did this for the one-parameter classes that we obtained. But, although progress has been made, our attempt to add more empirical realism and still obtain simple classes of parametric weighting functions compromises on other aspects. In particular, the problem of endogenizing the separation of the probability interval into regions of distinct probabilistic risk attitudes or distinct sensitivity, and thereby also reducing the number of parameters in the weighting functions (instead of employing differentiability), remains an open question.

9 Appendix: Proofs

Proof of Lemma 3: Assume that $\succcurlyeq$ satisfies vNM-independence and that for some $P, Q \in L$ we have $p_j \geqslant q_j$ for all $j = 1, \ldots, n$ but $P \neq Q$. We proceed by induction on the cardinality of the set $I = \{j : p_j > q_j\}$. Suppose $|I| = 1$ such that $P = (p_1, \ldots, p_n)$ and $Q = q_i P$ with $q_i < p_i$ for some $i \in \{1, \ldots, n\}$. We prove that $P \succ Q$. Instead of $x_i \succ x_{i-1}$ we use the equivalent notation for decumulative distributions $1_{\{1,\ldots,i\}}0 \succ 1_{\{1,\ldots,i-1\}}0$. The following equivalences follow from applying repeatedly vNM-independence

$$
\begin{aligned}
1_{\{1,\dots,i\}}0 &\succ 1_{\{1,\dots,i-1\}}0 \\
&\Leftrightarrow \\
(1-\frac{q_i}{p_i})1_{\{1,\dots,i\}}0+\frac{q_i}{p_i}(1,\dots,1,\frac{p_{i+1}}{q_i},\dots,\frac{p_n}{q_i}) &\succ (1-\frac{q_i}{p_i})1_{\{1,\dots,i-1\}}0+\frac{q_i}{p_i}(1,\dots,1,\frac{p_{i+1}}{q_i},\dots,\frac{p_n}{q_i}) \\
&\Leftrightarrow \\
(1,\dots,1,\frac{p_{i+1}}{p_i},\dots,\frac{p_n}{p_i}) &\succ (1,\dots,1,\frac{q_i}{p_i},\frac{p_{i+1}}{p_i},\dots,\frac{p_n}{p_i}).
\end{aligned}
$$

A subsequent application of vNM-independence by p_i-mixing both distributions with

$$(\frac{p_1-p_i}{1-p_i},\dots,\frac{p_{i-1}-p_i}{1-p_i},0,\dots,0)$$

gives $P \succ Q$.

Suppose now that $k > 1$ and that we have $P' \succ Q'$ whenever $p'_j \geqslant q'_j$ for all $j = 1,\dots,n$ but $P' \neq Q'$ and $|\{j : p_j > q_j\}| < k$. Assume further that for $P, Q \in L$ we have $p_j \geqslant q_j$ for all $j = 1,\dots,n$, $P \neq Q$, and $|I| = |\{j : p_j > q_j\}| = k$. We prove that $P \succ Q$. Let i be the smallest index such that $p_i > q_i$ (that is, for $j < i$ we have $p_j = q_j$). Let R, S be decumulative distributions defined as follows

$$R = (1,\dots,1,\frac{p_{i+1}}{q_i},\dots,\frac{p_n}{q_i}) \text{ and } S = (1,\dots,1,\frac{q_{i+1}}{q_i},\dots,\frac{q_n}{q_i}).$$

By the induction assumption it follows that $R \succ S$ because $p_j/q_i \geqslant q_j/q_i$ for all $j > i$, $R \neq S$

and $|\{j : p_j/q_i > q_j/q_i\}| = k - 1$. By vNM-independence the following equivalence holds

$$
\begin{array}{ccc}
R & \succ & S \\
& \Leftrightarrow & \\
(1 - \frac{q_i}{p_i})1_{\{1,\ldots,i\}}0 + \frac{q_i}{p_i}R & \succ & (1 - \frac{q_i}{p_i})1_{\{1,\ldots,i-1\}}0 + \frac{q_i}{p_i}S \\
& \Leftrightarrow & \\
(1, \ldots, 1, \frac{p_{i+1}}{p_i}, \ldots, \frac{p_n}{p_i}) & \succ & (1, \ldots, 1, \frac{q_i}{p_i}, \ldots, \frac{q_n}{p_i}).
\end{array}
$$

A further application of vNM-independence by p_i-mixing the latter distributions with

$$(\frac{p_1 - p_i}{1 - p_i}, \ldots, \frac{p_{i-1} - p_i}{1 - p_i}, 0, \ldots, 0)$$

gives $P \succ Q$.

Recall that P and Q were arbitrary with $p_j \geqslant q_j$ for all $j = 1, \ldots, n$, $P \neq Q$, and $|I| = |\{j : p_j > q_j\}| = k$. Therefore, by induction, it follows that $P \succ Q$ whenever $p_j \geqslant q_j$ for all $j = 1, \ldots, n$, $P \neq Q$. Hence, monotonicity is derived from weak order and vNM-independence, which concludes the proof. $\square$

Proof of Lemma 4: Assume that $\succcurlyeq$ satisfies vNM-independence and that $P, Q, P + R, Q + R \in L$. Suppose that $P \succcurlyeq Q$. We show, by contradiction, that $P + R \succcurlyeq Q + R$. Assume that $P + R \prec Q + R$. Then, vNM-independence implies that

$$
\begin{array}{ccc}
P & \succcurlyeq & Q \\
& \Leftrightarrow & \\
\frac{1}{2}P + \frac{1}{2}[Q + R] & \succcurlyeq & \frac{1}{2}Q + \frac{1}{2}[Q + R],
\end{array}
$$

and

$$P+R \quad \prec \quad Q+R$$

$$\Leftrightarrow$$

$$\frac{1}{2}[P+R]+\frac{1}{2}Q \quad \prec \quad \frac{1}{2}[Q+R]+\frac{1}{2}Q.$$

Using transitivity we get $\frac{1}{2}P+\frac{1}{2}[Q+R] \succ \frac{1}{2}[P+R]+\frac{1}{2}Q$ which contradicts completeness (or reflexivity) of the preference relation. Hence, $P \succcurlyeq Q \Rightarrow P+R \succcurlyeq Q+R$. The reversed implication follows by a similar argument. This completes the proof that weak order and vNM-independence imply additivity. $\square$

Proof of Lemma 5: Assume that the weak order $\succcurlyeq$ satisfies additivity. Further, suppose that for some $\alpha_i P, \alpha_i Q, \beta_i P, \beta_i Q \in L$ we have $\alpha_i P \succcurlyeq \alpha_i Q$ and $\beta_i P \prec \beta_i Q$. Obviously, by completeness $\alpha \neq \beta$. If $\alpha > \beta$, then, as $p_{i-1}, q_{i-1} > \alpha \Leftrightarrow p_{i-1}-\alpha+\beta, q_{i-1}-\alpha+\beta > \beta$, by additivity the following equivalence holds

$$\beta_i P \quad \prec \quad \beta_i Q$$

$$\Leftrightarrow$$

$$(p_1-\alpha+\beta,\dots,p_{i-1}-\alpha+\beta,\beta,p_{i+1},\dots,p_n) \quad \prec \quad (p_1-\alpha+\beta,\dots,p_{i-1}-\alpha+\beta,\beta,p_{i+1},\dots,p_n),$$

where we added $R=(\alpha-\beta)_{\{1,\dots,i-1\}}(0,\dots,0) \in L$ on both sides of the latter preference. A subsequent application of additivity with $R'=(\alpha-\beta)_{\{1,\dots,i\}}(0,\dots,0) \in L$ gives the equivalence

$$(p_1-\alpha+\beta,\dots,p_{i-1}-\alpha+\beta,\beta,p_{i+1},\dots,p_n) \quad \prec \quad (p_1-\alpha+\beta,\dots,p_{i-1}-\alpha+\beta,\beta,p_{i+1},\dots,p_n)$$

$$\Leftrightarrow$$

$$\alpha_i P \quad \prec \quad \alpha_i Q,$$

contradicting $\alpha_i P \succcurlyeq \alpha_i Q$. If $\alpha < \beta$, then a similar double application of additivity (with $R=(\beta-\alpha)_{\{1,\dots,i\}}(0,\dots,0), R'=(\beta-\alpha)_{\{1,\dots,i-1\}}(0,\dots,0) \in L$) is used to derive the latter contradiction. As $\alpha_i P, \alpha_i Q, \beta_i P, \beta_i Q \in L$ were chosen arbitrary we conclude that $\alpha_i P \succcurlyeq \alpha_i Q \Leftrightarrow \beta_i P \succcurlyeq \beta_i Q$ or equivalently that comonotonic independence holds. This completes the proof that weak order and additivity imply comonotonic independence. $\square$

PROOF OF LEMMA 7: Assume that the weak order $\succcurlyeq$ satisfies additivity and monotonicity. Further, suppose that for some $\alpha > 0$ we have $P = (p_1, \ldots, p_n), Q = (q_1, \ldots, q_n), \alpha P = (\alpha p_1, \ldots, \alpha p_n), \alpha Q = (\alpha q_1, \ldots, \alpha q_n) \in L$ and that $(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n)$. The case that $P = Q$ is trivial, so we assume $P \neq Q$.

Next we consider the case that $\alpha < 1$. The case that $\alpha = 1$ is trivial, while the case that $\alpha > 1$ is completely analogous to the case when $\alpha < 1$ is assumed, the difference being that the role of P, Q and $\alpha P, \alpha Q$, respectively, is reversed because in the analysis they are replaced by $1/\alpha(\alpha P), 1/\alpha(\alpha Q)$ and $\alpha P, \alpha Q$, respectively with $1/\alpha < 1$.

So, assume $\alpha < 1$, $P \sim Q$ and $\alpha P \succ \alpha Q$, (and note that the case $\alpha P \prec \alpha Q$ follows similarly if the role of P and Q are interchanged). Suppose that $(1-\alpha)P \succcurlyeq (1-\alpha)Q$. Then, a first application of additivity with $R = (1-\alpha)P$ implies

$$\begin{array}{rcl} \alpha P & \succ & \alpha Q \\ & \Leftrightarrow & \\ P & \succ & \alpha Q + (1-\alpha)P, \end{array}$$

and a second application of additivity with $R = \alpha Q$ gives

$$\begin{array}{rcl} (1-\alpha)P & \succcurlyeq & (1-\alpha)Q \\ & \Leftrightarrow & \\ \alpha Q + (1-\alpha)P & \succcurlyeq & Q. \end{array}$$

Using transitivity we observe that $P \succ \alpha Q + (1-\alpha)P \succcurlyeq Q \Rightarrow P \succ Q$, a contradiction to

$P \sim Q$. We conclude therefore that if $\alpha P \succ \alpha Q$ then $(1-\alpha)P \prec (1-\alpha)Q$ must hold.

Further, we observe that $\alpha \neq 1/2$. This follows from applying additivity twice to $1/2P \succ 1/2Q$ (first with $R = 1/2P$ and then with $R = 1/2Q$) and transitivity, and observing that this contradicts with $P \sim Q$. In particular, it follows that if $P \sim Q$ then also $P \sim 1/2P + 1/2Q \sim Q$. This observation obviously means that there exists a dense subset of $[0,1]$ such that for any value of α' in that subset $P \sim Q$ implies $\alpha' P \sim \alpha' Q$ ($\alpha' \in \{(2^l - 1)/2^k : k$ and l are natural numbers such that $k \geqslant l\}$). If $\succcurlyeq$ is continuous then the latter must hold for all $\alpha' \in [0,1]$. In particular, under the assumption of continuity, $P \sim Q$ implies $P \sim \alpha' P + (1-\alpha')Q \sim Q$ for any $\alpha' \in [0,1]$, a property called betweenness (see Chew 1983, Dekel 1986, Chew, Epstein and Segal 1991, Safra and Segal 1998 for analyses of this property). We did not assume continuity here. Instead our proof of this lemma relies on monotonicity of the preference relation.

If $\alpha \neq 1/2$ then either $\alpha < 1/2 < 1-\alpha$ or $\alpha > 1/2 > 1-\alpha$. Assume the former (and note that a similar argument applies if the latter case is assumed). Then monotonicity implies

$$\alpha P \succ \alpha Q \Rightarrow (1-\alpha)P \succ \alpha Q,$$

and additivity gives

$$\begin{aligned} (1-\alpha)P &\succ \alpha Q \\ &\Leftrightarrow \\ \alpha P + (1-\alpha)P &\succ \alpha P + \alpha Q \\ &\Leftrightarrow \\ P &\succ \alpha P + \alpha Q. \end{aligned}$$

Note further, that $(1-\alpha)P \prec (1-\alpha)Q$ and an application of additivity with $R = \alpha P$ gives

$$P \prec (1-\alpha)P + \alpha Q \in L.$$

Using transitivity, $P \prec (1-\alpha)P + \alpha Q$ and $P \succ \alpha P + \alpha Q$ implies $(1-\alpha)P + \alpha Q \succ \alpha P + \alpha Q$ contradicting monotonicity (as $\alpha < 1-\alpha$). This means that $\alpha P \succ \alpha Q$ and $(1-\alpha)P \prec (1-\alpha)Q$ cannot hold simultaneously. Therefore, $P \sim Q$ and $\alpha P \succ \alpha Q$ cannot hold jointly (and by a similar argument neither can $P \sim Q$ and $\alpha Q \succ \alpha P$ hold jointly). Hence, $P \sim Q$ implies $\alpha P \sim \alpha Q$. As P, Q and $0 < \alpha \leqslant 1$ were chosen arbitrary (and the case $\alpha > 1$ can be proven analogously), we conclude that $P \sim Q$ implies $\alpha P \sim \alpha Q$ whenever $P, Q, \alpha P, \alpha Q \in L$.

This completes the proof that weak order, monotonicity and additivity imply common ratio invariance. □

Proof of Theorem 8: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Common ratio invariance for decumulative distributions has been derived in the main text preceding Theorem 8.

Next we prove that statement (ii) implies statement (i). Obviously statement (ii) in Theorem 2 is satisfied, hence, there exists an additively separable functional representing the preference $\succcurlyeq$. We restrict the attention to the case that $p_1 < 1$ and $p_n > 0$ to avoid the problem of dealing with unbounded V_1, V_n. To show that our additive functional in fact is a RDU form with power weighting function we use results presented in Wakker and Zank (2002). Wakker and Zank did not have the restrictions that $p_1 < 1$ and $p_n > 0$ but permitted any non-negative rank-ordered real numbers $x_i, i = 1, \ldots, n$ because they worked in a setup with monetary outcomes instead of decumulative probabilities as we do here. But their results apply to our framework with minor modifications, in particular the restriction $p_1 \leqslant 1$ is not posing any difficulty. In their Lemma A2 they derived a similar additive representation as we have in Theorem 2, and then in their Lemma A3, using the analog of common ratio invariance for decumulative distributions, they showed that their additive representation in fact is a RDU form with common positive power function as "utility" and increasing "weighting function". To apply their results we just need to revert the roles of utility and weighting function. Further, because the functions $V_j, j = 1, \ldots, n$ are proportional they can continuously be extended to 0 and 1 (this follows from Wakker 1993, Proposition 3.5). Hence, we can conclude that there exist positive numbers s_j such that

$$V_j(p_j) = s_j w(p_j),$$

with $w(p) = a + c(p)^b$, for some real a, b, c. Monotonicity and continuity imply that b, c are positive, and requiring further that $w(0) = 0$ and $w(1) = 1$ shows that $a = 0$ and $c = 1$. Hence, $w(p) = p^b$ is established. We define utility iteratively as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \ldots, n$. Therefore, $V_j(p_j) = w(p_j)s_j = w(p_j)[u(x_j) - u(x_{j-1})]$ for $j = 1, \ldots, n$ with strictly monotonic utility u. We can conclude that the additive representation in Theorem 2 is RDU with a power weighting function and monotonic utility. Therefore statement (i) has been derived.

Uniqueness results follow from the joint cardinality of the functions V_j in Theorem 2, and the fact that they are proportional. These properties translate into the weighting function being unique because it assigns 0 to impossibility and 1 to certainty, and the utility being cardinal. This concludes the proof of Theorem 8. $\square$

Proof of Theorem 10: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Common ratio invariance has been shown in the main text preceding Theorem 10.

Next we prove that statement (ii) implies statement (i). As in the proof of Theorem 8, statement (ii) in Theorem 2 is satisfied, hence, there exists an additively separable functional representing the preference $\succcurlyeq$. Attention is initially restricted to the case that $p_1 < 1$ and $p_n > 0$ to exclude unbounded V_1 and V_n. To show that this additive functional is RDU with an exponential weighting function we use results presented in Zank (2001). Zank did allow for non-negative vectors with rank-ordered monetary outcomes in his Lemma 7 instead of probabilities as we have here. However, those results apply to the case considered here if we interchange the roles of utility and decision weights. Hence, we can conclude that in the representation of Theorem 2 the functions V_j are increasing exponential functions, i.e.,

$$V_j(p) = s_j[a \exp(cp) + b],$$

with $ac > 0$ and $s_j > 0$, and real b (or they are linear$V_j(p) = s_j[ap + b]$ with $a > 0$). As the functions are proportional, we can extend them continuously to all of $[0, 1]$ by Proposition 3.5 of Wakker (1993). We fix scale and location of the otherwise jointly cardinal V_j, i.e., $V_j(0) = 0, V_j(1) = 1$. Hence,

$$V_j(p) = s_j[\frac{e^{cp_j} - 1}{e^c - 1}],$$

with $c \neq 0$ (or $V_j(p) = s_j p$). We use the positive s_j's to define utility as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \ldots, n$. Therefore, the V_j's are exponential or linear for $j = 1, \ldots, n$ and u is strictly monotonic. Hence, statement (i) has been derived.

Uniqueness results follow by similar arguments as in the proof of Theorem 8. This concludes the proof of Theorem 10. □

Proof of Corollary 11: The proof follows directly by combining the results Theorems 8 and 10. That statement (i) implies statement (ii) is immediate. Conversely, note that the only weighting function that is a power function and also an exponential function must be linear, implying RDU with linear weighting function or expected utility. Uniqueness results are maintained as in Theorem 8 (or Theorem 10). This completes the proof of the corollary. □

Proof of Theorem 12: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Common ratio invariance for cumulative distributions can easily be demonstrated using a similar line of arguments as was used for the analog derivation of common ratio invariance for decumulative distributions presented in the main text preceding Theorem 8.

Next we prove that statement (ii) implies statement (i). Obviously statement (ii) in Theorem 2 is satisfied, hence, there exists an additively separable functional representing the preference $\succcurlyeq$. We restrict the attention to the case that $p_1 < 1$ and $p_n > 0$ to avoid the problem of dealing with unbounded V_1, V_n. To show that our additive functional is RDU with a dual power weighting function we use, similarly to the proof of Theorem 8, results of Wakker and Zank (2002). We define $W_j(1-p_j) = V_j(1-(1-p_j))$ $(= V_j(p_j))$ for $j = 1, \ldots, n$. These functions are decreasing in $(1-p_j)$ and they give an additive representation as we have in Theorem 2 but now on the set of cumulative distributions $\tilde{L}$. Using Lemma A3 of Wakker and Zank (2002) and common ratio invariance for cumulative distributions, this latter additive representation is in fact a RDU form with common positive power weighting function that is decreasing in cumulative probabilities. Further, because the functions $W_j, j = 1, \ldots, n$ are proportional they can continuously be extended to 0 and 1 (this follows from Wakker 1993, Proposition 3.5). Hence, there exist positive numbers s_j such that

$$\begin{aligned} W_j(1-p_j) &= s_j \tilde{w}(1-p_j) \\ &= V_j(p_j), \end{aligned}$$

with $\tilde{w}(1-p) = a - c(1-p)^d$,for some real a, c, d. Monotonicity and continuity imply that c, d are positive, and requiring further that $\tilde{w}(0) = 0$ and $\tilde{w}(1) = 1$ shows that $a = 0$ and $c = 1$. Hence, $\tilde{w}(1-p) = 1 - (1-p)^d$ is established, and we define utility iteratively as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \ldots, n$. Therefore, $V_j(p_j) = \tilde{w}(1-p_j)s_j = \tilde{w}(1-p_j)[u(x_j) - u(x_{j-1})]$ for $j = 1, \ldots, n$ with strictly monotonic utility u. We can conclude that the additive representation in Theorem 2 is RDU with dual a power weighting function and monotonic utility. Therefore statement (i) has been derived.

Uniqueness results follow by similar arguments as in the proof of Theorem 8. This concludes the proof of Theorem 12. □

PROOF OF COROLLARY 13: The proof follows directly by combining the results Theorems 10 and 12. That statement (i) implies statement (ii) is immediate. Conversely, note that the only weighting function that is a dual power function and also an exponential function must be linear, implying RDU with linear weighting function or expected utility. Uniqueness results are maintained as in Theorem 12 (or Theorem 10). This completes the proof of the corollary. □

PROOF OF COROLLARY 14: The proof follows directly by combining the results Theorems 8 and 12. That statement (i) implies statement (ii) is immediate. Conversely, note that statement (ii) in Theorems 8 and 12 hold, and therefore RDU with power weighting function $w(p) = p^b, b > 0$ holds and also RDU with dual power weighting function $w(p) = 1 - (1-p)^d, d > 0$ holds. As the weighting function under RDU is unique, we get

$$w(p) = p^b = 1 - (1-p)^d,$$

or equivalently

$$p^b + (1-p)^d = 1$$

for all $p \in [0,1]$. This implies that one of the following cases holds: (a) $b = 1 = d$, or (b) $b > 1 > d$, or (c) $b < 1 < d$. Considering the left derivative of w at 1 we observe that cases (b) and (c) cannot hold. In case (b) we have $\lim_{p\to 1} w'(p) = \lim_{p\to 1} bp^{b-1} = b$ and $\lim_{p\to 1} w'(p) = \lim_{p\to 1} d(1-p)^{d-1} = \infty$. The case (c) is similar by looking at the right derivative at 0. Therefore the only weighting function that is a power function and also a dual power function is the linear function, implying RDU with linear weighting function or expected utility. Uniqueness results are maintained as in Theorem 12 (or Theorem 8). This completes the proof of the corollary. □

PROOF OF THEOREM 15: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. For $\succcurlyeq$ restricted to $L_{\hat{p}}$ ($\tilde{L}_{\hat{p}}$), common ratio invariance comes down to common ratio invariance for decumulative (cumulative) distributions and can easily be derived by substitution of the specific RDU functional. Basically, a similar line of arguments is used as for the derivation of common ratio invariance for decumulative distributions, as was presented in the main text preceding Theorem 8.

Next we prove that statement (ii) implies statement (i). Obviously statement (ii) in Theorem 2 is satisfied, hence, there exists an additively separable functional representing the preference $\succcurlyeq$. We restrict the attention to the case that $p_1 < 1$ and $p_n > 0$ to avoid the problem of dealing with unbounded V_1, V_n. Similarly to the proof of Theorems 8 and 12, we use the results of Wakker and Zank (2002). The arguments used in the proof of Theorem 8 remain valid if we restrict the analysis to probability distributions in $L_{\hat{p}}$. We can conclude that the V_j's obtained in Theorem 2 are proportional power functions for decumulative probabilities not exceeding $\hat{p}$.

That is, there exist positive numbers s_j such that

$$V_j(p_j) = s_j w(p_j),$$

with $w(p) = cp^a$, for some positive a and c.

Similarly, the arguments used in the proof of Theorem 12 remain valid if we restrict the analysis to probability distributions in $\tilde{L}_{\hat{p}}$. We can conclude that the V_j's obtained in Theorem 2 are proportional dual power functions for cumulative probabilities not exceeding $1-\hat{p}$. That is, there exist positive numbers $\hat{s}_j$ such that

$$V_j(p_j) = \hat{s}_j w(p_j),$$

with $w(p) = 1 - d(1-p)^b$, for some positive d and b. Hence, statement (i) has been obtained.

Continuity at $\hat{p}$ implies that the parameters are related through $c = 1/\hat{p}^a - d(1-\hat{p})^b/\hat{p}^a$. Uniqueness results follow from the joint cardinality of the functions V_j in Theorem 2, and the fact that they are proportional. These properties translate into the weighting function being unique because it assigns 0 to impossibility and 1 to certainty, and that the s_j's and $\hat{s}_j$'s can be replaced only if re-scaled by a common positive number t. This concludes the proof of Theorem 15. $\square$

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