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The strategic Marshallian cross

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The strategic Marshallian cross

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Abstract

Our aim in this paper is to reveal some of the hidden power of Marshall's analysis of partial equilibrium, even when agents have opportunities to manipulate the market and choose to act strategically. Specifically, we explain how to construct strategic versions of Marshallian supply and demand curves. With a finite set of players on each side of the market, non-trivial Nash equilibria of the game are in 1–1 correspondence with intersections of the strategic supply and demand curves. This equivalence can be used to analyze properties of equilibria such as existence and uniqueness even when the players are heterogeneous. It also enables us to investigate which of the conventional comparative statics results of Marshallian analysis survive strategic price manipulation by buyers and sellers. Finally, we show that strategic manipulation is always achieved by withholding goods or money and that the possibility for such play diminishes under replication and vanishes in the limit.

Keywords: Strategic Marshallian cross, strategic market game, imperfect competition.

JEL codes: C72, D43, D50.

1 Introduction

Marshallian partial equilibrium analysis rests on several assumptions, price-taking by all agents being principal among them. To quote Marshall [18]: “Thus we assume that the forces of supply and demand have free play; that there is no combination of dealers on either side, but each acts for himself, and there is much free competition; that is buyers compete freely with buyers and sellers compete freely with sellers.”

This is perhaps most elegantly formalized in its general equilibrium version by Aumann [4], where the cooperative-game concept of core equivalence lends support to price taking. An alternative justification can be grounded in non-cooperative strategic market games of Shapley and Shubik [23], [24], [12], [17]. With a continuum of non-atomic players in such a game, strategic play leads to competitive outcomes; as Dubey, Mas-Colell and Shubik put it: “price-taking behavior is, in a mass market, the natural consequence of “message taking” behavior” [13]. When the number of agents is large but finite, competitive outcomes may often be regarded as approximate solutions, using limiting theorems as justification (see, for example, [14] and [23]). Here, we offer an alternative view: even when agents are finite in number and take advantage of their strategic opportunities, the Marshallian framework may still be used to study equilibria, provided the appropriate interpretation of supply and demand schedules is used. This approach can be extended to multi-market equilibrium, but we focus here on partial equilibrium, since definite results in comparative statics and uniqueness of equilibrium, for example, are more readily available in this setting, allowing us to make illuminating comparison with results obtained under price-taking assumptions.

We study the market for a single consumption good, payments being made with commodity money. All players are endowed with either the consumption good (sellers) or with money (buyers), but not both and have preferences that can be represented by a utility function. A further natural assumption, following Marshall is that preferences are quasi-linear in money [18]: “When a person buys anything for his own consumption, he generally spends on it a small part of his total resources;... In [such a case] there is no appreciable

change in his willingness to part with money.” (A formal study of this assumption has been conducted by Vives [26].) However, for our purposes we can use the weaker assumption of binormality: both consumption and money are normal goods.

We consider a simultaneous-move game in which sellers offer all or part of their endowment of the consumption good to buyers who make money bids, up to their endowment, for a share of the total offered by sellers. The share received by any buyer depends positively on their bid and, to respect anonymity, negatively on the aggregate bids of other buyers. The total money bid is divided amongst sellers in a similar way, as an increasing function of a seller’s offer and a decreasing function of the offers of other sellers. In Section 2, we discuss specific sharing rules. The key feature of this setup is that payoffs depend on other players’ strategies only through aggregate bids and offers. In *Game Theory in the Social Sciences* [25] Shubik commented on games with a single aggregate: “Games with the above property have more structure than a game selected at random. How this structure influences the equilibrium points has not yet been explored in depth.” An investigation of this issue has recently been conducted by Cornes and Hartley [11] building on earlier work of Novshek [20] in industrial economics and Andreoni [2] in public economics amongst others. The game considered here has two aggregates, which necessitates an extension of the approach in [11] closely based on competitive methodology.

We focus attention on non-autarkic Nash equilibria in which there is trade (total bids and offers are positive). Our aim is to show how such equilibria can be studied using a modified competitive approach. Competitive equilibrium has two essential components: (i) agents make best responses to the economic environment which frames their decision-making under the presumption that they are unable to influence that environment directly and (ii) these optimal decisions are consistent with that environment. In an exchange economy, the economic environment is summarized by a price vector, (i) becomes constrained utility maximization and (ii) is market clearing. When agents fully exploit the strategic possibilities open to them, the competitive approach can be adapted by modifying part (i) to allow for the direct effect

of agents' choices on the environment of the other agents. This requires, for any environment, that each agent $i \in I$ has a strategy \hat{s}_i that is a unique best response to every strategy profile of the other agents that, together with \hat{s}_i , forms the given environment. (When agents behave parametrically, \hat{s}_i is simply the conventional optimal choice in competitive equilibrium.) Condition (ii) requires that the strategy profile $\hat{\mathbf{s}} = (\hat{s}_i)_{i \in I}$ derived from an environment be consistent with that environment. This is equivalent to requiring $\hat{\mathbf{s}}$ to be a Nash equilibrium. In games of the type described above, the environment can be summarized by the levels of aggregate bids B and aggregate offers Q . Given an environment (B, Q) and an offer q'_i , the set of strategies forming the given environment all have aggregate offer $Q - q'_i$ and aggregate bid B . Since the payoffs of seller i depend only on q_i, B and Q , its best responses depend only on $Q - q_i$ and B . Binormality implies additionally that such best responses are unique; we refer to this best response as the replacement value of player i to (B, Q) . If the sum of replacement offers to (B, Q) is $Q (> 0)$ and the sum of replacement bids to (B, Q) is $B (> 0)$, then (B, Q) is a Nash equilibrium. Conversely, all non-autarkic pure strategy Nash equilibria can be obtained in this way.

It is insightful to apply this result by adopting an equivalent parametrization of the environment, which consists of strategic price $p = B/Q$ together with aggregate bids in the case of buyers and aggregate offers in the case of sellers. Then consistency can be decomposed into two steps. Firstly, one establishes the existence of $P^S \geq 0$ such that, for any level of the strategic price $p > P^S$, there is a unique level of aggregate offers Q that is the sum of replacement offers to (pQ, Q) . Such a Q is the only possible aggregate offer in any equilibrium in which strategic price is p . Viewed as a mapping from strategic prices to aggregate offers, we refer to this as the strategic supply schedule. This schedule is independent of buyers, depending only on the endowments and preferences of sellers. It takes fully into account the possibilities for strategic manipulation open to sellers. The terminology also reflects the fact that, as the number of sellers is increased by replication, the strategic supply curve approaches the competitive supply curve. Buyers can be treated in a similar fashion: there is a unique level of aggregate bids

that is the sum of all bidders replacement bids to $(B, B/p)$ provided $p < P^B$, where P^B is a (possibly infinite) threshold price. The strategic demand curve is obtained by dividing this aggregate bid by p and takes full account of the strategic opportunities available to buyers. The consistency condition for Nash equilibrium from the previous paragraph translates into our fundamental result: every strategy profile is a non-autarkic Nash equilibrium¹ if and only if the corresponding aggregate offer and strategic price lie on the intersection of supply and demand curves. This determines equilibrium at the level of aggregates: \widehat{B} and \widehat{Q} . Individual equilibrium strategies are given by the (unique) replacement bids and offers to $(\widehat{B}, \widehat{Q})$. Thus non-autarkic Nash equilibria are in 1–1 correspondence with intersections of the strategic supply and demand curves.

This result can be used to apply a competitive approach to economies in which agents exercise market power. For example, strategic supply and demand curves cross at least once if and only if $P^S < P^B$. Since it is possible to express P^S and P^B in terms of marginal rates of substitution at the initial endowments, we can derive necessary and sufficient conditions for a non-autarkic equilibrium². These show that, if the economy has no price-taking equilibrium, the only strategic equilibrium is autarky. However, it is possible to have only an autarkic strategic equilibrium even if there is a competitive equilibrium: strategic effects may prevent agents realizing any gains from trade even when these exist. Further restrictions on payoffs are required to rule out multiple non-autarkic equilibria. For example, if the difference between strategic demand and supply is monotonic these curves cross at most once. Since strategic demand is always decreasing in price, conditions on sellers that give increasing strategic supply ensure a unique equilibrium. One such condition is that each seller's *competitive* supply is

¹It is an artefact of the game that setting all bids and offers to zero is always an equilibrium. Indeed, the presence of such autarkic equilibria is a complicating factor in the conventional analysis of market games, since it precludes, for example, direct application of standard fixed point theorems when showing that non-autarkic equilibria exist. The competitive approach circumvents such difficulties by analyzing non-autarkic equilibria directly.

²A similar approach leads to conditions for determining whether individual players are active in equilibrium.

non-decreasing. If, in addition, $P^S < P^B$, the curves cross exactly once; this is the strategic Marshallian cross³ that ensures the existence of a unique non-autarkic equilibrium.

This competitive approach can also be used to study comparative statics. As an example, consider an economy in which all players have binormal preferences and suppose a quantity tax is imposed on sellers all of which have non-decreasing competitive supply. The conventional result that price rises and quantity traded falls is robust to strategic manipulation. This can be justified by first showing that the strategic supply curve (with price on the vertical axis) shifts upwards and then applying the standard argument to conclude that, in equilibrium, strategic price increases (possibly by more than the tax) and trade is reduced.

Another application of the competitive approach examines the effect of adding or removing buyers or sellers or changing endowments. For example, an extra active seller or increase in an incumbent seller's endowment increases strategic supply, shifting the supply curve to the right at the current equilibrium. With binormal preferences and increasing competitive supply, the result is a fall in strategic price and a rise in trade. Similarly, an extra active buyer shifts the strategic demand schedule upwards, resulting in an increase in strategic price and a reduction in trade. Thus, additional players (or increases in endowments) increase competitiveness on the side of the market they join, reflecting similar results under price-taking.

The construction of strategic supply and demand suggest that, when agents have little market power, these functions approach their competitive equivalents and indeed, under replication, strategic supply and strategic demand per replica do tend to their competitive equivalents. This implies that strategic price tends to competitive price and quantities per replica also converge. Furthermore, convergence is always from below: strategic supply and demand per replica are always less than the price-taking versions. Sellers exert market power by withholding the consumption good and buyers by withholding money.

Under proportional sharing the game above reduces to that of bilateral

³We respect the Marshallian precedent by putting price on the vertical axis.

oligopoly as presented by Gabszewicz and Michel [16]. Bloch and Ghosal [7] studied existence and uniqueness in this game under the additional assumption that a common utility function can be used to describe preferences of all players. The full symmetry assumptions were relaxed in Bloch and Ferrer [6] in their discussion of trade fragmentation, but these authors proved only existence of non-trivial equilibria. More recently, Amir and Bloch [1], focussing on comparative statics, extended the results of Bloch and Ghosal to games in which symmetry is only imposed on each side of the market, allowing buyers to have different preferences from sellers, and imposing weaker additional conditions on preferences; for example, they show that gross substitutes imply uniqueness as in general equilibrium. The competitive approach requires no symmetry assumptions and its conclusions for existence, uniqueness and comparative statics encompass all existing results known to the authors as well as clarifying the analysis.

After describing the details of the game and our assumptions on preferences in the next section, we investigate the existence of strategic supply and demand curves and prove the equivalence between their intersections and Nash equilibria in Section 3. The following four sections are devoted to applications of this result. We first derive necessary and sufficient conditions for the existence of a non-autarkic equilibrium and show that these imply that strategic effects may prevent a non-autarkic equilibrium even though gains from trade are possible. This section also discusses sufficient conditions for a unique non-autarkic equilibrium. The comparative statics studied in Section 6 include adding extra players, modifying endowments and imposing a tax. In the next section we study the relationship between strategic and competitive supply and demand curves and equilibria when the market is thickened by replication. After a conclusion, the final section (appendix) contains proofs.

2 Strategic trade

The setting for our partial equilibrium analysis is a market with two commodities, a consumption good and commodity money, in which the players

are partitioned into participants on either side of the market for the consumption good. Specifically, the set of players is $\mathcal{H} = H^S \cup H^B$, where $H^S \cap H^B = \emptyset$. Endowments of the two goods are $(e_h, 0)$ if $h \in H^S$ and $(0, e_h)$ if $h \in H^B$, where $e_h > 0$ for all $h \in \mathcal{H}$ and the first component refers to the consumption good. We interpret players in H^S as sellers and those in H^B as buyers of the consumption good. It is convenient to rule out monopoly and monopsony: $|H^S|, |H^B| \geq 2$, where $|X|$ denotes the cardinality of X . The strategy sets are the intervals $\mathcal{S}_h = [0, e_h]$ for all $h \in \mathcal{H}$. Sellers choose an *offer* $q_h \in \mathcal{S}_h$ for all $h \in H^S$ and we write Q for the *aggregate offer* $\sum_{h \in H^S} q_h$. Buyers choose a *bid* $b_h \in [0, e_h]$ for all $h \in H^B$ and we write B for the *aggregate bid* $\sum_{h \in H^B} b_h$. We write $\mathcal{S}^S = \prod_{h \in H^S} \mathcal{S}_h$ and $\mathcal{S}^B = \prod_{h \in H^B} \mathcal{S}_h$ for the sets of strategy profiles of sellers and buyers, respectively. Given a strategy profile $(\mathbf{q}, \mathbf{b}) \in \mathcal{S}^S \times \mathcal{S}^B$ in which $Q, B > 0$, the rules of the game must specify how the aggregate offer is distributed amongst the buyers and aggregate bid amongst the sellers. In general, this may be accomplished by sharing rules $\omega_h^S : \mathcal{S}^S \rightarrow [0, 1]$ for each seller $h \in H^S$ and $\omega_h^B : \mathcal{S}^B \rightarrow [0, 1]$ for each buyer $h \in H^B$, satisfying $\sum_{h \in H^S} \omega_h^S(\mathbf{q}) = \sum_{h \in H^B} \omega_h^B(\mathbf{b}) = 1$ for all \mathbf{q} and \mathbf{b} . The corresponding allocation is $\{x_h\}_{h \in \mathcal{H}}$, where

$$(x_{h1}, x_{h2}) = \begin{cases} (e_h - q_h, \omega_h^S(\mathbf{q}) B) & \text{if } h \in H^S \text{ or} \\ (\omega_h^B(\mathbf{b}) Q, e_h - b_h) & \text{if } h \in H^B. \end{cases} \quad (1)$$

If $Q = 0$ or $B = 0$ or both, we take $\{x_h\}_{h \in \mathcal{H}}$ to be the initial allocation; no trade takes place. Finally, player h evaluates allocations using the weak preference relation \succsim_h over non-negative bundles of the two goods.

We require the sharing rule $\omega_h^S(\mathbf{q})$ to be a function of q_h and Q (anonymity), increasing in q , decreasing in $q_{h'}$ for $h' \neq h$ (monotonicity) and satisfy $\sum_{h \in H^S} \omega_h^S(\mathbf{q}) = 1$ for all strategy profiles \mathbf{q} . A similar restriction is imposed on ω_h^B . Here, we focus on sharing rules of the form

$$\begin{aligned} \omega_h^S(\mathbf{q}) &= \lambda^S \frac{q_h}{Q} + (1 - \lambda^S) \theta_h \text{ for all } h \in H^S, \text{ and} \\ \omega_h^B(\mathbf{b}) &= \lambda^B \frac{b_h}{B} + (1 - \lambda^B) \theta_h \text{ for all } h \in H^B, \end{aligned}$$

where $0 \leq \theta_h \leq 1$ for all h . For such sharing rules, $\omega_h^S(\mathbf{q})$ depends only on q_h and Q and we abuse notation by writing it $\omega_h^S(q_h, Q)$. Similarly, we write the sharing rule for buyers as $\omega_h^B(b_h, B)$. When $\lambda^S = \lambda^B = 1$ we have fully proportional sharing and the game is equivalent to a Shapley-Shubik strategic market game [23] as applied to bilateral oligopoly by Gabsewicz and Michel [16]. With $\lambda^B < 1$, the aggregate offer is divided into two portions; the first is distributed in proportion to bids and the bidder h receives the exogenous share θ_h of the second, irrespective of their bid. Similar considerations apply to sellers.

We seek a Nash equilibrium of this game; that is, a strategy profile $\{q_h\}_{h \in H^S}, \{b_h\}_{h \in H^B}$ for which the corresponding allocation (1) has the property that, if (x'_{h1}, x'_{h2}) is the allocation resulting from a unilateral change of q_h to q'_h for some $h \in H^S$, or from b_h to b'_h for some $h \in H^B$, then $(x_{h1}, x_{h2}) \succsim_h (x'_{h1}, x'_{h2})$ for all $h \in \mathcal{H}$.

2.1 Preferences

We assume that preferences are convex, continuous and strictly increasing in both arguments. Thus, the upper preference set $P_h(x) = \{y \in \mathbb{R}_+^2 : y \succsim_h x\}$ is convex, closed and has recession cone \mathbb{R}_+^2 . It follows that there will be a supporting line to $P_h(x)$ at x with non-positive slope and we write $\partial_h(x)$ for the set of absolute values of such slopes. Thus, $\delta \in \partial_h(\tilde{x})$ if and only if the line

$$x_2 = \tilde{x}_2 - \delta(x_1 - \tilde{x}_1)$$

supports $P_h(\tilde{x})$ at \tilde{x} . If x lies on the vertical axis, there will be a vertical supporting line; to recognize this, we will include $+\infty$ in $\partial_h(x)$. When preferences are representable via a continuously differentiable utility function, $\partial_h(x)$ has a single member, the marginal rate of substitution at x , except possibly on the axes. The correspondence ∂_h can be regarded as the projection of the inverse of the demand correspondence onto prices. Since the demand correspondence is convex-valued and upper hemi-continuous (see, for example Arrow and Hahn [3]), this property also holds for ∂_h .

Partial equilibrium analysis typically assumes quasi-linear preferences.

However, for our purposes the weaker assumption of *binormality* will suffice: both goods are normal. In particular, this implies that marginal rates of substitution increase under moves to the north-west:

$$x_1 \leq x'_1, x_2 \geq x'_2, \delta \in \partial_h(x), \delta' \in \partial_h(x') \implies \delta \geq \delta', \quad (2)$$

where the final inequality is strict if $x_1 < x'_1$ and $x_2 > 0$.

Although our framework is that of an exchange economy, it can be reinterpreted as a model with production on the part of sellers, provided sellers have no desire to consume the output and have production opportunities exhibiting non-decreasing returns to scale.

3 Strategic supply and demand

The game described has an autarkic (no-trade) equilibrium in which $q_h = 0$ for all $h \in H^S$ and $b_h = 0$ for all $h \in H^B$. In the sequel, we focus on non-autarkic equilibria in which $Q, B > 0$.

In this section, we show how such equilibria can be studied using strategic versions of Marshallian supply and demand curves. We begin by focusing on the game played by sellers when aggregate bids are held fixed. We then study the buyers' side of the market and, in the final subsection, we examine consistency requirements for an equilibrium and define strategic supply and demand functions.

3.1 The sellers' partial game

For a given aggregate bid B , the game played by sellers can be viewed as dividing the output of a jointly owned production process, in which the share of B received by seller h is $\omega_h^S(\mathbf{q})$ when the offer profile is $\mathbf{q} \in \mathcal{S}^S$. We follow the techniques of Cornes and Hartley in analyzing such aggregative games (see, for example, [9], [10], [11] for applications to contest theory and common access resource games).

The essence of the approach is to circumvent some of the complications

that arise when working directly with best responses for heterogeneous players. Instead, we start by asking: what strategy choices by seller $h \in H^S$ are consistent with an equilibrium in which the aggregate offer and bid take the values Q and B ? We write $R_h^S(Q, B)$ for this set of strategies and observe that $q \in R_h^S(Q, B)$ if and only if $q \leq Q$ and q is a best response to $(Q - q, B)$. For $Q, B > 0$, this requires q to satisfy

$$(e_h - q, \omega_h^S(q, Q) B) \succsim_h x \text{ for all } x \in S,$$

where

$$S = \{(e_h - q', \omega_h^S(q', q' + Q - q) B) : q' \in [0, e_h]\}.$$

Noting that S is the upper boundary of a convex set, we have $q \in R_h^S(Q, B)$ if and only if $0 < q \leq \min\{Q, e_h\}$ and

$$\lambda^S \frac{B(Q - q)}{Q^2} \in \partial_h (e_h - q, \omega_h^S(q, Q) B), \quad (3)$$

or $q = 0$ and

$$\lambda^S \frac{B}{Q} \in \partial_h^- (e_h, \omega_h^S(0, Q) B), \quad (4)$$

where $\delta^- \in \partial_h^-(x)$ if and only if $\delta^- \leq \delta$ for some $\delta \in \partial_h(x)$. Note that our inclusion of $+\infty$ in $\partial_h(x)$ on the vertical axis avoids the need for an explicit boundary condition like (4) at $q = e_h$.

We can view R_h as a set-valued mapping defined on Q and we prove in the Appendix that (3) has a unique solution in q when preferences are binormal.

Lemma 3.1 *If $Q, B > 0$ and the preferences of seller $h \in H^S$ are binormal, $R_h^S(Q, B)$ is a singleton.*

Defining $p = B/Q$ to be the *strategic price*, this lemma implies that, for any $Q, B > 0$, the set $R_h(Q, pQ)$ contains a single element, q . We write $s_h^S(p; Q)$ for q/Q and refer to s_h^S as the *share function* of seller h . In the Appendix, we prove that, considered as a function of Q , the share function takes one of three forms depending on the values of p , λ^S and θ_h . To induce

seller h to make a positive offer, the strategic price must exceed a threshold,

$$p_h^* = \{\max \partial_h (e_h, 0)\} / \lambda^S.$$

For $p \leq p_h^*$, the share function is identically zero. For $p > p_h^*$, the form of the share function depends on the value of λ^S . If $\lambda^S = 1$ or $\theta_h = 0$, seller h will make a positive offer irrespective of the offers of the other sellers; the share function is positive for all $Q > 0$. If $\lambda^S < 1$ and $\theta_h > 0$, seller h will always receive a positive transfer and, if this is large enough, will prefer to make no offer. There will be a *dropout value* of the aggregate offer, $Q_h^*(p)$, and the share function is positive if and only if $Q < Q_h^*(p)$. Furthermore, $Q_h^*(p)$ is the minimum value of Q satisfying⁴

$$\lambda^S p \in \partial_h^- (e_h, (1 - \lambda^S) \theta_h p Q). \quad (5)$$

Whatever the form of the share function, it is continuous, strictly decreasing where positive and asymptotic to or meets the Q axis. Formally, we have the following result, proved in the Appendix.

Lemma 3.2 *Suppose the preferences of seller $h \in H^S$ are binormal.*

1. *If $0 < p \leq p_h^*$, then $s_h^S(p; Q) = 0$ for all $Q > 0$.*
2. *If $p > p_h^*$, $\lambda^S < 1$ and $\theta_h > 0$, then $s_h^S(p; Q)$ is zero for all $Q \geq Q_h^*(p)$ and positive and strictly decreasing for $0 < Q < Q_h^*(p)$. Furthermore, it is continuous for all $Q > 0$ and satisfies*

$$s_h^S(p; Q) \longrightarrow 1 - \frac{p_h^*}{p} \text{ as } Q \longrightarrow 0. \quad (6)$$

3. *If $p > p_h^*$ and, either $\lambda^S = 1$ or $\theta_h = 0$, then $s_h^S(p; Q)$ is positive, continuous and strictly decreasing for all $Q > 0$. Furthermore, it satisfies (6) and $s_h^S(p; Q) \longrightarrow 0$ as $Q \longrightarrow \infty$.*

⁴Note that $p > p_h^*$ implies that $Q_h^*(p) > 0$.

When $\lambda^S = 1$ or $\theta_h = 0$, it is convenient to define $Q_h^*(p) = +\infty$, which means that in all cases the share function is positive for $p > p_h^*$ and $Q < Q_h^*(p)$. Note that, if $\partial_h(e_h, 0) = \{0\}$, then $p_h^* = 0$ so the share function is always positive for $Q < Q_h^*(p)$ and approaches 1 as $Q \rightarrow 0$ for all p . In particular, this will be the case when indifference curves are asymptotic to the axes. (Such an assumption is frequently made to ensure interior equilibria in market games. See Peck et al. [21] for a precise statement.)

3.2 The buyers' partial game

The results we have described for sellers hold *mutatis mutandis* for buyers. Here, we simply summarize the results, omitting proofs which are essentially the same as those for sellers.

For any $h \in H^B$, we define $R_h^B(Q, B)$ to be the set of bids by h which are consistent with aggregate offer and bid: Q, B . We have $b \in R_h^B(Q, B)$ if and only if $0 \leq b \leq \min\{B, e_h\}$ and

$$\left[\lambda^B \frac{Q(B-b)}{B^2} \right]^{-1} \in \partial_h(\omega_h^B(b, B)Q, e_h - b) \quad (7)$$

or $b = 0$ and

$$\left[\lambda^B \frac{Q}{B} \right]^{-1} \in \partial_h^+(\omega_h^B(0, B)Q, e_h)$$

where $\delta^+ \in \partial_h^+(x)$ if and only if $\delta^+ \geq \delta$ for some $\delta \in \partial_h(x)$.

Lemma 3.3 *If $Q, B > 0$ and the preferences of buyer $h \in H^B$ are binormal, $R_h^B(Q, B)$ is a singleton.*

If $p > 0$ and $B > 0$, the set $R_h^B(B/p, B)$ contains a single element: b . We write $s_h^B(p; B)$ for b/B and refer to s_h^B as the *share function* of buyer h . For buyers, the threshold price is defined to be $p_h^* = \lambda^B \min \partial_h(0, e_h)$. (Note that if $\partial_h(0, e_h) = \{+\infty\}$ we take $p_h^* = +\infty$.) In this case, a buyer will only make a positive bid in equilibrium if the equilibrium price exceeds her threshold price and the aggregate bid of the other buyers does not exceed the dropout

value $B_h^*(p)$, which, for $\lambda^B < 1$ and $\theta_h > 0$, is the supremum of B satisfying

$$\frac{p}{\lambda^B} \in \partial_h^+ \left((1 - \lambda^B) \theta_h \frac{B}{p}, e_h \right).$$

If $\lambda^B = 1$ and $\theta_h = 0$, we take $B_h^*(p) = +\infty$.

Lemma 3.4 *Suppose the preferences of buyer $h \in H^B$ are binormal.*

1. *If $p \geq p_h^*$, then $s_h^B(p; B) = 0$ for all $B > 0$.*
2. *If $0 < p < p_h^*$, $\lambda^B < 1$, $\theta_h > 0$ and $B_h^*(p) < \infty$ then s_h^B is zero for all $B \geq B_h^*(p)$ and is positive and strictly decreasing for $0 < B < B_h^*(p)$. Furthermore, s_h^B is continuous for $B > 0$ and satisfies*

$$s_h^B(p; B) \longrightarrow 1 - \frac{p}{p_h^*} \text{ as } B \longrightarrow 0. \quad (8)$$

3. *If $0 < p < p_h^*$ and $\lambda^B = 1$ or $\theta_h = 0$ or $B_h^*(p) = \infty$ then $s_h^B(p; B)$ is positive, continuous and strictly decreasing for all $B > 0$. Furthermore, it satisfies (8) and $s_h^B(p; B) \longrightarrow 0$ as $B \longrightarrow \infty$.*

If $\partial_h(0, e_h) = \{+\infty\}$ and, in particular, if indifference curves are asymptotic to the axes, then $p_h^* = +\infty$. In this case, the share function is never identically zero and approaches 1 as $B \longrightarrow 0$.

3.3 Share functions and equilibria

Share functions can be used to characterize non-autarkic equilibria of the market game. The following lemma, which is proved by straightforward definition chasing, gives the required conditions.

Lemma 3.5 *The strategy profile $(\widehat{\mathbf{q}}, \widehat{\mathbf{b}}) \in \mathcal{S}^S \times \mathcal{S}^B$ is a Nash equilibrium if and only if*

$$\frac{\widehat{q}_h}{\widehat{Q}} = s_h^S(\widehat{p}; \widehat{Q}) \text{ for all } h \in H^S, \quad (9)$$

$$\frac{\widehat{b}_h}{\widehat{B}} = s_h^B(\widehat{p}; \widehat{B}) \text{ for all } h \in H^B, \quad (10)$$

where $\widehat{p} = \widehat{B}/\widehat{Q}$.

It is convenient to write

$$S^S(p; Q) = \sum_{h \in H^S} s_h^S(p; Q), \quad (11a)$$

$$S^B(p; B) = \sum_{h \in H^B} s_h^B(p; B), \quad (11b)$$

for the *aggregate share functions* of sellers and buyers respectively. Note that \widehat{B} and \widehat{Q} are positive equilibrium values of aggregate bids and offers if and only if $S^S(\widehat{p}; \widehat{Q}) = S^B(\widehat{p}; \widehat{B}) = 1$. The remainder of this subsection is devoted to elucidating the relationship between these equilibrium conditions and the Marshallian cross with strategic versions of supply and demand.

For any $h \in H^S$ with binormal preferences and $p > 0$, we can regard $s_h^S(p; Q)$ as a function of Q . Lemma 3.2 implies that this function is continuous, strictly decreasing where positive and vanishing or identically zero in the (large Q) limit. These properties are inherited by $S^S(p; Q)$. Therefore, $S^S(p; Q) = 1$ has a unique solution in Q if and only if

$$\lim_{Q \rightarrow 0} S^S(p; Q) = \sum_{h \in H^S} \max \left\{ 1 - \frac{p_h^*}{p}, 0 \right\} > 1.$$

Since the sum is an increasing function of p , this inequality holds if and only if $p > P^S$, where P^S denotes the unique solution of

$$\sum_{h \in H^S} \max \left\{ 1 - \frac{p_h^*}{P^S}, 0 \right\} = 1, \quad (12)$$

provided not all p_h^* are zero. (If $p_h^* = 0$ for all $h \in H^S$, the same conclusion is valid with $P^S = 0$.) The following lemma formalizes these observations. (The final assertion in the lemma is proved in the Appendix.)

Lemma 3.6 *If all sellers have binormal preferences, the equation $S^S(p; Q) = 1$ has no solution in $Q > 0$ for $p \leq P^S$ and a unique solution for $p > P^S$, which we write $\mathcal{Q}(p)$. Furthermore, $\mathcal{Q}(p)$ is continuous where defined.*

We have seen that, if the equilibrium price exceeds a threshold price, the aggregate offer must be sufficiently small to induce a seller to make a positive offer. The lemma shows that strategic price must exceed P^S to induce a positive offer in aggregate.

A similar result holds for the opposite side of the market in which we write P^B for the unique solution of

$$\sum_{h \in H^B} \max \left\{ 1 - \frac{P^B}{p_h^*}, 0 \right\} = 1, \quad (13)$$

unless $p_h^* = +\infty$ for all $h \in H^B$, in which case we define $P^B = +\infty$.

Lemma 3.7 *If all buyers have binormal preferences, the equation $S^B(p; B) = 1$ has no solution in $B > 0$ for $p \geq P^B$ and a unique solution for $p < P^B$, which we write $\mathcal{B}(p)$. Furthermore, $\mathcal{B}(p)$ is continuous where defined.*

The proof of this lemma is similar to that of Lemma 3.6 and omitted.

We define $\mathcal{X}_1^S(p) = \mathcal{Q}(p)$ and $\mathcal{X}_1^B(p) = \mathcal{B}(p)/p$ and refer to \mathcal{X}_1^S as the *strategic supply function* and \mathcal{X}_1^B as the *strategic demand function* for the consumption good. Strategic supply [demand] represents the total quantity of the consumption commodity that sellers [buyers] will offer [demand] when the price is p , fully taking into account all strategic possibilities open to players on both sides of the market. It follows from Lemma 3.5 that there will be an equilibrium with a positive price \hat{p} if and only if these functions take equal values: non-autarkic equilibria can be characterized by the intersection of the strategic supply and demand functions. The following fundamental proposition gives a precise statement.

Proposition 3.1 *Suppose that all players have binormal preferences. There is a Nash equilibrium with aggregate offer $\hat{Q} > 0$ and bid $\hat{B} > 0$ if and only if $\mathcal{X}_1^S(\hat{p}) = \mathcal{X}_1^B(\hat{p})$, where $\hat{p} = \hat{B}/\hat{Q}$.*

Lemmas 3.1 and 3.3 imply that, in equilibrium, aggregate bids and offers determine the complete strategy profile:

$$\begin{aligned}\widehat{q}_h &= \mathcal{X}_1^S(\widehat{p}) s_h^S(\widehat{p}; \widehat{Q}) \text{ for all } h \in H^S, \\ \widehat{b}_h &= \widehat{p} \mathcal{X}_1^B(\widehat{p}) s_h^B(\widehat{p}; \widehat{B}) \text{ for all } h \in H^B.\end{aligned}$$

Thus non-autarkic Nash equilibria are in (1–1) correspondence with intersections of the strategic supply and demand curves. This result allows us to address existence and uniqueness of Nash equilibria simply by counting the intersections of these curves. We can also use it to investigate comparative statics by determining the effect of parametric changes on the strategic supply and demand curves (through an examination of the effect of these changes on share functions). We start by looking at existence in the next section.

4 Existence

Proposition 3.1 implies that there is a non-autarkic equilibrium if and only if the strategic demand and supply curves intersect. Since \mathcal{X}_1^S is defined for $p \geq P^S$ and \mathcal{X}_1^B for $p \leq P^B$, it follows that, if $P^B \leq P^S$, then there can be no positive p for which $\mathcal{X}_1^S(p) = \mathcal{X}_1^B(p)$ and therefore only the autarkic equilibrium exists. To investigate the converse of this result requires more information on strategic demand and supply functions. The following lemma, proved in the Appendix, provides this for the buyers' side.

Lemma 4.1 *If all buyers have binormal preferences $p\mathcal{X}_1^B(p)$ is bounded and $\mathcal{X}_1^B(p) \rightarrow \infty$ as $p \rightarrow 0$. If $P^B < +\infty$, then $\mathcal{X}_1^B(p) \rightarrow 0$ as $p \rightarrow P^B$ from below.*

A similar result applies on the sellers' side; its proof is omitted.

Lemma 4.2 *If all sellers have binormal preferences $\mathcal{X}_1^S(p)$ is bounded and $p\mathcal{X}_1^S(p) \rightarrow \infty$ as $p \rightarrow \infty$. If $P^S > 0$, then $\mathcal{X}_1^S(p) \rightarrow 0$ as $p \rightarrow P^S$ from above.*

From these lemmas, if $0 < P^S < P^B < +\infty$, then $\mathcal{X}_1^B(P^S) > 0 = \lim_{p \rightarrow P^S} \mathcal{X}_1^S(p)$. In a neighborhood of P^S , strategic demand exceeds strategic supply. Similarly, in a neighborhood of P^B , this ordering is reversed. By continuity, there is $p \in (P^S, P^B)$ for which strategic demand and strategic supply are equal. This is illustrated in Figure 1, which shows an example with three equilibria with prices \hat{p}_1 , \hat{p}_2 and \hat{p}_3 . Note that, as drawn, price and aggregate offer are inversely related in equilibrium. We will see in the next section that this is always true when there are multiple equilibria. If $P^B = +\infty$, we can conclude from Lemmas 4.1 and 4.2 that $p\mathcal{X}_1^S(p) > p\mathcal{X}_1^B(p)$ and hence $\mathcal{X}_1^S(p) > \mathcal{X}_1^B(p)$ for sufficiently large p . Once again, we can deduce that the strategic demand and supply curves cross. Similarly, if $P^S = 0$ then $\mathcal{X}_1^B(p) > \mathcal{X}_1^S(p)$ for sufficiently small p . This establishes the following necessary and sufficient conditions for an equilibrium with positive trades.

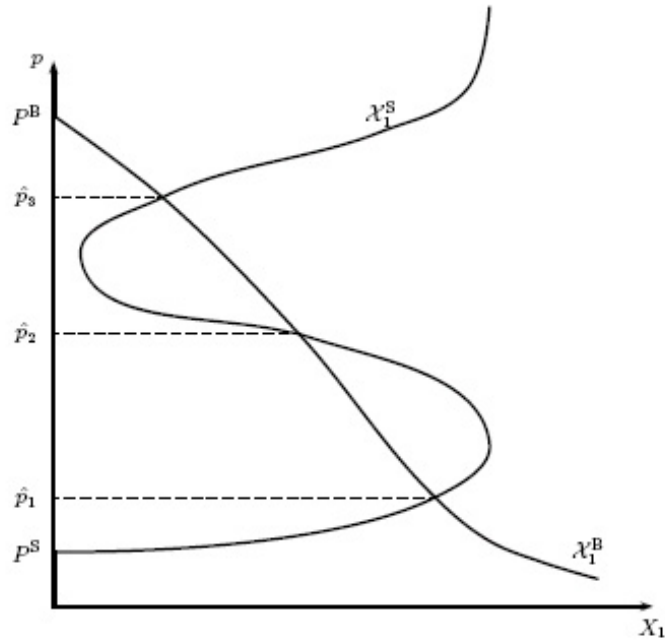


Figure 1: (Inverse) Strategic supply and demand and the existence of Nash equilibria.

Theorem 4.1 *Suppose all players have binormal preferences. There is a non-autarkic equilibrium if and only if $P^B > P^S$.*

In the next subsection, we discuss the implications of this theorem for gains from trade in bilateral oligopoly.

4.1 Autarkic equilibria and gains from trade in bilateral oligopoly

In this subsection, we apply Theorem 4.1 with⁵ $\lambda^S = \lambda^B = 1$ to a discussion of conditions under which gains from trade are possible but strategic opportunities prevent trade from taking place in equilibrium. In particular, we show that this occurs if and only if $P^B \leq P^S$ but

$$\max_{h \in H^B} p_h^* > \min_{h \in H^S} p_h^* \quad (14)$$

and demonstrate that this may be possible if the market is thin enough. We note that the ‘nice’ autarkic equilibrium of the example discussed by Cordella and Gabszewicz [8] has this property.

Intuitively, if there are no possible gains from trade, we would expect only the autarkic equilibrium. We can confirm this by noting that (12) implies that

$$1 = \sum_{h \in H^S} \max \left\{ 1 - \frac{p_h^*}{P^S}, 0 \right\} \leq \sum_{h \in H^S} \left(1 - \frac{\min_{h \in H^S} p_h^*}{P^S} \right),$$

which can be rearranged as

$$P^S \geq \frac{|H^S|}{|H^S| - 1} \min_{h \in H^S} p_h^*. \quad (15)$$

Similarly, the definition of P^B implies

$$P^B \leq \frac{|H^B| - 1}{|H^B|} \max_{h \in H^B} p_h^*. \quad (16)$$

⁵We impose this restriction to avoid over-complicating the presentation.

To have a non-autarkic equilibrium requires $P^B > P^S$ by Theorem 4.1, implying (14). This inequality says that there is a seller with a marginal rate of substitution exceeding that of some buyer at their endowment points and therefore that gains from trade are possible.

The converse to this result need not be true. This is most easily seen when $p_h^* = p_S^*$ for all $h \in H^S$ and $p_h^* = p_B^*$ for all $h \in H^B$. Then, it is straightforward to check that the inequalities (15) and (16) hold as equalities. This permits the possibility that $p_B^* > p_S^*$ but $P^B \leq P^S$, provided there are not too many buyers and sellers. So gains from trade are possible between any seller and any buyer but the only equilibrium is autarkic. Strategic interactions implicit in the market game prevent the players from realizing these gains. Note however that $p_B^* > p_S^*$ implies that $P^B > P^S$ if $|H^S|$ and $|H^B|$ are both large enough. Provided the market is sufficiently thick, even strategic players will realize gains from trade in equilibrium. However, this requires sufficient players on *both* sides of the market. It is possible to have oligopolistic sellers who could profitably trade with any individual buyer but are prevented by strategic considerations from doing so, however many buyers there are. This happens if

$$p_S^* < p_B^* < \frac{|H^S|}{|H^S| - 1} p_S^*.$$

That thick markets allow players to realize possible gains from trade is true in general if we thicken the market by replication. If we replicate the economy m times, (12) can be written $\phi [P^S(m)] = 1/m$, where

$$\phi [p] = \sum_{h \in H^S} \max \left\{ 1 - \frac{p_h^*}{p}, 0 \right\}$$

and $P^S(m)$ is the value of P^S in this economy. Since ϕ is a continuous and increasing function of p we deduce that $P^S(m)$ is decreasing in m and approaches $\min_{h \in H^S} p_h^*$ as $m \rightarrow \infty$. A similar argument shows that $P^B(m)$, the value of P^B in this economy, is increasing in m and approaches $\max_{h \in H^B} p_h^*$ as $m \rightarrow \infty$. We deduce that, if there are gains from trade:

$\max_{h \in H^B} p_h^* > \min_{h \in H^S} p_h^*$, then $P^B(m) > P^S(m)$ for all large enough m and Theorem 4.1 implies that there will be a non-autarkic equilibrium for all such m .

5 Uniqueness

When preferences are binormal, it follows from Proposition 3.1 that there will be a unique non-autarkic equilibrium if and only if the strategic demand and supply curves intersect once and only once. One straightforward condition for this is that the excess strategic demand: $\zeta(p) \triangleq \mathcal{X}_1^B(p) - \mathcal{X}_1^S(p)$ be a strictly decreasing function of p at least for $p \in (P^S, P^B)$. However, this condition is stated in terms of strategic excess demand, a derived concept, rather than the primitives of the game, i.e. players' preferences. We now consider sufficient conditions for excess demand to be a decreasing function and first observe that, under our continuing assumption of binormality, strategic demand falls with price. This is proved in the Appendix.

Lemma 5.1 *If all buyers have binormal preferences, then \mathcal{X}_1^B is a strictly decreasing function of $p \in (0, P^B)$.*

It follows from the lemma that non-decreasing strategic supply is a sufficient condition for a unique equilibrium. Just as with price-taking sellers, binormality of preferences alone does not allow us to draw such a conclusion; additional restrictions on preferences are required. Indeed, suppose that seller h has competitive supply correspondence:

$$\tilde{X}_h^c(p; e) = \{q \in [0, e] : p \in \partial_h(e - q, pq)\},$$

for price p and endowment e . We say that seller h exhibits *increasing competitive supply* if the player's supply correspondence is upwards sloping. That is, for any $e > 0$, $p' > p > 0$, if $x \in \tilde{X}_h^c(p; e)$ and $x' \in \tilde{X}_h^c(p'; e)$ we have $x' \geq x$.

If all sellers have increasing competitive supply, total competitive supply with any given set of endowments is increasing. The following result, proved

in the Appendix, shows that, under the same assumption, strategic supply is also increasing.

Lemma 5.2 *If all players have binormal preferences and all sellers have increasing competitive supply, \mathcal{X}_1^S is a non-decreasing function of $p \in (P^S, \infty)$.*

The following result, which encompasses the existence and uniqueness theorem proved by Amir and Bloch [1] for market games which are symmetric on each side of the market, combines the results of these observations.

Theorem 5.1 *Suppose all players have binormal preferences and all sellers have increasing competitive supply. There is a unique non-autarkic equilibrium if and only if $P^B > P^S$.*

A sufficient condition for increasing competitive supply is that preferences are quasi-linear in money, for then $\partial_h(x_1, x_2)$ is independent of x_2 and convexity implies that competitive supply is increasing. Alternatively, when preferences can be represented by a smooth quasi-concave utility function, the first of the inequalities in the gross substitutes condition of Amir and Bloch [1] also implies that competitive supply is increasing.

Corollary 5.1 *If $P^B > P^S$, all buyers have binormal preferences and either (i) all sellers have quasi-linear preferences, or (ii) the preferences of all sellers satisfy the gross substitutes condition, there is a unique non-autarkic equilibrium.*

Proposition 3.1, which underpins these results, can be viewed as “strategic market clearing” in the goods market. A similar result can be obtained by focussing on the market for commodity money. It is readily verified that, with the natural definitions of strategic demand and supply for money, Walras’ law holds. Consequently, strategic market clearing for one good implies the same property for the other. It follows that uniqueness can be established by showing that there is a unique point at which strategic demand and supply for money intersect. By symmetry, we conclude that, if preferences are binormal and satisfy the appropriate modification of increasing

supply (reverse the roles of x_1 and x_2), the equilibrium is unique. Hence, this additional assumption on buyers' preferences also guarantees uniqueness and can be used to avoid placing additional restrictions on sellers' preferences. Indeed, the gross substitutes conditions of [1] entail both original and modified versions of increasing competitive supply. However, the modified version is not typically valid for preferences that are quasilinear in goods.

5.1 Coalition proofness

Under the assumptions of Theorem 5.1, the game has two equilibria; one autarkic and one non-autarkic: $(\hat{\mathbf{q}}, \hat{\mathbf{b}})$. Bernheim et al's refinement of coalition-proofness [5] allows us to select the latter. The argument rests on three fundamental observations (of which we omit detailed proofs). Let \mathcal{K} be a proper subset of the players and use $\bar{\mathcal{K}}$ to denote the complement of \mathcal{K} . Fix all the strategies of players in \mathcal{K} at their equilibrium values and consider the game with player set $\bar{\mathcal{K}}$ in which outcomes are determined by the strategies chosen in $\bar{\mathcal{K}}$ together with the strategies (\hat{q}_h, \hat{b}_h) for $h \in \mathcal{K}$ via (1) and preferences are $\{\succsim_h\}_{h \in \bar{\mathcal{K}}}$. The competitive approach can be modified to establish our first observation: under the assumptions of Theorem 5.1, if $\hat{q}_h > 0$ for at least one $h \in \mathcal{K}$, or $\hat{b}_h > 0$ for at least one $h \in \mathcal{K}$ (or both) this game has a unique equilibrium $(\hat{q}_h, \hat{b}_h)_{h \in \bar{\mathcal{K}}}$. A similar approach also proves that, if $\hat{q}_h = 0$ for all $h \in \mathcal{K}$ and $\hat{b}_h = 0$ for all $h \in \mathcal{K}$ (or both), this game has two equilibria: $(\hat{q}_h, \hat{b}_h)_{h \in \bar{\mathcal{K}}}$ and the autarkic equilibrium. Finally, we observe that a seller for whom $\hat{q}_h > 0$ or a bidder for whom $\hat{b}_h > 0$ strictly prefers the outcome in $(\hat{\mathbf{q}}, \hat{\mathbf{b}})$ to that in the autarkic equilibrium, whereas an inactive player is indifferent between the equilibria or prefers the former. This is a consequence of players choosing best responses and of convexity of preferences.

It follows from these three observations that, under the assumptions of Theorem 5.1, there is no non-empty subset of players all of whom will wish to deviate from the non-autarkic equilibrium. Such an equilibrium is therefore robust to coalitional deviations. By contrast, the autarkic equilibrium is not robust to such deviations; the subset of players active in the non-autarkic equilibrium can profitably deviate to their strategies in that equilibrium, but

this is not susceptible to further deviations. Of course, if there is no non-autarkic equilibrium, the autarkic equilibrium is trivially coalition proof.

Theorem 5.2 *If all players have binormal preferences and all sellers have increasing competitive supply, there exists a unique coalition-proof equilibrium. This is autarkic if and only if $P^B \leq P^S$.*

6 Comparative Statics

In this section, we demonstrate how the competitive approach may be applied to comparative statics. We make no attempt to be exhaustive and consider only three illustrative cases: changes in the number of players, changes in endowments and the imposition of quantity taxes. Throughout this section, we assume that preferences are binormal and that all sellers have increasing competitive supplies.

6.1 Adding players

Suppose that an additional buyer joins a game which has non-autarkic equilibrium with strategic price $\hat{p} > 0$ and aggregate bid \hat{B} . We assume that the new player, k , has binormal preferences and, to avoid changing payoffs of incumbents, that $\theta_k = 0$. The equilibrium value of the aggregate share function for buyers increases by $s_k^B(\hat{p}; \hat{B})$; this increase is strict if and only if $\hat{p} < p_k^*$ and $0 < \hat{B} < B_h^*(\hat{p})$, by Lemma 3.4. With the additional buyer, $S^B(\hat{p}; \hat{B}) > 1$ and, as Lemma 3.4 implies that $S^B(\hat{p}; B)$ is decreasing in B , we deduce that $\mathcal{B}(p)$ increases and the increase is strict at $p = \hat{p}$. We conclude that the strategic demand function: \mathcal{B}/p increases in the same way. This is illustrated in Figure 2 for the case in which p_k^* lies between the existing equilibrium price and P^B . The monotonicity properties specified in Lemmas 5.1 and 5.2 allow us to conclude that the equilibrium price and quantity traded both increase. The following proposition includes as a special case a result of Amir and Bloch [1] for symmetric bilateral oligopoly games.

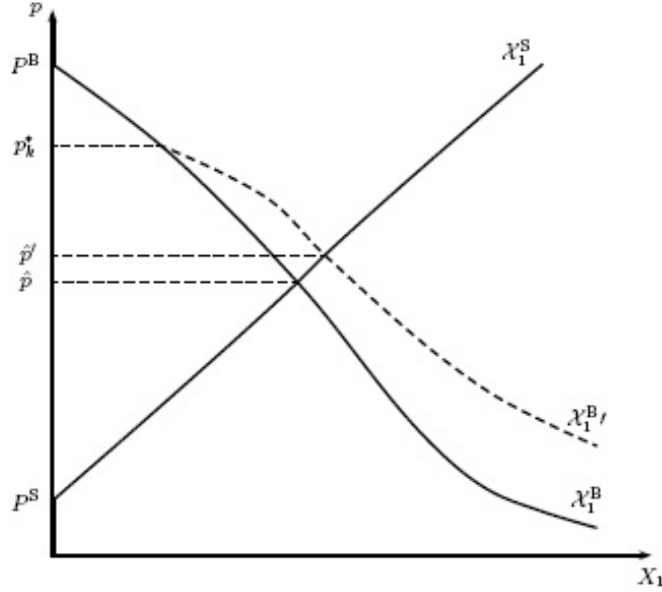


Figure 2: The effect of adding a buyer for whom $p_k^* > \hat{p}$ (primes denote new levels).

Proposition 6.1 *Suppose a new buyer with binormal preferences and $\theta_h = 0$ joins a game in which all players have binormal preferences and all sellers have increasing competitive supply. Then the aggregate bid, offer and strategic price do not decrease and increase strictly if the new buyer is active in equilibrium.*

The increases in aggregate bid and offer are immediate from $\mathcal{Q}(\hat{p}) = \mathcal{X}_1^S(\hat{p})$ and $\mathcal{B}(\hat{p}) = \hat{p}\mathcal{X}_1^B(\hat{p})$ since both p and $\mathcal{X}_1^B(p)$ rise. Of course, if $p_k^* \leq \hat{p}$ or $\hat{B} \geq B_k^*(\hat{p})$, then $s_k^B(\hat{p}; \hat{B})$ is zero; the additional buyer does not bid and the equilibrium is otherwise unchanged.

A similar approach can be applied to an additional seller, k' . In this case, the strategic supply function increases and the increase is strict if $\hat{p} > p_{k'}^*$ and $0 < Q < Q_{k'}^*(\hat{p})$. Consequently, under these conditions, the equilibrium price falls and the equilibrium quantity traded rises, whereas there is no change if $p_{k'}^* \geq \hat{p}$ or $Q \geq Q_{k'}^*(\hat{p})$. In this case, aggregate offers also increase but we cannot draw a definite conclusion for aggregate bids, since the price falls.

The asymmetry with the case of an additional buyer arises because we only suppose that supply of the consumption good is increasing in the price. If we additionally assume that the supply of commodity money by each buyer is increasing, we could conclude that aggregate bids would increase, leading to a fall in price if the new player is active.

Finally, observing that whether a buyer or a seller is added to the game, trade in the consumption good increases, leads to the following corollary.

Corollary 6.1 *Suppose new players all satisfying $\theta_k = 0$ are added to a game and, in the enlarged game, all players have binormal preferences and all sellers have increasing competitive supply. Then, if any of the new players are active in the new equilibrium, trade in the consumption good increases.*

6.2 Changes in endowments

A similar approach can be used to study changes in endowments. For example, consider the change in equilibrium when the endowment of the consumption good of seller h increases from e_h to e'_h . (We use primes to refer to the new game.) It follows immediately from the definition that $p_h^{*'} \leq p_h^*$; an increase in endowment may induce a positive offer from a seller who did not previously trade, but the opposite is never true. Furthermore, $Q_h^{*'}(p) \geq Q_h^*(p)$. When the seller makes a positive offer in both cases, the share function increases.

Lemma 6.1 *Suppose the preferences of seller $h \in H^S$ are binormal. If $p > p_h^{*'}$ and $0 < Q < Q_h^{*'}(p)$, then $s_h^{S'}(p; Q) > s_h^S(p; Q)$.*

This lemma is proved in the Appendix and has the consequence that strategic supply $\mathcal{X}_1^S(p)$ increases for $p > p_h^{*'}$. We conclude, as in the previous subsection that, if the original equilibrium price exceeds $p_h^{*'}$, then at the new equilibrium, price falls and trade increases.

A similar result holds for a change in some buyer's endowment of money from e_h to $e'_h > e_h$. Here $p_h^{*'} \geq p_h^*$. Lemma 6.2 is the equivalent of Lemma 6.1 for buyers; its proof is similar and omitted.

Lemma 6.2 *Suppose the preferences of buyer $h \in H^B$ are binormal. If $p < p_h^*$ and $0 < B < B_h^*(p)$, then $s_h^{B'}(p; B) > s_h^B(p; B)$.*

As above, we conclude that, if the original equilibrium price is less than p_h^* , then at the new equilibrium, price rises and trade increases.

Proposition 6.2 *Consider a game in which all players have binormal preferences and all sellers have increasing competitive supply. If the endowments of some or all players increase, either the equilibrium is unchanged or the aggregate quantity traded increases. In the latter case, (i) if the endowments of all buyers remain unchanged, the strategic price falls, (ii) if the endowments of all sellers remain unchanged, the strategic price rises.*

6.3 Taxes

In this subsection, we apply the competitive approach to study the effects of taxation when players have market power. The sharing rules ω_h^S and ω_h^B can be viewed as taxation and redistribution within the model. However, our aim here is to analyze quantity taxation to fund public goods and services and so we set $\lambda^S = \lambda^B = 1$. (Gabzsewicz and Grazzini [15] examine the use of taxes to restore efficiency in bilateral oligopoly.) Suppose that a specific tax is imposed on the consumption good and paid in units of commodity money. If the level of the tax is t , the allocation corresponding to a strategy profile of bids and offers, (\mathbf{q}, \mathbf{b}) , is

$$(x_{h1}, x_{h2}) = \begin{cases} (e_h - q_h, \max\{p - t, 0\} q_h) & \text{if } h \in H^S, \\ (b_h/p, e_h - b_h) & \text{if } h \in H^B, \end{cases}$$

where $p = B/Q$. This formulation supposes that the tax is collected from sellers and that each seller's tax liability is limited above by their revenue. This limit will not bind in a non-autarkic equilibrium; it cannot be a best response to make a positive offer resulting in negative revenue.

The tax will leave buyers' share functions unchanged. Share functions for sellers are defined as before. Extending our previous notation, we let

$R_h^S(Q, B; t)$ denote the set of strategies of seller h compatible with an equilibrium in which aggregate offers and bids are Q and B , respectively. If $B > 0$ and preferences are binormal, $R_h^S(Q, B; t)$ is a singleton. This is established by essentially the same proof as given in the Appendix for the case $t = 0$ (Lemma 3.1).

For any $p > 0$ and $Q > 0$, the set $R_h^S(Q, pQ; t)$ contains a single element: q and we write $s_h^S(p; Q; t)$ for q/Q . Lemma 3.2 continues to hold provided we replace p_h^* with $p_h^* + t$ and set $\lambda^S = 1$.

We wish to compare the strategic supply functions with and without the tax and this entails comparing the share functions. This is done in the next lemma, proved in the Appendix.

Lemma 6.3 *Suppose $\lambda^S = 1$ and the preferences of seller $h \in H^S$ are binormal. If $p > p_h^* + t$, then*

$$s_h^S(p; Q; t) \leq s_h^S(p - t; Q; 0),$$

for all $Q > 0$. If $\partial_h(x)$ is a singleton in the interior of \mathbb{R}_+^2 , the inequality is strict.

Note that the final condition in the lemma will hold if preferences can be represented by a differentiable utility function.

Using an obvious extension of our previous notation, the aggregate offer satisfies

$$1 = S^S(p; \mathcal{Q}(p; t); t) \leq S^S(p - t; \mathcal{Q}(p; t); 0),$$

by Lemma 6.3. Since aggregate share functions are decreasing in Q , we deduce that

$$\mathcal{Q}(p; t) \leq \mathcal{Q}(p - t; 0).$$

Thus, the strategic supply function shifts upwards by at least t . We conclude that the equilibrium price rises and the quantity traded falls.

Proposition 6.3 *Consider a game in which all players have binormal preferences, all sellers have increasing competitive supply and $\lambda^S = \lambda^B = 1$.*

Imposing a specific tax will raise the price paid by buyers and reduce the quantity traded.

This proposition would appear to suggest that the possibility of strategic manipulation makes no difference to the qualitative effects of imposing a tax. However, this would be misleading. For example, observe that, in contrast to the competitive case, the strategic supply function moves up by at least t and strictly more than t if ∂_h is everywhere a singleton. This raises the possibility of “tax overshifting”: a price rise exceeding the tax. This phenomenon is well recognized in conventional oligopoly. See, for example, the account in Vives [27] of work by Seade [22]. In the game studied here, tax overshifting will occur for any set of smooth suppliers’ preferences provided the strategic demand curve is steep enough.

7 Strategic and competitive equilibria

In this section, we compare strategic with Marshallian supply and demand curves. To clarify the exposition by permitting a direct comparison between strategic and competitive demand and supply, we assume $\lambda^S = \lambda^B = 1$ throughout this section.

Cournot oligopolists exercise monopoly power by restricting output. We will show that the same is true of players who can manipulate prices in partial equilibrium. First, we need some new notation. If $h \in H^S$, we will write $z_h^S(p)$ for the quantity of the consumption commodity which seller h would supply if the price ratio were p and she were a price taker. That is $z_h^S(p) = \tilde{X}_h^c(p; e)$. Note that $\zeta = z_h^S(p)$ satisfies

$$p \in \partial_h(e_h - \zeta, p\zeta). \quad (17)$$

When preferences are binormal, it is straightforward to show that z_h^S is continuous and $z_h^S(p) = 0$ for $p \leq p_h^*$, whereas z_h^S is strictly positive and strictly increasing for $p > p_h^*$. The following lemma, proved in the Appendix, implies that strategic behavior entails restricting supply.

Lemma 7.1 *Suppose $\lambda^S = 1$, seller h has binormal preferences and $p > p_h^*$, $Q > 0$. Then*

$$Qs_h^S(p, Q) < z_h^S(p).$$

Summing the inequalities in the lemma and, using the equilibrium requirement that the aggregate share function (11a) equals unity, gives

$$\mathcal{X}_1^S(p) = \mathcal{Q}(p) \sum_{h \in H^S} s_h^S(p, \mathcal{Q}(p)) < Z^S(p),$$

where $Z^S(p)$ denotes the aggregate Marshallian supply of the consumption commodity.

Corollary 7.1 *Suppose $\lambda^S = 1$ and all sellers have binormal preferences, then $\mathcal{X}_1^S(p) < Z^S(p)$ for all $p > P^S$.*

Similar conclusions apply to buyers. If $h \in H^B$, we write $z_h^B(p)$ for the quantity of the consumption good which price-taking buyer h would demand if the price ratio were p . The following lemma for buyers has a similar proof to Lemma 7.1.

Lemma 7.2 *Suppose $\lambda^B = 1$, buyer h has binormal preferences and $0 < p < p_h^*$, $Q > 0$. Then*

$$Bs_h^B(p, B) < pz_h^B(p).$$

Summing the inequalities in the lemma and using the equilibrium condition,

$$\mathcal{X}_1^B(p) = \frac{\mathcal{B}(p)}{p} \sum_{h \in H^B} s_h^B(p, \mathcal{B}(p)) < Z^B(p),$$

where $Z^B(p)$ is aggregate Marshallian demand.

Corollary 7.2 *Suppose $\lambda^B = 1$ all buyers have binormal preferences, then $\mathcal{X}_1^B(p) < Z^B(p)$ for all $0 < p < P^B$.*

These corollaries show that Marshallian supply and demand curves lie to the right of the corresponding strategic curves⁶. It follows that strategic behavior reduces the quantity traded.

⁶With quantity on the horizontal axis.

Theorem 7.1 *If $\lambda^S = \lambda^B = 1$, all players have binormal preferences and all sellers have increasing competitive supply, the quantity traded in Nash equilibrium cannot exceed that in competitive equilibrium.*

Note that, as shown in Section 4, this does not preclude positive competitive trade, and yet no trade when agents act strategically.

7.1 Competitive limit

As markets get thicker, we expect the Nash equilibrium of the market game to approach the competitive equilibrium of the corresponding limit economy. A discussion of competitive limits in strategic market games with interior solutions is given by Dubey and Shubik [14] and Mas-Colell [19].

In this subsection, we study strategic supply and demand schedules in large games and show that they approach competitive versions in the limit. Formally, we examine m -fold replications of the basic game and investigate the limits of strategic supply and demand per replica, as $m \rightarrow \infty$. Note that the equilibrium requirements can then be written

$$\begin{aligned} S^S(p; Q) &= \sum_{h \in H^S} s_h^S(p; Q) = \frac{1}{m}, \\ S^B(p; B) &= \sum_{h \in H^B} s_h^B(p; B) = \frac{1}{m}, \end{aligned}$$

where the sum is over players in the basic game. For any p , we let $\mathcal{Q}^m(p)$ and $\mathcal{B}^m(p)$ denote the solutions of these equations, as a function of p . Let p^m be a solution in p of $p\mathcal{Q}^m(p) = \mathcal{B}^m(p)$, which is unique if sellers have increasing competitive supply, and let $Q^m = \mathcal{Q}^m(p^m)$. Thus (mQ^m, p^m) is the intersection point of the strategic supply and demand curves of the m -fold replicated game. The Marshallian supply and demand functions in the corresponding economy are simply multiplied by m and therefore the competitive price and quantity are p^C and mQ^C , where p^C and Q^C are the competitive price and quantity in the unreplicated economy.

Lemma 7.3 *Suppose $\lambda^S = 1$ and all sellers have binormal preferences and $p > 0$, then*

$$\begin{aligned} \frac{Q^m(p)}{m} &\longrightarrow Z^S(p) \\ ms_h^S(p, Q^m(p)) &\longrightarrow \frac{z_h^S(p)}{Z^S(p)} \text{ for all } h \in H^S \end{aligned}$$

as $m \longrightarrow \infty$ for all $p > \min_{h \in H^S} p_h^*$.

This result is proved in the Appendix. The first limit shows that strategic supply per replica approaches Marshallian supply pointwise. A similar result holds for buyers, but the proof is omitted.

Lemma 7.4 *Suppose $\lambda^B = 1$ and all buyers have binormal preferences, then*

$$\begin{aligned} \frac{B^m(p)}{pm} &\longrightarrow Z^B(p) \\ ms_h^B(p, B^m(p)) &\longrightarrow \frac{z_h^B(p)}{Z^B(p)} \text{ for all } h \in H^B \end{aligned}$$

as $m \longrightarrow \infty$ for all $0 < p < \max_{h \in H^B} p_h^*$.

That strategic supply and demand curves per replica approach the corresponding competitive curves suggests that the same is true for equilibria. Note, however, that we have only proved pointwise convergence whereas to draw this conclusion directly requires uniform convergence. Nevertheless, we can use a more indirect argument, given in the Appendix, to show convergence of the Nash equilibrium (per replica) to the competitive solution.

Theorem 7.2 *If $\lambda^S = \lambda^B = 1$, all players have binormal preferences and all sellers have increasing competitive supply, then $p^m \longrightarrow p^C$ and $Q^m/m \longrightarrow Q^C$ as $m \longrightarrow \infty$.*

8 Conclusions

We have shown how much of Marshallian competitive supply and demand analysis remains applicable even when agents have significant opportunities

for strategic manipulation of the price. The analysis raises the question of whether this competitive approach to strategic behavior is more widely applicable, for example to general equilibrium with multiple markets. A further question is whether market power of buyers in a Cournot oligopoly can be analyzed by replacing competitive with strategic demand in a two-stage game. Both of these topics are currently under investigation.

A Appendix (proofs)

Proof of Lemma 3.1. Note that the locus of points $(e_h - q, \omega_h^S(q, Q) B)$ is a downwards sloping curve. On such a curve, the mapping from q to marginal rates of substitution is typically multi-valued. In the first part of the proof we show (in two steps) that its inverse is single valued.

The first step is to use upper hemi-continuity to show that, for any $\delta \geq 0$, there exists a $q \in [0, e_h]$ for which

$$\delta \in \tilde{\partial}_h(e_h - q, \omega_h^S(q, Q) B), \quad (18)$$

where $\tilde{\partial}_h(x) = \partial_h(x)$ for $x_1 < e_h$ and $\tilde{\partial}_h(x) = \partial_h^-(x)$ for $x_1 = e_h$. Clearly, upper hemi-continuity extends from ∂_h to $\tilde{\partial}_h$. Now consider the set

$$\Delta = \left\{ q : 0 \leq q \leq e_h, \tilde{\partial}_h(e_h - q, \omega_h^S(q, Q) B) \cap [0, \delta] \neq \emptyset \right\}.$$

Since $\tilde{\partial}_h$ has a closed graph, Δ is closed. Hence, Δ contains its maximum in q , which we write \bar{q} . If $\bar{q} = e_h$, (18) holds with $q = \bar{q}$, since $\tilde{\partial}_h(0, \omega_h^S(e_h, Q) B)$ is unbounded above. If $\bar{q} < e_h$, we can find a sequence $\{q^n\}$ in the complement of Δ convergent to \bar{q} , together with a convergent sequence $\{\delta^n\}$, which satisfies $\delta^n \geq \delta$ and

$$\delta^n \in \tilde{\partial}_h(e_h - q^n, \omega_h^S(q^n, Q) B)$$

for all n . Taking the limit $n \rightarrow \infty$, using upper hemi-continuity, we deduce

that $\tilde{\partial}_h(e_h - \bar{q}, \omega_h^S(\bar{q}, Q) B)$ contains both $\lim_{n \rightarrow \infty} \delta^n \geq \delta$ and $\delta' \leq \delta$ (the latter since $\bar{q} \in \Delta$). So, (18) holds by convexity of ∂_h .

The second step is to show that the solution of (18) is unique in q . This follows by observing that, if $0 \leq q' < q \leq e_h$ and (18) holds for primed and unprimed variables, binormality implies $\delta' < \delta$.

This result allows us to define a function $\phi : \mathbb{R}^+ \rightarrow [0, e_h]$, where $\phi(\delta)$ is the unique q satisfying (18). We observe that upper hemi-continuity implies that ϕ is continuous and binormality (2) implies that ϕ is non-decreasing. Furthermore, $\phi(0) = 0$, since $0 \in \tilde{\partial}_h(e_h, \omega_h^S(0, Q) B)$. The optimality condition (3) is equivalent to q being a fixed point of φ , where

$$\varphi(q) = \phi\left(\lambda^S \frac{B(Q - q)}{Q^2}\right).$$

Since, φ is continuous, non-increasing and $\varphi(Q) = 0$, there is a unique fixed point of φ in $[0, Q]$. ■

Proof of Lemma 3.2. Optimality conditions (3) and (4) imply that $\sigma = s_h^S(p; Q)$ is the unique solution of

$$\lambda^S p(1 - \sigma) \in \tilde{\partial}_h(e_h - \sigma Q, \omega_h^S(\sigma Q, Q) pQ), \quad (19)$$

where $\tilde{\partial}_h$ is defined in the preceding proof.

First, consider the case $p \leq p_h^*$. Then, if we had $\sigma > 0$ for some $Q > 0$, (19) and binormality would imply

$$\lambda^S p \geq \lambda^S p(1 - \sigma) > \lambda^S p_h^*,$$

since $\lambda^S p_h^* \in \partial_h(e_h, 0)$, $e_h - \sigma Q < e_h$ and $\omega_h^S(\sigma Q, Q) pQ > 0$. This contradicts binormality and means that $s_h^S(p, Q) = 0$ for all $Q > 0$.

For the rest of the proof, we will assume $p > p_h^*$. Observe that $s_h^S(p; Q) = 0$ if and only if (5) and, by definition, this holds if and only if $Q \geq Q_h^*(p)$.

We prove that $s_h^S(p; Q)$ is decreasing in Q where positive by supposing

that $0 < Q' < Q < Q_h^*(p)$ and $s_h^S(p; Q) = \sigma \geq \sigma' = s_h^S(p; Q')$. Then,

$$\begin{aligned} e_h - \sigma Q &< e_h - \sigma' Q', \\ \omega_h^S(\sigma Q, Q) pQ &> \omega_h^S(\sigma Q', Q') pQ', \\ \lambda^S p(1 - \sigma) &\in \partial_h(e_h - \sigma Q, \omega_h^S(\sigma Q, Q) pQ), \\ \lambda^S p(1 - \sigma') &\in \partial_h(e_h - \sigma' Q', \omega_h^S(\sigma Q', Q') pQ' pQ'), \end{aligned}$$

and so binormality implies $\sigma < \sigma'$, a contradiction.

The preceding result implies that $\bar{\sigma} = \lim_{Q \rightarrow 0} s_h^S(p; Q)$ exists. Taking limits in (19) and using the upper hemi-continuity of ∂_h shows that

$$\lambda^S p(1 - \bar{\sigma}) \in \partial_h(e_h, 0).$$

By definition of p_h^* , we have $p(1 - \bar{\sigma}) \leq p_h^*$. Strict inequality would imply the existence of $Q > 0$ satisfying

$$\lambda^S p[1 - s_h^S(p; Q)] \in \partial_h(e_h, 0),$$

which would contradict (19) and binormality. Hence, $p(1 - \bar{\sigma}) = p_h^*$ which establishes the first limit. The second limit (in the third part of the lemma) follows directly from $s_h^S(p; Q) \leq e_h/Q$.

It remains to establish continuity, so suppose $Q \rightarrow Q^0$ and assume first that $Q^0 < Q_h^*(p)$. Since shares lie between 0 and 1, there is a sequence $\{Q^n\}_{n=1}^\infty$, convergent to Q^0 and such that $\{s_h^S(p; Q^n)\}$ converges to σ^0 , say. Taking limits in (19), using the upper hemi-continuity of ∂_h and exploiting (19) again, shows that $\sigma^0 = s_h^S(p; Q^0)$. Since all such sequences have this limit, continuity is established. To prove continuity at $Q^0 = Q_h^*(p)$ entails showing that, for any sequence $\{Q^n\}_{n=1}^\infty$ converging to $Q_h^*(p)$ from below, $\sigma^0 = 0$. Taking limits in (19), using upper hemi-continuity shows that $\lambda^S p(1 - \sigma^0) \in \partial_h(x^0)$, where

$$x^0 = (e_h - \sigma^0 Q_h^*(p), \omega_h^S(\sigma^0 Q_h^*(p), Q_h^*(p)) pQ_h^*(p)).$$

But $\sigma^0 > 0$ would imply that $x_1^0 < e_h$ and $x_2^0 > (1 - \lambda^S) \theta_h p Q_h^*(p)$ and

therefore, by (5) and binormality, $\lambda^S p < \lambda^S p (1 - \sigma^0)$ giving a contradiction. Hence, $\sigma^0 = 0$. ■

Completion of proof of Lemma 3.6. We first note that the share function $s_h^S(p; Q)$ of seller h is a continuous function of p . This follows from the optimality conditions (3) and (4) by a similar argument to that in the last paragraph of the proof of Lemma 3.2. Hence, $S^S(p; Q)$ is also continuous in p .

Next, we observe that, if $\lambda^S = 1$ or $\theta_h = 0$, then $ps_h^S(p; Q)$ is a strictly increasing function of p for fixed Q . This follows by re-writing (3) as

$$(p - \{p\sigma\}) \in \partial_h \left(e_h - \frac{\{p\sigma\}Q}{p}, \{p\sigma\}Q \right).$$

By an argument similar to that in the proof of Lemma 3.2, we can see that $\{p\sigma\}$ non-increasing with p is incompatible with binormality (2).

We are now in a position to prove continuity, so suppose that $p \rightarrow p^0 > 0$. If we did not have $\mathcal{Q}(p) \rightarrow \mathcal{Q}(p^0)$, there would be a sequence $\{p^n\}_{n=1}^\infty$ convergent to p^0 and satisfying either (a) $\mathcal{Q}(p^n) \rightarrow \tilde{Q} \neq \mathcal{Q}(p^0)$ or (b) $\mathcal{Q}(p^n) \rightarrow \infty$ as $n \rightarrow \infty$. But, in case (a) we could take limits in the equilibrium equation

$$S^S(p^n; \mathcal{Q}(p^n)) = 1$$

to obtain $S^S(p^0; \tilde{Q}) = 1$, which would imply $\tilde{Q} = \mathcal{Q}(p^0)$, a contradiction. In case (b), if $\lambda^S = 1$ or $\theta_h = 0$, we could define $\bar{p} = \max p_n$ and $\underline{p} = \min p_n > 0$. Then, for any $h \in H^S$ and positive integer n , we can use the fact that ps_h^S is increasing, to deduce

$$0 \leq s_h^S(p^n; \mathcal{Q}(p^n)) \leq \frac{\bar{p}}{p^n} s_h^S(\bar{p}; \mathcal{Q}(p^n)) \leq \frac{\bar{p}}{\underline{p}} s_h^S(\bar{p}; \mathcal{Q}(p^n)).$$

It follows from Lemma 3.2 that $s_h^S(p^n; \mathcal{Q}(p^n)) \rightarrow 0$ as $n \rightarrow \infty$. This also holds for $\lambda^S < 1$ and $\theta_h > 0$, since Lemma 3.2 implies that $s_h^S(p^n; \mathcal{Q}(p^n))$ is identically zero for large enough n . It follows that $S^S(p^n; \mathcal{Q}(p^n)) \rightarrow 0$. In particular, $S^S(p^n; \mathcal{Q}(p^n)) < 1$ for all large enough n , which would contradict

the definition of $\mathcal{Q}(p^n)$, establishing continuity. ■

Proof of Lemma 4.1. Boundedness is immediate from the definition: $p\mathcal{X}_1^{\mathcal{B}}(p) = \mathcal{B}(p)$ and the feasibility condition that \mathcal{B} is bounded above by the aggregate endowment of buyers.

We establish that $\mathcal{X}_1^{\mathcal{B}}(p) \rightarrow +\infty$ as $p \rightarrow 0$ by contradiction. If not, we could find a sequence $\{p^n\}$, such that, as $n \rightarrow \infty$, we would have $p^n \rightarrow 0$ and

$$\begin{aligned} \mathcal{X}_1^{\mathcal{B}}(p^n) &\rightarrow \alpha > 0, \\ \sigma_h^n &= s_h^{\mathcal{B}}(p^n, \mathcal{B}(p^n)) \rightarrow \overleftarrow{\sigma}_h \text{ for all } h \in H^{\mathcal{B}}, \end{aligned}$$

for some $\overleftarrow{\sigma}_h \in [0, 1]$. We have used the fact that the vector $(\sigma_h^n)_{h \in H^{\mathcal{B}}}$ lies in a compact set (a simplex) for all n . Noting that, for all $h \in H^{\mathcal{B}}$, the optimality condition (7) can be written

$$\frac{p^n}{\lambda^{\mathcal{B}}(1 - \sigma_h^n)} \in \tilde{\partial}_h([\lambda^{\mathcal{B}}\sigma_h^n + (1 - \lambda^{\mathcal{B}})\theta_h] \mathcal{X}_1^{\mathcal{B}}(p^n), e_h - \mathcal{X}_1^{\mathcal{B}}(p^n) p^n \sigma_h^n),$$

where $\tilde{\partial}_h(x) = \partial_h(x)$ for $x_2 < e_h$ and $\tilde{\partial}_h(x) = \partial_h^+(x)$ for $x_2 = e_h$. We can take limits and use hemi-continuity of $\tilde{\partial}_h$ (inherited from that of ∂_h) to deduce that $0 \in \partial_h^+(x_1, e_h)$ where $x_1 = [\lambda^{\mathcal{B}}\overleftarrow{\sigma}_h + (1 - \lambda^{\mathcal{B}})\theta_h] \alpha$, the claimed contradiction of strictly increasing preferences.

Finally, suppose that $p \rightarrow P^{\mathcal{B}} < +\infty$ from below and $p\mathcal{X}_1^{\mathcal{B}}(p) \rightarrow 0$. Since $\mathcal{B}(p) = p\mathcal{X}_1^{\mathcal{B}}(p)$ is bounded, we could find a sequence approaching $P^{\mathcal{B}}$ from below on which \mathcal{B} had a limit $\overline{\mathcal{B}} > 0$. But taking the limit of the equilibrium condition $S^{\mathcal{B}}(p; \mathcal{B}(p)) = 1$ on the sequence would give $S^{\mathcal{B}}(P^{\mathcal{B}}; \overline{\mathcal{B}}) = 1$. This contradiction with Lemma 3.7 proves that \mathcal{B} and therefore $\mathcal{X}_1^{\mathcal{B}}$ vanish in the limit. ■

Proof of Lemma 5.1. We first rewrite the optimality condition for

buyers (7) as

$$\frac{p}{\lambda^B(1-\sigma)} \in \tilde{\partial}_h \left(\left[\frac{B}{p} \right] [\lambda^B \sigma + (1-\lambda^B) \theta_h], e_h - \left[\frac{B}{p} \right] p\sigma \right), \quad (20)$$

where $\sigma = s_h^B(p; B)$ and $\tilde{\partial}_h$ is as defined in the preceding proof. We will show, by contradiction, that s_h^B is non-decreasing in $p \in (0, p_h^*)$ and strictly decreasing where positive, provided B/p is held fixed. For suppose, $p' > p$ and either (i) $\sigma' > \sigma$ or (ii) $\sigma' \geq \sigma$, $\sigma' > 0$, where $\sigma' = s_h^B(p'; Bp'/p)$. We would have

$$\begin{aligned} \left[\frac{B}{p} \right] [\lambda^B \sigma + (1-\lambda^B) \theta_h] &\leq \left[\frac{B}{p} \right] [\lambda^B \sigma' + (1-\lambda^B) \theta_h], \\ e_h - \left[\frac{B}{p} \right] p\sigma &\geq e_h - \left[\frac{B}{p} \right] p'\sigma', \\ \frac{p}{1-\sigma} &< \frac{p'}{1-\sigma'}. \end{aligned}$$

Note that in case (i) the first inequality is strict and in case (ii) the second inequality is strict. This is the claimed contradiction with binormality (2). If $p < P^B$ we can deduce that $S^B(p; B)$ strictly decreases with p provided B/p is held fixed.

Hence, if $0 < p' < p < P^B$,

$$S^B \left(p'; \frac{\mathcal{B}(p)}{p} p' \right) > S^B \left(p; \frac{\mathcal{B}(p)}{p} p \right) = 1 = S^B(p'; \mathcal{B}(p')).$$

Since $S^B(p; B)$ is strictly decreasing in B , where positive (using Lemma 3.4), we deduce that $\mathcal{B}(p)p'/p < \mathcal{B}(p')$, proving the lemma. ■

Proof of Lemma 5.2. Our first step is to show that, if all sellers have binormal preferences and increasing competitive supply, then

$$x'_2 > x_2 > 0, \delta \in \partial_h(x_1, x_2), \delta > 0 \Rightarrow \exists \delta' \in \partial_h(x_1, x'_2) \text{ s.t. } \frac{\delta'}{x'_2} < \frac{\delta}{x_2}, \quad (21)$$

The argument is by contradiction, so suppose that we have $x'_2 > x_2 > 0, \delta \in$

$\partial_h(x_1, x_2)$ and

$$\frac{\delta'}{x'_2} \geq \frac{\delta}{x_2} > 0 \text{ for all } \delta' \in \partial_h(x_1, x'_2).$$

By definition of competitive supply, we have $x_2/\delta \in \tilde{X}_h^c(\delta; e)$, where $e = x_1 + x_2/\delta$. Now define $p^* = x'_2\delta/x_2 > \delta$ and choose any $q \in \tilde{X}_h^c(p^*; e)$. The definition of competitive supply puts

$$p^* \in \partial_h\left(x_1 + \frac{x_2}{\delta} - q, p^*q\right).$$

If we had $q > x_2/\delta$, then

$$\begin{aligned} x_1 + \frac{x_2}{\delta} - q &< x_1, \\ p^*q &> \frac{x_2}{\delta} \frac{x'_2\delta}{x_2} = x'_2 \end{aligned}$$

which implies $p^* > \delta'$, since seller h has binormal preferences. We must therefore have $q \leq x_2/\delta$, which is the claimed contradiction with increasing competitive supply (as $\tilde{X}_h^c(p; e)$ increasing in p and $p^* > \delta$ implies $q > x_2/\delta$).

The second step is to use the optimality condition (19), where $\sigma = s_h^S(p; Q)$, to show, by contradiction, that s_h^S is non-decreasing in p for fixed Q . For, suppose $p' > p$ and $\sigma' = s_h^S(p'; Q) < \sigma$. There are two possibilities (i) $p'\sigma' \leq p\sigma$, or (ii) $p'\sigma' > p\sigma$.

In case (i) and using the fact that ω_h^S is non-decreasing in its first argument, we have

$$\begin{aligned} e_h - \sigma'Q &> e_h - \sigma Q, \\ \omega_h^S(\sigma'Q, Q) p'Q &\leq \omega_h^S(\sigma Q, Q) pQ, \\ \lambda^S(1 - \sigma') p' &> \lambda^S(1 - \sigma) p, \end{aligned} \tag{22}$$

contradicting binormality (2).

In case (ii), it follows from the definition of the sharing rules that

$$\omega_h^S(\sigma'Q, Q) p'Q > \omega_h^S(\sigma Q, Q) pQ$$

and therefore (21) implies the existence of

$$\delta' \in \tilde{\partial}_h (e_h - \sigma Q, \omega_h^S(\sigma' Q, Q) p' Q)$$

such that

$$\frac{\delta'}{\omega_h^S(\sigma' Q, Q) p' Q} < \frac{\lambda^S p (1 - \sigma)}{\omega_h^S(\sigma Q, Q) p Q}. \quad (23)$$

Since

$$\lambda^S p' (1 - \sigma') \in \tilde{\partial}_h (e_h - \sigma' Q, \omega_h^S(\sigma' Q, Q) p' Q)$$

and (22) also holds in case (ii), binormality would imply

$$\lambda^S p' (1 - \sigma') \leq \delta'.$$

Substituting this inequality in (23) and using the expression ω_h^S , shows, after some manipulation that $\sigma' > \sigma$, contradicting our initial supposition.

Hence, if $p' > p > P^S$,

$$S^S(p'; \mathcal{Q}(p)) \geq S^S(p; \mathcal{Q}(p)) = 1 = S^S(p'; \mathcal{Q}(p')),$$

from which we can deduce that $\mathcal{Q}(p') \geq \mathcal{Q}(p)$, by Lemma 3.2. It follows that strategic supply is non-decreasing in p . ■

Proof of Lemma 6.1. If $Q \geq Q_h^*(p)$ the result is trivial, so assume $Q < Q_h^*(p)$. If $p_h^* < p \leq p_h^*$, the share function is identically zero for the original endowment and positive for the increased endowment (by Lemma 3.2) so the result is obvious. When $p > p_h^*$, the proof is by contradiction. Fixing $p, Q > 0$ and using the optimality condition (19), suppose we had $e'_h > e_h$ and

$$\sigma' = s_h^{S'}(p; Q) \leq \sigma = s_h^S(p; Q).$$

Then, we would also have

$$\begin{aligned} e'_h - \sigma'Q &> e_h - \sigma Q, \\ \omega_h^S(\sigma'Q, Q)pQ &\leq \omega_h^S(\sigma Q, Q)pQ, \\ \lambda^S p(1 - \sigma') &\geq \lambda^S p(1 - \sigma). \end{aligned}$$

Note that $\sigma > 0$ since $p > p_h^*$ and $Q < Q_h^*(p)$, so these inequalities contradict binormality (2). ■

Proof of Lemma 6.3. Writing $\sigma = s_h^S(p; Q; t)$ and $\sigma' = s_h^S(p - t; Q; 0)$, and recall that with $\lambda^S = 1$, $\omega_h^S(\sigma Q, Q) = \sigma$ and $\tilde{\partial}_h = \partial_h$. Then the first order conditions (19) imply

$$\begin{aligned} p(1 - \sigma) - t &\in \partial_h(e_h - \sigma Q, [p - t]\sigma Q), \\ (p - t)(1 - \sigma') &\in \partial_h(e_h - \sigma'Q, [p - t]\sigma'Q). \end{aligned}$$

The proof is by contradiction, so suppose we had $\sigma > \sigma'$. Then, we would also have

$$\begin{aligned} e_h - \sigma Q &< e_h - \sigma'Q, \\ [p - t]\sigma Q &> [p - t]\sigma'Q, \\ p(1 - \sigma) - t &< (p - t)(1 - \sigma'). \end{aligned}$$

The last inequality is justified by noting that it follows from $\sigma > \sigma'$ and $t > 0$. This set of inequalities contradicts binormality (2).

If $\partial_h(x_1, x_2)$ is a singleton, except possibly on the boundary, we could not have $\sigma = \sigma'$, for this would imply $p(1 - \sigma) - t$ and $(p - t)(1 - \sigma)$ were distinct members of

$$\partial_h(e_h - \sigma Q, [p - t]\sigma Q),$$

a contradiction. ■

Proof of Lemma 7.1. The proof is by contradiction. Let $\zeta = z_h(p)$

and $\sigma = s_h^S(p, Q)$. Suppose we had $Q\sigma \geq \zeta$. Then, we would have

$$\begin{aligned} e_h - \sigma Q &\leq e_h - \zeta, \\ p\sigma Q &\geq p\zeta. \end{aligned}$$

Binormality (2) and conditions (17) and (19) imply that $p(1 - \sigma) \geq p$ and hence $\sigma = 0$. This contradicts Lemma 3.2 since $p > p_h^*$. ■

Proof of Lemma 7.3. Fix $p > \min_{h \in H^S} p_h^*$. From the preamble to the lemma and Corollary 7.1, we have $0 \leq \mathcal{Q}^m(p) \leq mZ^S(p)$. It follows that the sequence $\{\mathcal{Q}^m(p)/m\}$ lies in a compact set. The lemma is proved by showing that all convergent subsequences converge to $Z^S(p)$.

Suppose that $\mathcal{Q}^m(p)/m$ converges to Q^* on a subsequence. For any m and $h \in H^S$, let $\sigma_h^m = s_h(p, \mathcal{Q}^m(p))$ and observe that the equilibrium condition

$$mS^S(p; \mathcal{Q}^m(p)) = 1$$

implies that the vector $(m\sigma_h^m)_{h \in H^S}$ lies in the $|H^S| - 1$ -dimensional simplex. Since this is a compact set, there is a sub-subsequence on which $(m\sigma_h^m)_{h \in H^S}$ converges to $(\sigma_h^*)_{h \in H^S}$, say, in the simplex. Optimality condition (19) can be written

$$p(1 - \sigma_h^m) \in \partial_h \left(e_h - m\sigma_h^m \frac{\mathcal{Q}^m(p)}{m}, pm\sigma_h^m \frac{\mathcal{Q}^m(p)}{m} \right),$$

for all $h \in H^S$. Taking the limit on the sub-subsequence and using the upper hemicontinuity of ∂_h we have

$$p \in \partial_h (e_h - \sigma_h^* Q^*, p\sigma_h^* Q^*),$$

for all $h \in H^S$, where we use the fact that $\sigma_h^m \rightarrow 0$. This states that $\sigma_h^* Q^* = z_h^S(p)$ for all $h \in H^S$. Summing over $h \in H^S$ gives $Q^* = Z^S(p)$, completing the proof of the first limit. The second result follows since all subsequences of $\{(m\sigma_h^m)_{h \in H^S}\}$ converge to $(z_h^S(p)/Q^*)_{h \in H^S}$. ■

Proof of Theorem 7.2. First consider prices. For any $\varepsilon > 0$, let

$$\tilde{Q} = \max \{ Z^S (p^C - \varepsilon), Z^B (p^C + \varepsilon) \}.$$

Since Z^S is strictly increasing and Z^B strictly decreasing, the former lies below the latter on the vertical line $Q = \tilde{Q}$ and Lemmas 7.3 and 7.4 imply that, for all large enough m , the graph of $Q^m(p)/m$ cuts this line below the graph of $B^m(p)/pm$. This means that their intersection: $(Q^m/m, p^m)$ lies to the right of the line (from their monotonicity properties) and therefore

$$p - \varepsilon < p^m < p + \varepsilon,$$

proving the first assertion. This argument is illustrated in Figure 3.

A similar argument applied to quantities with $\tilde{Q} = Q^C - \varepsilon$ shows that $Q^C - \varepsilon < Q^m/m < Q^C$, for all large enough m . ■

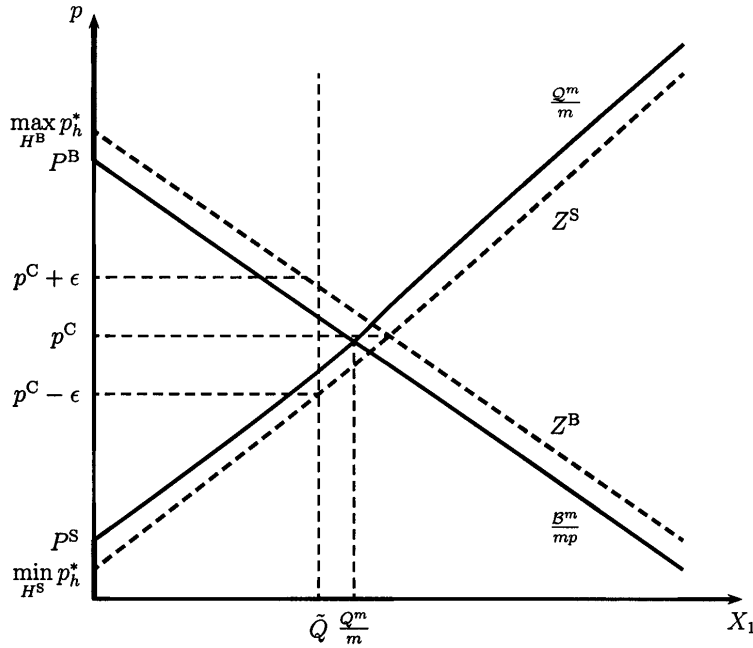


Figure 3: Competitive equilibrium versus Nash equilibrium.

References

- [1] Amir, R. and F. Bloch; 2003; “Comparative statics in strategic market games”; *Mimeo*.
- [2] Andreoni, J.; 1988; “Privately provided public goods in a large economy: the limits of altruism”; *Journal of Public Economics*; **35** pp 57-73.
- [3] Arrow, K. J. and F. H. Hahn; *General competitive analysis*; Holden-Day, San Francisco, 1971.
- [4] Aumann, R.; 1964; “Markets with a continuum of traders”; *Econometrica*; **32** pp 32-50.
- [5] Bernheim, D., B. Peleg and M. Whinston; “Coalition-proof Nash equilibria 1. Concepts”; *Journal of Economic Theory*; **42** pp 1-12.
- [6] Bloch, F. and H. Ferrer; 2001; “Trade fragmentation and coordination in strategic market games”; *Journal of Economic Theory*; **101** pp 301–16.
- [7] Bloch, F. and S. Ghosal; 1997; “Stable trading structures in bilateral oligopolies”; *Journal of Economic Theory*; **74** pp 368–84.
- [8] Cordella, T. and J. J. Gabszewicz; 1998; ““Nice” trivial equilibria in strategic market games”; *Games and Economic Behavior*; **22** pp 162–9.
- [9] Cornes, R. and R. Hartley; 2002; “Joint production games with mixed sharing rules”; Keele Economics Research Papers, 2002/16.
- [10] Cornes, R. and R. Hartley; 2003; “Risk aversion, heterogeneity and contests”, *Public Choice*; **117** pp 1–25.
- [11] Cornes, R. and R. Hartley; 2005; “Asymmetric contests with general technologies”, *Economic Theory*; **26** pp 923-46.
- [12] Dubey, P.; 1994; “Strategic market games: a survey of some results”; in: J-F Mertens, S. Sorin (Eds.), *Game-theoretic methods in general equilibrium analysis*, Kluwer, Amsterdam, pp209-24.

- [13] Dubey, P., A. Mas-Colell and M. Shubik; 1980; “Efficiency properties of strategic market games: an axiomatic approach”, *Journal of Economic Theory*; **22** pp 339-362.
- [14] Dubey, P. and M. Shubik; 1978; “A theory of money and financial institutions. The non-cooperative equilibria of a closed trading economy with market supply and bidding strategies”; *Journal of Economic Theory* **17** pp 1–20.
- [15] Gabszewicz, J. J. and L. Grazzini; 1999; “Taxing market power”; *Journal of Public Economic Theory*; **1** pp 475–97.
- [16] Gabszewicz, J. J. and P. Michel; 1997; “Oligopoly equilibrium in exchange economies”; in: B.C. Eaton, R. G. Harris (Eds.), *Trade, technology and economics: Essays in honor of Richard G. Lipsey*, Elgar, Cheltenham, UK, pp 217-40.
- [17] Giraud, G.; (2003) “Strategic market games: an introduction”; *Journal of Mathematical Economics*; **39** pp 355-75.
- [18] Marshall, A.; *Principles of Economics*; 8th Ed., MacMillan, London, UK, 1920.
- [19] Mas-Colell, A.; 1982; “The Cournotian foundations of Walrasian equilibrium theory: an exposition of recent theory”, in: W. Hildenbrand (Ed.), *Advances in Economic Theory*, Cambridge University Press, Cambridge, UK, pp 183-224.
- [20] Novshek, W.; 1985; “On the existence of Cournot equilibrium”; *Review of Economic Studies*; **52** pp 85-98.
- [21] Peck, J., K. Shell and S. Spear; 1992; “The market game: existence and structure of equilibrium”, *Journal of Mathematical Economics*; **21** pp 271-99.
- [22] Seade, J.; 1987; “Profitable cost increases and the shifting of taxation”; *Mimeo*.

- [23] Shapley, L. and M. Shubik; 1977; “Trade using one commodity as a means of payment”; *Journal of Political Economy*; **85** pp 937–68.
- [24] Shubik, M.; 1973; “Commodity money, oligopoly, credit and bankruptcy in a general equilibrium model”; *Western Economics Journal*; **11** pp 24–38.
- [25] Shubik, M.; *Game theory in the social sciences: concepts and solutions*; MIT Press, Cambridge, MA, 1980.
- [26] Vives, X.; 1987; “Small income effects: A Marshallian theory of consumer surplus and downward sloping demand”; *Review of Economic Studies*; **54** pp 87–103.
- [27] Vives, X; *Oligopoly Pricing*; MIT Press, Cambridge, MA, 1999.