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Chaotic Footloose Capital

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Abstract

This paper examines the long-term behaviour of a discrete-time Footloose Capital model, where capitalists, who are themselves immobile between regions, move their physical capital between regions in response to economic incentives. The spatial location of industry can exhibit cycles of any periodicity or behave chaotically. Long-term behaviour is highly sensitive to transport costs and to the responsiveness of capitalists to profit differentials. The concentration of industry in one region can result from high transport costs or from rapid responses by capitalists. In terms of possible dynamical behaviours, the discrete-time model is much richer than the standard continuous-time Footloose Capital model.

JEL classification: F20 ; R1 ; R12

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This paper examines the long-term behaviour of a discrete-time Footloose Capital model, where capitalists, who are themselves immobile between regions, move their physical capital between regions in response to economic incentives. The spatial location of industry can exhibit cycles of any periodicity or behave chaotically. Long-term behaviour is highly sensitive to transport costs and to the responsiveness of capitalists to profit differentials. The concentration of industry in one region can result from high transport costs or from rapid responses by capitalists. In terms of possible dynamical behaviours, the discrete-time model is much richer than the standard continuous-time Footloose Capital model.

1. Introduction

The Footloose Capital model focuses on the spatial location of industry, where capitalists, who are themselves immobile, move their physical capital between countries or regions in response to economic incentives. Introduced by Martin and Rogers (1995), the Footloose Capital (FC) model is a variant of the influential Core-Periphery (CP) model developed by Krugman (1991a, 1991b). The FC model plays a prominent role in *Economic Geography and Public Policy* by Baldwin *et al*. (2003). Its attractions are two-fold. First, it is more applicable to the case where the two regions are different nations than is the standard CP model in which the footloose factor is labour. Accordingly, the FC model features particularly in the trade chapters of Baldwin *et al*. Second, it is much more tractable analytically than the standard CP model. This is due to the fact that capital earnings are repatriated and spent in the region in which the capital owners live. Since production changes brought about by the movements of capital are not accompanied by expenditure switching, the demand-linked circular causality that features in the Core-Periphery model does not arise. Furthermore, since costs-of-living are irrelevant to the production location decisions of capitalists, the cost-linked circular causality of the Core-Periphery model is eliminated.

Currie and Kubin (2005) show that reformulating the standard continuous-time symmetric Core-Periphery (CP) model in discrete time has profound implications for its dynamical behaviour. In this paper, we explore the implications of specifying the much simpler FC model in discrete time. We consider not only the case of symmetric regions but also some implications of regional asymmetries. We set out the assumptions of the discrete-time FC model in Section 2. In Section 3, we characterize a short-run general equilibrium contingent on the regional allocation of capital. In Section 4, we complete the model by specifying the capital migration process, and we identify the fixed points of the dynamical model. In Section 5, we explore the complex dynamical behaviour for the case of symmetric regions, where each region has the same number of workers and where the owners of half the capital are located in each region. In Section 6, we identify some dynamical implications of exogenous regional asymmetries. Section 7 concludes.

2. Assumptions

There are two regions, each with a monopolistically competitive manufacturing sector and a perfectly competitive agricultural sector. Capital is used only in manufacturing. There are, in total, K units of physical capital. The share of physical capital that is *owned* by capitalists located in region 1 is denoted by s_K. Capital is mobile between the regions at the transitions between time periods in response to economic incentives. Labour – the only other factor of production – is used in both sectors. There are, in total, L workers, who are immobile between regions but instantaneously mobile between sectors within a region. The share of workers located in region 1 is denoted by s_L. Each worker provides one unit of labour per period.

Consumers in both regions have Cobb-Douglas preferences over the homogeneous agricultural good and a quantity index that is a CES function of the varieties of manufactured goods. The exponents of the agricultural good and of the manufacturing composite in the common utility function – and hence the invariant shares of income devoted to the agricultural good and to manufactures – are $(1-\mu)$ and μ, respectively. The constant elasticity of substitution between the manufactured varieties is denoted by $\sigma > 1$; the lower σ, the greater the consumers' taste for variety.

With labour being the sole agricultural input, one unit of labour yields one unit of the agricultural product. We assume that neither region has enough labour to satisfy the total demand of both regions for the agricultural good. Thus both regions always produce the agricultural commodity – the so-called non-full-specialization condition. Transportation of the agricultural product between regions is costless.

Manufacturing involves increasing returns: each manufacturer requires a fixed input of 1 unit of capital to operate and has a constant marginal labour requirement β. Transport costs for manufactures take an iceberg form: if 1 unit is shipped between the regions, $1/T$ arrives where $T \geq 1$. 'Trade freeness' is defined as $\phi \equiv T^{1-\sigma}$ where $0 < \phi \leq 1$, with $\phi = 1$ representing no trade cost and with trade cost becoming prohibitive as $\phi \to 0$. The manufacturing sectors involve Dixit-Stiglitz monopolistic competition. Given the consumers' preference for variety and the increasing returns, a firm would always produce a variety different from the varieties produced by other firms. Thus the number of varieties is always the same as the number of firms. Furthermore, since 1 unit of capital is required for each manufacturing firm, the total number of firms / varieties, *n*, is always equal to the total supply of capital:

$$n = K \tag{1}$$

The number of varieties produced in period *t* in region *r*, where $r = 1, 2$, is:

$$n_{1,t} = \lambda_t n = \lambda_t K \qquad n_{2,t} = \left(1 - \lambda_t\right) n = \left(1 - \lambda_t\right) K \tag{2}$$

where $0 \leq \lambda_t \leq 1$ denotes the share of physical capital *used* in region 1 in period *t*.

As with most economic geography models, the primary focus of the FC model is the spatial location of manufacturing industry, governed here by the endogenous regional allocation of capital. In Section 3, we characterize a short-run general equilibrium in period *t* contingent on λ_t. In Section 4, we complete the dynamical model by specifying the capital migration process. An accepted tension in the standard CP model, where the inter-regionally mobile factor is labour, concerns the labour migration assumption. All workers are assumed identical; yet at the transition between periods some but not all workers migrate in response to wage differences. This issue has been addressed by Puga (1998), in terms of the arrival of opportunities to migrate and random migration costs, and by Baldwin *et al.* (2003), on the basis of utility maximizing households who, facing costly migration, divide their working time between the regions. Whereas the idea of workers dividing their time between regions is somewhat disconcerting, the notion of capitalists diversifying their physical capital between regions is perfectly natural. Accordingly, when considering the location of physical capital, we adopt the device of a representative capitalist, who alters the regional allocation of his capital in response to economic incentives. Thus λ_t is the share of the representative capitalist's physical capital that is located in region 1 in period *t*.

3. Short-run General Equilibrium

With the instantaneous establishment of equilibrium in the agricultural market and no transport costs, the agricultural price is the same in both regions. Since competition results in zero agricultural profits, the equilibrium nominal wage of workers in period t is equal to the agricultural product price and is therefore always the same in both regions. We take this wage / agricultural price as the numeraire. Since manufacturers in both regions face that same wage in every period, all set the same mill price p, using the Dixit-Stiglitz pricing rule. Given that the wage is 1, the local price of every variety is:

$$p = \frac{\beta\sigma}{\sigma - 1} \qquad (3)$$

The effective price paid by consumers for a variety produced in the other region is pT.

Short-run general equilibrium in period *t* requires that each manufacturer meets the demand for its variety.[1] For a variety produced in region *r*:

$$q_{r,t} = d_{r,t} \qquad (4)$$

[1] As a result of Walras' Law, simultaneous equilibrium in all product markets implies equilibrium in the regional labour markets.

where $q_{r,t}$ is the output of each manufacturer in region r and $d_{r,t}$ is the demand for that manufacturer's variety. From (3), the short-run equilibrium profit per variety in region r is:

(5) $$\pi_{r,t} = pq_{r,t} - \beta q_{r,t} = \frac{pq_{r,t}}{\sigma} = \left[\frac{\beta}{\sigma - 1}\right] q_{r,t}$$

This profit per variety constitutes the regional profit or rental per unit of capital. Total ('world') income, denoted by Y_{1+2}, comprises wages and profits and is invariant over time. Since total expenditure is equal to total income, expenditure on the agricultural product is $(1-\mu)Y_{1+2}$; the non-full-specialization condition for agriculture is that the latter exceed $\max\{s_L L, (1-s_L)L\}$. Expenditure on manufactures is μY_{1+2}. Since, from (5), profit equals the value of sales times $1/\sigma$, the total profit received by capitalists is $\mu Y_{1+2}/\sigma$. Therefore, given the unit wage, total income is $Y_{1+2} = L + (\mu/\sigma) Y_{1+2}$, so that:

(6) $$Y_{1+2} = \frac{\sigma L}{\sigma - \mu}$$

Total profit is therefore $\mu L/(\sigma - \mu)$. Given that all profits are repatriated to capitalists and given that each capitalist receives the average profit, the regional incomes are:

(7) $$\begin{aligned} Y_1 &= s_L L + s_K \frac{\mu}{\sigma - \mu} L = \frac{s_L(\sigma - \mu) + s_K \mu}{\sigma - \mu} L = \left(s_L \frac{\sigma - \mu}{\sigma} + s_K \frac{\mu}{\sigma} \right) Y_{1+2} = s_Y Y_{1+2} \\ Y_2 &= (1-s_L)L + (1-s_K)\frac{\mu}{\sigma - \mu} L = \left((1-s_L)\frac{\sigma - \mu}{\sigma} + (1-s_K)\frac{\mu}{\sigma} \right) Y_{1+2} = (1-s_Y) Y_{1+2} \end{aligned}$$

where Y_r is the nominal income in region r and where region 1's share in total income (and in total expenditure) is:

(8) $$s_Y = s_L \frac{\sigma - \mu}{\sigma} + s_K \frac{\mu}{\sigma}$$

Whereas these nominal regional incomes are invariant over time, *real* regional incomes depend on costs-of-living and, thus, indirectly on the spatial location of industry. The regional manufacturing price indices facing consumers are given by:

$$
\begin{aligned}
G_{1,t} &= \left[n_{1,t} p^{1-\sigma} + n_{2,t} p^{1-\sigma} T^{1-\sigma} \right]^{\frac{1}{1-\sigma}} = \left[\lambda_t + (1-\lambda_t)\phi \right]^{\frac{1}{1-\sigma}} n^{\frac{1}{1-\sigma}} p \\
G_{2,t} &= \left[n_{1,t} p^{1-\sigma} T^{1-\sigma} + n_{2,t} p^{1-\sigma} \right]^{\frac{1}{1-\sigma}} = \left[\lambda_t \phi + (1-\lambda_t) \right]^{\frac{1}{1-\sigma}} n^{\frac{1}{1-\sigma}} p
\end{aligned}
\tag{9}
$$

With less than perfectly free trade, the cost-of-living is lower in region 1 than in region 2 [i.e., $G_{1,t} < G_{2,t}$] if and only if it has a larger manufacturing sector [i.e., $\lambda_t > 1/2$]. The demand per variety in each region is:

$$
\begin{aligned}
d_{1,t} &= \left[\mu Y_1 G_{1,t}^{\sigma-1} + \mu Y_2 G_{2,t}^{\sigma-1} \phi \right] p^{-\sigma} = \left[s_Y G_{1,t}^{\sigma-1} + (1-s_Y) G_{2,t}^{\sigma-1} \phi \right] p^{-\sigma} \mu Y_{1+2} \\
d_{2,t} &= \left[\mu Y_1 G_{1,t}^{\sigma-1} \phi + \mu Y_2 G_{2,t}^{\sigma-1} \right] p^{-\sigma} = \left[s_Y G_{1,t}^{\sigma-1} \phi + (1-s_Y) G_{2,t}^{\sigma-1} \right] p^{-\sigma} \mu Y_{1+2}
\end{aligned}
\tag{10}
$$

From (4), (9) and (10):

$$
\begin{aligned}
q_{1,t} = d_{1,t} &= \left[\frac{s_Y}{\lambda_t + (1-\lambda_t)\phi} + \frac{1-s_Y}{\lambda_t \phi + (1-\lambda_t)} \phi \right] \frac{\mu Y_{1+2}}{np} \\
q_{2,t} = d_{2,t} &= \left[\frac{s_Y}{\lambda_t + (1-\lambda_t)\phi} \phi + \frac{1-s_Y}{\lambda_t \phi + (1-\lambda_t)} \right] \frac{\mu Y_{1+2}}{np}
\end{aligned}
\tag{11}
$$

From (1), (5), (6) and (11), the short-run equilibrium profits per variety / per unit of capital are:

$$
\begin{aligned}
\pi_{1,t} &= \left[\frac{s_Y}{\lambda_t + (1-\lambda_t)\phi} + \frac{1-s_Y}{\lambda_t \phi + (1-\lambda_t)} \phi \right] \Psi \\
\pi_{2,t} &= \left[\frac{s_Y}{\lambda_t + (1-\lambda_t)\phi} \phi + \frac{1-s_Y}{\lambda_t \phi + (1-\lambda_t)} \right] \Psi
\end{aligned}
\tag{12}
$$

where Ψ is the average profit per unit of capital:

(13) $$\Psi = \frac{\mu L}{(\sigma - \mu) K}$$

Given the regional allocation of capital λ_t, expression (12) determines uniquely the short-run equilibrium regional profits.[2] In a core-periphery state where $\lambda_t = 0$ or $\lambda_t = 1$, the profit per unit of capital in the core is Ψ and the 'virtual' profit in the periphery is $\left[s_Y \phi^{-1} + (1 - s_Y)\phi\right]\Psi$. With perfectly free trade [$\phi = 1$], location does not matter: $\pi_{1,t} = \pi_{2,t} = \Psi$ for $0 \le \lambda_t \le 1$.

4. Capital Movements and the Complete Dynamical Model

The representative capitalist alters his allocation of physical capital in response to the regional difference in profits. Since profit is spent in the region in which the capitalist concerned lives, costs-of-living do not impact on the capital allocation decision. From (12), the difference in (nominal) profits is determined uniquely by the allocation of capital at the beginning of period *t*:

(14) $$\pi_{1,t} - \pi_{2,t} = \Pi(\lambda_t) = \left[\frac{s_Y}{\lambda_t + (1 - \lambda_t)\phi} - \frac{1 - s_Y}{\lambda_t \phi + (1 - \lambda_t)} \right] (1 - \phi)\Psi$$

For $\phi < 1$, $\Pi(\lambda_t)$ is monotonically declining:

(15) $$\Pi'(\lambda_t) = -\left[\frac{s_Y}{\left[\lambda_t + (1 - \lambda_t)\phi\right]^2} + \frac{1 - s_Y}{\left[\lambda_t \phi + (1 - \lambda_t)\right]^2} \right] (1 - \phi)^2 \Psi < 0$$

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[2] In contrast, for the CP model, the dependence of regional expenditures on the spatial allocation of the mobile factor means that closed-form solutions are not possible.

As with the standard continuous-time core-periphery model and its variants, our specification of the migration process is essentially *ad hoc*. We invoke a discrete-time counterpart of the process assumed by Baldwin *et al*. (2003, p. 72) in their continuous-time FC model. Taking into account the constraint $0 \le \lambda_{t+1} \le 1$, the piecewise smooth one-dimensional map whereby λ_{t+1} is determined by λ_t is:

$$(16) \qquad \lambda_{t+1} = Z(\lambda_t) = \begin{cases} 0 & \text{if} \quad M(\lambda_t) < 0 \\ M(\lambda_t) & \text{if} \quad 0 \le M(\lambda_t) \le 1 \\ 1 & \text{if} \quad M(\lambda_t) > 1 \end{cases}$$

where λ_t is in [0,1] implies that λ_{t+1} is in [0,1] and where

$$(17) \qquad M(\lambda_t) = \lambda_t + \lambda_t(1-\lambda_t)\gamma(\pi_{1,t} - \pi_{2,t}) = \lambda_t + \lambda_t(1-\lambda_t)\gamma\Pi(\lambda_t)$$

with $\gamma > 0$. We refer to γ as the 'speed' with which the representative capitalist alters his regional allocation of capital in response to economic incentives. The supposition that he does not always immediately shift all his capital to the region with the higher profit – however small the differential – could be justified in terms of adjustment costs. Thus, given static expectations, the representative capitalist's maximand would be:

$$(18) \qquad \pi_{1,t}\lambda_{t+1}\kappa + \pi_{2,t}(1-\lambda_{t+1})\kappa - \frac{(\lambda_{t+1} - \lambda_t)^2}{2c\lambda_t(1-\lambda_t)}$$

where κ denotes the capital owned by the representative capitalist and where adjustment costs are quadratic with $c > 0$.[3] Rearranging the first-order condition for an interior

[3] This rationale follows that applied by Baldwin *et al*. (2003, p. 55) to the labour allocation decisions of a household in the CP model where labour is the inter-regionally mobile factor.

maximum gives (17), where $\gamma = \kappa c$. However, since these adjustment costs do not involve the use of real resources, we would not wish to rely on this interpretation of the migration process. Alternatively, γ could be interpreted as a behavioural parameter that reflects the degree of cautiousness of the representative capitalist.

If $Z(\lambda^*) = \lambda^*$, then λ^* is a fixed point for the dynamical system. Such fixed points correspond to stationary long-run equilibria. The local stability properties of a fixed point λ^* depend on the stability coefficient:

(19) $$Z'(\lambda^*) = M'(\lambda^*) = 1 + \lambda^*(1-\lambda^*)\gamma\,\Pi'(\lambda^*) + (1-2\lambda^*)\gamma\,\Pi(\lambda^*)$$

i.e., the first derivative of $Z(\lambda_t)$ evaluated at λ^*. With perfectly free trade [i.e., $\phi = 1$], all regional capital allocations are fixed points. Therefore, we assume henceforth that $\phi < 1$. Since capital does not move to a region with no manufacturing sector in the previous period, the concentration of all manufacturing in one (either) region is necessarily a fixed point. That is, from (16), $Z(0) = 0$ and $Z(1) = 1$. We refer to $\lambda = 0$ and $\lambda = 1$ as the *core-periphery fixed points*. Since capital migration does not occur when regional profits are equal, λ^* is a fixed point if $\Pi(\lambda^*) = 0$. Since $\Pi(\lambda_t)$ is monotonically declining, there can be at most one interior fixed point λ^*. If an interior fixed point λ^* exists, then, from (14), equality of profits implies that the long-run equilibrium share is:

(20) $$\lambda^* = \frac{1}{2} + \frac{1+\phi}{1-\phi}\left(s_Y - \frac{1}{2}\right)$$

Using (8), λ^* depends on the regional shares in factor endowments, s_L and s_K:

(21) $$\lambda^* = \frac{1}{2} + \frac{1+\phi}{1-\phi}\left(\left(\frac{\sigma-\mu}{\sigma}\right)\left(s_L - \frac{1}{2}\right) + \frac{\mu}{\sigma}\left(s_K - \frac{1}{2}\right)\right)$$

An interior fixed point need not exist if there is an asymmetry in regional incomes. Suppose that region 1 is larger, i.e., $s_Y > 1/2$. As trade freeness increases, region 1's long-run equilibrium share of capital use increases until $\lambda^* = 1$. Solving (20) for $\lambda^* = 1$ gives the critical level of ϕ at and above which there does not exist an interior fixed point:

$$\phi_{CP1} = \frac{1 - s_Y}{s_Y} \tag{22}$$

Analogously, if $s_Y < 1/2$, solving (20) for $\lambda^* = 0$ gives the critical level of trade freeness, $\phi_{CP2} = s_Y / (1 - s_Y)$, at and below which there does not exist an interior fixed point.

The discrete-time map $Z(\lambda_t)$ can exhibit complex dynamical behaviour. $Z(\lambda_t)$ is non-invertible, i.e., in general, λ_t cannot be uniquely determined from λ_{t+1}. Given the initial condition λ_0, the orbit of the system is uniquely determined. The first iterate is $\lambda_1 = Z(\lambda_0)$; the second iterate is $\lambda_2 = Z(\lambda_1) = Z(Z(\lambda_0))$; and so on. Letting $Z^{[n]}(\lambda_0)$ denote $Z\left(Z^{[n-1]}(\lambda_0)\right)$, the system's orbit is $\lambda_0, Z(\lambda_0), Z^{[2]}(\lambda_0), Z^{[3]}(\lambda_0), \ldots$ As noted above, if $Z(\lambda^*) = \lambda^*$, then λ^* is a fixed point. If $Z^{[k]}(\tilde{\lambda}) = \tilde{\lambda}$ and if *k* is the smallest such positive integer, $\tilde{\lambda}$ is a periodic point of period *k* and the orbit with initial point $\tilde{\lambda}$ is a period-*k* orbit.[4] A chaotic orbit is a bounded, non-periodic orbit that displays sensitive dependence on the initial condition.[5] Sensitive dependence means that orbits that begin as close together

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[4] A fixed point of $Z(\lambda_t)$ is a period-1 orbit. If $\tilde{\lambda}$ is a periodic point of period *k*, it is a fixed point of $Z^{[k]}(\lambda_t)$. Where we refer to a 'fixed point', we always have in mind a fixed point of $Z(\lambda_t)$.

[5] See Alligood *et al.* (1996, Chapter 1).

as desired eventually move apart. The basin of attraction of an attractor is the set of initial conditions λ_0 with orbits that approach the attractor.

5. Symmetric Regions

5.1. Implications of Regional Symmetry

Symmetry of the regions means here that they have the same factor endowments, which implies equal nominal regional incomes and expenditures. Given $s_L = s_K = s_Y = 1/2$, the profit difference (14) simplifies to:

(23) $$\pi_{1,t} - \pi_{2,t} = \Pi(\lambda_t) = \left[\frac{1}{\lambda_t + (1-\lambda_t)\phi} - \frac{1}{\lambda_t\phi + (1-\lambda_t)}\right]\frac{(1-\phi)\Psi}{2}$$

For $\phi < 1$, regional symmetry implies:

(24) $$\Pi(0) = \frac{(1-\phi)^2\,\Psi}{2\phi} > 0 \qquad \Pi(1/2) = 0 \qquad \Pi(1) = -\frac{(1-\phi)^2\,\Psi}{2\phi} < 0$$

From (23), regional symmetry also implies:

(25) $$\Pi(1-\lambda_t) = -\Pi(\lambda_t)$$

Since the numbering of regions is irrelevant:

(26) $$(1-\lambda_{t+1}) = M(1-\lambda_t) \qquad (1-\lambda_{t+1}) = Z(1-\lambda_t)$$

It is easily confirmed that (25) and (26) imply the following symmetry property:

(27) $$M(\lambda_t) = 1 - M(1-\lambda_t) \qquad Z(\lambda_t) = 1 - Z(1-\lambda_t)$$

The symmetry of $Z(\lambda_t)$ has crucial implications for the system's dynamical properties.

Symmetry Proposition

Given a period-*k* orbit, either that orbit is symmetric with respect to $\lambda = 1/2$ or there exists a period-*k* orbit that is symmetric to it. In the former case, the orbit's basin of attraction is symmetric with respect to $\lambda = 1/2$; in the latter case, the basins of attraction of the two orbits are symmetric with respect to each other.

With symmetric regions, the map $Z(\lambda_t)$ possesses the same three fixed points for any $\phi < 1$. In addition to the core-periphery fixed points $\lambda = 0$ and $\lambda = 1$, there is necessarily a *symmetric fixed point* $\lambda = 1/2$, i.e., $\Pi(1/2) = 0$ implies $Z(1/2) = 1/2$. Thus the Symmetry Proposition is confirmed for $k = 1$: a fixed point is either symmetric, i.e., $\lambda = 1/2$, or the symmetric one also exists, i.e., $Z(\lambda^*) = \lambda^*$ implies $Z(1-\lambda^*) = 1-\lambda^*$ for the core-periphery fixed points.

The core-periphery fixed points $\lambda = 0$ and $\lambda = 1$ are unstable for $\phi < 1$, since (19), (24) and (25) imply:

(28) $$Z'(0) = 1 + \gamma\,\Pi(0) = 1 - \gamma\,\Pi(1) = Z'(1) > 1$$

From (15), (19) and (24), the symmetric fixed point $\lambda = 1/2$ is locally stable if and only if:

(29) $$Z'(1/2) = 1 + \frac{\gamma}{4}\Pi'(1/2) = 1 - \left(\frac{1-\phi}{1+\phi}\right)^2 \gamma\,\Psi \geq -1$$

5.2. Comparative Dynamics

With symmetric regions, the system's dynamical behaviour depends on the migration speed γ, on trade freeness ϕ and on the average profit per unit of capital Ψ. That the migration speed matters in a discrete-time system is natural. Figure 1 shows the map $Z(\lambda_t)$ corresponding to two speeds, γ_P and γ_A, where $\gamma_A > \gamma_P$. Graphically, the symmetry property $Z(\lambda_t) = 1 - Z(1-\lambda_t)$ means that rotating the diagram through 180^0 results in the same map. An increase in the speed 'stretches' the map, without affecting the fixed points. The maps in Figure 1 are based on critical speeds.[6] For $\gamma < \gamma_P$, the symmetric fixed point is an attractor. As γ increases through γ_P, where $Z'(1/2) = -1$, the symmetric fixed point becomes unstable and a period-doubling bifurcation occurs. Further increases in speed give rise to orbits of every periodicity and to chaotic behaviour. Long-term behaviour is hyper-sensitive with respect both to the speed γ and to the initial point λ_0. At $\gamma = \gamma_A$, the iterate of the interior maximum of (16) is 1 and the iterate of the interior minimum is 0. For $\gamma > \gamma_A$, the responses of capitalists are sufficiently rapid (adjustment costs are sufficiently low) that agglomeration in one of the regions occurs for almost all initial points. At the resulting core-periphery fixed point, the virtual profit in the periphery exceeds the profit in the core.

Instead of elaborating on the foregoing claims regarding the impact of the migration speed, we consider in more detail the impact of trade freeness, since the latter is the main pre-occupation in core-periphery-type models. Figure 2 is a bifurcation diagram (or orbit diagram) that shows the impact of trade freeness on the qualitative nature of the system's

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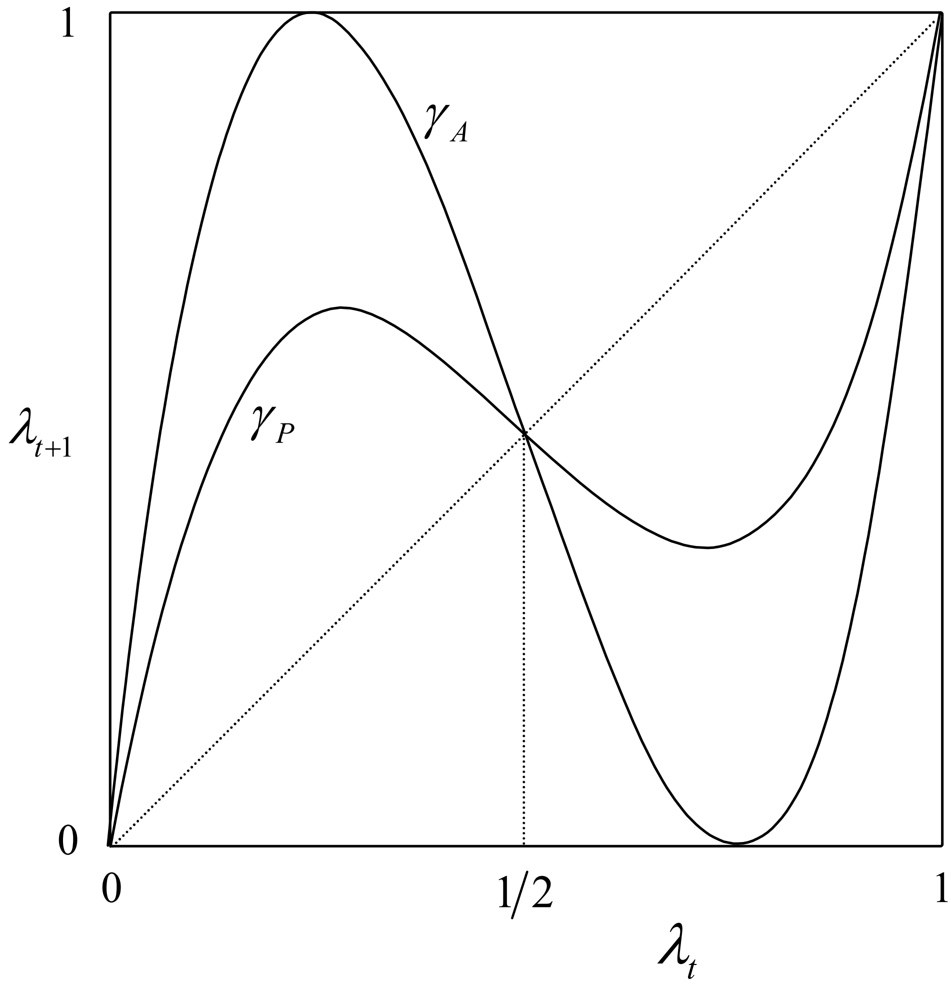
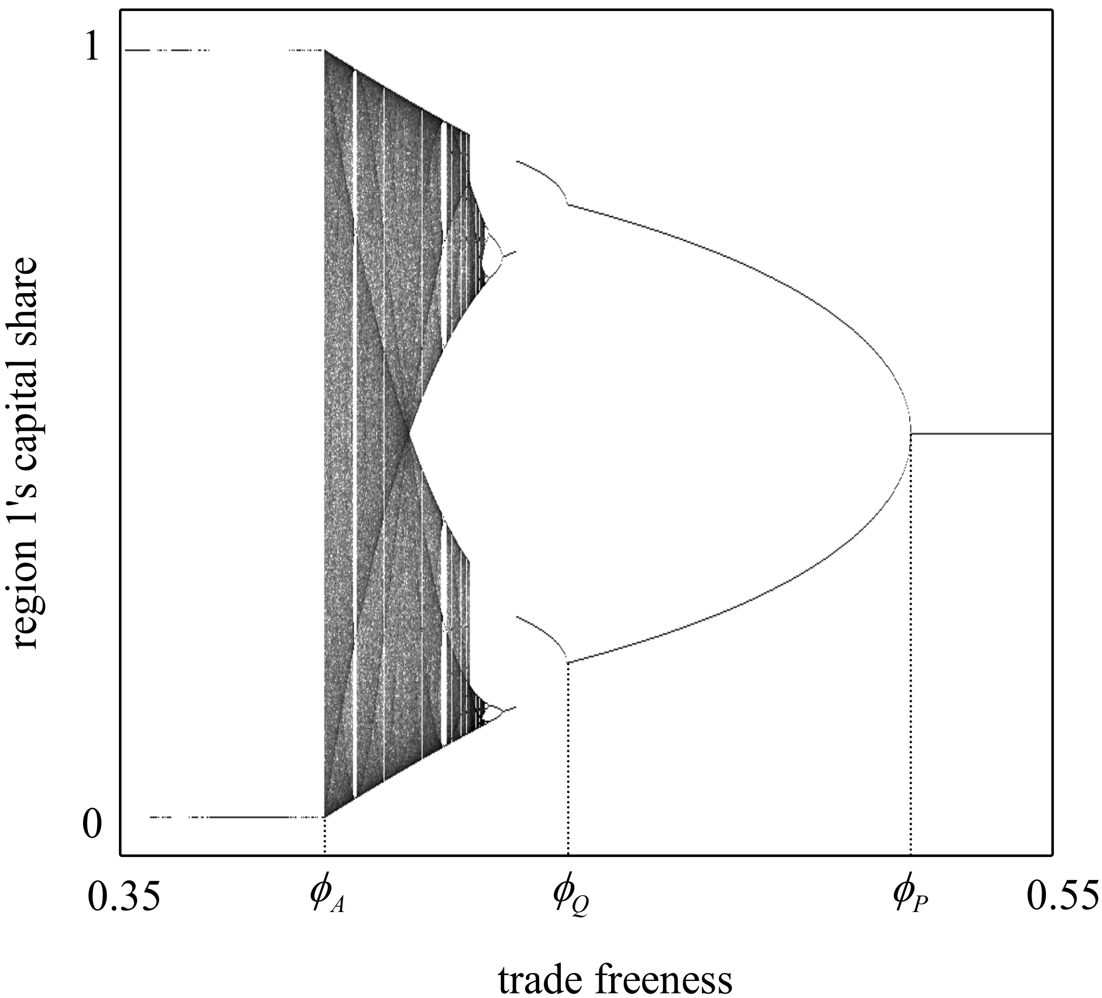
[6] In Figure 1, which is based on $\phi = 0.5$ and $\Psi = 40$, the critical speeds are $\gamma_P = 0.45$ and $\gamma_A \approx 0.874$.

Figure 1



With symmetric regions, the maps $\lambda_{t+1} = Z\left(\lambda_t\right)$ show the dependence of region 1's share of capital use in period $t+1$ on its share in period t for two capital mobility speeds, γ_P and γ_A, where $\gamma_A > \gamma_P$.

Figure 2

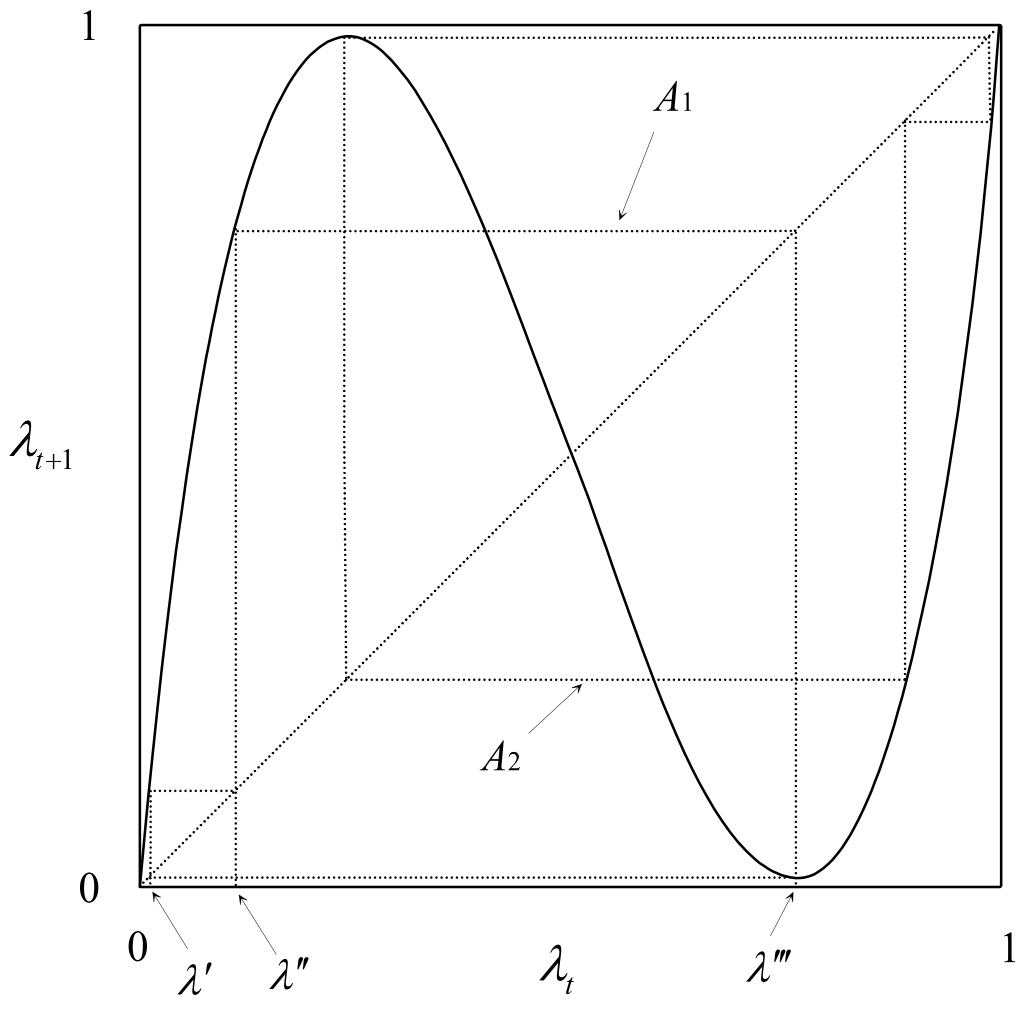


Bifurcation diagram showing the impact of trade freeness ϕ on the long-run behaviour of region 1's share of capital use λ_t.

orbit. For levels of trade freeness in the range $0.35 \leq \phi \leq 0.55$, it plots the orbit $Z^{[t]}(\lambda_0)$ for $1001 \leq t \leq 3000$, where the first 1000 iterations are discarded in order to focus on *long-term* behaviour. It is based on $\Psi = 40$, $\gamma = 0.5$, and $\lambda_0 = 0.499$, i.e., each orbit starts close to the symmetric fixed point. In the ensuing discussion of the impact of a *ceteris paribus* change in $\phi \equiv T^{1-\sigma}$, the latter change should be interpreted as being due to a change in *T*, since a change in σ would also change Ψ. For $\phi > \phi_P \approx 0.519$, the symmetric fixed point $\lambda = 1/2$ is an attractor for any λ_0 in $(0,1)$. As ϕ falls through ϕ_P, where $Z'(1/2) = -1$, $\lambda = 1/2$ becomes unstable and a period-doubling (or flip) bifurcation gives rise to a period-2 orbit symmetric around $\lambda = 1/2$. As ϕ falls through ϕ_Q, a pitchfork bifurcation of the second iterate implies that the period-2 orbit becomes unstable and two attracting period-2 orbits emerge. As ϕ falls further, orbits of every periodicity and chaotic behaviour occur.

That the long-term asymmetric coexistence of manufacturing in both regions is possible is confirmed by Figure 3. It shows two attracting period-3 orbits, A_1 and A_2, for $\phi = 0.39587$. The existence of a period-3 orbit is particularly significant: it guarantees that there are periodic orbits of all (integer) periods. In accordance with the Symmetry Proposition, the attractors A_1 and A_2 are symmetric with respect to each other. That is, for A_1, region 1 cycles between λ', λ'' and λ''' and region 2 cycles between $(1-\lambda')$, $(1-\lambda'')$ and $(1-\lambda''')$, whereas the converse applies for A_2. In terms of their real incomes, workers and capitalists in region 1 would be significantly better-off on A_2 than on A_1. The basins of attraction of A_1 and A_2 are symmetric with respect to each other; that is, if the orbit from initial point λ_0 is attracted to A_1, the orbit from initial point $(1-\lambda_0)$ must be attracted to A_2. Each basin comprises an infinite number of intervals separated by the periodic points of the repelling

Figure 3



With symmetric regions, two period-3 attractors that are symmetric with respect to each other.

period-k orbits ($k \neq 3$) and by their preimages of any rank. There are infinitely many initial points which exhibit sensitive dependence on initial conditions.[7]

That long-term behaviour can be hyper-sensitive to small changes in trade freeness is confirmed by the presence of 'windows' in the bifurcation diagram: a miniscule change in ϕ can abruptly alter long-term behaviour from, say, an orbit of very low periodicity to an orbit of very high periodicity or even to chaotic behaviour. Furthermore, a small reduction in ϕ can result in an abrupt change from chaotic behaviour to agglomeration in one region. This occurs in Figure 2 when ϕ falls through $\phi_A \approx 0.393$, the latter being the level of ϕ below which agglomeration occurs for almost all initial conditions. At $\phi = \phi_A$, for which the iterate of the interior maximum of (16) is 1 and the iterate of the interior minimum is 0, there exist periodic orbits of every period, and every point in [0,1] has sensitive dependence on initial conditions.[8] For $\phi < \phi_A$, trade costs are sufficiently high that, for almost all initial points (i.e., except for an invariant Cantor set of points of zero measure[9]), the system sooner or later converges on either $\lambda = 0$ or $\lambda = 1$. At such a core-periphery fixed point, the profit in the core is less than the virtual profit in the periphery. If such a core-periphery equilibrium is subjected to a small disturbance, the system may be attracted back to that particular fixed point or it may be attracted to the other core-periphery fixed points. Figure

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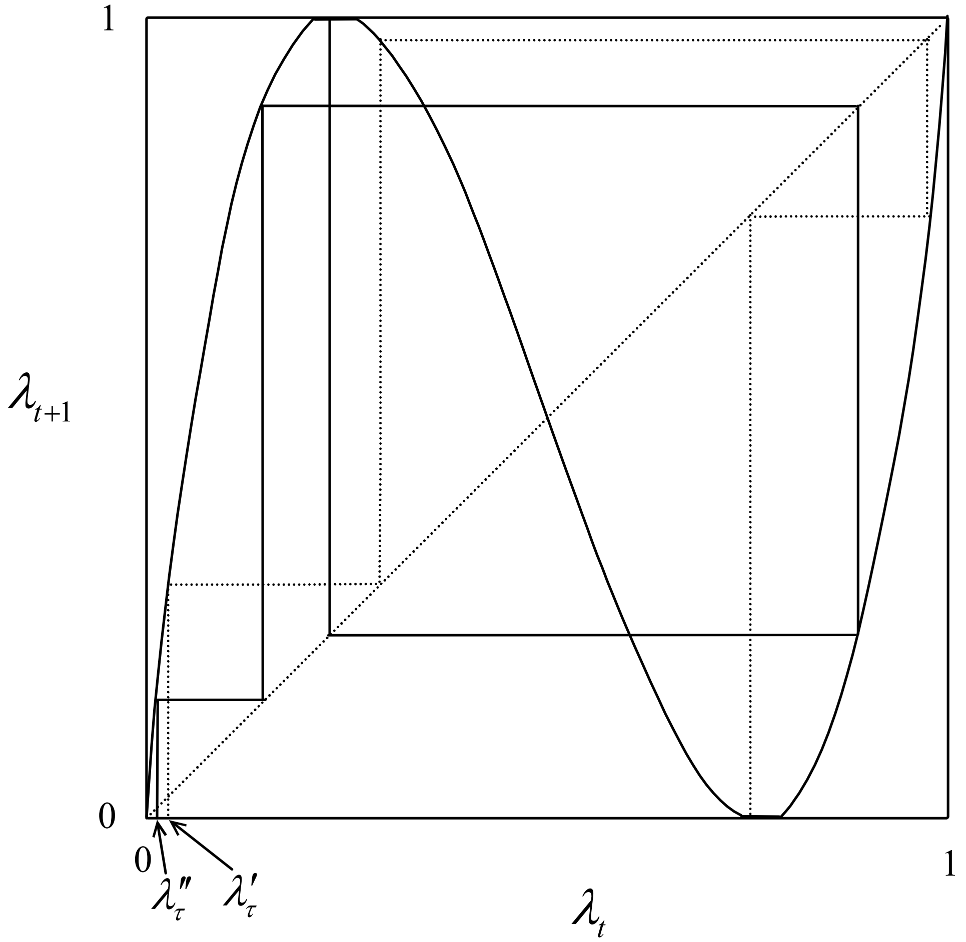
[7] On the implications of the existence of a period-3 orbit, see Alligood *et al.* (1996, Chapter 1).

[8] This can be shown by applying to the FC map for $\phi = \phi_A$ the same methods that Alligood *et al.* (1996, Chapter 1) use to demonstrate these properties for the logistic map $G(x) = 4x(1-x)$.

[9] Identifying the set of initial points whose orbits do not converge on a core-periphery fixed point would be similar to identifying the set of points whose orbits remain in $(0,1)$ for the logistic $G(x) = \mu x(1-x)$ where $\mu > 4$. On the latter, see Devaney (1989, p. 35).

Figure 4



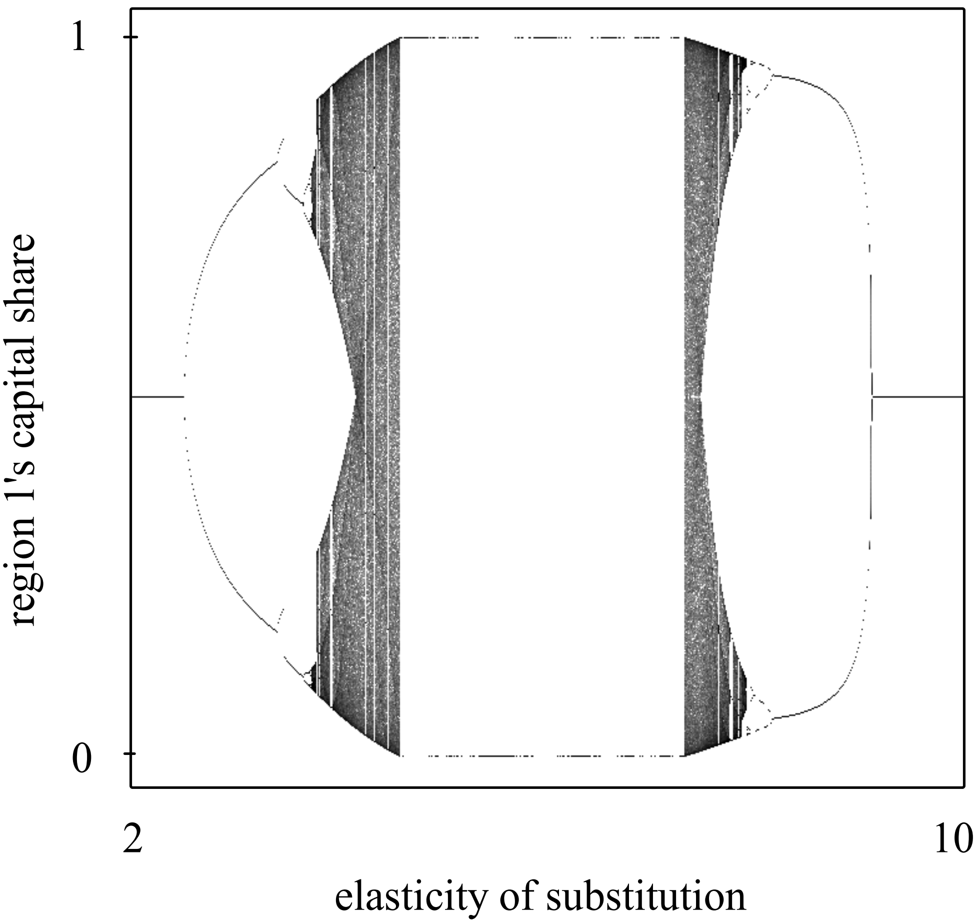
Map $\lambda_{t+1} = Z\left(\lambda_t\right)$ for a case where $\phi > \phi_A$. From λ'_τ, agglomeration occurs in region 2; from λ''_τ, agglomeration occurs in region 1.

4, based on $\phi = 0.3917$, assumes that all manufacturing is concentrated in region 2 but that, for whatever reason, a 'small' proportion of the capital is moved to region 1 at the beginning of period τ. Figure 4 compares the outcome of a shift to λ'_τ with the outcome of a shift to λ''_τ. Both orbits initially move further away from $\lambda = 0$. However, the dotted orbit from λ'_τ snaps back to $\lambda = 0$, whereas the solid orbit starting from λ''_τ results in $\lambda = 1$.

Increases in the capital migration speed and reductions in trade freeness are broadly 'de-stabilizing'. The equivalence follows from the fact that higher trade barriers increase the importance of location and therefore enhance any incentive to move capital. For any level of trade freeness, there exist capital migration speeds sufficiently rapid that long-term behaviour exhibits periodic or chaotic coexistence or that all industry is concentrated in one region. Similarly, for any speed, there exist transport costs sufficiently high for long term behaviour to exhibit complexity or for agglomeration to occur. From (14) and (17), a *ceteris paribus* increase in the average profit per unit of capital, Ψ, has the same qualitative impact as an increase in γ. Thus, from (13), an increase in manufactures' share of expenditure μ or in the labour-capital ratio L/K are de-stabilizing. The impact of a change in the consumers' preference for variety is more complex: an increase in σ reduces trade freeness (which is de-stabilizing) but also reduces the average profit per unit of capital (which is stabilizing). In Figure 5, which is based on $\mu = 0.5$, $L/K = 100$, $T = 2$ and $\gamma = 0.35$, the symmetric fixed point is an attractor for $\sigma < 2.5$ and for $\sigma > 9.13$.

An important question is the robustness of the comparative dynamical propositions with respect to the specification of the capital migration process. An alternative specification, for example, could involve capitalists responding to the *ratio* of regional profits. Since Ψ does not affect the profit ratio, μ and L/K would have no impact on the dynamical behaviour;

Figure 5



Bifurcation diagram showing the impact of the elasticity of substitution σ on the long-run behaviour of region 1's share of capital use λ_t.

and σ would have an effect only via trade freeness. However, the impacts of γ and of ϕ would be qualitatively similar and, more generally, the system would still exhibit complex dynamical behaviour, with multiple attractors and hyper-sensitivity to initial conditions and to parameters.

5.3. Comparison of Models

The contrast between the discrete-time and continuous-time symmetric FC models could not be more dramatic. For the continuous-time model, the instantaneous short-run general equilibrium corresponds to the short-run general equilibrium identified in Section 3. Baldwin *et al*. assume a capital migration process $\dot{\lambda} = \lambda(1-\lambda)(\pi_1 - \pi_2)$. In fact, provided only that capital migrates to the region with the higher profit, the precise specification of the migration process is irrelevant for the long-term behaviour of the continuous-time system. Indeed, for $\phi < 1$, the only stable long-run equilibrium for the continuous-time symmetric FC model is the symmetric equilibrium $\lambda = 1/2$, since the core-periphery equilibria are necessarily unstable. For this reason, Baldwin *et al.* (2003, p. 80) describe the continuous-time symmetric FC model as "rather uninteresting". In contrast, the dynamical behaviour of the discrete-time model is extremely interesting.

As noted above, the analytical tractability of the FC model derives from the fact that the repatriation of the earnings from capital eliminates both the demand-linked and cost-linked circularities that feature in the CP model. Whereas the symmetric FC model possesses three fixed points, the CP model has parameter ranges for which there are five fixed points. Nevertheless, the discrete-time symmetric FC model retains much of the dynamical complexity of the discrete-time symmetric CP model examined by Currie and Kubin (2005). In both models, long-term behaviour can involve periodic or chaotic coexistence of manufacturing in both regions; multiple attractors, sensitive dependence on parameters and sensitive dependence on initial conditions are pervasive. Moreover, whereas discussions of the continuous-time CP model portray high transport costs as stabilizing, high transport costs are de-stabilizing in both the discrete-time FC and CP models.

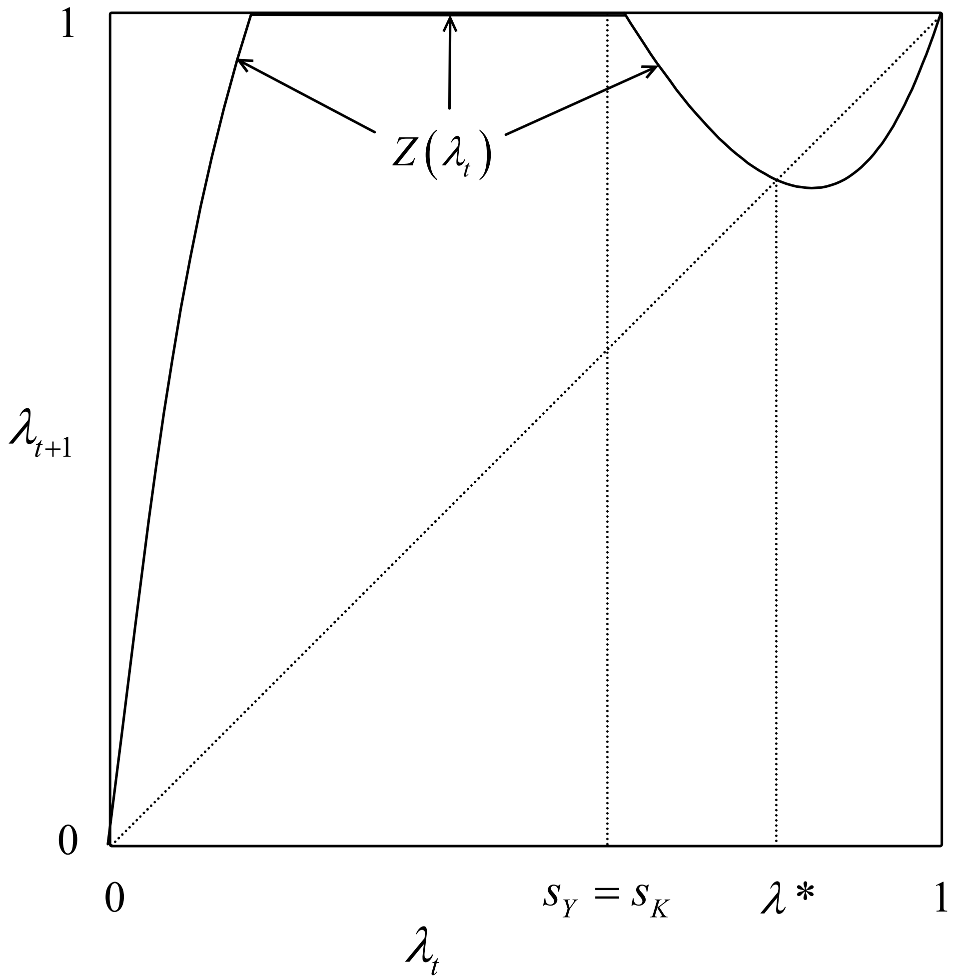
6. Asymmetric Regions

6.1. Comparative Dynamics

Exogenous regional asymmetries imply, in general, that $Z(1-\lambda_t) \neq 1-Z(\lambda_t)$; in other words, rotating $Z(\lambda_t)$ through 180^0 does not result in the same map. In particular, from (20), $s_Y \neq 1/2$ implies $Z(1/2) \neq 1/2$. Figure 6, for which $\Psi = 40$, $\gamma = 0.5$, $\phi = 0.5$ and $s_L = s_K = s_Y = 0.6$, shows that a relatively modest difference simply in the size of regions can result in a highly asymmetric map. In accordance with (20), at the interior fixed point $\lambda^* = 0.8$, the larger region's share in capital use exceeds its share in income (and in capital ownership). Although the interior fixed point λ^* is locally stable, many initial points result in agglomeration in the larger region. Indeed, with an initial allocation equal to the ownership share [i.e., $\lambda_0 = s_K = 0.6$], the profit difference would be sufficient to induce capitalists to move immediately all their capital to the larger region.

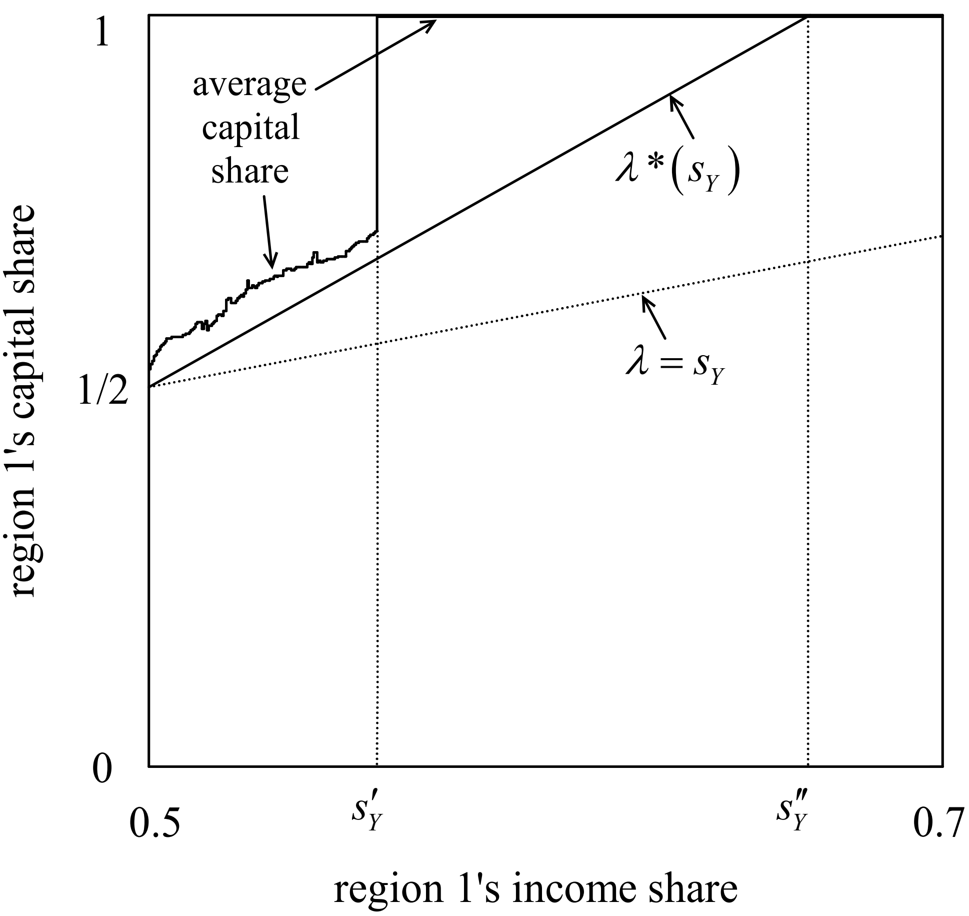
Figure 7, for which $\Psi = 40$, $\phi = 0.5$ and $\gamma = 0.65$, focuses on the significance of region 1's share of income, assuming that it is the larger region. For $0.5 \leq s_Y \leq s_Y''$, $\lambda^*(s_Y)$ represents the linear relationship (20): it shows the share of capital use at which regional profits are equal. The profit in region 1 (the core) equals the virtual profit in region 2 (the periphery) at $s_Y'' = 1/(1+\phi) = 2/3$, which is equivalent to $\phi = (1-s_Y)/s_Y$. The capital migration speed is

Figure 6



Map $\lambda_{t+1} = Z(\lambda_t)$ with asymmetric regions, where λ^* is the locally stable interior fixed point, s_Y is region 1's share in total expenditure and s_K is its share in capital ownership.

Figure 7

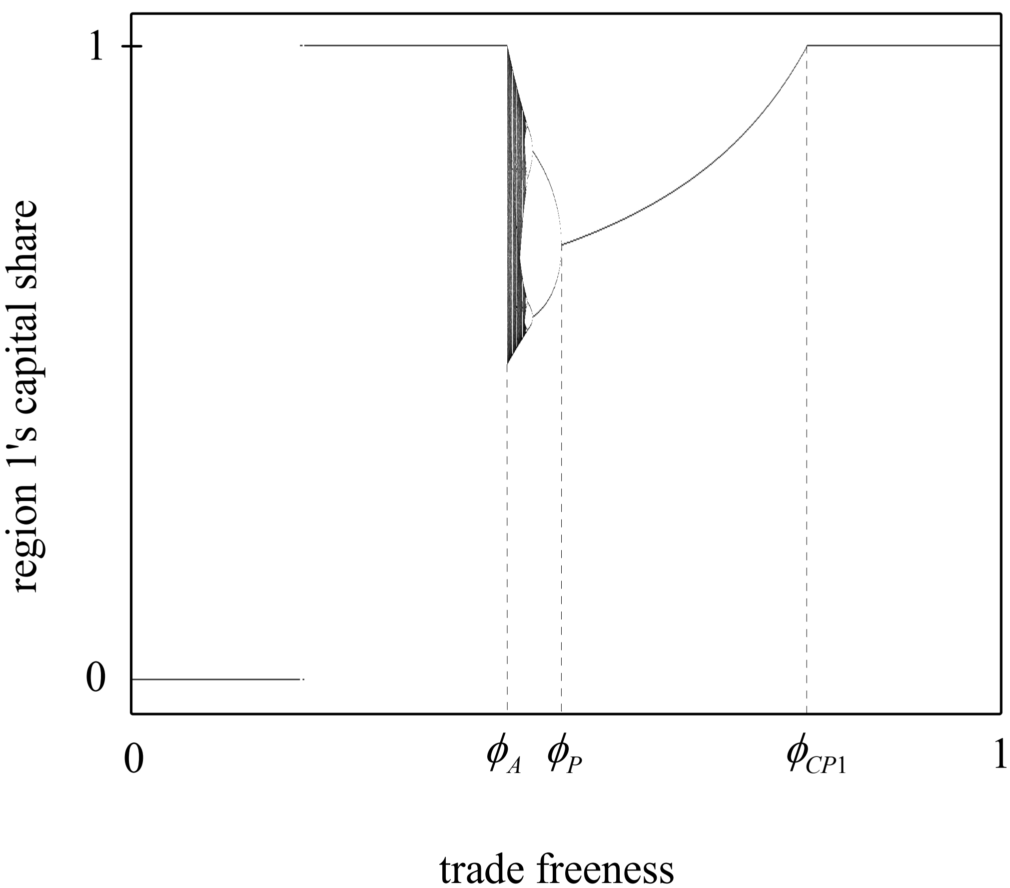


The dependence of region 1’s unstable equilibrium share of capital use and of its actual average share of capital use on region 1’s income share.

sufficiently high (or trade freeness is sufficiently low) that the interior fixed point λ^* is repelling for an income share below s''_Y. For $s_Y \le s'_Y$, region 1's capital use exhibits periodic or chaotic behaviour but, *on average*, it is systematically higher than λ^* (and *a fortiori* greater than the income share). Above income share s'_Y, all manufacturing is concentrated in region 1. Compared to the virtual profit in region 2, profit per unit of capital in region 1 is lower for $s'_Y < s_Y < s''_Y$ but higher for $s_Y > s''_Y$.

Figure 8 illustrates the possible impact of the level of trade freeness where relative factor endowments differ. Both regions have the same labour endowments but all the capitalists live in region 1 [i.e., $s_K = 1$]. The simulations are based on $\Psi = 40$, $\gamma = 0.5$, $\sigma = 4$ and $\mu = 0.5$; λ_0 is set close to the relevant fixed point; and the level of ϕ ranges from just above 0 to just below 1. The values of σ and μ imply $s_Y \approx 0.563$, so that, even with $s_K = 1$, income shares are not very different. For $\phi < \phi_A$, the volatility that results from the relatively high trade costs leads to the concentration of all industry in region 1. For $\phi_A \le \phi < \phi_P$, the interior fixed point is repelling and the system exhibits periodic or chaotic long-term behaviour, with region 1 exporting capital. For $\phi_P \le \phi < \phi_{CP1}$, the system is attracted to the interior fixed point, where, as implied by (20), λ^* increases as ϕ rises. That is, region 1's exports of capital fall as trade in manufactures becomes less costly. For $\phi \ge \phi_{CP1} = (1 - s_Y)/s_Y$, trade is sufficiently free that region 1 does not export any capital. From (14), $\phi > (1 - s_Y)/s_Y$ implies $\Pi(1) > 0$, i.e., the profit in region 1 exceeds the virtual profit in region 2.

Figure 8



Bifurcation diagram showing the impact of trade freeness ϕ with all capitalists living in region 1.

6.2. Comparison with Continuous-time FC model

With asymmetric regions, the discrete-time FC model is again dynamically much richer than its continuous-time counterpart. For the latter, if, say, $s_Y > 1/2$, the only stable long-term behaviour involves $\lambda = \lambda^*$ for $\phi \le s_Y \big/ \left(1 - s_Y\right)$ and $\lambda = 1$ otherwise. The divergence between λ^* and s_Y is attributed to the so-called 'home market effect', whereby the region with the larger home market has a more than proportionately larger manufacturing sector, and therefore exports manufactures.[10] In Figure 7, the impact of the home market effect is on average greater in the discrete-time model than in the continuous-time model. The abrupt agglomeration that occurs in the discrete-time model as s_Y increases through s'_Y does not occur in the continuous-time model.

The sole dynamical feature that Baldwin *et al*. highlight in the continuous-time model is the possibility of a very 'rapid' rate of relocation – which they call 'near-catastrophic agglomeration' (2003, p. 88)[11] – when the regions are only slightly different in size and ϕ approaches ϕ_{CP1}. In the discrete-time FC model, sensitive dependence on parameters is much more pervasive. Moreover, in contrast to the continuous-time FC model, the system displays sensitive dependence on initial conditions.

7. Some Final Comments

Reformulating the Footloose Capital model in terms of a discrete-time framework has astonishing implications for its possible long-run dynamical behaviours. Whereas the long-

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[10] For a detailed examination of the home-market effect in the continuous-time FC model, see Baldwin *et al*. (2003, ch. 3).

[11] On the notion of near-catastrophic agglomeration, see also Baldwin (1999).

run behaviour of the continuous-time FC model is confined to a unique stationary equilibrium, long-run behaviour in the discrete-time model can involve the asymmetric coexistence of manufacturing in both regions, even with symmetric regions possessing the same factor endowments. The system possesses multiple attractors, exhibits chaotic behaviour, and displays sensitive dependence both on initial conditions and on parameters. The discrete-time model is much richer in terms of its potential for explaining and understanding economic phenomena relating to capital movements between regions and countries. But it also cautions against relying on simple comparative static propositions.

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