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Inequality in the Relative Differential Sense and Predation in a Primitive Economy

by

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Inequality in the Relative Differential Sense and Predation in a Primitive Economy

by

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GROSSMAN [1998] and GROSSMAN and KIM [2002] study the level of predation and production in a primitive economy where individual resource endowments follow a 2-class distribution. Here we allow endowments to follow a general continuous distribution, and we study the impact of changes in inequality in this distribution. General comparative static results are obtained using the Relative Differential Inequality concept, whose properties in the continuous distribution setting are detailed, complementing known results for the discrete distribution case. (JEL: D31, D63, D50, D74)

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1. *Introduction*

The paper has two purposes. The first objective is to study the impact of changing inequality in resource endowments on the level of predation and production in a primitive economy, specifically that of GROSSMAN [1998] and GROSSMAN and KIM [2002]. The second aspect (following MOYES [1994] in the discrete distribution case) is to detail properties of the "relative differential inequality (RDI)" concept for the case of continuous distributions, and to show how (following CHIU and MADDEN [1998]) this concept can be brought to bear on the primitive economy inequality analysis.

There is a common, if usually loose belief that increases in inequality in a society exacerbate problems with criminal and other destructive or rent-seeking behaviour by individuals. One precise statement supporting this belief is CHIU and MADDEN [1998], who study the impact of income inequality in a neighbourhood on the housing market and the level of burglary. Our main objective is to provide an analogous precise statement in the context of a primitive economy where there is no public provision of defence of property, and private provision is insufficient to deter predatory acts against the property of others. The model we use is that of GROSSMAN [1998] and GROSSMAN and KIM [2002], except that we replace Grossman and Kim's two-class distribution of resource endowments across the society with a general continuous distribution. We confirm, in various ways, the expected consequences of increasing inequality.

Properties of the RDI concept in the context of discrete distributions have been extensively studied by MOYES [1994], CHATEAUNEUF [1996], and SAVAGLIO [2000, 2001], following the initial idea of MARSHALL, WALKUP and WETS [1967]. In Section 2 we define RDI in the continuous distribution setting and prove its main properties in relation to Lorenz inequality, analogous to discrete results in MOYES [1994], CHATEAUNEUF [1996], and SAVAGLIO [2000, 2001]; we also show how RDI manifests itself within the well-known classes of income distribution functions of the uniform and Pareto families. The usefulness of RDI then emerges in Section 3, with the detailed insights it provides regarding inequality in the primitive society model.

2. *Relative Differential Inequality and Lorenz Inequality*

Consider first two discrete distributions $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)$ both ranked with $0<x_1<x_2<...<x_n$ and $0<y_1<y_2<...<y_n$. Then x is said to Lorenz dominate y (see, e.g. LAMBERT [2001]) if and only if:

$$\sum_{i=1}^{k} x_i \Big/ \sum_{i=1}^{n} x_i > \sum_{i=1}^{k} y_i \Big/ \sum_{i=1}^{n} y_i\,,\ k=1,2,...,n-1.$$

On the other hand x dominates y in the relative differential inequality (RDI) sense if and only if:

$x_{i+1}/y_{i+1} < x_i/y_i$, $i = 1,2,...,n-1$.

Following MARSHALL ET AL. [1967] and MARSHALL and OLKIN [1979], MOYES [1994], CHATEAUNEUF [1996] and SAVAGLIO [2000,2001] have studied RDI in the discrete setting. Two particular properties are:

(a) RDI induces a sub-ordering of that created by Lorenz dominance—RDI dominance implies Lorenz dominance, but the converse is not generally true (see MOYES [1994], CHATEAUNEUF [1996], and SAVAGLIO [2001]).

(b) RDI dominance is equivalent to Lorenz dominance on every subset of $\{1,2,...,n\}$ (see MOYES [1994], Remark 2.4, p.279).

A third property follows, noting that $x_i \Big/ \sum_{j=1}^{n} x_j$ and $y_i \Big/ \sum_{j=1}^{n} y_j$ are the increments to the ordinates of the Lorenz curves for x and y between $i-1$ and i, and defining "the Lorenz curve for x is everywhere less curved than that for y" to mean that the ratio of these increments, $x_i \sum_{j=1}^{n} y_j \Big/ y_i \sum_{j=1}^{n} x_j$ is decreasing in i. Immediately we have:

(c) x dominates y in the RDI sense is equivalent to the Lorenz curve for x being everywhere less curved than that for y.

Our objective in this section is to report various properties of RDI, including parallels to (a), (b) and (c), in the continuum, rather than discrete setting.

We model the continuum case via its inverse distribution function $K:[0,1] \to R_+ \cup \{+\infty\}$, where K is a continuous, non-decreasing function, where $\int_0^1 K(i)di$ is finite and where $K(i) > 0$ some $i \in [0,1]$. Some examples that will recur are:

<u>The equal distribution</u>: $K(i) = k > 0,\ i \in [0,1]$.

<u>The uniform distribution family with parameters *n* and *m*</u>: $K(i) = n + (m-n)i$ where $m \geq n > 0$.

<u>The Pareto distribution family with parameters *b* and α</u>: $K(i) = b(1-i)^{-\alpha}$ where $b > 0,\ 0 \leq \alpha < 1$.

The Lorenz curve for K is the graph of $L_K:[0,1] \to [0,1]$, where

$L_K(j)=\int_0^j K(i)di \Big/ \int_0^1 K(i)di$, and is differentiable as K is continuous[1]. The Lorenz comparison of 2 distributions K and M is:

Definition 1 *K* Lorenz dominates *M* if and only if $L_K(j)>L_M(j)$ for all $j\in(0,1)$.

The relative differential inequality comparison is:
Definition 2 *K* dominates *M* in the relative differential inequality (RDI) sense if and only if $K(i)/M(i)<K(j)/M(j)$ for all $i,j\in[0,1]$ where $i>j$.

The link between these two concepts stems from:

Definition 3 *K* Lorenz dominates *M* on the subset $S=[a,b]\subset[0,1]$ where $a<b$, if and only if $\int_a^j K(i)di \Big/ \int_a^b K(i)di > \int_a^j M(i)di \Big/ \int_a^b M(i)di$ for all $j\in(a,b)$.

Theorem 2.1 *K* dominates *M* in the RDI sense if and only if *K* dominates *M* on any subset $S=[a,b]\subset[0,1]$, $a<b$.

Proof Only if Suppose *K* RDI dominates *M*, and $j\in(a,b)$, $i\in(a,j)$, $[a,b]\subset[0,1]$, $a<b$.

Then for all $i\in(a,j)$,

$$K(j)/M(j)<K(i)/M(i)\Rightarrow K(j)\int_a^j M(i)di<M(j)\int_a^j K(i)di$$
$$\Rightarrow \int_a^j K(i)di \Big/ \int_a^j M(i)di > K(j)/M(j) \tag{2.1}$$

We need to show for all $j\in(a,b)$:

$$\int_a^j K(i)di \Big/ \int_a^b K(i)di > \int_a^j M(i)di \Big/ \int_a^b M(i)di, \text{ or,}$$

$$\int_a^j K(i)di \Big/ \left[\int_a^j K(i)di+\int_j^b K(i)di\right] > \int_a^j M(i)di \Big/ \left[\int_a^j M(i)di+\int_j^b M(i)di\right], \text{ or:}$$

$$\int_a^j K(i)di \Big/ \int_a^j M(i)di > \int_j^b K(i)di \Big/ \int_j^b M(i)di. \tag{2.2}$$

Now with $k\in(j,b)$ and following analogously the derivation of (2.1):

$$K(k)/M(k)<K(j)/M(j)\Rightarrow K(j)/M(j)>\int_j^b K(k)dk \Big/ \int_j^b M(k)dk. \tag{2.3}$$

(2.1) and (2.3) imply (2.2), as required.

If Suppose *K* Lorenz dominates *M* on any $[a,b]\subset[0,1]$, $a<b$. We need to show

[1] The more usual definition (e.g. LAMBERT [2001]) is in terms of the direct (rather than inverse) distribution function, $\int_{\underline{y}}^{y} xdJ(x) \Big/ \int_{\underline{y}}^{\overline{y}} xdJ(x)$ where $J=K^{-1}$, $\underline{y}=K(0)$, $\overline{y}=K(1)$, $y=K(j)$. Integration by parts shows the equivalence: $\int_{\underline{y}}^{y} xdJ(x)=yJ(y)-\int_{\underline{y}}^{y} J(x)d(x)=\int_0^j K(i)di$.

$K(i)/M(i)$ is decreasing on $[0,1]$. Suppose not. Then $K(i)/M(i)$ is weakly increasing on some interval $[a,b]<[0,1]$, $a<b$; for $j\in(a,b)$, $i\in(a,j)$ and analogous to "only if",

$$K(j)/M(j)\geq K(i)/M(i)\Rightarrow K(j)/M(j)\geq \int_a^j K(i)\,di \Big/ \int_a^j M(i)\,di\,. \tag{2.4}$$

And for $k\in(j,b)$:

$$K(k)/M(k)\geq K(j)/M(j)\Rightarrow \int_j^b K(k)\,dk \Big/ \int_j^b M(k)\,dk \geq K(j)/M(j)\,. \tag{2.5}$$

(2.4) and (2.5) imply:

$$\frac{\int_a^b K(k)\,dk-\int_a^j K(k)\,dk}{\int_a^b M(k)\,dk-\int_a^j M(k)\,dk}\geq \frac{\int_a^j K(i)\,di}{\int_a^j M(i)\,di}$$, which implies

$$\int_a^j M(i)\,di \Big/ \int_a^b M(i)\,di \geq \int_a^j K(i)\,di \Big/ \int_a^b K(i)\,di\,,$$

which contradicts the supposed Lorenz dominance of *K* over *M* on $[a,b]$. ■

Theorem 2.1 characterizes RDI in terms of the Lorenz curves for each subset of the population. Theorem 2.2 provides an alternative RDI characterization, solely in terms of the standard (whole population) Lorenz curve and its curvature.

Definition 4 The Lorenz curve for *K* is everywhere less curved than that for *M* if and only if $dL_K(j)/dj \big/ dL_M(j)/dj$ is decreasing in *j*, $j\in(0,1)$.

Of course, since $L_K(0)=L_M(0)=0$ and $L_K(1)=L_M(1)=1$, this curvature property implies that $L_K(j)>L_M(j),\ j\in(0,1)$ and so *K* Lorenz dominates *M*. It characterizes RDI exactly however:

Theorem 2.2 *K* dominates *M* in the RDI sense if and only if the Lorenz curve for *K* is everywhere less curved than that for *M*.

Proof $$\frac{dL_K(j)/dj}{dL_M(j)/dj}=\frac{K(j)}{M(j)}\cdot\frac{\int_0^1 M(i)\,di}{\int_0^1 K(i)\,di},$$

which shows that the curvature is decreasing if and only if $K(j)/M(j)$ is decreasing, or *K* RDI dominates *M*. ■

It is clear that RDI is in general a more demanding criterion than Lorenz. In the rest of this section we offer a number of remarks and results which elaborate on the extent of these extra demands, focusing first on the uniform and Pareto examples. It turns out that, for each of these families, any 2 distributions within the family are RDI comparable, and the RDI comparison always coincides with Lorenz:

Theorem 2.3 For two uniform distribution K (with parameters m, n and $\rho = m/n$) and M (with parameters m', n' and $\rho' = m'/n'$) the following three statements are equivalent:

(I) K Lorenz dominates M;

(II) K dominates M in the RDI sense;

(III) $\rho < \rho'$.

Proof:

Let K and M defines 2 uniform distributions as described in the statement of Theorem 2.3. The Lorenz curve ordinates for K and M are, respectively;

$$L_K(i) = \left[2i + (\rho - 1)i^2\right] / (1+\rho), \quad L_M(i) = \left[2i + (\rho' - 1)i^2\right] / (1+\rho')$$

K Lorenz dominates M if and only if, for all $i \in (0,1)$,

$$2i(1+\rho') + (1+\rho')(\rho - 1)i^2 > 2i(1+\rho) + (1+\rho)(\rho' - 1)i^2$$

which becomes $2i(\rho' - \rho) > 2(\rho' - \rho)$ and holds if and only if $\rho' > \rho$, establishing the equivalence of (I) and (III).

Define $Q(i) = \left(dL_K(i)/di\right) / \left(dL_M(i)/di\right)$. Then:

$$Q(i) = \frac{1+\rho'}{1+\rho} \cdot \frac{1+(\rho-1)i}{1+(\rho'-1)i}$$

and $dQ(i)/di < 0$ if and only if $\rho' > \rho$. Thus the Lorenz curve for K is everywhere less curved than that for M if and only if $\rho' > \rho$. The equivalence of (II) and (III) follows from Theorem 2.2. ■

Theorem 2.4 For two Pareto distributions K (with parameters b, α) and M (with parameter b', α'), the following three statements are equivalent:

(I) K Lorenz dominates M;

(II) K dominates M in the RDI sense;

(III) $\alpha < \alpha'$.

Proof:

Let K and M define 2 Pareto distributions as described in the statement of Theorem 2.4. The Lorenz curve ordinates for K and M are, respectively:

$$L_K(i) = 1 - (1-i)^{\frac{\alpha-1}{\alpha}}, \quad L_M(i) = 1 - (1-i)^{\frac{\alpha'-1}{\alpha'}}$$

Now $L_K(i) > L_M(i)$ for all $i \in (0,1)$ if and only if $\alpha > \alpha'$, establishing the equivalence of

(I) and (III). Define:

$$Q(i)=\left(dL_K(i)/di\right)\Big/\left(dL_M(i)/di\right)=\frac{\alpha-1}{\alpha}\cdot\frac{\alpha'}{\alpha'-1}\cdot(1-i)^{\frac{\alpha-\alpha'}{\alpha\alpha'}}$$

and $dQ(i)/di<0$ if and only if $\alpha<\alpha'$, establishing the equivalence of (II) and (III), via Theorem 2.2. ■

In the uniform case $\rho=1$ corresponds to complete equality, higher values indicating RDI dominated distributions. Correspondingly in the Pareto case, $\alpha=0$ is complete equality and increasing α generates RDI dominated distributions. However comparisons between these families are quite different:

Theorem 2.5 Suppose *K* is a uniform distribution (with parameters $m,\ n,\ \rho=m/n$) and *M* is a Pareto distribution (with parameters $b,\ \alpha$).

(a) *K* dominates *M* in the RDI sense if $\rho-1\in[0,\alpha]$;

(b) *K* and *M* are not RDI comparable if $\rho-1\in(\alpha,\infty)$;

(c) *K* Lorenz dominates *M* if $\rho-1\in[0,2\alpha)$.

Proof

(a) For K and M as described in the statement of Theorem 2.5, $K(i)/M(i)=n\left[1+(\rho-1)i\right](1-i)^{\alpha}/b$. Hence:

$$\frac{d}{di}\frac{K(i)}{M(i)}=\frac{n}{b}\left[(\rho-1)(1-i)^{\alpha}-\alpha(1-i)^{\alpha-1}\left(1+(\rho-1)i\right)\right]<0 \quad\text{iff}\quad \alpha^{-1}(1-i)<(\rho-1)^{-1}+i$$

which holds for all $i\in(0,1)$, comparing the linear functions of i on the right and left, iff $\rho-1\le\alpha$. Thus, when $\rho-1\le\alpha$, $K(i)/M(i)$ is decreasing throughout $[0,1]$, establishing (a).

(b) When $\rho-1>\alpha$, $d\left[M(i)/K(i)\right]/di>(<)0$ for $i>(<)j$ where $j=(1+\alpha)^{-1}\left[1+\alpha(\rho-1)^{-1}\right]\in(0,1)$, precluding RDI comparability.

(c) The Lorenz ordinates for K and M are:

$$L_M(j)=\left[2j+(\rho-1)j^2\right]/(1+\rho)\quad L_M(j)=1-(1-j)^{1-\alpha}$$

Consider (α,ρ) pairs along the line $\alpha=\beta(\rho-1)$ where $\rho\in\left[1,1+\frac{1}{\beta}\right)$ (so $\alpha\in[0,1)$)

and where $\beta > \frac{1}{2}$ (so $\rho - 1 < 2\alpha$). The required Lorenz dominance is then that, for any $\rho \in \left[1, 1 + \frac{1}{\beta}\right]$, $\beta > \frac{1}{2}$:

$\phi(j) = 2j + (\rho - 1) j^2 + (\rho + 1)(1 - j)^{1-\beta(p-1)} - (\rho + 1) > 0$, for all $j \in (0,1)$. When $\rho = 1$, $\phi(j) = 0$ for all $j \in (0,1)$. We show that for all $j \in (0,1)$, $\partial\phi/\partial\rho > 0$ if $\rho \in \left[1, 1 + \frac{1}{\beta}\right)$ and $\beta > \frac{1}{2}$, which will establish the result.

$\partial\phi/\partial\rho = j^2 - 1 + (1 - j)^{1-\beta(p-1)} - \beta(1 + \rho)(1 - j)^{1-\beta(\rho-1)} \cdot \ln(1 - j) > 0$

iff $f(j) > g(j) = \ln(1 - j)$, where $f(j) = \left[1 - (1 + j)(1 - j)^{\beta(\rho-1)}\right] / \beta(1 + \rho)$.

Now $f'(j) = (1 - j)^{\beta(p-1)-1}\left[\beta(\rho - 1)(1 + j) - (1 - j)\right] / \beta(1 + \rho)$.

Hence $f'(0) > g'(0) = -1$ as $\rho \geq 1$, $\beta > \frac{1}{2}$.

Moreover $f''(j) = \rho(\rho - 1)(1 - j)^{\beta(p-1)-2}\left[3 - j - \beta(\rho - 1)(1 + j)\right] / \beta(1 + \rho) \geq 0$ for all $j \in (0,1)$ as $\beta(\rho - 1) < 1$. Thus f is convex, g is strictly concave, $f(0) = g(0) = 0$, $f'(0) > g'(0)$, and so $\partial\phi/\partial\rho > 0$ as required. ■

Theorem 2.5 implies that, when $\rho - 1 \in (\alpha, 2\alpha)$, the uniform distribution Lorenz dominates, but does not RDI dominate, the Pareto distribution, thus presenting a continuous distribution example (see MOYES [1994] for a discrete example) of how the corollary to Theorem 2.1 is only a one-way implication.

Finally an empirical pointer to the extra demands of RDI compared with Lorenz dominance. Using SHORROCKS' [1983] income distribution by decile for 19 countries as a discrete distribution, and computing curvature of Lorenz curves for the discrete case as described in (c) at the start of this section, we found 20 cases of pairwise RDI comparability, compared to 80 cases of Lorenz comparability. Specifically, in the RDI sense, Denmark dominates Columbia, Finland and Malaysia, Japan dominates Brazil, Sri Lanka, Finland, Panama, Netherlands, Malaysia and Columbia, New Zealand and Sri Lanka dominate Finland, Sweden dominates Malaysia, Columbia, Panama and Finland, Tanzania dominates Brazil, UK dominates Netherlands, Malaysia and Columbia (see GU [2005]).

3 A Model of a Primitive Economy

GROSSMAN [1998] and GROSSMAN and KIM [2002] have studied a primitive economy model where individuals decide whether to become producers or predators, where producers acquire an endowment of resources and allocate part of this resource to its defence (the rest to the production of consumables), and where predators prey on the consumables of producers. They assume that there is a given distribution of potential resource endowments that follows a (discontinuous) two-class distribution, and study the equilibrium level of predation, and related matters, in this setting. In what follows we assume that the resource endowment distribution is a general, continuous distribution, showing, in particular, that the RDI concept allows strong statements about how the level of inequality in resource endowments affects the level of predation and other aspects of the primitive economy.

We think of a continuum of agents $i \in [0,1]$ who face the choice of becoming a producer or a predator. We let r denote the fraction of predators, so $1-r$ will be producers, and we denote $R = r/(1-r)$. If agent i decides to be a producer he acquires a resource endowment of $F(i)$, where F is a continuous distribution function, and allocates a fraction of this endowment $g(i,r)$ to its defence, investing the rest in the production of consumables. We assume a unit of resources produces a unit of consumables, for simplicity, and $G(i,r)\left(= g(i,r)/(1-g(i,r))\right)$ denotes the ratio of endowments producer i allocates to defence to the resources he invests in production. After predation producer i retains the fraction $p(i,r)$ of the consumables that he has produced, where

$$p(i,r) = \frac{1}{1+\theta R/G(i,r)} \tag{3.1}$$

and where $\theta > 0$ is a parameter measuring the effectiveness of the predation technology in the economy.

Thus producer i loses the fraction $1-p(i,r)$ of his consumables to predation. If individual i chooses to be a predator then we assume that he does not acquire his own (potential) resource endowment $F(i)$, and instead receives an equal share $(1/r)$ of the aggregate consumables lost to predation by producers.

As a result of these assumptions, the (final) consumption of producer i, $C(i,r)$ is:

$$C(i,r) = p(i,r)\cdot F(i)/(1+G(i,r)) \tag{3.2}$$

where $1/(1+G(i,r)) = 1-g(i,r)$ is the proportion of resources invested in production by producer i, so $F(i)/(1+G(i,r))$ is the total consumables produced by i.

On the other hand each predator ends up with the same amount of consumables, namely:

$$D(r)=\int_{i\in NP}[1-p(i,r)]\cdot\frac{F(i)}{1+G(i,r)}di\Big/r \qquad (3.3)$$

where "NP" denotes the set of non-predators (i.e. producers). Here the numerator is the aggregate amount of consumables lost to predation by producers.[2]

Producer i chooses $G(i,r)$ to maximize $C(i,r)$, taking r (and so R) as given. The solution to this problem shows that all produces choose the same $G(i,r)$:

$$G(i,r)=\sqrt{\theta R}=G(r) \qquad (3.4)$$

Thus producer and predator consumption become:

$$C(i,r)=\frac{F(i)}{\left(1+\sqrt{\theta r/(1-r)}\right)^2} \qquad (3.5)$$

$$D(r)=\frac{\sqrt{\theta r/(1-r)}}{\left(1+\sqrt{\theta r/(1-r)}\right)^2}\int_{j\in NP}F(j)\,dj/r\,. \qquad (3.6)$$

4 Equilibrium and Inequality in the Primitive Economy

Suppose a fraction of predators $r\in(0,1)$ satisfies $C(r,r)=D(r)$. From (3.5), $C(i,r)$ is weakly increasing in i, so that $C(i,r)\le D(r),\ i\le r$ and $C(i,r)\ge D(r),\ i\ge r$. Hence r has the property that for individuals $i\in[0,r]$ predation (rather than production) is an optimal choice and for $i\in[r,1]$ production is optimal, creating the fraction r of predators and $1-r$ of producers with $NP=[r,1]$. This defines the natural equilibrium concept for the model:

Definition 4.1 $r\in(0,1)$ is an interior equilibrium level of predation if and only if

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[2] If there is a large finite number of individuals n whose endowment distribution is approximately $F(i)$, then the numerator is $1/n$ of aggregate consumables lost to predators, but then the denominator is also $1/n$ of the total number of predators, leaving (3.3) unchanged. Other aggregates used in this paper also ignore n, as this causes no qualitative difference, as here.

$C(r,r) = D(r)$ with the set of producers $NP = [r,1]$.

Rearranging the equation $C(r,r) = D(r)$ and using $NP = [r,1]$, the equilibrium condition is equivalently $r \in (0,1)$ where $\phi(r) = \psi(r)$ and :

$$\phi(r) = \frac{1}{\sqrt{\theta}}\sqrt{r(1-r)}, \;\; \psi(r) = \int_r^1 F(i)\,di \Big/ F(r)$$

Here $\phi(r)$ defines a strictly concave function on the interval $[0,1]$, with $\phi(0) = \phi(1) = 0$, maximum at $r = \frac{1}{2}$ and $\phi'(0) = +\infty$, $\phi'(1) = -\infty$. $\psi(r)$ also defines a continuous function on $[0,1]$, with $\psi(0) > 0 \left(= +\infty \text{ if } F(0) = 0\right)$, $\psi(1) = 0$ (by definition if $F(1) = +\infty$), and where ψ is decreasing and $\psi(r) \geq 1 - r$ for all $r \in [0,1]$ (since $\int_r^1 F(i)di \geq (1-r)F(r)$ as F is weakly increasing).

In the case of an equal distribution ($F(i) = k > 0$ for all $i \in [0,1]$), $\psi(r) = 1 - r$ and there is a unique interior equilibrium with $r = \theta/(1+\theta)$. Since $\psi(r) \geq 1 - r$ for any other distribution, $r = \theta/(1+\theta)$ is a lower bound on equilibrium predation levels. For a Pareto distribution, $\psi(r) = (1-\alpha)^{-1}(1-r)$ and it is straightforward to calculate the unique interior equilibrium as $r = \theta\Big/\left[\theta + (1-\alpha)^2\right]$, which coincides of course with the equal distribution when $\alpha = 0$, and also shows that the equilibrium level of predation increases monotonically with α. In particular, from Theorem 2.4, if K and M are two Pareto distributions the equilibrium level of predation is higher under M than under K if and only if K RDI dominates M.

Figure 1 illustrates $\psi(r)$ for a Pareto distribution, $\phi(r)$, and the unique interior equilibrium at r^*.

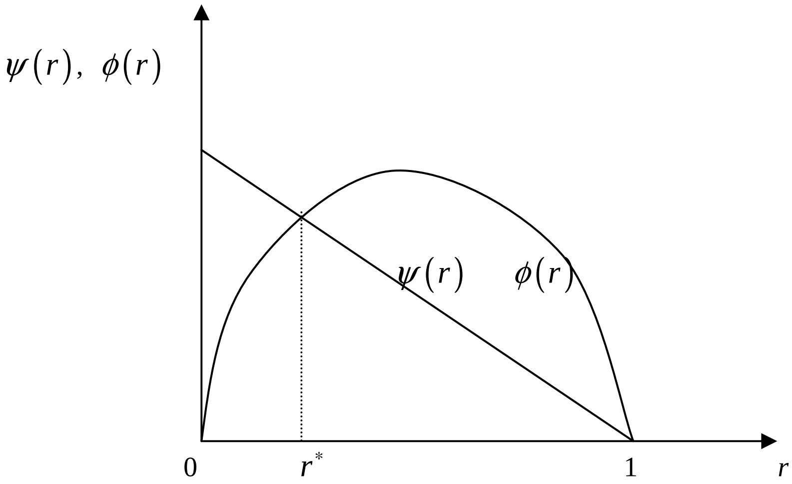

Figure 1

The equilibrium also has a natural stability property. For $r < r^*$, $\psi(r) > \phi(r)$, which implies $D(r) > C(r,r)$ so that if $i \in [0,r]$ were predators, individual $i \in [r, r+\varepsilon)$ some $\varepsilon > 0$ would prefer to become predators, increasing r towards r^*. Conversely with $r > r^*$, the same dynamic would lead to r falling towards r^*. Hence we say that an equilibrium $r^* \in (0,1)$ is (globally) stable when $\psi(r) > (<) \phi(r)$ for $r < (>) r^*$, clearly true in Figure 1.

The following provides a neat and more general sufficient condition for existence, uniqueness and stability.

<u>Theorem 4.1</u> Suppose $\psi(r) = \int_r^1 F(i)\,di / F(r)$ is convex. Then there exists a unique interior equilibrium level of predation, and the equilibrium is stable.

<u>Proof</u> Let $\eta(r) = \phi(r) - \psi(r)$. η is strictly concave with $\eta(1) = 0$ and $\eta(0) = -\psi(0) < 0$. Moreover for r sufficiently close to 1, $\eta(r) > 0$ since $\lim_{r\to1} \eta(r)/(1-r) = +\infty$, as $\lim_{r\to1} \phi(r)/(1-r) = +\infty\ (\phi'(1) = -\infty)$ and $-\psi(r)/(1-r) \geq -\psi(0)$ for all $r < 1$ (ψ is convex). It follows that there exists a unique interior equilibrium $r^* \in (0,1)$ where $\eta(r) > 0 (< 0)$ for $r > r^*\ (r < r^*)$ which also ensures stability. ■

For a uniform distribution, $\psi(r)=\left[1-r+\frac{1}{2}(\rho-1)(1-r^2)\right]\Big/\left[1+(\rho-1)r\right]$, which is easily shown to be convex, guaranteeing a unique stable equilibrium in this case also. Moreover $\partial\psi(r)/\partial\rho$ has the sign of $(1-r)$ ensuring that the equilibrium level of predation increases (from the equal distribution value of $\theta/(1+\theta)$ at $\rho=1$) as ρ increases. Hence, via Theorem 2.3, there is the same link between RDI dominance and the equilibrium level of predation as in the Pareto case. In fact this link is quite general.

Theorem 4.2 Suppose *K* and *M* are two RDI comparable resource distributions which generate unique, interior and stable equilibrium levels of predation r_K and r_M respectively, in two primitive economies with the same predation technology parameter θ. Then $r_K < r_M$ if and only if *K* dominates *M* in the RDI sense.

Proof Define $\psi_K(r)=\int_r^1 K(i)\,di/K(r_K)$ and $\psi_M(r)=\int_r^1 M(i)\,di/M(r_M)$, and suppose K dominates M in the RDI sense. Then $K(i)/M(i)<K(r)/M(r)$ for all $r\in(0,1)$, $i\in(r,1)$, so $\int_r^1 K(i)\,di<\left[K(r)/M(r)\right]\int_r^1 M(i)\,di$ and $\psi_K(r)<\psi_M(r)$ for all $r\in(0,1)$. It follows that the assumed unique, stable equilibrium under K (r_K, where $\psi_K(r_K)=\phi(r_K)$) must be smaller than the assumed unique, stable equilibrium under M (r_M, where $\psi_M(r_M)=\phi(r_M)$). ■

The importance and role of RDI in this result is now brought out further with an example which shows that the statement in Theorem 4.2 does not remain true if one replaces "RDI dominance" by "Lorenz dominance".

Let *M* be a Pareto distribution with parameters $b>0$ and $\alpha=\frac{1}{2}$. Then $\psi_M(r)=2(1-r)$ and $r_M=\theta/(\theta+1/4)$ is the unique level of predation. Assume $\theta<1/4$ so that $r_M<1/2$ (and the equilibrium is on the upward sloping part of the $\phi(r)$ graph—see figure 1). *K* is defined as follows, where $0<\delta<M(r_M)$:

$$K(i)=\begin{cases}M(i)-\delta,\ i\ge r_M\\ M(r_M)-\delta,\ i\le r_M\end{cases}$$

The transition from M to (the continuous) K involves a lump sum fall in income (by δ) for all $i\in[r_M,1]$, with equal incomes for $i\in[0,r_M]$. Although $K(i)/M(i)$ is decreasing for $i\in[0,r_M]$, it is increasing for $i\in[r_M,1]$ so K and M are not RDI comparable. However for δ small enough K Lorenz dominates M, as follows.

The Lorenz curves for K and M are defined by:

$$L_M(j)=1-(1-j)^{1/2}$$

$$L_K(j)=\begin{cases}\left[M(r_M)-\delta\right]j/\mu \text{ if } j\le r_M\\ \left\{r_M\left[M(r_M)-\delta\right]+2b\left[(1-r_M)^{1/2}-(1-j)^{1/2}\right]-\delta(j-r_M)\right\}/\mu \text{ if } j\ge r_M\end{cases}$$

where $\mu=r_M\left[M(r_M)-\delta\right]+2b(1-r_M)-\delta(1-r_M)$. For $j\le r_M$, calculations reveal that $L_K(j)>L_M(j)$ if $L_K(r_M)>L_M(r_M)$, which requires:

$$\delta<b\left[(1+4\theta)^{1/2}-1\right] \tag{4.1}$$

Similar (tedious) calculations reveal that $L_K(j)>L_M(j)$ for $j\in[r_M,1]$ if:

$$\delta<2b\left[\frac{\theta+\frac{1}{2}}{\left(\theta+\frac{1}{4}\right)^{1/2}}-1\right] \tag{4.2}$$

In fact (4.1) implies (4.2), so K Lorenz dominates M if (4.1) holds. Nevertheless we now see there is a unique equilibrium under K, r_K, where $r_K>r_M$.

For K:

$$\psi_K(r)=\begin{cases}\dfrac{2b(1-r)^{1/2}-\delta(1-r)}{b(1-r)^{-1/2}-\delta} \text{ if } r\ge r_M\\[2ex] r_M-r+\dfrac{2b(1-r_M)^{1/2}-\delta(1-r_M)}{M(r_M)-\delta} \text{ if } r\le r_M\end{cases}$$

It is straightforward to check that $\psi_K(r_M)>\psi_M(r_M)$ for all $\delta>0$ and since $\psi_K(r)$ is linear with slope-1 when $r\le r_M$, there is certainly no equilibrium under K with $r\in[0,r_M]$

(since $r_M < \frac{1}{2}$). Moreover (again tedious) calculations show that $\psi_K(r)$ is convex on $[r_M, 1]$ if $\delta < 3b(1+4\theta)^{1/2}$, which is again ensured by (4.1). Thus (4.1) ensures that there is a unique equilibrium under *K*, r_K, with $r_K > r_M$, although K Lorenz dominates *M*.

Moreover it is straightforward to check that the means of *K* and *M* coincide if:

$$\delta/b = \frac{4\theta+2}{(4\theta+1)^{1/2}} - 2 \qquad (4.3)$$

For any $\theta > 0$, the δ/b in (4.3) satisfies (4.1) also. In this case the unique equilibrium under *K*, r_K, has $r_K > r_M$ although *K* dominates *M* in the generalized Lorenz sense (or equivalently in the second-order stochastic dominance sense—see SHORROCKS [1983], LAMBERT [2001]). Hence the statement of Theorem 4.2 does not remain true if one replaces RDI dominance by either Lorenz dominance or generalised Lorenz dominance.

The economic intuition behind Theorem 4.2 and the above conterexample is as follows. If r_M is the equilibrium under *M*, then $C(r_M, r_M) = D(r_M)$, so the "marginal individual" at r_M is indifferent between predation and production. Suppose the set of predators stays at $[0, r_M]$ for now, but *M* gives way to a new distribution *K*. For the previously marginal individual the effect on *C* depends on the change from $M(r_M)$ to $K(r_M)$ and the effect on *D* depends on the change from $\int_{r_M}^1 M(j)dj$ to $\int_{r_M}^1 K(j)dj$. These effects are only from the upper tail of the distributions, and increases in inequality within this upper tail cause D to increase more than C for the marginal individual so that, individuals in $[r_M, r_M+\varepsilon]$, some $\varepsilon > 0$ would become predators under *K*, increasing the equilibrium level of predation. Thus increases in inequality in this way, in the upper tail of the distribution, increase equilibrium predation, and if *M* RDI dominates *K* such an increase emerges. The same increase in inequality in the upper tail occurs in the counterexample, but there, in the lower tail, the increase in equality leads to Lorenz dominance of *K* over *M*.

5. Further Comparative Satics of the Primitive Economy

If r is an equilibrium level of predation, we have the following aggregates and shares:

(i) The aggregate resources realised in the economy are $\int_r^1 K(i)di$;

(ii) From (3.4) a fraction $g = \sqrt{\theta R}\Big/\left(1+\sqrt{\theta R}\right)$, where $R = r/(1-r)$, of aggregate resources are devoted to defence of property, so $g\int_r^1 K(i)\,di$ are aggregate resources devoted to defence;

(iii) Hence aggregate consumption (=production of consumables) is $U = (1-g)\int_r^1 K(i)di$, which is also aggregate utilitarian social welfare;

(iv) From (3.1) and (3.4) the producer share of aggregate consumption is $p = 1-g$ whilst $1-p$ is the predator share;

(v) Hence aggregate producer consumption (or welfare) is $U_1 = p(1-g)\int_r^1 K(i)\,di$ whilst that of predation is $U_2 = (1-p)(1-g)\int_r^1 K(i)\,di$.

Clearly g is an increasing function of R and r, hence:

Theorem 5.1 Suppose *K* and *M* are two RDI comparable resource distributions which generate unique and stable equilibrium levels of predation r_K and r_M respectively, in two primitive economies with the same technology parameter θ. Then the fraction g of aggregate resources devoted to defence of property is higher in *M* than in *K*, and the predator (producer) share of aggregate consumption $1-p$ (p) is higher (lower) in M than in K if and only if *K* dominates *M* in the RDI sense.

Proof Immediate from Theorem 4.2 since $g = 1-p$ is increasing in r. ■

Thus, in the more unequal economy (*M*, in the RDI sense) not only is the level of predation higher, but also the fraction of aggregate resources devoted to defence and the predator share of aggregate consumption are higher. These fractions and shares depend only on the extent of inequality (in the RDI sense) in the underlying potential resource distribution (given the common θ). However movements of aggregates (given the common θ) depend in general on other features of the resource distribution. We use:

Definition 5.1 *K* dominates *M* in the first-order stochastic dominance (FSD) sense if and only if $K(i) > M(i)$, $i \in [0,1]$.

Theorem 5.2 Suppose *K* dominates *M* in both the RDI and FSD senses, where *K* and *M* are resource distributions which generate unique and stable equilibrium levels of predation, r_K and

r_M respectively, in two primitive economies with the same technology parameter θ. Then:

(a) Aggregate resources realised are higher in *K* than in *M*;
(b) Aggregate consumption (U) is higher in *K* than in *M*;
(c) Aggregate producer consumption (U_1) is higher in *K* than in *M*.

<u>Proof</u>

(a) From Theorem 4.2, $r_K < r_M$, so FSD ensures $\int_{r_K}^{1} K(i)di > \int_{r_M}^{1} M(i)\,di$;

(b) From Theorem 5.1, g in *K* (g_K say) is lower than in M (g_M), so, from (a),

$$(1-g_K)\int_{r_K}^{1} K(i)di > (1-g_M)\int_{r_M}^{1} M(i)di;$$

(c) From Theorem 5.1, p in *K* (p_K) is higher than in M (p_M), so, from (b),

$$p_K(1-g_K)\int_{r_K}^{1} K(i)di > p_M(1-g_M)\int_{r_M}^{1} M(i)di. \qquad \blacksquare$$

Thus aggregate realised resources, consumption (=utilitarian social welfare) and producer consumption are absolutely higher in richer, more equal economies (*K*, in the FSD and RDI senses).

6. Conclusion

We have conducted two exercises in this paper. In the first exercise, we bridged a gap in the literature by examining the properties of RDI in continuous distributions and the equivalence of RDI and Lorenz inequality in some frequently used special distributions. In the second one, we extended the primitive economy model in GROSSMAN [1998] and GROSSMAN and KIM [2002] to allow individual endowments to follow a general continuous distribution. For a distribution to RDI dominate another requires Lorenz dominance over any subset of the population. In the primitive economy model the level of predation is essentially dictated by resource inequality in upper tails of the distribution, which is picked up by the RDI concept, but not by Lorenz (or generalised Lorenz) dominance. This allows a number of precise and quite general comparative static statements relating to the effect of inequality on the level of predation and other features of the primitive economy. We hope to have demonstrated the usefulness of RDI in economic modelling, which we expect could be much more widespread.

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