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by

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Pure and randomized equilibria in the stochastic von Neumann–Gale model

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Abstract

The paper examines the problem of the existence of equilibrium for the stochastic analogue of the von Neumann-Gale model of economic growth. The mathematical framework of the model is a theory of set-valued random dynamical systems defined by positive stochastic operators with certain properties of convexity and homogeneity. Existence theorems for equilibria in such systems may be regarded as generalizations of the Perron-Frobenius theorem on eigenvalues and eigenvectors of positive matrices. The known results of this kind are obtained under rather restrictive assumptions. We show that these assumptions can be substantially relaxed if one allows for randomization. The main result of the paper is an existence theorem for randomized equilibria. Some special cases (models defined by positive matrices) are considered in which the existence of pure equilibria can be established.

1 Introduction

This paper examines questions related to the stochastic version of the von Neumann-Gale growth model (von Neumann (1937), Gale (1956)). Historically, this was one the first models in Mathematical Economics that led to a rich and interesting theory (see e.g. Rockafellar (1967) and Makarov and Rubinov (1977)). Originally, the focus of the theory was on economic

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growth. It has recently been observed (Evstigneev and Taksar (2000), Dempster, Evstigneev and Taksar (2005)) that a stochastic generalization of the von Neumann–Gale framework can be fruitfully applied to the analysis of fundamental problems in Mathematical Finance (asset pricing under transaction costs, capital growth theory). This observation gave a new momentum to studies in this area, raised new questions and revived interest to long-standing open problems.

The classical version of the von Neumann–Gale model is purely deterministic. The importance of its stochastic generalizations was realized early on. First steps in this direction were made in the pioneering contributions by Dynkin (1971) (see also Dynkin and Yushkevich (1979) Chapter 4), Radner (1971, 1972) and their research groups in the early 1970s. However, the initial attack on these problems left many questions unanswered, and a substantial progress in this area was achieved only in the last decade. An account of these achievements is given in the survey by Evstigneev and Schenk-Hoppé (2006), where one can find further references.

The mathematical basis of the von Neumann-Gale model is a theory of multivalued dynamical systems (e.g. Rockafellar (1967), Akin (1993), Aubin and Frankowska (1990)). The set-valued operators defining such systems possess certain properties of positivity, convexity and homogeneity. In the stochastic case, these operators become random. They map elements of certain cones of random vectors into convex subsets of such cones.

A central role in the theory is played by the notion of a von Neumann equilibrium. This notion may be regarded as a generalization of some fundamental concepts in the theory of positive operators. Roughly speaking, a von Neumann equilibrium is a counterpart of a triplet (x, p, α), where x and α are the Perron-Frobenius eigenvector and eigenvalue of a positive operator A and p is the Perron–Frobenius eigenvector of the dual to A. For some classes of stochastic von Neumann-Gale models—the analysis of which can be reduced to the analysis of single-valued positive random operators—the existence of an equilibrium follows from appropriate stochastic versions of the Perron-Frobenius theorem; see Section 6 and Appendix B of the present paper.

Currently, the existence of a von Neumann equilibrium is established under conditions that also guarantee its stability (the turnpike property)—see Evstigneev and Schenk-Hoppé (2006) for the statements of the results. Very few results are available pertaining to stochastic analogues of the classical *linear* von Neumann model (von Neumann (1937)). One can argue that the von Neumann equilibrium is "useful" mainly when it is stable—then the turnpike provides a good approximation for all rapidly growing paths. Nevertheless, it is of interest and importance to obtain general existence the-

orems independent of the issues of stability. The present paper focuses on this problem.

A well-known way of dealing with existence problems for equilibrium and optimal solutions is to introduce randomization. A classical example is the concept of Nash equilibrium in mixed strategies. A whole range of similar notions and related results are known in control, optimization and games, e.g. Kreps (1990) and Young (1969). We show that by an appropriate extension of the model at hand, one can establish the existence of a randomized von Neumann equilibrium under quite general conditions. This extension is constructed by using an auxiliary "sunspot" process, serving as an additional source of randomness.

The structure of the paper is as follows. Section 2 describes the general stochastic analogue of the von Neumann-Gale model and introduces key concepts associated with it. Section 3 focuses on a class of stationary models defined by random cones. Section 4 contains statements of the main results of this work pertaining to randomized equilibria and provides formulations of some previous results used in the present study. Proofs of the theorems on randomized equilibria are given in Section 5. Section 6 concentrates on a special class of models defined in terms of positive random matrices. Two Appendices, A and B, assemble some general mathematical facts used in this work.

2 The stochastic version of the von Neumann-Gale model

2.1. The basic model.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $... \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}$ a non-decreasing sequence of σ-algebras. Sets in $\mathcal{F}_t$ are interpreted as events observable prior to time t. Vector functions of $\omega \in \Omega$ measurable with respect to $\mathcal{F}_t$ are construed as random vectors whose realizations become known by time t. For each t, we denote by $L_1^n(t)$ the space $L_1(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ consisting of (equivalence classes of) n-dimensional $\mathcal{F}_t$-measurable vector functions $x(\omega)$ with $E|x| = E\sum_i |x^i| < \infty$. The letter E stands for the expectation with respect to the given probability measure P. We denote by $L_\infty^n(t) = L_\infty(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ the space of essentially bounded functions in $L_1^n(t)$ and by $\mathcal{X}_t$ the cone of nonnegative elements in $L_\infty^n(t)$.

The stochastic version of the von Neumann–Gale model is specified by the above probabilistic structure and a family of cones $Z_t \subseteq \mathcal{X}_{t-1} \times \mathcal{X}_t$ $(t = 1, 2, ...)$. A sequence of random vectors $\{x_t\}_{t=0}^N$, $x_t \in \mathcal{X}_t$ $(N \leq \infty)$, is called

a *path* (*trajectory*) in the model if $(x_{t-1}, x_t) \in Z_t$ for all t. Equivalently, the model can be described by a family of multivalued operators $x \mapsto A_t(x)$ $(t = 1, 2, ...)$, where $A_t(x) = \{y : (x, y) \in Z_t\}$. It will be assumed that $A_t(x) \neq \emptyset$ for each $x \in \mathcal{X}_{t-1}$, which means that the projection of the cone Z_t on the first factor in the product $\mathcal{X}_{t-1} \times \mathcal{X}_t$ coincides with $\mathcal{X}_{t-1}$. It is easily seen that the graph Z_t of the operator $x \mapsto A_t(x)$ is a cone if and only if the following homogeneity and convexity conditions hold:

$$\lambda A_t(x) \subseteq A_t(\lambda x), \ \lambda \in [0, \infty), \ x \in \mathcal{X}_{t-1}; \tag{1}$$

$$\theta A_t(x) + (1 - \theta) A_t(x') \subseteq A_t(\theta x + (1 - \theta) x'), \ x, x' \in \mathcal{X}_{t-1}, \ \theta \in [0, 1]. \tag{2}$$

(A linear combination of two sets in a vector space is defined as the set of pairwise linear combinations of their elements: $A + A' = \{a + a' : a \in A, a' \in A'\}$ and $\lambda A = \{\lambda a : a \in A\}$.) Clearly, the condition $(x_{t-1}, x_t) \in Z_t$ involved in the definition of a path $\{x_t\}$ of the dynamical system under consideration is equivalent to $x_t \in A_t(x_{t-1})$.

In addition to the assumption that the sets Z_t are cones, we will always suppose that these sets satisfy the following condition: *if* $(x, y), (x', y') \in Z_t$ *and* $\Gamma \in \mathcal{F}_{t-1}$, *then* $\mathbf{1}_\Gamma(x, y) + (1 - \mathbf{1}_\Gamma)(x', y') \in Z_t$, where $\mathbf{1}_\Gamma$ is the indicator function equal to 1 on Γ and 0 outside Γ. If this condition holds, the cone Z_t is called $\mathcal{F}_{t-1}$-*decomposable*. This property expresses a possibility of *choice* between (x, y) and (x', y') depending on information contained in $\mathcal{F}_{t-1}$. Examples of such cones will be considered below.

2.2. Rapid paths.

The main focus of the theory is on paths that grow faster in a sense than others. A path $\{x_t\}_{t=0}^N$ $(N \leq \infty)$ is said to be *rapid* if there exists a sequence of nonnegative random vectors $\{p_t\}_{t=0}^N$ such that $p_t \in L_1^n(t)$,

$$p_t x_t = 1 \text{ (a.s.)} \tag{3}$$

for all $t \geq 0$, and

$$E(p_t y / p_{t-1} x) \leq 1 \tag{4}$$

for all $t \geq 1$ and all $(x, y) \in Z_t$ with $p_{t-1} x > 0$. A rapid path maximizes in each period the expected value of the *growth rate* $(p_t y_t - p_{t-1} y_{t-1})/p_{t-1} y_{t-1}$ among all paths $\{y_t\}_{t=0}^N$ for which $p_{t-1} y_{t-1} \neq 0$.

The typical interpretation of $\{p_t\}$ in economic contexts is that of prices depending on the (random) state of the economic environment. A rapid path achieves the highest expected growth rate of the aggregate value $p_t x_t$. The

fact that $p_t x_t$ is supposed to be equal to 1 is just a matter of convenience; instead of the constant 1 we could take any constant, independent of time and of ω.

There are several equivalent ways to define a rapid path. It can be shown (see Evstigneev and Flåm (1998) Proposition 2.2) that if (3) holds, then condition (4), involved in the definition of a rapid path, can be replaced by any of the following requirements:

$$E \ln \left(\frac{p_t y}{p_{t-1} x} \right) \leq 0; \tag{5}$$

$$E p_t y \leq E p_{t-1} x; \tag{6}$$

$$E \left(p_t y \mid \mathcal{F}_{t-1} \right) \leq p_{t-1} x, \tag{7}$$

where $t \geq 1$, $(x, y) \in Z_t$ and, additionally, $p_{t-1} x \neq 0$ in (5). A rapid path therefore maximizes the expected logarithmic growth rate, and it maximizes the one-period expected gain in aggregate value (both in the sense of unconditional and conditional expectation).

A sequence $\{p_t\}_{t=0}^N$ of nonnegative random vectors is called a *dual path* if $p_t \in L_1^n(t)$ and any of the equivalent conditions (6) and (7) holds. (The equivalence of these conditions follows from the $\mathcal{F}_{t-1}$-decomposability of Z_t.) It follows from (7) that, for any dual path $\{p_t\}_{t=0}^N$ and any trajectory $\{y_t\}_{t=0}^N$, the sequence $\{p_t y_t\}_{t=0}^N$ is a *supermartingale* with respect to the filtration $\{\mathcal{F}_t\}_{t=0}^N$. We say that a dual path $\{p_t\}_{t=0}^N$ *supports* the trajectory $\{x_t\}_{t=0}^N$ if $p_t x_t = 1$ for all t. Thus, *a trajectory $\{x_t\}_{t=0}^N$ is rapid if and only if there exists a dual path $\{p_t\}_{t=0}^N$ supporting it.*

2.3. Stationary models.

In this paper, we will focus on stationary models. This is the framework where the notion of a von Neumann equilibrium is defined. The stationary version of the stochastic von Neumann–Gale model is defined as follows. Suppose that, in addition to the above data, we are given a one-to-one mapping $T : \Omega \to \Omega$ (the *time shift*). The model is called *stationary* if the following invariance conditions hold:

(**Inv.1**) The mappings T and T^{-1} are $\mathcal{F}$-measurable and preserve the measure P, i.e., $P(\Gamma) = P(T^{-1}\Gamma) = P(T\Gamma)$ for each $\Gamma \in \mathcal{F}$ (that is, the transformation T is an *automorphism* of the probability space $(\Omega, \mathcal{F}, P)$).

(**Inv.2**) We have

$$\mathcal{F}_{t+1} = T^{-1}\mathcal{F}_t \ (= \{T^{-1}\Gamma : \Gamma \in \mathcal{F}_t\}) \ (t = 0, \pm 1, \pm 2, ...). \tag{8}$$

(**Inv.3**) A pair of vector functions $(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t$ belongs to the set Z_t if and only if the pair $(Tx, Ty) \in \mathcal{X}_t \times \mathcal{X}_{t+1}$ belongs to the set Z_{t+1}.

Here and in what follows, the letter T is used to denote both the transformation of Ω and the induced transformation $(Tx)(\omega) = x(T\omega)$ acting on functions of $\omega \in \Omega$. The transformation T may be thought of as a shift of the time scale one unit of time forward. By virtue of (8), a random variable ξ is $\mathcal{F}_t$-measurable if and only if $T\xi$ is $\mathcal{F}_{t+1}$-measurable. Condition (**Inv.3**), stated in terms of the cones Z_t, can equivalently be formulated in terms of the operators $A_t(\cdot)$ as follows:

$$A_{t+1}(Tx) = TA_t(x),\ x \in \mathcal{X}_{t-1}. \tag{9}$$

2.4. Balanced paths and von Neumann equilibria.

An infinite path $\{x_t\}_{t=0}^{\infty}$ is called *balanced* if there exists a random vector $x \in \mathcal{X}_0$ and a random scalar $0 < \alpha \in L_\infty^1(1)$ such that almost surely[1]

$$x_t = \alpha_1\alpha_2...\alpha_t\bar{x}_t \text{ for all } t \geq 1,\ x_0 = x,\ \text{and } |x_0| = 1, \tag{10}$$

where

$$\alpha_t = T^{t-1}\alpha,\ \text{and } \bar{x}_t = T^t x. \tag{11}$$

Each component x_t^i $(i = 1, 2, ..., n)$ of a balanced path grows at the same stationary random rate determined by the growth factor $x_t^i(\omega)/x_{t-1}^i(\omega) = \alpha(T^t\omega)$ and with stationary proportions $x_t^i(\omega)/x_t^j(\omega) = x^i(T^t\omega)/x^j(T^t\omega)$. Clearly, a pair (x, α) generates a balanced path if and only if

$$x \in \mathcal{X}_0,\ |x| = 1,\ 0 < \alpha \in L_\infty^1(1),\ \text{and } (x, \alpha Tx) \in Z_1. \tag{12}$$

A balanced path (10) maximizing the expected logarithmic growth rate $E\ln\alpha_t$ (independent of t in view of stationarity) is called a *von Neumann path.*

A triple of nonnegative functions

$$(x, \alpha, p),\ 0 \leq x \in L_\infty^n(0),\ 0 < \alpha \in L_\infty^1(1),\ 0 \leq p \in L_1^n(0), \tag{13}$$

is said to form a *von Neumann equilibrium* if the following conditions hold:

[1]All equalities and inequalities between scalar- and vector-valued functions of ω are supposed to hold almost surely (a.s.) and coordinatewise. We will usually omit "a.s." where this does not lead to ambiguity.

(a) the sequence $x_t = \alpha_1 ... \alpha_t \bar{x}_t$, $x_0 = \bar{x}_0$ (where $\alpha_t = T^{t-1}\alpha$ and $\bar{x}_t = T^t x$) is a balanced path; and

(b) the sequence $p_t = (\alpha_1 ... \alpha_t)^{-1} \bar{p}_t$, $p_0 = \bar{p}_0$ (where $\bar{p}_t = T^t p$) is a dual path supporting $\{x_t\}$.

If the above requirements are met, $\{x_t\}$ is called an *equilibrium path* and $\{p_t\}$ an *equilibrium dual path*. The stationary process $\alpha_1, ..., \alpha_t, ...$ is the sequence of random *equilibrium growth factors*. Dual paths of the form described in (b) are called *balanced*.

Under the assumptions we impose on the cones Z_t, it can be shown (Arnold, Evstigneev and Gundlach (1999) Section 3) that a triple (x, α, p) of the form (13) is a von Neumann equilibrium if and only if

$$(x, \alpha T x) \in Z_1, \ |x| = 1, \ px = 1, \tag{14}$$

and

$$E\left(\alpha^{-1}(Tp)y \mid \mathcal{F}_0\right) \leq px \text{ for all } (x, y) \in Z_1. \tag{15}$$

According to the above definition, a von Neumann equilibrium defines a balanced path growing at a stationary rate and a balanced dual path supporting it and decreasing at the *same* rate.

2.5. Assumptions for stationary models.

We introduce assumptions on the cones Z_t (or, equivalently, on the operators $A_t(x) = \{y \in \mathcal{X}_t : (x, y) \in Z_t\}$) that are used in the analysis of the stationary stochastic version of the von Neumann–Gale model. In hypotheses (**Z.0**)–(**Z.4**), the subscript t ranges over $\{1, 2, ...\}$. In (**Z.4**) and (**Z.5**), γ stands for some fixed strictly positive number.

(**Z.0**) If $(x, y) \in Z_t$ and λ is a $\mathcal{F}_{t-1}$-measurable random variable with non-negative real values, then $(\lambda x, \lambda y) \in Z_t$, provided λx and λy are essentially bounded.

(**Z.1**) The set Z_t is closed in $\mathcal{X}_{t-1} \times \mathcal{X}_t$ with respect to a.s. convergence of sequences uniformly bounded in the norm $|| \cdot ||_\infty = \operatorname{ess\,sup} | \cdot |$.

(**Z.2**) There is a constant M such that $|y| \leq M|x|$ for any $(x, y) \in Z_t$.

(**Z.3**) If $(x, y) \in Z_t$, $x' \in \mathcal{X}_{t-1}$, $y' \in \mathcal{X}_t$, $x' \geq x$ and $y' \leq y$, then $(x', y') \in Z_t$.

(**Z.4**) For some $(\check{x}_{t-1}, \check{y}_t) \in Z_t$, we have $\check{y}_t \geq \gamma e$, where $e = (1, 1, ..., 1)$.

(**Z.5**) There exists an integer $m \geq 1$ such that, for every $i \in \{1, ..., n\}$ and every $t \in \{0, 1, 2, ...\}$, one can find random vectors $y_t \in \mathcal{X}_t, ..., y_{t+m} \in \mathcal{X}_{t+m}$ satisfying

$$y_t = e_i, \ (y_t, y_{t+1}) \in Z_{t+1}, ..., (y_{t+m-1}, y_{t+m}) \in Z_{t+m}, \ y_{t+m} \geq \gamma e. \tag{16}$$

Condition (**Z.0**) is a version of the hypothesis of $\mathcal{F}_{t-1}$-decomposability of the cone Z_t. Note that the random variable λ involved in this condition is not necessarily bounded. In view of stationarity, if the above hypotheses hold for some t, they hold for all t.

3 Stationary models defined by random cones

3.1. The process of "states of the world" and random cones.

Classes of models most important for the applications in economics and finance are defined in terms of an exogenous process of random factors. Let $..., s_{-1}, s_0, s_1, ...$ be a stationary random process with values in some measurable space (s_t is the "state of the world" at time t). Let Ω be the space whose elements are sequences $\omega = (..., s_{-1}, s_0, s_1, ...)$. Denote by $\mathcal{F}$ the σ-algebra defining the product measurable structure on Ω and by P the probability measure on $\mathcal{F}$ induced by the given stochastic process. Define $\mathcal{F}_t$ $(t = 0, \pm 1, ...)$ as the σ-algebra generated by the "history" $s^t(\omega) = (..., s_{t-1}(\omega), s_t(\omega))$ of the process $..., s_{-1}, s_0, s_1, ...$ up to time t. (We write $s_t(\omega)$ for the tth element of the sequence $\omega = (..., s_{-1}, s_0, s_1, ...)$.) Suppose the shift operator T is given by $s_t(T\omega) = s_{t+1}(\omega)$. Then conditions (**Inv.1**) and (**Inv.2**) are fulfilled.

For each $s^t = (...s_{t-1}, s_t)$, let $G(s^t)$ be a closed convex cone whose elements are pairs on nonnegative n-dimensional vectors $(a, b) \in \mathbb{R}^{2n}_+$. Suppose $G(s^t)$ depends measurably on s^t, i.e., the Euclidean distance $d(c, G(s^t))$ is a measurable function of $s^t \in ... \times S \times S$ for each $c \in \mathbb{R}^{2n}_+$ (the space $... \times S \times S$ is endowed with the product measurable structure). Assume that, for each s^t, the following conditions hold:

(**G.1**) for any $a \in \mathbb{R}^n_+$, the set $\{b : (a, b) \in G(s^t)\}$ is non-empty;

(**G.2**) the set $G(s^t)$ contains with every (a, b) all (a', b') such that $a' \geq a$ and $0 \leq b' \leq b$;

(**G.3**) the set $G(s^t)$ is contained in $\{(a, b) : |b| \leq M|a|\}$ where M does not depend on t and s^t;

(**G.4**) there exists a pair of random vector functions $(\check{a}(s^{t-1}), \check{b}(s^t))$ such that $(\check{a}(s^{t-1}), \check{b}(s^t)) \in G(s^t)$ for all s^t and $\check{b}(s^t) \geq \gamma e$.

(**G.5**) There is an integer $m \geq 1$ such that, for every $i \in \{1, ..., n\}$ one can find measurable vector functions $y_1(s^1), ..., y_m(s^m)$ satisfying

$$\big(e_i, y_1(s^1)\big) \in G(s^1), \big(y_1(s^1), y_2(s^2)\big) \in G(s^2), ...,$$

$$\big(y_{m-1}(s^{m-1}), y_m(s^m)\big) \in G(s^m),\ y_m(s^m) \geq \gamma e.$$

The number γ involved in (**G.4**) and (**G.5**) is supposed to be non-random and strictly positive.

Define

$$G_t(\omega) = G(s^t(\omega)),$$

$$Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t :\ (x(\omega), y(\omega)) \in G_t(\omega) \text{ a. s.}\}. \tag{17}$$

For each t, the set Z_t is an $\mathcal{F}_{t-1}$-decomposable convex cone in $\mathcal{X}_{t-1} \times \mathcal{X}_t$, and so the sets Z_t, $t = 1, 2, ...$, define a von Neumann-Gale model, which we will denote by $\mathcal{M}$. This model is stationary because conditions (**Inv.1**)–(**Inv.3**) hold, the latter being valid in view of the relation $G_t(\omega) = G_{t-1}(T\omega)$. Furthermore, it satisfies conditions (**Z.0**) and (**Z.1**), which follow immediately from (17) and the closedness of $G_t(\omega)$. Properties (**Z.2**)–(**Z.5**) follow from (**G.2**)–(**G.5**), respectively.

Any $\mathcal{F}_t$-measurable function of ω can be represented as a measurable function of $s^t(\omega)$. Therefore paths in the model $\mathcal{M}$ can be identified with sequences $\{x_0(s^0), x_1(s^1), x_2(s^2), ...\}$ of essentially bounded measurable vector functions satisfying

$$(x_0(s^0), x_1(s^1)) \in G(s^1),\ (x_1(s^1), x_2(s^2)) \in G(s^2),\ (x_2(s^2), x_3(s^3)) \in G(s^3), ... \tag{18}$$

(a.s.). Dual paths can be identified with sequences $\{p_0, p_1, ...\}$ of integrable vector functions $p_t(s^t)$ such that

$$Ep_t(s^t)y(s^t) - Ep_{t-1}(s^{t-1})x(s^{t-1}) \leq 0 \tag{19}$$

for all pairs of bounded measurable vector functions $(x(s^{t-1}), y(s^t)) \in G(s^t)$ (a.s.). A dual trajectory $\{p_t\}$ supports a trajectory $\{x_t\}$ if

$$p_0(s^0)x_0(s^0) = p_1(s^1)x_1(s^1) = p_2(s^2)x_2(s^2) = ... = 1 \text{ (a.s.)}. \tag{20}$$

3.2. Balanced paths and von Neumann equilibria in the model $\mathcal{M}$.

Recall that a pair (x, α) such that $x \in \mathcal{X}_0$, $|x| = 1$ and $0 < \alpha \in L^1_\infty(1)$ generates a balanced path in a stationary von Neumann-Gale model if and only if $(x, \alpha Tx) \in Z_1$. A sequence $y_t(\omega)$ $(t = 0, 1, ...)$ of functions of ω such that $y_t(\omega)$ is $\mathcal{F}_t$-measurable and $y_t = y_{t-1}(T\omega)$ can be represented as $y_t(\omega) = y(s^t(\omega))$ for some measurable function $y(s^t)$ on $... \times S \times S$. (Indeed, we can represent $y_0(\omega)$ as $y(s^0(\omega))$, which implies $y(s^t(\omega)) = T^t y(s^0(\omega)) =$

$T^t y_0(\omega) = y_t(\omega)$). Thus a balanced path in $\mathcal{M}$ is generated by a pair $x(s^0), \alpha(s^1)$ such that

$$(x(s^0), \alpha(s^1)x(s^1)) \in G(s^1) \text{ and } |x(s^0)| = 1 \text{ (a.s.).} \tag{21}$$

Balanced paths in $\mathcal{M}$ are sequences of the form

$$x_0 = x(s^0),\ x_1 = \alpha(s^1)x(s^1),\ x_2 = \alpha(s^2)\alpha(s^1)x(s^2), ...$$

Suppose a pair x, α generates a balanced path in $\mathcal{M}$. Let $p(s^0) \geq 0$ be an integrable vector function with values in $\mathbb{R}^n_+$. By virtue of (21), (14) and (15), the triplet $x(s^0), \alpha(s^1), p(s^0)$ forms a von Neumann equilibrium in the model $\mathcal{M}$ if and only if

$$p(s^0)x(s^0) = 1 \text{ (a.s.)} \tag{22}$$

and

$$E[\frac{p(s^1)y(s^1)}{\alpha(s^1)}|s^0] \leq p(s^0)x(s^0) \text{ for all } (x(s^0), y(s^1)) \in G(s^1) \text{ (a.s.),} \tag{23}$$

where $x(s^0)$ and $y(s^1)$ are measurable and bounded.

4 Existence problem for a von Neumann equilibrium

4.1. Previous results.

The problem of the existence of a von Neumann equilibrium is central to the theory under consideration. Up to now, the key result obtained in this direction has been as follows. The existence of a von Neumann equilibrium has been established under the assumption that a von Neumann path exists. More precisely, let $\mathcal{B}$ denote the class of those pairs (x, α) which generate balanced paths, i.e. satisfy (12). Consider the variational problem:

$(\mathcal{P})$ Maximize $E \ln \alpha$ over all $(x, \alpha) \in \mathcal{B}$.

Theorem 1. *The following properties of $(x, \alpha) \in B$ are equivalent.*

(a) (x, α) is a solution to problem $(\mathcal{P})$.

(b) There exists a $p \in L^n_1(0)$, $p \geq 0$, such that (x, α, p) is a von Neumann equilibrium.

The result pertains to the general model described in Section 2.1 and holds under assumptions (**Z.0**)–(**Z.5**). For a proof see Arnold, Evstigneev and Gundlach (1999) Theorem 1.

Theorem 1 leaves the question of the existence of a von Neumann path open. In the deterministic case, one can easily provide quite general conditions under which the answer to this question is positive: a von Neumann path exists if the (non-random) cone Z describing the model is closed and the inclusion $(0, y) \in Z$ implies $y = 0$. However, in the stochastic case this issue becomes a substantial problem. Up to now, the existence results were obtained for models in which the cones Z_t satisfy certain conditions of strict convexity (see Evstigneev and Taksar (2001)). A special class of models where these conditions are not fulfilled, but the existence theorem can be obtained, is analyzed in this paper. This class is characterized by the assumption that the cones Z_t are of the form $Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : y \leq D_t(\omega)x\}$, where $D_t(\omega)$ $(t = 0, 1, ...)$ is a stationary process of positive random matrices. In this case, the existence of a von Neumann path reduces to a stochastic analogue of the Perron-Frobenius theorem—see Appendix B.

All the previous results have been obtained under assumptions guaranteeing not only the existence of a von Neumann path, but also its uniqueness and, moreover, its stability (the turnpike property). In this paper, we relax these assumptions and establish a general existence theorem, independent of the issues of stability. To this end we use randomization. By introducing an additional "source of randomness" into the model at hand, we obtain the existence of a randomized von Neumann equilibrium under very general conditions.

4.2. Assumptions and definitions related to randomization.

Suppose a stationary process $..., s_{-1}, s_0, s_1, ...$ of states of the world and a random closed cone $G(s^t)$ are given. Assume that $G(\cdot)$ satisfies conditions (**G.1**)–(**G.3**) and the following assumption:

(**G.6**) there exists a measurable function $\gamma(s^t) > 0$ such that the cone $G(s^t)$ contains $(e, \gamma(s^t)e)$ for all s^t and $E|\ln \gamma(s^t)| < \infty$.

The last condition guarantees the existence of a balanced path whose expected logarithmic growth rate is finite.

Conditions (**G.4**) and (**G.5**) are not supposed to be fulfilled anymore, unless otherwise stated.

Suppose that together with the process $\{s_t\}_{-\infty}^{\infty}$, we are given another stochastic process $\{\zeta_t\}_{-\infty}^{\infty}$ with values in some measurable space, i.e. we are given a probability measure Q on the space of paths

$$\{\sigma_t\}_{-\infty}^{\infty},\ \sigma_t = (s_t, \zeta_t), \tag{24}$$

such that the projection of Q on Ω coincides with the original measure P. The process $\{\sigma_t\}_{-\infty}^{\infty}$ is called *non-anticipative* (with respect to the given process $\{s_t\}_{-\infty}^{\infty}$) if, for each bounded measurable function $g(\sigma^t)$, we have

$$E^Q[g(\sigma^t)|..., s_{-1}, s_0, s_1, ...] = E^Q[g(\sigma^t)|s^t],\ t = 0, \pm 1, ..., \tag{25}$$

where $\sigma^t = (..., \sigma_{t-1}, \sigma_t)$ and E^Q stands for the expectation with respect to the measure Q. Equality (25) means that if we wish to predict σ^t based on information about $..., s_{-1}, s_0, s_1, ...$, then what matters is only s^t—the past and the present of the process $\{s_t\}$, the probabilistic forecast of σ^t being independent of the future $s_{t+1}, s_{t+2}, ...$ of the process $\{s_t\}$.

Given a stationary non-anticipative process $\{\sigma_t\} = \{(s_t, \zeta_t)\}$, $t = 0, \pm 1, \pm 2, ...$, define

$$\bar{G}(\sigma^t) = G(s^t). \tag{26}$$

Consider the model $\overline{\mathcal{M}}$ specified in terms of the stochastic process (24) and the random cone (26). This model will be called the *extension* of $\mathcal{M}$ corresponding to the stationary non-anticipative process $\{\sigma_t\}$. Note that, in the extended model, the random cone $\bar{G}(\sigma^t) = G(s^t)$ does not depend on the process $\{\zeta_t\}$. However, the class of paths in $\overline{\mathcal{M}}$ is larger than in $\mathcal{M}$: these paths might depend not only on $\{s_t\}$, but also on $\{\zeta_t\}$.

4.3. Results of this work.

Central results related to randomization are contained in the following theorem.

Theorem 2. *There exists a stationary non-anticipative extension $\{\sigma_t\} = \{(s_t, \zeta_t)\}$ of the process $\{s_t\}$ such that the extended model $\overline{\mathcal{M}}$ defined in terms of $\{\sigma_t\}$ has a von Neumann path. If, additionally, the cone $G(s^t)$ satisfies conditions (**G.1**)–(**G.5**), then the model $\overline{\mathcal{M}}$ possesses a von Neumann equilibrium.*

If $G(s^t)$ satisfies conditions (**G.1**)–(**G.5**), then, clearly, $\bar{G}(\sigma^t)$ satisfies the same conditions, which implies that the model $\overline{\mathcal{M}}$ meets requirements (**Z.0**)–(**Z.5**). By virtue of Theorem 1, this model possesses a von Neumann equilibrium. Thus the second assertion of Theorem 2 is a consequence of the first assertion and Theorem 1.

The above results are based on the concept of an extension of the original model. We now will slightly change the angle of consideration. We assume that the data of the model are the same (not extended), while the same time we enlarge the class of paths in the model. Let us say that a stochastic

process $\{\xi_0, \xi_1, ...\}$ with values in $\mathbb{R}^n_+$ defines a *randomized path* in the model $\mathcal{M}$ if the process $\{(s_t, \xi_t)\}$ is non-anticipative and for each t the following conditions hold:

$$\operatorname{ess\,sup} |\xi_t| < \infty, \ (\xi_{t-1}, \xi_t) \in G(s^t) \text{ (a.s.)}$$

for all t. When speaking of randomized paths, we will assume that the joint distribution of the processes $\{\xi_0, \xi_1, ...\}$ and $\{..., s_{-1}, s_0, s_1, ...\}$ is given. The notion of non-anticipativity for a process $\{\xi_0, \xi_1, ...\}$ starting from $t = 0$ is fully analogous to the notion of non-anticipativity of a process $\{...\zeta_{-1}, \zeta_0, \zeta_1, ...\}$, we only have to define the "history" of the former as $\xi^t = (\xi_0, \xi_1, ..., \xi_t)$. A randomized path is called *balanced* if $|\xi_0| = 1$, $|\xi_t| > 0$ for all t, and the pairs of scalars and vectors

$$(|\xi_{t+1}|/|\xi_t|, \xi_t/|\xi_t|), \ t \geq 0,$$

form a stationary random sequence (compare with Radner (1971)). The process $\{\xi_0, \xi_1, ...\}$ is called a *randomized von Neumann path* if it maximizes the expected logarithmic growth rate $E \ln(|\xi_{t+1}|/|\xi_t|)$ among all balanced randomized paths.

The following result can be deduced from Theorem 2 (for details see the next section).

Theorem 3. *Let $\overline{\mathcal{M}}$ be the extension of the model $\mathcal{M}$ described in Theorem 2 and let $\{\bar{\xi}_0, \bar{\xi}_1, ...\}$ be the von Neumann path in $\overline{\mathcal{M}}$. If conditions (**G.1**)–(**G.5**) hold, then $\{\bar{\xi}_0, \bar{\xi}_1, ...\}$ is a randomized von Neumann path in $\mathcal{M}$ and for any (not necessarily balanced) randomized path $\{\xi_0, \xi_1, ...\}$ in $\mathcal{M}$, we have*

$$\sup[E \ln |\xi_t| - E \ln |\bar{\xi}_t|] < \infty. \tag{27}$$

Property (27) expresses a quasioptimality property of the randomized path $\{\bar{\xi}_0, \bar{\xi}_1, ...\}$: it cannot be "infinitely worse" asymptotically than any other randomized path.

5 Results on randomization: proofs

5.1. The characteristic function of a cone.

Define $\Delta = \{x \in \mathbb{R}^n_+ : |x| = 1\}$ and put

$$f(s^1, a, b) = \max\{r \in \mathbb{R}_+ : \ (a, rb) \in G(s^1)\} \ (a, b \in \Delta). \tag{28}$$

The maximum in (28) is attained and does not exceed M in view of the closedness of $G(s^1)$ and condition (**G.3**). The function $f(s^1, a, b)$ of $a, b \in \Delta$ is called the *characteristic function* of the cone $G(s^1)$. Put

$$\phi(s^1, a, b) = \ln f(s^1, a, b).$$

Denote by $\mathcal{S}^1$ the σ-algebra defining the measurable structure on the space S^1 of sequences $s^1 = (..., s_0, s_1)$ and by $\overline{\mathcal{S}}^1$ its universal completion (i.e. the intersection of all its completions with respect to all finite measures). Let $\mathcal{B}(X)$ denote the Borel σ-algebra on X.

Proposition 1. *The function $\phi(s^1, a, b)$ possesses the following properties:*
(i) ϕ is measurable with respect to $\overline{\mathcal{S}}^1 \times \mathcal{B}(\Delta) \times \mathcal{B}(\Delta)$;
(ii) ϕ is upper semicontinuous with respect to (a, b);
(iii) the set of values of the function ϕ is contained in $[-\infty, \ln M]$.

Proof. Assertion (i) follows from Lemma 3 in Evstigneev (1976). Assertions (ii) and (iii) are straightforward. □

5.2. A maximization problem involving the characteristic function.

Denote by $\mathfrak{X}$ the set of all stationary non-anticipative extensions $\sigma = \{\sigma_t\} = \{(s_t, \zeta_t)\}$ of the process $\{s_t\}$, $t = 0, \pm 1, \pm 2, ...$ such that ζ_t takes values in the unit simplex Δ. For each $\sigma \in \mathfrak{X}$ define

$$\Phi(\sigma) = E^Q \phi(s^t, \zeta_{t-1}, \zeta_t), \tag{29}$$

where Q is the probability measure induced by the process $\sigma = \{\sigma_t\}$ on the space of its paths. We will identify processes $\sigma = \{\sigma_t\}$ with such measures Q and write $\Phi(\sigma)$ and $\Phi(Q)$ interchangeably.

Consider the spaces $\Omega = ... \times S \times S \times ...$ and $X = ... \times \Delta \times \Delta \times ...$ endowed with the product measurable structures. Put $\tilde{\Omega} = \Omega \times X$. Denote by $\widetilde{\mathcal{F}}$ the product σ-algebra on $\tilde{\Omega}$ and by $\widetilde{\mathcal{F}}_t$ ($t = 0, \pm 1, \pm 2, ...$) the σ-algebra generated by $\{(s_i, x_i)\}$, $i \leq t$. Let $\mathcal{G}$ denote the set of all bounded $\widetilde{\mathcal{F}}$-measurable real-valued functions $g(\omega, x)$ on $\tilde{\Omega}$ that are continuous with respect to x in the product topology of X and by $\mathcal{G}_t$ the analogous set of functions measurable with respect to $\widetilde{\mathcal{F}}_t$. Let $\tilde{T}$ denote the transformation of the space $\tilde{\Omega} := \Omega \times X$ shifting $\tilde{\omega} = (\omega, x) = \{(s_t, x_t)\}$ into $\tilde{T}\tilde{\omega} = \{(s_{t+1}, x_{t+1})\}$. Random processes σ in $\mathfrak{X}$ are represented by probability measures Q on $\tilde{\Omega} = \Omega \times X$ satisfying the following conditions:

$$Q(\Gamma \times X) = P(\Gamma), \ \Gamma \in \mathcal{F} \tag{30}$$

(P is a projection of Q);

$$E^Q(\tilde{T}g) = E^Q(g),\ g \in \mathcal{G} \tag{31}$$

(stationarity);

$$E^Q[g(s^t,\zeta^t)|...,s_{-1},s_0,s_1,...] = E^Q[g(s^t,\zeta^t)|s^t],\ g \in \mathcal{G}_t\ ,\ t = 0,\pm 1,..., \tag{32}$$

(non-anticipativity). The latter condition holds if and only if

$$E^Q[g(s^t,\zeta^t)h(\omega)] = E^Q\{g(s^t,\zeta^t)E^P[h(\omega)|s^t]\}, \tag{33}$$

for all $t = 0,\pm 1,...$, for all $g \in \mathcal{G}_t$ and all bounded measurable $h(\omega)$. Indeed, if (32) holds, then we have

$$E^Q[g(s^t,\zeta^t)h(\omega)] = E^Q\{E^Q[g(s^t,\zeta^t)|\omega]h(\omega)\} =$$

$$E^Q\{E^Q[g(s^t,\zeta^t)|s^t]h(\omega)\} = E^Q\{g(s^t,\zeta^t)E^Q[h(\omega)|s^t]\},$$

where $E^Q[h(\omega)|s^t] = E^P[h(\omega)|s^t]$. Conversely, it follows from (33) that

$$E^Q\{E^Q[g(s^t,\zeta^t)|\omega]h(\omega)\} = E^Q[g(s^t,\zeta^t)h(\omega)] =$$

$$E^Q\{g(s^t,\zeta^t)E^Q[h(\omega)|s^t]\} = E^Q\{E^Q[g(s^t,\zeta^t)|s^t]h(\omega)\},$$

which implies (32).

Proposition 2. *The functional $\Phi(\sigma)$ attains its maximum on $\mathfrak{X}$.*

Proof. The proof is based on Theorem A.1 presented in Appendix A. Conditions (Q.1)–(Q.3), needed in order to apply Theorem A.1, follow from (30), (31) and (33). □

5.3. Proofs of Theorems 2 and 3.

We will obtain Theorem 2 as a consequence of the following proposition.

Proposition 3. *Let $\{(s_t,\zeta_t)\}$ be a stationary non-anticipative process in the class $\mathfrak{X}$ maximizing the functional (29) among all such processes. Let $\overline{\mathcal{M}}$ be the extension of the model $\mathcal{M}$ corresponding to the process $\{(s_t,\zeta_t)\}$. Then the model $\overline{\mathcal{M}}$ possesses a von Neumann path.*

Proof. Define $\sigma_t = (s_t,\zeta_t)$, $\sigma^t = (...,\sigma_{t-1},\sigma_t)$. Put $x(\sigma^0) = \zeta_0$ and consider a measurable function $\alpha(\sigma^1)$ such that

$$\alpha(\sigma^1) = f(s^1,\zeta_0,\zeta_1)$$

almost surely with respect to the measure Q corresponding to the process $\{\sigma_t\}$. The existence of a measurable version $\alpha(\sigma^1)$ of $f(s^1, \zeta_0, \zeta_1)$ follows from assertion (i) of Proposition 1.

We claim that the pair $x(\sigma^0), \alpha(\sigma^1)$ generates a von Neumann path in $\overline{\mathcal{M}}$. By the definition of the function f (see (28)), we have

$$(\zeta_0,\ f(s^1, \zeta_0, \zeta_1)\zeta_1) \in G(s^1),$$

which implies

$$(x(\sigma^0),\ \alpha(\sigma^1)x(\sigma^1)) \in G(s^1) \text{ (a.s.)}.$$

Since $x(\sigma^0) = \zeta_0 \in \Delta$, we obtain that the pair $x(\sigma^0), \alpha(\sigma^1)$ generates a balanced path in the model $\overline{\mathcal{M}}$.

Consider any other pair $y(\sigma^0), \beta(\sigma^1)$ generating a balanced path in the model $\overline{\mathcal{M}}$. We have

$$(y(\sigma^0), \beta(\sigma^1)y(\sigma^1)) \in G(s^1) \text{ (a.s.)},$$

and so

$$\beta(\sigma^1) \leq f(s^1, y(\sigma^0), y(\sigma^1)) \text{ (a.s.)}.$$

Put $\zeta'_t = y(\sigma^t)$. The process $\{\sigma'_t\} = \{(s_t, \zeta'_t)\} = \{(s_t, y(\sigma^t))\}$ is stationary and non-anticipative because for any bounded measurable function $g(\cdot, \cdot)$ in $\mathcal{G}_t$ we have

$$g(s^t, \zeta'_t) = g(s^t, y(\sigma^t)) = h(\sigma^t)$$

for some bounded measurable $h(\cdot)$, and so we have

$$E[g(s^t, \zeta'_t)|..., s_{-1}, s_0, s_1, ...] = E[g(s^t, \zeta'_t)|s^t],\ t = 0, \pm 1, ...,$$

by virtue of (25). Consequently,

$$E \ln \beta(\sigma^1) \leq E \ln f(s^1, y(\sigma^0), y(\sigma^1)) = E \ln f(s^1, \zeta'_0, \zeta'_1) \leq$$

$$E \ln f(s^1, \zeta_0, \zeta_1) = E \ln \alpha(\sigma^1),$$

which completes the proof. □

Proof of Theorem 2: Follows from Proposition 3. □

Proof of Theorem 3. Consider any randomized path $\{\xi_0, \xi_1, ...\}$ in $\mathcal{M}$. The process $\{(s_t, \xi_t)\}$ is non-anticipative, and so

$$x_t := E(\xi_t | s^t) = E(\xi_t | \omega),$$

where $\omega = (..., s_{-1}, s_0, ...)$. Therefore

$$(x_{t-1}, x_t) = E[(\xi_{t-1}, \xi_t)|\omega] \in G(s^t)$$

(see Arkin and Evstigneev (1987) Appendix II, Lemma 1). Consequently, $\{x_t\}$ is a path in $\mathcal{M}$ and hence in $\overline{\mathcal{M}}$. It is known that any rapid path is quasioptimal (see Evstigneev and Flåm (1987) Proposition 2.5). In particular, a von Neumann path is rapid if it is supported by a balanced dual path, which is the case under conditions (**G.1**)–(**G.5**) (see Theorem 1). Consequently,

$$\sup[E \ln |x_t| - E \ln |\bar{\xi}_t|] < \infty. \tag{34}$$

It remains to observe that

$$E \ln |x_t| = E \ln E(|\xi_t||s^t) \geq E[E(\ln |\xi_t||s^t)] = E \ln |\xi_t|,$$

which in view of (34) yields (27).

To prove that $\{\bar{\xi}_0, \bar{\xi}_1, ...\}$ is a randomized von Neumann path, we take any randomized balanced path $\{\xi_0, \xi_1, ...\}$ and observe that, for each N and t,

$$N(E \ln \frac{|\xi_t|}{|\xi_{t-1}|} - E \ln \frac{|\bar{\xi}_t|}{|\bar{\xi}_{t-1}|}) = \sum_{i=1}^{N} E \ln \frac{|\xi_i|}{|\xi_{i-1}|} - \sum_{i=1}^{N} E \ln \frac{|\bar{\xi}_i|}{|\bar{\xi}_{i-1}|} =$$

$$E \sum_{i=1}^{N} \ln \frac{|\xi_i|}{|\xi_{i-1}|} - E \sum_{i=1}^{N} \ln \frac{|\bar{\xi}_i|}{|\bar{\xi}_{i-1}|} =$$

$$E \ln \frac{|\xi_N|}{|\xi_0|} - E \ln \frac{|\bar{\xi}_N|}{|\bar{\xi}_0|} = E \ln |\xi_N| - E \ln |\bar{\xi}_N| \leq C$$

for some constant C (see (27)). Consequently,

$$E \ln \frac{|\xi_t|}{|\xi_{t-1}|} - E \ln \frac{|\bar{\xi}_t|}{|\bar{\xi}_{t-1}|} \leq \frac{C}{N} \to 0 \text{ as } N \to \infty,$$

which implies

$$E \ln \frac{|\xi_t|}{|\xi_{t-1}|} - E \ln \frac{|\bar{\xi}_t|}{|\bar{\xi}_{t-1}|} \leq 0.$$

Thus $\{\bar{\xi}_0, \bar{\xi}_1, ...\}$ is a randomized von Neumann path. □

6 Stochastic von Neumann-Gale model defined in terms of positive random matrices

6.1. Model description and the statement of the result.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T : \Omega \to \Omega$ a mapping and $\mathcal{F}_t$ $(t = 0, \pm 1, \pm 2, ...)$ σ-algebras satisfying the conditions listed in Section 4.2. For each $t = 0, 1, ...$, let $D_t(\omega)$ be a nonnegative random $n \times n$ matrix measurable with respect to $\mathcal{F}_t$. Define

$$Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \ y \leq D_t(\omega)x\}, \tag{35}$$

and assume that $D_t(T\omega) = D_{t+1}(\omega)$ for all $t \geq 0$. Formula (35) defines a von Neumann–Gale model specified in terms of the random cone $G_t(\omega) = \{(a, b) \geq 0 : \ b \leq D_t(\omega)a\}$ (see Dempster, Evstigneev and Schenk-Hoppé (2003) for some applications of such models in Finance).

Put $D(\omega) = D_1(\omega)$ and suppose that $D(\omega)$ is uniformly bounded and there exists a constant $\gamma > 0$ such that for some $\check{x}_0 \in \mathcal{X}_0$, we have $D\check{x}_0 \geq \gamma e$. Further, assume that for some $m \geq 1$, the smallest element of the matrix $C(m, \omega)$ (see (50) in Appendix B) is greater than γ. Then conditions (**Z.0**)–(**Z.5**) are satisfied and the following theorem is valid.

Theorem 4. *The model (35) possesses a unique von Neumann equilibrium (x, α, p), where $0 < x \in L^n_\infty(0)$ and $0 < \alpha \in L^1_\infty(1)$ are the solutions to*

$$\alpha(\omega)x(T\omega) = D(\omega)x(\omega), \ \ |x(\omega)| = 1 \ \ \textit{(a.s.)}$$

(cf. (52) in Theorem B.1, Appendix B) and $0 < p \in L^n_1(0)$ is the (unique) solution to

$$E(\alpha^{-1}(Tp)D|\mathcal{F}_0) = p, \ px = 1. \tag{36}$$

By virtue of (14) and (15), a triple (x, α, p) of functions satisfying (13) is a von Neumann equilibrium if and only if

$$\alpha Tx \leq Dx, \ |x| = 1, \ px = 1, \tag{37}$$

and

$$E\left(\alpha^{-1}(Tp)Dy \mid \mathcal{F}_0\right) \leq py \tag{38}$$

for all $0 \leq y \in L^n_\infty(0)$.

6.2. Several lemmas.

Before proving Theorem 4, it will be convenient to establish the following three lemmas.

Lemma 1. *If functions $x(\omega) \geq 0$ and $\alpha(\omega) \geq 0$ satisfy (52), then they are bounded above and bounded away from zero.*

Proof. Since $D(\omega)$ is uniformly bounded, there exists a constant M such that

$$|D(\omega)b| \leq M \text{ for all } b \geq 0 \text{ with } |b| = 1. \tag{39}$$

By virtue of (52), we get $\alpha = |\alpha Tx| = |Dx| \leq M$. Define

$$x_t(\omega) = x(T^t\omega),\ \alpha_t = \alpha(T^{t-1}\omega). \tag{40}$$

From (52), we obtain by induction that

$$D_m\, D_{m-1} \ldots D_1\, x_0 = \alpha_m\, \alpha_{m-1} \ldots \alpha_1\, x_m. \tag{41}$$

Since the smallest element of $D_m \ldots D_1$ is not less than $\gamma > 0$ and $|x_m| = 1$, equation (41) implies that every coordinate of x_m is not less than γM^{-m} and $\alpha_m \geq \gamma M^{-m+1}$. Consequently, $x \geq \gamma M^{-m} e$ and $\alpha \geq \gamma M^{-m+1}$. □

Lemma 2. *If functions $x(\omega) \geq 0$, $p(\omega) \geq 0$ and $\alpha(\omega) > 0$ satisfy (37) and (38), then they are bounded above and bounded away from zero, and the inequalities in (37), (38) hold as equalities.*

Proof. In view of (39) and (37), we have $\alpha = |\alpha Tx| \leq |Dx| \leq M$. Further, we observe that $|p| \geq 1$ because $1 = px \leq |p||x|$ and $|x| = 1$. Define

$$p_t(\omega) = p(T^t\omega). \tag{42}$$

It follows from (38) that $E\left(\alpha_t^{-1} p_t D_t y \mid \mathcal{F}_{t-1}\right) \leq p_{t-1} y$ for all $0 \leq y \in L_\infty^n(t-1)$ and hence for all $\mathcal{F}_{t-1}$-measurable $y(\omega) \geq 0$. By induction, this implies

$$E\left(\frac{p_t D_t D_{t-1} \ldots D_1 y}{\alpha_t \ldots \alpha_1} \mid \mathcal{F}_0\right) \leq p_0 y \text{ for all } 0 \leq y \in L_\infty^n(0). \tag{43}$$

By using the inequalities $\alpha \leq M$, $|p_t| \geq 1$ and inequality (43) applied to $y = e_i$ and $t = m$, we find $p_0^i \geq \gamma M^{-m}$, as the smallest element of the matrix $D_m \ldots D_1$ exceeds γ. Setting $y = x_0\,(= x)$ in (38), we get $Ep_1\alpha_1^{-1} D_1 x_0 \leq Ep_0 x_0 = Ep_1 x_1 = 1$. On the other hand, $\alpha_1^{-1} D_1 x_0 \geq x_1$ by virtue of (37). Therefore the nonnegative random variable $g := p_1(\alpha_1^{-1} D_1 x_0 - x_1)$ satisfies $Eg \leq 0$. This implies $g = 0$ (a.s.) and hence $\alpha_1^{-1} D_1 x_0 = x_1$ because $p_1 > 0$.

Consequently, the inequality in (37) holds as equality (a.s.), i.e. formula (52) is valid. By virtue of Lemma 1, x and α are bounded away from zero and infinity. From the equality $px = 1$, we obtain that p is bounded above.

To complete the proof of the lemma, it is sufficient to establish the first equality in (36). From (38) we can see that the random variable $h := [p_0 - E\left(\alpha_1^{-1} p_1 D_1 \mid \mathcal{F}_0\right)]x_0$ is nonnegative. Furthermore, we have $Eh = Ep_0x_0 - E\alpha_1^{-1}p_1D_1x_0 = 1 - Ep_1x_1 = 0$. Consequently, $h = 0$ (a.s.), which implies $p_0 - E\left(\alpha_1^{-1}p_1D_1 \mid \mathcal{F}_0\right) = 0$ because $x_0 > 0$ (a.s.). This proves (36). □

Lemma 3. *Let $x \geq 0$ be an $\mathcal{F}_0$-measurable vector function and $\alpha \geq 0$ an $\mathcal{F}_1$-measurable scalar function satisfying (52). Then there is a unique $\mathcal{F}_0$-measurable random vector p satisfying (36).*

Proof of Lemma 3. We first note that, by virtue of Lemma 1, x and p are strictly positive and bounded. Let us apply Theorem B.1 to the random matrix $B(\omega) = D^*(\omega)$ (the conjugate of $D(\omega)$) and to the automorphism T^{-1} of the probability space $(\Omega, \mathcal{F}, P)$. By virtue of this theorem, there exist a measurable vector function $q(\omega) > 0$ and a measurable scalar function $\beta(\omega) > 0$ such that

$$\beta(\omega)q(T^{-1}\omega) = B(\omega)q(\omega), \;\; |q(\omega)| = 1. \tag{44}$$

In view of Remark B.1, we can choose versions of $q(\cdot)$ and $\beta(\cdot)$ such that $q(\cdot)$ is measurable with respect to the σ-algebra generated by $D(T\omega)$, $D(T^2\omega), ...$ and $\beta(\cdot)$ is measurable with respect to the σ-algebra generated by $D(\omega)$, $D(T\omega), ...$

Define $q'(\omega) = q(T^{-1}\omega)$. It follows from (44) that

$$\beta(\omega)q'(\omega) = q'(T\omega)D(\omega), \;\; |q'(\omega)| = 1 \tag{45}$$

because $D(\omega) = B^*(\omega)$. (In the notation used in (44), $q(\omega)$ is a column vector, while in (45), $q'(\omega)$ is a row vector.) Put

$$p'(\omega) = \frac{q'(\omega)}{q'(\omega)x(\omega)}$$

The random vector $p'(\omega)$ is well-defined as $|q'(\omega)| = 1$ and $x(\omega) > 0$. By using (45), we obtain

$$\beta'(\omega)p'(\omega) = p'(T\omega)D(\omega), \; p'(\omega)x(\omega) = 1, \tag{46}$$

where

$$\beta'(\omega) = \beta(\omega)\frac{q'(\omega)x(\omega)}{q'(T\omega)x(T\omega)}.$$

Observe that (52) and (46) imply $\beta'(\omega) = \alpha(\omega)$. Indeed, we have

$$\alpha(\omega) = \alpha(\omega)p'(T\omega)x(T\omega) = p'(T\omega)D(\omega)x(\omega) = \beta'(\omega)p'(\omega)x(\omega) = \beta'(\omega).$$

Define $p(\omega) = E[p'(\omega)|\mathcal{F}_0]$. The second equality in (46) implies that $p'(\omega)$ is bounded (because $x(\omega)$ is bounded away from zero.) From the same equality, we also find that $p(\omega)x(\omega) = 1$ as $x(\omega)$ is $\mathcal{F}_0$-measurable. Since $\beta'(\omega) = \alpha(\omega)$, the first relation in (46) gives

$$p(\omega) = E[\frac{p'(T\omega)D(\omega)}{\alpha(\omega)}|\mathcal{F}_0] = E[\frac{E[p'(T\omega)|\mathcal{F}_1]D(\omega)}{\alpha(\omega)}|\mathcal{F}_0] = E[\frac{p(T\omega)D(\omega)}{\alpha(\omega)}|\mathcal{F}_0],$$

and so p is a solution to (36).

Let us establish the uniqueness of a solution to (36). Consider some such solution, p. From (36), we get

$$E[\frac{p_{t+1}D_{t+1}}{\alpha_{t+1}}|\mathcal{F}_t] = p_t,$$

which implies

$$E[\frac{p_{t+1}D_{t+1}D_t...D_1}{\alpha_{t+1}...\alpha_1}|\mathcal{F}_t] = \frac{p_tD_t...D_1}{\alpha_t...\alpha_1}.$$

Consequently, the random sequence $\hat{p}_0, \hat{p}_1, ...$, where

$$\hat{p}_t = \frac{p_tD_t...D_1}{\alpha_t...\alpha_1} \ (t \geq 1), \ \hat{p}_0 = p_0, \tag{47}$$

is a nonnegative martingale. Note that this sequence is bounded because $\hat{p}_t x = p_t x_t = 1$ (see (36)) and x is bounded away from zero. Thus there exists a limit $\hat{p} = \lim_{t\to\infty} \hat{p}_t$ (a.s.). By virtue of (47), we have

$$\hat{p}_t(T\omega)D_1(\omega) = \frac{p_{t+1}D_{t+1}...D_2D_1}{\alpha_{t+1}...\alpha_2} = \hat{p}_{t+1}\alpha_1.$$

By passing to the limit in this equality and in the equality $\hat{p}_t x = 1$, we get

$$\hat{p}(T\omega)D(\omega) = \hat{p}\alpha, \ \hat{p}x = 1. \tag{48}$$

Consequently,

$$\bar{p}(T\omega)D(\omega) = \bar{p}\bar{\alpha}, \ |\bar{p}| = 1, \tag{49}$$

where $\bar{p} = \hat{p}/|\hat{p}|$ and $\bar{\alpha} = \alpha|T\hat{p}|^{-1}|\hat{p}|$. By virtue of Theorem B.1 (applied to the operator T^{-1} and the matrix $B(\omega) = D^*(\omega)$), the solution $\bar{p}$ to equations

(49) is unique. Consequently, the solution $\hat{p}$ to (48) is unique as well because $\hat{p} = \bar{p}|\hat{p}|$ and $|\hat{p}| = \bar{p}(\bar{p}x)^{-1}$. The uniform boundedness of the sequence $\hat{p}_t$ makes it possible to pass to the limit in the equality $E[\hat{p}_t|\mathcal{F}_0] = p$, which yields $E[\hat{p}|\mathcal{F}_0] = p$. Since the solution $\hat{p}$ to equations (48) is unique, the solution $p = E[\hat{p}|\mathcal{F}_0]$ to (36) is unique too. □

6.3. Proof of Theorem 4.

Existence. Consider the solutions $x > 0$ and $\alpha > 0$ to (52). By virtue of Remark B.1, we can select versions of the functions $x(\omega)$ and $\alpha(\omega)$ such that x is $\mathcal{F}_0$-measurable and α is $\mathcal{F}_1$-measurable. By virtue of Lemma 1, we have $x \in L^n_\infty(0)$ and $\alpha \in L^1_\infty(1)$. Consider the $\mathcal{F}_0$-measurable random vector p satisfying (36) (its existence was established in Lemma 3). The functions x, α and p satisfy (37), (38). In view of Lemma 2, p is bounded, hence $p \in L^n_1(0)$. Thus conditions (13), (37) and (38) hold, and so (x, α, p) is a von Neumann equilibrium.

Uniqueness. Consider a triplet (x, α, p) satisfying (13), (37) and (38). According to Lemma 2, the inequalities in (37), (38) hold as equalities. Thus the functions x and α are solutions to (52), hence they are strictly positive by virtue of Lemma 1, and consequently they are uniquely determined by virtue of Theorem A.1. The uniqueness of p follows from Lemma 3. □

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Appendix A

Let $(\Omega, \mathcal{F}, P)$ be a probability space, X a compact metric space and $\mathcal{B}$ the Borel σ-algebra on X. Consider the following convergence of measures on $\mathcal{F} \times \mathcal{B}$. Let us say that a sequence of probability measures Q_m on $\mathcal{F} \times \mathcal{B}$ converges to a measure Q on $\mathcal{F} \times \mathcal{B}$ and write $Q_m \rightarrowtail Q$ if $\int Q_m(d\omega, dx) g(\omega, x) \to \int Q(d\omega, dx) g(\omega, x)$ for each bounded $\mathcal{F} \times \mathcal{B}$-measurable function continuous with respect to x. The set of such functions g will be denoted by $\mathcal{G}$.

Let $\mathcal{Q}$ be a set of probability measures Q on $\mathcal{F} \times \mathcal{B}$, possessing the following properties.

(Q.1) The projection of Q on Ω is P, i.e. $Q(\Gamma \times X) = P(\Gamma)$, $\Gamma \in \mathcal{F}$.

(Q.2) The set $\mathcal{Q}$ is convex.

(Q.3) The set $\mathcal{Q}$ is closed under the convergence of measures defined above.

Denote by $\mathcal{F}^P$ the completion of $\mathcal{F}$ with respect to the measure P. Let $f(\omega, x)$ be a function on $\Omega \times X$ taking values in $[-\infty, \infty)$, upper semi-continuous in x, measurable with respect to $\mathcal{F}^P \times \mathcal{B}$, and bounded above by $M(\omega) \geq 0$ with $\bar{M} := \int M(\omega) P(d\omega) < \infty$. For each $Q \in \mathcal{Q}$, define $\Phi(Q) = \int Q(d\omega, dx) f(\omega, x)$. The functional $\Phi(Q)$ is well-defined (because f is $(\mathcal{F} \times \mathcal{B})^Q$-measurable) and takes values in $[-\infty, \bar{M}]$.

Theorem A.1. *The functional $\Phi(Q)$ attains its maximum on $\mathcal{Q}$.*

This theorem follows from known results on Young measures—see Balder (1988) Theorems 2.2 and 2.3.

Appendix B

We formulate a stochastic analogue of the Perron–Frobenius theorem for positive matrices. Consider a probability space $(\Omega, \mathcal{F}, P)$ and a one-to-one mapping $T : \Omega \to \Omega$ such that T and T^{-1} are measurable and preserve the measure P. Let $D(\omega)$ be a measurable function taking values in the set of nonnegative $n \times n$ matrices. Define

$$C(t, \omega) = D(T^{t-1}\omega) D(T^{t-2}\omega) ... D(\omega), \ t = 1, 2, ..., \tag{50}$$

and $C(0, \omega) = \mathrm{Id}$ (the identity matrix). Then we have

$$C(t, T^s\omega) C(s, \omega) = C(t + s, \omega), \ t, s \geq 0, \tag{51}$$

i.e., the matrix function $C(t, \omega)$ is a *cocycle* over the dynamical system $(\Omega, \mathcal{F}, P, T)$ (see, e.g., Arnold (1998)).

For a matrix $D > 0$, denote by $\kappa(D)$ the ratio of the smallest element of the matrix to its greatest element. Let the following condition hold.

(*) There is a (non-random) integer $m > 0$ for which $C(m, \omega) > 0$ and $E|\ln \kappa(C(m, \omega))| < \infty$.

Theorem B.1. *There exists a measurable vector function $x(\omega) > 0$ and a measurable scalar function $\alpha(\omega) > 0$ such that*

$$\alpha(\omega) x(T\omega) = D(\omega) x(\omega), \ |x(\omega)| = 1 \ \ (a.s.). \tag{52}$$

The pair of functions $(\alpha(\cdot), x(\cdot)) > 0$ satisfying (52) is determined uniquely up to the equivalence with respect to the measure P. If $t \to \infty$, then

$$\frac{C(t, T^{-t}\omega) a}{|C(t, T^{-t}\omega) a|} \to x(\omega) \ (a.s.), \tag{53}$$

where convergence is uniform in $a \geq 0$, $a \neq 0$.

The above result may be regarded as a generalization of the Perron–Frobenius theorem on eigenvalues and eigenvectors of positive matrices: $x(\cdot)$ and $\alpha(\cdot)$ play the roles of an "eigenvector" and an "eigenvalue" of the cocycle $C(t,\omega)$. Theorem B.1 is a special case of the result in Evstigneev (1974) Theorem 1; see also Arnold, Gundlach and Demetrius (1994) Theorem 3.1.

Remark B.1. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be sub-σ-algebras of $\mathcal{F}$ such that the random matrices $D(T^{-1}\omega), D(T^{-2}\omega), ...$ are $\mathcal{F}_0$-measurable and the random matrices $D(T\omega), D(T^{-1}\omega), ...$ are $\mathcal{F}_1$-measurable. We can see from (53) and (50) that $z(\cdot) \in \mathcal{F}_0^P$ and $\phi(\cdot) \in \mathcal{F}_1^P$, i.e., the functions $z(\cdot)$ and $\phi(\cdot)$ are measurable with respect to the σ-algebras $\mathcal{F}_0$ and $\mathcal{F}_1$ completed by all sets of measure zero. From this it follows that we can select versions of $z(\cdot)$ and $\phi(\cdot)$, satisfying (52), which are $\mathcal{F}_0$- and $\mathcal{F}_1$-measurable, respectively.

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