Asset pricing and hedging under transaction costs: An approach based on the von Neumann-Gale model

Igor V. Evstigneev, University of Manchester
Michael I. Taksar, University of Missouri

Introduction

The paper analyses questions of asset pricing and hedging under transaction costs and trading constraints. A new approach to these questions is proposed. The approach is suggested by analogies between dynamic models of financial markets and (stochastic analogues of) the von Neumann–Gale model of economic growth [17], [38].

We obtain, in the general context of transaction costs and trading constraints, analogues of the classical results of Harrison and Pliska [18], Dalang, Morton and Willinger [8] and others related to the Fundamental Theorem of Asset Pricing (FTAP), providing criteria for the absence of arbitrage. Also, we generalize the known principle of valuation of contingent claims based on the notion of superhedging (e.g. Pliska [32]). According to this principle, the minimum initial level of wealth needed to superhedge a contingent claim represents a "fair price" of the claim.

We look at the topics in question from the classical mathematical economics standpoint. We regard the FTAP as a version of various results in general equilibrium theory and multiobjective programming dealing with the characterization of Pareto-optimal states – see, e.g., Aliprantis, Brown and Burkinshaw [2]. These results describe such states as maxima of positive linear functionals (supporting prices). In different specialized models, assertions of this kind take on different specific forms. In the conventional model of a frictionless financial market, they are expressed in terms of the concept of an equivalent martingale measure. In more realistic settings, however, this concept requires substantial generalization and refinement.

A key role in this work is played by the observation that dynamic models of financial markets can formally be described by using the framework of homogeneous convex random dynamical systems. These dynamical systems are defined by multivalued stochastic operators satisfying certain conditions of convexity and homogeneity. Such multivalued operators have been studied, primarily, in connection with stochastic analogues of the von Neumann–Gale model of economic growth (see von Neumann [38] and Gale [17] for the deterministic theory, and Radner [33], Dynkin [11], Evstigneev and Kabanov [14], Arnold, Evstigneev and Gundlach [4], Anoulova, Evstigneev and Gundlach [3] for stochastic generalizations). The methodology and the technical tools developed in that field turn out to be helpful for understanding the general structure of no-arbitrage criteria and hedging results in models with transaction costs and trading constraints.

In the present work, we concentrate primarily on the conceptual aspects and on the economic content of the theory under review. Therefore we try to reduce technical considerations as much as possible. We focus on finite-dimensional cases, assuming that the time parameter takes on a finite number of values and the underlying probability space is finite.

We initiated this study in 2000, and the preliminary results were set out in our paper [15]. In the course of further work, the paper has been substantially revised. In particular, a more general model for a financial market with frictions, comprising a broader range of examples and applications, has been suggested.

The paper is organized as follows. In Section 1, we present the classical model for the valuation of hedgeable contingent claims in a frictionless financial market with a riskless asset. Section 2 provides a version of the theory applicable to superhedgeable (non necessarily hedgeable) contingent claims. Section 3 describes a general dynamic model for a securities market with transaction costs and trading constraints. In Section 4, we examine the central notion of the theory under consideration – the notion of a consistent price system (playing a role similar to the role of an equivalent martingale measure in the classical setting). Section 5 contains a discussion of analogies between the dynamic securities market model we study and the von Neumann-Gale model of economic growth. Sections 6 and 7 review a number of examples and applications. The Appendix assembles some general facts in convex analysis employed in this study.

1 The classical model of a frictionless financial market

1.1 The basic elements of the model

Let $A = \{a^1, ..., a^L\}$ be a finite set, elements of which are interpreted as possible states of the world. At each time $t = 1, 2, ..., T$, any of these states can be realized. That state of the world which is realized at time t is denoted by a_t. A sequence $\omega = (a_1, ..., a_T)$ is called a *history* (scenario) of the market over the time period $1, 2, ..., T$. For each $t = 1, 2, ..., T-1$, the sequence $\omega^t = (a_1, ..., a_t)$ is called the partial history or partial scenario (up to time t).

For each $\omega = (a_1, ..., a_T) \in \Omega$, we are given a probability measure[1] P on the set Ω of all market scenarios. Thus, to each $\omega \in \Omega$, a positive number $P(\omega)$ is assigned representing the probability that the market will develop according to the scenario ω. The sum of all these probabilities is 1:

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

In this work, we always assume that P is strictly positive: $P(\omega) > 0$ for each ω. We denote by the letter E the expectation with respect to the probability measure P:

$$EX = \sum_{\omega \in \Omega} X(\omega) P(\omega),$$

where $X(\omega)$ is any function on Ω.

There are $N + 1$ securities (assets) $i = 0, 1, ..., N$. The vector of their prices at time $t = 0, ..., T$ is denoted by

$$\mathbf{S}_t = (\mathbf{S}_t^0, \mathbf{S}_t^1, ..., \mathbf{S}_t^N).$$

We assume that the vector $\mathbf{S}_t$ $(t \geq 1)$ is a function of the sequence $\omega^t = (a_1, ..., a_t)$ of the states of the world up to time t:

$$\mathbf{S}_t = \mathbf{S}_t(a_1, ..., a_t).$$

In other words, every coordinate S_t^i of the vector $\mathbf{S}_t$ – the price of asset i at time $t \geq 1$ – is the given function

$$S_t^i = S_t^i(a_1, ..., a_t)$$

of the sequence $\omega^t = (a_1, ..., a_t)$. The vector $\mathbf{S}_0 = (S_0^0, S_0^1, ..., S_0^N)$ is constant (non-random).

The 0th asset $i = 0$, plays a special role. Its price S_t^0 at time t (i.e., the 0th coordinate of the vector $\mathbf{S}_t$) is supposed to be non-random and equal to $(1 + r)^t$, where $r \geq 0$ is some

[1] The terms "probability", "probability measure" and "probability distribution" are used interchangeably.

given number. Asset $i = 0$ may be interpreted as cash, r being the interest rate (the same for lending and borrowing).

Any real-valued function $X(\omega)$ of $\omega \in \Omega$ will be called a *contingent claim*. A contingent claim is interpreted as a contract that allows its owner to receive the specified amount of money $X(\omega)$ at time T. This amount might be both positive and negative. In general, $X(\omega)$ might depend on the whole market history $\omega = (a_1, a_2, ..., a_T)$ – on the current state of the world, a_T, at time T and on the previous states $a_1, a_2, ..., a_{T-1}$. A contingent claim X can be regarded as a security (asset) which can, as well the basic securities $i = 0, 1, 2, ..., N$, traded on the market. The price of this security at time T is equal to $X(\omega)$. The amount $X(\omega)$, and hence the price, might be negative if the contract at hand involves the obligation to pay, rather than the right to get, the prescribed sum of money.

What is the fair price of a contingent claim at time $t = 0$? The general informal principle in this model is that the price of X is the minimum level of wealth at time 0 which is needed to hedge the contingent claim X. "Hedging" means that we can generate the payoff of the contingent claim by a self-financing trading strategy. In the next subsection, we will give precise definitions of the these notions.

1.2 Trading strategies

A *trading strategy* is a sequence $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ where $\mathbf{h}_t = (h_t^0, h_t^1, ..., h_t^N)$ is the investor's *portfolio* at time $t = 0, 1, ..., T$. Portfolio positions are expressed in terms of units of assets, so that h_t^i is the number of units of asset i in the portfolio $\mathbf{h}_t$. The investor can choose a portfolio $\mathbf{h}_t$ $(t \geq 1)$ based on the observation of the history $\omega^t = (a_1, ..., a_t)$. Thus, in general, $\mathbf{h}_t$ is a function of ω^t:

$$\mathbf{h}_t = \mathbf{h}_t(\omega^t),\ t = 1, 2, ..., T.$$

To emphasize this fact the term *contingent portfolio* is used. The portfolio $\mathbf{h}_0 = (h_0^0, h_0^1, ..., h_0^N)$ is constant (non-random). By selecting a trading strategy, the investor specifies what portfolio he/she is going to have at each time $t = 0, 1, ..., T$ in each possible contingency.

A central role in this theory is played by *self-financing* trading strategies, i.e. those satisfying the following condition[2]:

$$\langle \mathbf{S}_t, \mathbf{h}_{t-1} \rangle = \langle \mathbf{S}_t, \mathbf{h}_t \rangle,\ t = 1, 2, ..., T. \tag{1}$$

Condition (1) means that the investor, at each time period, rebalances his/her portfolio (replaces the old one, $\mathbf{h}_{t-1}$, by the new one, $\mathbf{h}_t$) remaining within the budget constraints.

[2]If $u = (u^0, ..., u^N)$ and $v = (v^0, ..., v^N)$ are vectors, then we write

$$\langle u, v \rangle = uv = \sum_{i=0}^{N} u^i v^i$$

for the scalar product of these vectors.

According to (1), the value $\langle \mathbf{S}_t, \mathbf{h}_t \rangle$ of the new portfolio $\mathbf{h}_t$, expressed in terms of the new price system $\mathbf{S}_t = (S_t^0, S_t^1, ..., S_t^N)$, is equal to the value $\langle \mathbf{S}_t, \mathbf{h}_{t-1} \rangle$ of the old portfolio $\mathbf{h}_{t-1}$ expressed in terms of this price system. Self-financing strategies exclude consumption and do not involve an inflow of external funds. During the time period $0 \leq t \leq T$, the investor "plays" on price changes with the view of obtaining at the end of the period a portfolio $\mathbf{h}_T$ that is most preferred to him/her in the sense of one criterion or another.

We say that a contingent claim $X(\omega)$ can be *replicated*, or *hedged*, if there exists a self-financing trading strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ such that

$$\langle \mathbf{S}_T, \mathbf{h}_T \rangle = X(\omega)$$

for all ω (note that $\mathbf{S}_T$ and $\mathbf{h}_T$ depend on ω). This means that the strategy $\mathbf{H}$ yields exactly the same payoff at time T as the contingent claim $X(\omega)$. To price such contingent claims we will use the general principle outlined above: the price of X is the minim level of initial wealth needed to hedge X. Formally, this price can be defined as

$$\inf_{\mathbf{H}} \langle \mathbf{S}_0, \mathbf{H}_0 \rangle, \tag{2}$$

where the infimum is taken over all self-financing strategies $\mathbf{H} = (\mathbf{h}_{0,} \mathbf{h}_{1,...,} \mathbf{h}_T)$ replicating X, i.e., satisfying

$$\langle \mathbf{S}_T, \mathbf{H}_T \rangle = X(\omega) \tag{3}$$

for all ω. Typically, the greatest lower bound in this minimization problem is attained, and so we can replace "inf" by "min".

1.3 The pricing of contingent claims by no arbitrage

In the present model, one can formulate a very natural assumption under which the value of the objective function $\langle \mathbf{S}_0, \mathbf{H}_0 \rangle$ in the minimization problem (2), (3) is *the same* for all self-financing strategies $\mathbf{H}$ satisfying (3). This remarkable fact lies in the basis of the principle of asset pricing by no arbitrage. To formulate the assumption, let us introduce the following definition. For a trading strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$, denote by $V_0^{\mathbf{H}}$ and $V_T^{\mathbf{H}}$ the values of its initial and final portfolios:

$$V_0^{\mathbf{H}} = \langle \mathbf{S}_0, \mathbf{h}_0 \rangle, \ V_T^{\mathbf{H}} = \langle \mathbf{S}_T, \mathbf{h}_T \rangle.$$

(Note that $V_T^{\mathbf{H}} = V_T^{\mathbf{H}}(\omega)$ is random, while $V_0^{\mathbf{H}}$ is not.) Let us say that there is an *arbitrage opportunity* if there exists a self-financing trading strategy $\mathbf{H}$ such that

$$V_0^{\mathbf{H}} \leq 0,$$

$$V_T^{\mathbf{H}}(\omega) \geq 0 \text{ for all } \omega, \text{ and } V_T^{\mathbf{H}}(\omega) > 0 \text{ for at least one } \omega.$$

It is said here that the self-financing strategy $\mathbf{H}$ allows, starting from some non-positive wealth at time 0, to get at time T non-negative wealth always and strictly positive wealth sometimes (for at least one ω). The fundamental assumption we impose is this:

(NA) Arbitrage opportunities do not exist.

If (NA) holds then the level of initial wealth needed to replicate or hedge X is defined uniquely. In other words, the following assertion holds.

Theorem 1.1. *Let hypothesis (NA) hold. Let* $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ *and* $\mathbf{H}' = (\mathbf{h}'_0, ..., \mathbf{h}'_T)$ *be two self-financing trading strategies hedging the same contingent claim* X*. Then*

$$\langle \mathbf{S}_0, \mathbf{h}_0 \rangle = \langle \mathbf{S}'_0, \mathbf{h}'_0 \rangle .$$

Proof. Suppose the contrary: $\langle \mathbf{S}_0, \mathbf{h}_0 \rangle \neq \langle \mathbf{S}_0, \mathbf{h}'_0 \rangle$. We may assume without loss of generality that $\langle \mathbf{S}_0, \mathbf{h}'_0 \rangle > \langle \mathbf{S}_0, \mathbf{h}_0 \rangle$, i.e., the difference $\Delta = \langle \mathbf{S}_0, \mathbf{h}'_0 \rangle - \langle \mathbf{S}_0, \mathbf{h}_0 \rangle$ is strictly positive. Consider the trading strategy $\mathbf{H}^\Delta = (\mathbf{h}_0^\Delta, ..., \mathbf{h}_T^\Delta)$, where

$$\mathbf{h}_t^\Delta = (\Delta, 0, 0, ..., 0)$$

for each t (the 0th coordinate of the vector $\mathbf{h}_t^\Delta$ is Δ, and all the other coordinates are equal to zero). Clearly the strategy $\mathbf{h}_t^\Delta$ is self-financing – because the portfolio $\mathbf{h}_t^\Delta$ does not change in time. Consequently, the strategy $\bar{\mathbf{H}} = (\bar{\mathbf{h}}_0, ..., \bar{\mathbf{h}}_T)$ defined by

$$\bar{\mathbf{h}}_t = \mathbf{h}_t - \mathbf{h}'_t + \mathbf{h}_t^\Delta$$

is self-financing. For this strategy, we have

$$V_0^{\bar{\mathbf{H}}} = \langle \mathbf{S}_0, \bar{\mathbf{h}}_0 \rangle = \langle \mathbf{S}_0, \mathbf{h}_0 \rangle - \langle \mathbf{S}_0, \mathbf{h}'_0 \rangle + \Delta = -\Delta + \Delta = 0$$

and

$$V_T^{\bar{\mathbf{H}}} = \langle \mathbf{S}_T, \bar{\mathbf{h}}_T \rangle = \langle \mathbf{S}_T, \mathbf{h}_T \rangle - \langle \mathbf{S}_T, \mathbf{h}'_T \rangle + \Delta(1+r)^T =$$

$$X - X + \Delta(1+r)^T = \Delta(1+r)^T > 0$$

(we have used the fact that $S_t^0 = (1+r)^t$, $t = \overline{0, T}$). Thus the strategy $\bar{\mathbf{H}}$ provides an arbitrage opportunity. A contradiction. □

1.4 Risk-neutral probabilities

We have shown that, under the no arbitrage hypothesis (NA), the initial values $V_0^{\mathbf{H}}$ and $V_0^{\mathbf{H}'}$ of two self-financing trading strategies $\mathbf{H}$ and $\mathbf{H}'$ hedging the same contingent claim X coincide. This fact leads to the following *no-arbitrage pricing principle:*

$$\boxed{\begin{array}{c}\text{The fair price } l(X) \text{ of a contingent claim } X \\ \text{is equal to the initial value } V_0^{\mathbf{H}} \text{ of any self-financing} \\ \text{strategy } \mathbf{H} \text{ replicating } X.\end{array}}$$

Now a question arises: how can we compute $l(X)$? To answer this question we introduce the notion of a risk-neutral probability measure. A probability Q on Ω is called *risk-neutral* if the expectation with respect to this measure of the net present value of any self-financing trading strategy is zero:

$$E^Q V^{\mathbf{H}} = 0. \tag{4}$$

The symbol $E^Q X$ denotes the expectation of a random variable X with respect to Q:

$$E^Q X = \sum_{\omega \in \Omega} X(\omega) Q(\omega).$$

The *net present value* for a trading strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ is defined by

$$V^{\mathbf{H}} = \frac{\langle \mathbf{S}_T, \mathbf{h}_T \rangle}{(1+r)^{-T}} - \langle \mathbf{S}_0, \mathbf{h}_0 \rangle = \frac{V_T^{\mathbf{H}}}{(1+r)^T} - V_0^{\mathbf{H}}.$$

Since $\mathbf{S}_0, \mathbf{h}_0$ and r are non-random, formula (4) can be written

$$\frac{E^Q V_T^{\mathbf{H}}}{(1+r)^T} - V_0^{\mathbf{H}} = 0$$

or

$$V_0^{\mathbf{H}} = \frac{E^Q V_T^{\mathbf{H}}}{(1+r)^T}. \tag{5}$$

Thus the value $V_0^{\mathbf{H}}$ of the initial portfolio $\mathbf{h}_0$ of any self-financing trading strategy $\mathbf{H}$ is equal to the expectation – with respect to the risk-neutral measure Q - of the discounted value

$$\frac{V_T^{\mathbf{H}}}{(1+r)^T}$$

of the final portfolio $\mathbf{h}_T$ of this strategy. The property we have just formulated is an equivalent form of the definition of a risk-neutral probability measure: formula (5) is equivalent to formula (4).

We will consider only those risk-neutral measures Q which are strictly positive,

$$Q(\omega) > 0 \text{ for all } \omega. \tag{6}$$

Property (6) will be included into the definition of a risk-neutral probability measure.

Recall that a contingent claim $X(\omega)$ is hedgeable if there exists a self-financing trading strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ such that

$$X(\omega) = V_T^{\mathbf{H}}(\omega).$$

If Q is a risk-neutral probability, we have

$$E^Q \left[\frac{V_T^{\mathbf{H}}}{(1+r)^T} - V_0^{\mathbf{H}} \right] = 0,$$

or, equivalently,

$$E^Q \frac{X}{(1+r)^T} = E^Q \frac{V_T^{\mathbf{H}}}{(1+r)^T} = V_0^{\mathbf{H}}.$$

Thus we obtain the following formula for the fair price $l(X)$ of contingent claim X :

$$l(X) = E^Q \frac{X}{(1+r)^T}.$$

This formula leads to the celebrated *risk-neutral pricing principle:*

The fair price $l(X)$ of a contingent claim X is equal to the discounted risk-neutral expected value $(1+r)^{-T} E^Q X$ of the contingent claim X.

1.5 Fundamental theorem of asset pricing

We have defined the notion of a risk-neutral probability and we have shown how such measures can be used for asset pricing. To employ the risk-neutral pricing principle, we have to be sure that risk-neutral probabilities exist. Fortunately, this is the case under the very general assumption we have imposed – the no arbitrage hypothesis (NA).

Theorem 1.2. *A risk-neutral probability measure exists if and only if the market does not allow for arbitrage opportunities.*

This result, going back to Harrison, Kreps and Pliska, is called the *Fundamental Theorem of Asset Pricing* (FTAP).

Before proving this theorem, we formulate an equivalent version of hypothesis (NA).

(NA1) There is no self-financing trading strategy $\mathbf{H}$ for which the net present value $V^{\mathbf{H}}(\omega)$ is non-negative for all ω and strictly positive for some ω.

Proposition 1.1. *Hypotheses (NA) and (NA1) are equivalent.*

Proof. (NA)$\Rightarrow$(NA1). Suppose (NA1) does not hold, i.e., there exists a self-financing trading strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ such that $V^{\mathbf{H}}(\omega) \geq 0$ for all ω and $V^{\mathbf{H}}(\omega) > 0$ for at least one ω. Recall that

$$V^{\mathbf{H}} = (1+r)^{-T} \langle \mathbf{S}_T, \mathbf{h}_T \rangle - \langle \mathbf{S}_0, \mathbf{h}_0 \rangle.$$

Put $\Delta = \langle \mathbf{S}_0, \mathbf{h}_0 \rangle$ and consider the trading strategy $\mathbf{H}^\Delta = (\mathbf{h}_0^\Delta, ..., \mathbf{h}_T^\Delta)$, where $\mathbf{h}_t^\Delta = (\Delta, 0, 0, ..., 0)$ for each t (the 0th coordinate of $\mathbf{h}_t^\Delta$ is Δ; all the others are equal to zero). The strategy $\mathbf{H}^\Delta$ is self-financing since $\mathbf{h}_t^\Delta$ does not change in time. Hence the strategy $\tilde{\mathbf{H}} = (\tilde{\mathbf{h}}_0, ..., \tilde{\mathbf{h}}_T)$, where $\tilde{\mathbf{h}}_t = \mathbf{h}_t - \mathbf{h}_t^\Delta$, is self-financing. We have

$$V_0^{\tilde{\mathbf{H}}} = \langle \mathbf{S}_0, \mathbf{h}_0 \rangle - \langle \mathbf{S}_0, \mathbf{h}_0^\Delta \rangle = 0$$

because $\langle \mathbf{S}_0, \mathbf{h}_0^\Delta \rangle = S_0^0 \cdot \Delta = \Delta = \langle \mathbf{S}_0, \mathbf{h}_0 \rangle$. Further,

$$V_T^{\tilde{\mathbf{H}}} = \langle \mathbf{S}_T, \mathbf{h}_T \rangle - \langle \mathbf{S}_T, \mathbf{h}_T^\Delta \rangle = \langle \mathbf{S}_T, \mathbf{h}_T \rangle - (1+r)^T \Delta =$$

$$(1+r)^T [\frac{\langle \mathbf{S}_T, \mathbf{h}_T \rangle}{(1+r)^T} - \Delta] =$$

$$(1+r)^T [\frac{\langle \mathbf{S}_T, \mathbf{h}_T \rangle}{(1+r)^T} - \langle \mathbf{S}_0, \mathbf{h}_0 \rangle] = (1+r)^T V^{\mathbf{H}}.$$

We can see that the strategy $\tilde{\mathbf{H}}$ is an arbitrage opportunity. Indeed, $V_0^{\tilde{\mathbf{H}}} = 0$, while $V_T^{\tilde{\mathbf{H}}} = (1+r)^T V^{\mathbf{H}} \geq 0$ for all ω and $V_T^{\tilde{\mathbf{H}}} = (1+r)^T V^{\mathbf{H}} > 0$ for some ω. This contradicts (NA).

(NA1)$\Rightarrow$(NA) Suppose (NA) does not hold. Then there is $\mathbf{H}$ satisfying $V_0^{\mathbf{H}} \leq 0$, $V_T^{\mathbf{H}} \geq 0$ for all ω and $V_T^{\mathbf{H}} > 0$ for some ω. But then the last two inequalities hold for $V^{\mathbf{H}} = (1+r)^{-T} V_T^{\mathbf{H}} - V_0^{\mathbf{H}}$, which contradicts (NA1). □

Proof of Theorem 1.2. "Only if". Let Q be a risk-neutral probability. Suppose (NA) does not hold. Then, as we have shown, (NA1) does not hold, i.e., for some $\mathbf{H}$, we have $V^{\mathbf{H}}(\omega) \geq 0$ for all ω and $V^{\mathbf{H}}(\omega^*) > 0$ for some ω^*. But in view of the definition of a risk-neutral probability (see (1)), we have $0 = E^Q V^{\mathbf{H}} = \sum_{\omega \in \Omega} Q(\omega) V^{\mathbf{H}}(\omega) \geq Q(\omega^*) V^{\mathbf{H}}(\omega^*) > 0$, which is a contradiction $(0 > 0)$.

"If". Suppose (NA), and hence (NA1), holds. Consider the following maximization problem:

$$\max_{\mathbf{H}} EV^{\mathbf{H}}$$

subject to

$$V^{\mathbf{H}}(\omega) \geq 0 \text{ for all } \omega \in \Omega.$$

Here the maximum is taken with respect to all self-financing trading strategies $\mathbf{H} = (\mathbf{h}_0, \mathbf{h}_1, ..., \mathbf{h}_T)$ satisfying constraints (4). We regard $\mathbf{H}$ as a vector whose coordinates are

$$h_t^i(\omega),\ i = 0, 1, ..., N,\ t = 0, 1, 2, ..., T,\ \omega \in \Omega.$$

For each $\omega \in \Omega$, the net present value $V^{\mathbf{H}}(\omega)$ is a linear function of $\mathbf{H}$, and so the constraints in (4) are linear. (The number of the constraints is equal to the number of points in Ω). Also, $EV^{\mathbf{H}}$ is a linear function of $\mathbf{H}$, and so the maximization problem (3), (4) is a linear programming problem.

Observe that the maximum value in the problem

$$\max_{\mathbf{H}} EV^{\mathbf{H}},$$

$$V^{\mathbf{H}}(\omega) \geq 0 \text{ for all } \omega \in \Omega,$$

is zero (it is attained at $\mathbf{H} = 0$). Suppose the contrary. Then there exists $\mathbf{H}$ such that $V^{\mathbf{H}}(\omega) \geq 0$ for all $\omega \in \Omega$ and $EV^{\mathbf{H}} > 0$. But then $V^{\mathbf{H}}(\omega) > 0$ for at least one ω. This contradicts (NA1). $\square$

Definition 1.1. The market under consideration is called *complete* if every contingent claim $X(\omega)$ can be hedged.

Theorem 1.3. The following assertions are equivalent:

(i) *For every* $t = 0, 1, ..., T-1$ *and for every contingent portfolio* $h_t = h_t(\omega^t)$ *of risky assets* $i = 1, 2, ..., N$, *we have*

$$E^Q \langle h_t, s_{t+1} - s_t \rangle = 0.$$

(ii) *The probability measure* Q *is risk-neutral.*

Condition above means that the expectation E^Q of the discounted profit $\langle h_t, s_{t+1} - s_t \rangle$ is equal to zero for each contingent portfolio h_t. In the applications of the risk-neutral pricing principle, we will construct risk neutral probabilities Q by verifying condition (12). This is much easier than dealing with the net present value directly. We do not provide a proof of Theorem 1.3 here; it will follow from the results in Section 7 (Theorem 7.1).

2 Superhedging of (non-hedgeable) contingent claims in a frictionless market

2.1 Superhedging

Consider the model of a frictionless financial market described in the previous section. We have presented a theory for the pricing of hedgeable contingent claims in the framework of this model. How to deal with non-hedgeable contingent claims? To this end we will use the following definition. Let $X(\omega)$ be a contingent claim. Let us say that a trading strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ *superhedges* (*superreplicates*) X if

$$\langle \mathbf{S}_T,\ \mathbf{h}_T\rangle \geq\ X(\omega) \tag{7}$$

for all ω. According to the general pricing principle used in this work, it is natural to define the fair price $l(X)$ of the contingent claim X as the minimum level of initial wealth v such that there exists a self-financing trading strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ superhedging X and satisfying

$$v \geq \langle \mathbf{S}_0, \mathbf{h}_0\rangle\,.$$

Equivalently, we can write

$$l(X) = \min_{\mathbf{H}} \langle \mathbf{S}_0, \mathbf{h}_0\rangle\,, \tag{8}$$

where the minimum is taken over all self-financing strategies $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ satisfying (7). Here, $\langle \mathbf{S}_0, \mathbf{h}_0\rangle$ (which we also denoted by $V_0^{\mathbf{H}}$) is regarded as a function of trading strategy $\mathbf{H}$.

Note that the minimum in (8) is, indeed, attained (and finite) if the no-arbitrage hypothesis (NA) holds. To prove this, observe that in (8), we deal with a linear programming problem [3]. It is known that the minimum in such problems is attained if and only if the objective function is bounded below. To provide a lower bound on the objective function consider a risk-neutral probability Q, which exists by virtue of the FTAP. Then we have

$$\langle \mathbf{S}_0,\ \mathbf{h}_0\rangle = E^Q \frac{\langle \mathbf{S}_T,\ \mathbf{h}_T\rangle}{(1+r)^T}$$

for any self-financing strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$, and if $\mathbf{H}$ superreplicates X, then (7) yields

$$\langle \mathbf{S}_0, \mathbf{h}_0\rangle\ \geq\ E^Q \frac{\langle \mathbf{S}_T, \mathbf{h}_T\rangle}{(1+r)^T}\ \geq\ E^Q \frac{X}{(1+r)^T}. \tag{9}$$

[3]This problem involves the inequality constraints (7) and the equality constraints arising from the self-financing condition $\langle \mathbf{S}_t, \mathbf{h}_{t-1}\rangle = \langle \mathbf{S}_t, \mathbf{h}_t\rangle$.

Consequently, there is a lower bound for the initial values of all self-financing strategies superreplicating X.

The above definition can be applied to each contingent claim in our model because, for each contingent claim X, there is a self-financing strategy supperreplicating X. Indeed, since Ω is finite, we can consider the maximum value of X,

$$M = \max_{\omega \in \Omega} X(\omega),$$

and consider the strategy $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ for which

$$\mathbf{h}_t = (M, 0, 0, ..., 0). \text{ for all } t = 0, ..., T.$$

This is a buy-and-hold strategy (the portfolio does not change in time), and so it is of course self-financing. Furthermore,

$$\langle \mathbf{S}_T,\ \mathbf{h}_T \rangle = (1+r)^T \cdot M \geq X(\omega)$$

for all ω, which means that $\mathbf{H}$ superreplicates X.

Assume that the no arbitrage hypothesis (NA) holds and suppose that the contingent claim X is hedgeable. Then the new definition of $l(X)$, given in this section, coincides with the old one, given in the previous section. This follows from the proposition below.

Proposition 2.1. Let $\mathbf{H} = (\mathbf{h}_0, ..., \mathbf{h}_T)$ be a self-financing strategy such that $\langle \mathbf{S}_T,\ \mathbf{h}_T \rangle = X(\omega)$ for all ω. Let $\mathbf{H}' = (\mathbf{h}'_0, ..., \mathbf{h}'_T)$ be a self-financing strategy such that $\langle \mathbf{S}_T,\ \mathbf{h}'_T \rangle \geq X(\omega)$ for all ω. Then

$$\langle \mathbf{S}_0,\ \mathbf{h}_0 \rangle \leq \langle \mathbf{S}_0,\ \mathbf{h}'_0 \rangle$$

Proof. Suppose $\langle \mathbf{S}_0,\ \mathbf{h}_0 \rangle > \langle \mathbf{S}_0,\ \mathbf{h}'_0 \rangle$. Then $\langle \mathbf{S}_T,\ \mathbf{h}'_T \rangle > X(\omega)$ for at least one ω. Define $\bar{\mathbf{H}} = \mathbf{H}' - \mathbf{H}$. The strategy $\mathbf{H}$ is self-financing strategy and we have

$$V_0^{\bar{\mathbf{H}}} < 0,\ V_0^{\bar{\mathbf{H}}} \geq 0 \text{ for all } \omega,$$

$$V_0^{\bar{H}} \geq 0 \text{ for some } \omega.$$

Consequently and there exits an arbitrage opportunity. This contradicts (NA).

We can see from Proposition 2.1 that the minimum in the optimization problem (8) is attained at a self-financing strategy $\mathbf{H}$ such that inequality (7) holds as equality. But if so, the initial value of such a strategy is defined uniquely (under the no arbitrage hypothesis (NA)), and it coincides with the price $l(X)$ defined in the previous section.

2.2 Hedging constraints

Throughout this subsection, hypothesis (NA) is supposed to hold. How can we compute the minimum level of wealth needed to superhedge contingent claim X ? First of all, observe that inequality (9) implies

$$l(X) = \min_{\mathbf{H}} \langle \mathbf{S}_0,\ \mathbf{h}_0 \rangle \geq E^Q \frac{X}{(1+r)^T}$$

($\mathbf{H}$ runs through the set of all self-financing strategies superreplicating X). It turns out that the following formula holds

$$l(X) = \sup_{Q \in \mathcal{Q}} E^Q \frac{X}{(1+r)^T}, \tag{10}$$

where Q ranges over the set $\mathcal{Q}$ of all risk-neutral probabilities. This formula provides a method for computing the fair price of any, not necessarily hedgeable, contingent claim. Recall that the notion of "fairness" is defined in terms of the minimum level of wealth needed to superreplicate X.

To establish formula (10), we will include the dynamic security market model into the abstract mathematical framework described in the Appendix and use Theorem A.5 to derive (10). To this end define

$$\mathcal{A} = \left\{ (v_0, v_{T)} : v_0 \geq V_0^{\mathbf{H}},\ v_T \leq V_T^{\mathbf{H}} \text{ for some self-financing strategy } \mathbf{H} \right\},$$

Here, v_0 is a number, and v_T is a function of ω: $v_T = v_T(\omega)$. The inequality $v_T \leq V_T^{\mathbf{H}}$ is supposed to hold for each $\omega \in \Omega$. The meaning of the set $\mathcal{A}$ is as follows: a pair (v_0, v_T) belongs to $\mathcal{A}$ if and only if we can superreplicate the contingent claim v_T starting from the initial endowment v_0. Thus the set $\mathcal{A}$ specifies the *hedging constraints* in the market under consideration: it determines the totality of those contingent claims which can be superreplicated starting from any given initial endowment.

Further, we define

$$\mathcal{K} = \{(v_0, v_T) : v_0 \leq 0,\ v_T \geq 0\},$$

where v_0 is a number and $v_T = v_T(\omega)$ is a function of $\omega \in \Omega$. As above, the inequality $v_T(\omega) \geq 0$ is supposed to hold for each ω.

It is easy to verify that $\mathcal{A}$ and $\mathcal{K}$ are cones. Let us show this first for $\mathcal{A}$. Take any

$$(v_0, v_T) \in \mathcal{A}, (v_0', v_T') \in \mathcal{A} \text{ and } \alpha, \beta \in R_+^1.$$

Consider self-financing strategies $\mathbf{H}$ and $\mathbf{H}'$ such that $v_0 \geq V_0^{\mathbf{H}}$, $v_T \leq V_T^{\mathbf{H}}$, $v_0' \geq V_0^{\mathbf{H}'}$, $v_T' \leq V_T^{\mathbf{H}'}$, and put $\bar{\mathbf{H}} = \alpha\mathbf{H} + \beta\mathbf{H}'$. clearly $\bar{\mathbf{H}}$ is a self-financing strategy and

$$\alpha v_0 + \beta v_0' \geq \alpha V_0^{\mathbf{H}} + \beta V_0^{\mathbf{H}'} = V_0^{\bar{\mathbf{H}}},$$

$$\alpha v_T + \beta v'_T \le \alpha V_T^{\mathbf{H}} + \beta V_T^{\mathbf{H}'} = V_T^{\bar{\mathbf{H}}}.$$

The last two inequalities mean that

$$(\alpha v_0 + \beta v'_0,\ \alpha v_T + \beta v'_T) \in \mathcal{A}.$$

Consequently, we have proved that $\mathcal{A}$ is a cone.

Note that the cone $\mathcal{A}$ is polyhedral, and hence closed. This is so because we can represent $\mathcal{A}$ as the projection on the space of pairs (v_0, v_T) of the polyhedral cone

$$\widetilde{\mathcal{A}} = \{(v_0, v_T, \mathbf{H}) : v_0 \ge V_0^{\mathbf{H}},\ v_T \le V_T^{\mathbf{H}}\}$$

in the space of triples $(v_0, v_T, \mathbf{H})$, where v_0 is a number, $v_T(\omega)$ is a function of ω and $\mathbf{H}$ is a self-financing strategy. The set $\widetilde{\mathcal{A}}$ is indeed a cone because if $(v_0, v_T, \mathbf{H}) \in \widetilde{\mathcal{A}}$ and $(v'_0, v'_T, \mathbf{H}') \in \widetilde{\mathcal{A}}$, then

$$\alpha(v_0, v_T, \mathbf{H}) + \beta(v'_0, v'_T, \mathbf{H}') \in \widetilde{\mathcal{A}}$$

for all $\alpha,\ \beta \ge 0$. The cone $\tilde{\mathcal{A}}$ is polyhedral because it is determined by a finite set of linear equalities and inequalities.

Now let us prove that $\mathcal{K}$ is a cone. Consider any

$$(v_0, v_T) \in \mathcal{K}, (v'_0, v'_T) \in \mathcal{K} \text{ and } \alpha, \beta \in R^1_+.$$

Then

$$v_0, v'_0 \le 0,\ v_T, v'_T \ge 0$$

and so

$$\alpha v_0 + \beta v'_0 \le 0,\ \alpha v_T + \beta v'_T \ge 0.$$

Thus $(\alpha v_0 + \beta v'_0,\ \alpha v_T + \beta v'_T) \in \mathcal{K}$, which proves that $\mathcal{K}$ is a cone. It is clear that $\mathcal{K}$ is polyhedral since it is determined by a finite family of linear (non-strict) inequalities.

2.3 An equivalent version of the no arbitrage hypothesis

Proposition 2.2. *The market does not have arbitrage opportunities (i.e. hypothesis (NA) holds) if and only if the following assertion is valid:*

(NA) $\mathcal{A} \cap \mathcal{K} = \{0\}$.

Proof. (NA)$\Longrightarrow$**(NA)**. Suppose **(NA)** does not hold. Let us show that (NA) fails. If **(NA)** does not hold, then there is (v_0, v_T) such that

$$(v_0, v_T) \ne 0,\ (v_0, v_T) \in \mathcal{A},\ v_0 \le 0,\ v_T \ge 0.$$

Since $(v_0, v_T) \neq 0$, then either $v_0 \neq 0$ or $v_T \neq 0$.

Consider first the case where $v_T \neq 0$. Then $v_T \geq 0$, $v_T(\omega) > 0$ for at least one ω. By the definition of $\mathcal{A}$, there is a self-financing $\mathbf{H}$ such that

$$0 \geq v_0 \geq V_0^{\mathbf{H}},$$

$$V_T^{\mathbf{H}} \geq v_T \geq 0 \text{ for all } \omega$$

and

$$V_T^{\mathbf{H}} \geq v_T > 0 \text{ for some } \omega.$$

But this is exactly an arbitrage opportunity in the sense of (NA).

Now consider the case where $v_0 \neq 0$. Then $v_0 < 0$. Define the following strategy

$$\mathbf{H}' = (\mathbf{h}'_0, ..., \mathbf{h}'_T), \ \mathbf{h}_t = (-\frac{v_0}{2}, 0, 0, ..., 0), \ t = 0, ..., T.$$

Clearly, $\mathbf{H}'$ is self-financing, and so the strategy $\bar{\mathbf{H}} = \mathbf{H} + \mathbf{H}'$ is self-financing too. We have

$$V_0^{\bar{\mathbf{H}}} = V_0^{\mathbf{H}} + V_0^{\mathbf{H}'} \leq v_0 - \frac{v_0}{2} = \frac{v_0}{2} < 0,$$

and

$$V_0^{\bar{\mathbf{H}}} = V_0^{\mathbf{H}} + V_0^{\mathbf{H}'} \geq v_T - (\frac{v_0}{2})(1+r)^T > 0 \text{ for each } \omega.$$

Consequently, $\bar{\mathbf{H}}$ is an arbitrage opportunity.

Thus we have proved that (NA)$\Longrightarrow$ **(NA)**. Now we will prove the converse implication.

(NA) $\Longrightarrow$(NA). Suppose (NA) does not hold. Then there is a self-financing strategy $\mathbf{H}$ such that

$$V_0^{\mathbf{H}} \leq 0, \ V_T^{\mathbf{H}} \geq 0 \text{ for all } \omega, \ V_T^{\mathbf{H}} > 0 \text{ for some } \omega.$$

Define $v_0 = V_0^{\mathbf{H}}$ and $v_T = V_T^{\mathbf{H}}$. Then $(v_0, v_T) \neq 0$, which means that $\mathcal{A} \cap \mathcal{K} \neq \{0\}$. Consequently, **(NA)** fails to hold.

The proof is complete. □

2.4 No arbitrage criteria and consistent discount factors

We will apply Theorem A.3 to the cones $\mathcal{A}$ and $\mathcal{K}$ defined in the previous subsection, and this will lead to a result about necessary and sufficient conditions for the validity of the no-arbitrage hypothesis **(NA)**. This result (Theorem 2.1below) will provide a no arbitrage criterion in terms of "stochastic discount factors". It will yield a new proof of the FTAP (different from that already given in Section 1).

According to Theorem A.3, **(NA)** holds if and only if there exist a linear functional $q = q(v)$ $[v = (v_0, v_T)]$ such that

$$q \in \mathcal{K}^+ \tag{11}$$

and

$$q(v) \leq 0 \text{ for all } v \in \mathcal{A}. \tag{12}$$

Observe that any linear functional $q(v)$ can be written as

$$q(v) = -q_0 v_0 + E\ q_T v_T \quad [v = (v_0, v_T)]\,, \tag{13}$$

where q_0 is some number, $q_T(\omega)$ is some real-valued function of ω, and

$$E\ q_T v_T = E\ q_T(\omega) v_T(\omega) = \sum_{\omega \in \Omega} q_T(\omega) v_T(\omega) P(\omega).$$

Indeed, we can regard a pair $v = (v_0, v_T)$ as a vector of dimension $1 + L^T$, where L^T is the number of points in Ω (the number of market scenarios). For each ω, the value of the functional $v_T(\omega)$ at the point ω is regarded as a coordinate of vector v. A linear function $q(v)$ of such a vector can be represented as a scalar product of the form

$$q(v) = l_0 v_0 + \sum_{\omega \in \Omega} l_T(\omega) v_T(\omega) \tag{14}$$

for some vector $l = (l_0, l_T)$ of the same dimension as $v = (v_0, v_T)$. Now if we define

$$q_0 = -l_0,\ q_T(\omega) = \frac{l_T(\omega)}{P(\omega)},$$

we can write (14) as

$$q(v) = -q_0 v_0 + \sum_{\omega \in \Omega} q_T(\omega) v_T(\omega) \cdot P(\omega),$$

which leads to formula (13).

Theorem 2.1. *The no arbitrage hypothesis* **(NA)** *holds if and only if the following assertion is valid*:

($\mathcal{Q}$) *There exists a strictly positive number q_0 and a strictly positive real-valued function $q_T(\omega)$such that*

$$E\ q_T V_T^{\mathbf{H}} = q_0 V_0^{\mathbf{H}} \tag{15}$$

for all self-financing strategies **H**.

The ratio q_T/q_0 may be interpreted as a *stochastic discount factor* with the following property: the expected discounted value of the final portfolio of any self-financing strategy $\mathbf{H}$ is equal to the value of the initial portfolio of this strategy. If this property holds, the discount factor q_T/q_0 is called *consistent.*

Proof. We have already noted that, by virtue of Theorem A.3, **(NA)** holds if and only if relations (11) and (12) are valid for some $q = (q_0, q_T)$, where q_0 is a number and $q_T(\omega)$ is a function of ω.

Observe that $q \in \mathcal{K}^*$ if and only if

$$q_0 \geq 0,\ q_T(\omega) \geq 0 \text{ for all } \omega;$$

and $q \in \mathcal{K}^+$ if and only if

$$q_0 > 0,\ q_T(\omega) > 0 \text{ for all } \omega.$$

Consequently, (11) means that q_0 and q_T are strictly positive.

Let us examine when inequality (12) holds. Recall that $v = (v_0, v_T) \in \mathcal{A}$ if and only if

$$v_0 \geq V_0^{\mathbf{H}} \text{ and } v_T \leq V_T^{\mathbf{H}}$$

for some self-financing strategy $\mathbf{H}$. It follows from (12)and (13) that

$$-q_0 V_0^{\mathbf{H}} + E\, q_T V_T^{\mathbf{H}} \leq 0 \tag{16}$$

for any self-financing $\mathbf{H}$. Conversely (16) implies (12), and so (12) is equivalent to (16).

Since $-\mathbf{H}$ is also a self-financing strategy, we have

$$-q_0 V_0^{(-\mathbf{H})} + E\, q_T V_T^{(-\mathbf{H})} \leq 0,$$

but

$$V_T^{(-\mathbf{H})} = -V_T^{\mathbf{H}} \text{ and } V_0^{(-\mathbf{H})} = -V_0^{\mathbf{H}},$$

and so

$$q_0 V_0^{\mathbf{H}} - E\, q_T V_T^{\mathbf{H}} \leq 0. \tag{17}$$

by combining (16) and (17), we obtain that (12) is equivalent to (15), which completes the proof. □

Now let us apply Theorem 2.1 to obtain a new proof of the FTAP. Consider the self-financing strategy $\bar{\mathbf{H}} = (\bar{\mathbf{h}}_0, ..., \bar{\mathbf{h}}_T)$, where $\bar{\mathbf{h}}_t = (1, 0, 0, ..., 0)$. For the strategy $\bar{\mathbf{H}}$, we have

$$V_0^{\bar{\mathbf{H}}} = 1,\ V_T^{\bar{\mathbf{H}}} = (1 + r)^T.$$

Then (15) yields

$$E\,\frac{q_T}{q_0}(1+r)^T = 1. \tag{18}$$

Now we can write (15) as

$$E\left[(1+r)^T\right]\cdot\frac{V_T^{\mathbf{H}}}{(1+r)^T} = V_0^{\mathbf{H}}. \tag{19}$$

Define

$$q(\omega) = \frac{q_T(\omega)}{q_0}(1+r)^T \tag{20}$$

and consider the probability measure

$$Q(\omega) = q(\omega)\cdot P(\omega),\ \omega \in \Omega \tag{21}$$

The last formula indeed defines a probability measure because $q(\omega) > 0$ (and so $Q(\omega) > 0$) and

$$\sum_{\omega\in\Omega} Q(\omega) = \sum_{\omega\in\Omega} q(\omega)P(\omega) = E\ q(\omega) = 1$$

by virtue of (18) and (20).

For any function $Y(\omega)$,

$$E^Q Y = \sum_{\omega\in\Omega} Q(\omega)Y(\omega) = \sum_{\omega\in\Omega} q(\omega)P(\omega)Y(\omega) = E\ qY.$$

consequently, (19) can be written as

$$E^Q\frac{V_T^{\mathbf{H}}}{(1+r)^T} = V_0^{\mathbf{H}}.$$

Thus we obtain that the probability measure Q defined by (19) and (20) is risk-neutral. The above arguments lead to another proof of the FTAP.

2.5 Minimum wealth needed for superhedging

We are going to prove the following theorem.

Theorem 2.2. *The minimum level of wealth, $l(X)$, needed to superhedge a contingent claim X is equal to the supremum*

$$\sup_Q E^Q\frac{X}{(1+r)^T}$$

of the discounted expected values of X with respect to all risk-neutral measures.

Proof. To prove this assertion, we will apply Theorem A.5 to the cones $\mathcal{A}$ and $\mathcal{K}$ defined in subsection 2.2. first of all, we observe that condition **(NA)** holds (this has been proved in Proposition 2.2).

Denote by $\mathcal{Q}$ the set of those pairs $q = (q_0, q_T)$ which satisfy (11) and (12). Recall that, with every pair of the form $q = (q_0,\ q_T)$, we associate the linear functional

$$q(v) = -q_0 v_0 + E q_T v_T$$

of $v = (v_0, v_T)$. The property indicated in (11) means that

$$q_0 > 0 \text{ and } q_T(\omega) > 0 \text{ for each } \omega. \tag{22}$$

As we proved in Proposition 2.3, (12) holds if and only if

$$E\,\frac{q_T}{q_0} V_T^{\mathbf{H}} = V_0^{\mathbf{H}} \tag{23}$$

for each self-financing $\mathbf{H}$. By virtue of Theorem A.5, $v = (v_0, v_T) \in \mathcal{A} - \mathcal{K}$ if and only if

$$q(v) = -q_0 v_0 + E\ q_T v_T \leq 0 \tag{24}$$

for all $q = (q_0, q_T)$ satisfying (22) and (23). It follows from the definitions of the cones $\mathcal{A}$and $\mathcal{K}$ that $\mathcal{A} - \mathcal{K} = \mathcal{A}$. Consequently $v = (v_0, v_T) \in \mathcal{A}$ if and only if (24) holds for all (q_0, q_T) satisfies (22) and (23).

Let us write (24) in the following form

$$E\,\frac{q_T}{q_0} v_T \leq v_0. \tag{25}$$

It was demonstrated in the previous subsection (see (18) - (21)) that q_0 and q_T satisfy (23) if and only if the probability

$$Q(\omega) = \frac{q_T(\omega)}{q_0}(1+r)^T \cdot P(\omega) \tag{26}$$

is risk-neutral. We can write (25) as

$$E^Q \frac{v_T}{(1+r)^T} \leq v_0, \tag{27}$$

where Q is given by (26). Consequently, $(v_0,\ v_T) \in \mathcal{A}$ if and only if (27) holds for all risk-neutral probabilities $Q(\omega) > 0$. In other words, $(v_0,\ v_T) \in \mathcal{A}$ if and only if

$$v_0 \geq \sup_{Q \in \mathcal{Q}} E^Q \frac{v_T}{(1+r)^T},$$

where the supremum is taken over the set $\mathcal{Q}$ of all risk-neutral probabilities.

The proof is complete. $\square$

3 Dynamic securities model with transaction costs and trading constraints

3.1 The data of the model

We describe a model of a financial market influenced by random factors. Let $A = \{a^1, ..., a^L\}$ be a finite set, elements of which are interpreted as possible states of the world. At each time $t = 1, 2, ..., T$, any of these states can be realized. The state of the world which is realized at time t is denoted by a_t. A sequence $\omega = (a_1, ..., a_T)$ is called a *history* (*scenario*) of the market over the time period $1, 2, ..., T$. For each $t = 1, 2, ..., T-1$, the sequence $\omega^t = (a_1, ..., a_t)$ is called the *partial history* or *partial scenario* (up to time t). For every $\omega = (a_1, ..., a_T) \in \Omega$, we are given a number $P(\omega) > 0$ such that $\sum_{\omega \in \Omega} P(\omega) = 1$. The numbers $P(\omega)$, $\omega \in \Omega$, define a probability measure on the set Ω of all market histories: $P(\omega)$ is the probability of the realization of the scenario ω. The situation at the market at time t might depend on the realization $\omega^t = (a_1, ..., a_t)$ of the random states of the world at present (at time t) and in the past (at times $1, 2, ..., t-1$).

Trading at the securities market is possible at any of the dates $t = 0, 1, ..., T$. At time t, N_t *assets* (*securities*) $i = 1, 2, ..., N_t$ are traded. A *portfolio* of assets at time 0 is a vector $h_0 \in R^{N_0}$. A *(contingent) portfolio* of assets at time $t = 1, 2, ..., T$ is a vector function

$$h_t(\omega^t) = (h_t^1(\omega^t), ..., h_t^{N_t}(\omega^t)) \tag{28}$$

of dimension N_t depending on the observed market history $\omega^t = (a_1, ..., a_t)$ up to time t. The coordinate h_t^i of the vector h_t stands for the number of units of asset i in the portfolio h_t. The set of all contingent portfolios (28), i.e., the linear space of vector functions $h_t(\omega^t)$ with values in R^{N_t}, will be denoted by $\mathcal{X}_t$ $(t = 1, 2, ..., T)$. For $t = 0$, we will write $\mathcal{X}_0 = R^{N_0}$.

Two linear spaces $\mathcal{V}_0$ and $\mathcal{V}_T$ are given – the spaces of *initial endowments* and *contingent claims*. Elements of $\mathcal{V}_0$ are m_0-dimensional non-random vectors (interpreted as investor's initial endowments). Elements of $\mathcal{V}_T$ are all m_T-dimensional vector functions of ω (interpreted as contingent claims). Generally, both initial endowments and contingent claims can be vectors, rather than scalars, which is the case when there are several currencies in the market under consideration. An important special case is $m_0 = m_T = 1$; in this case, initial endowments and contingent claims are measured in terms of a single currency (cash).

In the model, a sequence of cones

$$Z_t \subseteq \mathcal{X}_{t-1} \times \mathcal{X}_t, \ t = 1, 2, ..., T, \tag{29}$$

describing *trading constraints* is given. Elements of Z_t are pairs (h_{t-1}, h_t) of contingent portfolios such that h_t can be obtained at time t by *rebalancing* h_{t-1} without an inflow of external funds. When rebalancing, one can buy new assets for the portfolio h_t only at the expense of selling some assets contained in h_{t-1}. Generally, the operations of buying and selling assets involve transaction costs. If there are no transaction costs and no constraints on admissible portfolios h_t at each time t, then

$$Z_t = \{(h_{t-1}, h_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \langle S_t, h_{t-1} \rangle \le \langle S_t, h_t \rangle\}, \tag{30}$$

where $S_t = S_t(\omega^t)$ is the vector of the market prices of the assets.

The model at hand allows to take into account *proportional* transaction costs. The assumption of proportionality of transaction costs is reflected by the assumption that each of the sets Z_t is a cone, and so if (h_{t-1}, h_t) is a pair of contingent portfolios in Z_t, then $(\lambda h_{t-1}, \lambda h_t) \in Z_t$ for all $\lambda \ge 0$. Furthermore, if portfolios h_t and h'_t can be obtained by rebalancing h_{t-1} and h'_{t-1}, respectively, then $h_t + h'_t$ can be obtained by rebalancing $h_{t-1} + h'_{t-1}$. Various models of proportional transaction costs considered in the literature lead to constraint sets Z_t possessing these properties.

Further, in the model we are given two cones

$$W_0 \subseteq \mathcal{V}_0 \times \mathcal{X}_0 \text{ and } W_T \subseteq \mathcal{X}_T \times \mathcal{V}_T.$$

The cone W_0 describes possibilities of constructing an initial portfolio h_0 starting from some initial endowment $v_0 \in \mathcal{V}_0$. It is supposed that an investor with initial endowment v_0 can construct a portfolio h_0 at time 0 if and only if $(v_0, h_0) \in W_0$. In the case of a frictionless market with $m_0 = 1$,

$$W_0 = \{(v_0, h_0) \in \mathcal{V}_0 \times \mathcal{X}_0 : S_0 h_0 \le v_0\}, \tag{31}$$

where S_0 is the vector of asset prices at time 0. Clearly (31) means that an investor can buy those and only those portfolios of assets at time 0 whose values, expressed in terms of the price vector S_0, do not exceed the initial endowment v_0. The cone W_T describes possibilities of portfolio liquidation and superhedging contingent claims. Given a contingent claim v_T, an investor with contingent portfolio h_T at time T can superhedge v_T by liquidating the portfolio h_T if and only if $(v_T, h_T) \in W_T$. If the market is frictionless and $m_T = 1$, then

$$W_T = \{(v_T, h_T) : S_T h_T \le v_T\}. \tag{32}$$

Finally, we assume that proper cones $\mathcal{M}_0 \subseteq \mathcal{V}_0$ and $\mathcal{M}_T \subseteq \mathcal{V}_T$ are given. These cones define partial orderings $\le_t$ ($t = 0, T$) in the spaces $\mathcal{V}_0$ and $\mathcal{V}_T$ according to the formula

$$v \le_t v' \Longleftrightarrow v' - v \in \mathcal{M}_t$$

($t = 0, T$). We write $v' \ge_t v$ if and only if $v \le_t v'$. Important examples of the cones $\mathcal{M}_t$ are the standard cones in the spaces $\mathcal{V}_t$ ($t = 0, T$):

$$\mathcal{M}_0 = R_+^{m_0},\ \mathcal{M}_T = \{v_T \in \mathcal{V}_T : v_T(\omega) \ge 0\}.$$

We write $\geq$ (or $\leq$) between two vector functions of ω if the corresponding inequality holds for each ω and coordinatewise. For most of examples we have in mind, the consideration of the standard partial orderings in the spaces of initial endowments and contingent claims is sufficient. There are some other settings, however, where it is convenient to deal with more general, not necessarily standard, partial orderings in $\mathcal{V}_0$ and $\mathcal{V}_T$ (see Kabanov [23] and Kabanov and Stricker [25]).

3.2 The problem of superhedging contingent claims

The general question we are going to consider is as follows. Suppose a contingent claim $v_T \in \mathcal{V}_T$ is given. What is the set of initial endowments $v_0 \in \mathcal{V}_0$ starting from which an investor, trading in the financial market, can superhedge v_T at time T? It is supposed that at time 0, the investor can construct a portfolio h_0 satisfying $(v_0, h_0) \in W_0$, then follow some feasible trading strategy

$$H = (h_0, h_1, ..., h_T), \tag{33}$$

satisfying the constraints

$$(h_{t-1}, h_t) \in Z_t,\ t = 1, ..., T, \tag{34}$$

and finally liquidate the portfolio h_T to superhedge the contingent claim v_T. The last step (liquidating h_T and superhedging v_T) can be implemented if and only if $(h_T, v_T) \in W_T$.

A *feasible trading strategy* $H = (h_0, h_1, ..., h_T)$ is defined as a sequence of contingent portfolios $h_t \in \mathcal{X}_t$ $(t = 0, ...T)$ satisfying the trading constraints (34). In the case of a frictionless market, where the cones Z_t are defined by (30), those and only those strategies are feasible which are self-financing. In the general case, the trading constraints (34) express the same idea of self-financing, but, in contrast with (30), they may take into account transaction costs and portfolio constraints. A number of examples will be analyzed in detail in the sequel.

3.3 Hedging constraints and no arbitrage

The general problem outlined in the previous subsection is concerned with the analysis of pairs $(v_0, v_T) \in \mathcal{V}_0 \times \mathcal{V}_T$ such that the contingent claim v_T can be superhedged starting from the initial endowment v_0. Such pairs (v_0, v_T) are elements of the set

$$\mathcal{A} = \{(v_0, v_T) : (v_0, h_0) \in W_0 \text{ and}$$

$$(h_T, v_T) \in W_T \text{ for some feasible strategy } H = (h_0, ..., h_T)\}. \tag{35}$$

In other words, $(v_0, v_T) \in \mathcal{A}$ if and only if there is a feasible strategy $H = (h_0, ..., h_T)$ for which $(v_0, h_0) \in W_0$ and $(h_T, v_T) \in W_T$. Thus, $\mathcal{A}$ provides a description of the *hedging constraints* in our model.

Define

$$\mathcal{K} = \{(v_0, v_T) \in \mathcal{V}_0 \times \mathcal{V}_T : v_0 \leq_0 0 \text{ and } v_T \geq_T 0\}. \tag{36}$$

We will examine the above-mentioned general hedging problem under the following fundamental hypothesis.

($\mathcal{NA}$) $\mathcal{A} \cap \mathcal{K} = \{0\}$.

This hypothesis expresses the idea of the *absence of arbitrage opportunities* in the market under consideration. Suppose for the moment that the partial orderings $\leq_t$ $(t = 0, T)$ are standard (in this case we drop the subscript "t'). Then the fact that ($\mathcal{NA}$) fails to hold means the existence of a pair $(v_0, v_T) \in \mathcal{A}$ such that

$$v_0 \leq 0, \;\; v_T \geq 0$$

and $(v_0, v_T) \neq 0$. Then either $v_0 \neq 0$ or $v_T \neq 0$. In the former case, we can superhedge a nonnegative contingent claim starting from an initial endowment vector that is strictly negative in at least one component and non-positive in all its components. In the latter case, starting from a non-positive initial endowment, we can superhedge a contingent claim that is always non-negative and strictly positive in at least one component for at least one ω. Recall that, in the current model, initial endowments and contingent claims are, generally, vectors, which reflects the possibility of several currencies available in the market.

Central results of this work we will be obtained under the following assumption:

(I) The cones $\mathcal{A}$ and $\mathcal{K}$ are closed.

We note that if the cones Z_t $(t = 1, ..., T)$ and W_t $(t = 0, T)$ are polyhedral, then $\mathcal{A}$ is polyhedral as well, because $\mathcal{A}$ can be represented as the projection on the space $\mathcal{V}_0 \times \mathcal{V}_T$ of the polyhedral cone

$$\widetilde{\mathcal{A}} = \{(v_0, v_T, H) : H = (h_0, ..., h_T) \text{ is feasible}, \; (v_0, h_0) \in W_0, \;\; (h_T, V_T) \in W_T\}.$$

Consequently, in this case, $\mathcal{A}$ is closed. The cone $\mathcal{K}$ can be represented as

$$\mathcal{K} = \{(v_0, v_T) \in \mathcal{V}_0 \times \mathcal{V}_T : v_0 \in -\mathcal{M}_0, \;\; v_T \in \mathcal{M}_T\} = (-\mathcal{M}_0) \times \mathcal{M}_T.$$

consequently, if $\mathcal{M}_0$ and $\mathcal{M}_T$ are closed (in particular, if they are polyhedral), then $\mathcal{K}$ is closed. Since we have assumed that $\mathcal{M}_0$ and $\mathcal{M}_T$ are proper, $\mathcal{K}$ is proper as well.

3.4 Criteria for no-arbitrage and hedging in terms of consistent discount factors

We will provide a necessary and sufficient condition for the validity of hypothesis ($\mathcal{NA}$) similar to that given in Theorem 2.1. Let us write $q_0 \in \mathcal{M}_0^+$ if q_0 is a vector in $\mathcal{V}_0$ such that

$q_0 v_0 > 0$ for each $v_0 \in \mathcal{M}_0$, $v_0 \neq 0$. The notation $q_T \in \mathcal{M}_T^+$ will mean that $E q_T v_T > 0$ for all $v_T \in \mathcal{M}_T$, $v_T \neq 0$. The notation for $\mathcal{M}_0^+$ and $\mathcal{M}_T^+$ corresponds to that introduced in the Appendix (Section A.4).

In this subsection, we will assume that hypothesis (I) holds.

Theorem 3.1. *The no arbitrage hypothesis ($\mathcal{NA}$) is valid if and only if the following assertion holds.*

($\mathcal{Q}$) *There exist $q_0 \in \mathcal{M}_0^+$ and $q_T \in \mathcal{M}_T^+$ such that*

$$E q_T v_T \leq q_0 v_0 \tag{37}$$

for all $(v_0, v_T) \in \mathcal{A}$.

The set of all pairs $(q_0, q_T) \in \mathcal{M}_0^+ \times \mathcal{M}_T^+$ satisfying (37) will be denoted by $\mathcal{Q}$.

We will call any pair $q_0 \in \mathcal{M}_0^*$, $q_T \in \mathcal{M}_T^*$ a pair of *discount factors* (on initial endowments and contingent claims) respectively. Recall that "*" refers to a dual cone, and so the inclusions $q_0 \in \mathcal{M}_0^*$ and $q_T \in \mathcal{M}_T^*$ mean that $q_0 v_0 \geq 0$ for all $v_0 \in \mathcal{M}_0$ and $E q_T v_T \geq 0$ for all $v_T \in \mathcal{M}_T$. Discount factors $q_0, q_T \in \mathcal{M}_0^* \times \mathcal{M}_T^*$ satisfying (37) will be called *consistent.* Condition (37) says that the expected discounted value $E q_T v_T$ is not greater than the discounted value $q_0 v_0$ for all contingent claims v_T that can be superhedged starting from the initial endowment v_0. In ($\mathcal{Q}$) the existence of strictly positive (i.e., satisfying the condition $(q_0, q_T) \in \mathcal{M}_0^+ \times \mathcal{M}_T^+$) consistent discount factors is claimed. Such discount factors will be called *strictly consistent.* Since v_0 and v_T are, generally, vectors in our model, q_0 and q_T are vectors as well. If components v_0^i, $i = 1, ..., m_0$, and v_T^j, $j = 1, ..., m_T$, of the initial endowment v_0 and the contingent claim v_T are measured in several currencies, then the respective components q_0^i, $i = 1, ..., m_0$ and q_T^j, $j = 1, ..., m_T$ of the vectors q_0 and q_T are discount factors for these currencies.

Proof of Theorem 3.1. We apply Theorem A.3 to the cones $\mathcal{A}$ and $\mathcal{K}$. Since $\mathcal{A}$ and $\mathcal{K}$ are closed (as was assumed) and since $\mathcal{K}$ is proper (which follows from the assumption that $\mathcal{M}_0$ and $\mathcal{M}_T$ are proper), all the requirements needed for the validity of Theorem A.3 are satisfied. Note that the cones $\mathcal{A}$ and $\mathcal{K}$ are contained in the linear space $\mathcal{V}_0 \times \mathcal{V}_T$. Linear functionals $q(v)$ on this space can be written as

$$q(v) = -q_0 v_0 + E q_T v_T \ \ [v = (v_0, v_T)], \tag{38}$$

where $q_0 \in R^{m_0}$ and $q_T(\omega)$ is a function of $\omega \in \Omega$ with values in R^{m_T}. The existence of a linear functional involved in assertion (**Q**) of Theorem A.3 (applied to the cones $\mathcal{A}$ and $\mathcal{K}$) is equivalent to the existence of $(q_0, q_T) \in \mathcal{M}_0^+ \times \mathcal{M}_T^+$ satisfying (37). Indeed, if $q(v)$ is given by (38), then the inequality $q(v) \leq 0$ (see (**Q**)) is equivalent to (37) and the inclusion $q \in \mathcal{K}^+$ is equivalent to $(q_0, q_T) \in \mathcal{M}_0^+ \times \mathcal{M}_T^+$. $\square$

We now will give an answer to the general question posed in subsection 3.2. Theorem 3.2 below provides a necessary and sufficient condition for a contingent claim v_T to be superhedgeable starting from an initial endowment v_0. This condition is formulated in terms of the set $\mathcal{Q}$ of pairs of strictly positive consistent discount factors (q_0, q_T) (see (37)). In addition to (I) we impose the following hypothesis.

(II) (a) If $(v_0, h_0) \in W_0$, $v_0' \in \mathcal{V}_0$ and $v_0 \leq_0 v_0'$, then $(v_0', h_0) \in W_0$. (b) If $(h_T, v_T) \in W_T$, $v_T' \in \mathcal{V}_T$ and $v_T' \leq_T v_T$, then $(h_T, v_T') \in W_T$.

This hypothesis relates the partial orderings $\geq_t$ and the cones W_T $(t = 0, T)$. According to assertion (a) in (II), if we can construct a portfolio h_0 starting from same initial endowment v_0, then we can construct h_0 starting from any $v_0' \geq_0 v_0$. By virtue of (b), if a contingent claim v_T can be superhedged by liquidating a portfolio h_T, then any $v_T' \leq_T v_T$ can also be superhedged by liquidating this portfolio.

Theorem 3.2 *Let hypothesis ($\mathcal{NA}$) hold. Then for any $(v_0, v_T) \in \mathcal{V}_0 \times \mathcal{V}_T$ the following conditions are equivalent.*

($\mathcal{H}$) $(v_0, v_T) \in \mathcal{A}$.

($\mathcal{D}$) *For all $(q_0, q_T) \in \mathcal{Q}$, we have $Eq_T v_T \leq q_0 v_0$.*

Condition ($\mathcal{H}$) means that v_T can be superhedged starting from v_0. Consequently, Theorem 3.2 yields the following *hedging criterion.* If we wish to check whether we can superhedge a contingent claim v_T starting from initial endowment v_0, we have to perform the following test. Take any strictly consistent discount factors $(q_0, q_T) \in \mathcal{Q}$. If the expected value $Eq_T v_T$ is not greater than the value $q_0 v_0$, then v_T can indeed be superhedged starting from v_0. If for some strictly consistent discount factors this is not the case, v_T cannot be superhedged starting from v_0.

Proof of Theorem 3.2 The result is a consequence of Theorem A.5 and the fact that $\mathcal{A} - \mathcal{K} = \mathcal{A}$. To prove this equality we observe that $\mathcal{A} \subseteq \mathcal{A} - \mathcal{K}$ because $0 \in \mathcal{K}$. Conversely, suppose $(v_0, v_T) \in \mathcal{A}$ and $(k_0, k_T) \in \mathcal{K}$. Let us show that $(v_0 - k_0, v_T - k_T) \in \mathcal{A}$. Since $(v_0, v_T) \in \mathcal{A}$, we have $(v_0, h_0) \in W_0$, $(h_T, v_T) \in W_T$ for some feasible strategy $H = (h_0, ..., h_T)$. Then $(v_0 - k_0, h_0) \in W_0$ and $(h_T, v_T - k_T) \in W_T$ by virtue of (II). Consequently, $(v_0 - k_0, v_T - k_T) \in \mathcal{A}$.

3.5 Bank accounts and consistent discount factors

Let us introduce the following condition.

($\mathcal{B}$) The dimensions m_0 and m_T coincide: $m_0 = m_T = m$. There exists a strictly positive vector

$$B_T(\omega) = (B_T^1(\omega), ..., B_T^m(\omega))$$

depending on $\omega \in \Omega$ such that, for any $v_0 = (v_0^1, ..., v_0^m) \in \mathcal{V}_0$, we have $(v_0, v_T) \in \mathcal{A}$, where

$$v_T = (v_T^1, ..., v_T^m) = (B_T^1(\omega)v_0^1, ..., B_T^m(\omega)v_0^m). \tag{39}$$

Proposition 3.1. *If condition ($\mathcal{B}$) holds, then for any pair q_0, q_T of consistent discount factors, we have*

$$q_0^j = EB_T^j q_T^j, \ j = 1, 2, ..., m, \tag{40}$$

where q_t^j is the jth coordinate of q_t^j $(t = 0, T)$.

Proof. Consider any $v_0 \in \mathcal{V}_0$, define v_T according to (39) and substitute $(v_0, v_T) \in \mathcal{A}$ into (37). Since $-v_0 \in \mathcal{V}_0$, we have $(-v_0, -v_T) \in \mathcal{A}$, and so (37) holds for $(-v_0, -v_T)$ as well. This yields

$$q_0 v_0 = E q_T v_T = E \sum_{j=1}^{m} B_T^j q_T^j v_0^j. \tag{41}$$

Since v_0 is an arbitrary element of the linear space $\mathcal{V}_0 = R^m$, (41) implies (40).

To explain the meaning of condition $(\mathcal{B})$ suppose that initial endowments and contingent claims are both measured in terms of m currencies which are traded assets. Specifically, assume that $N_t \geq m$ and the first m components of portfolios $h_t \in \mathcal{X}_t$ correspond to these m currencies. Further, suppose that any amount v_0^j of each currency can be deposited with a bank account at time 0, which will yield the amount $B_T^j(\omega)v_0^j$ at time T ($B_T^j - 1$ being the cumulative interest rate over the time period from 0 to T). The amount v_0^j might be both positive and negative, the latter case reflecting a possibility of borrowing currency j from the jth account with the same interest rate $B_T^j(\omega)$. Formally, the situation described corresponds to the assumption that, for each $v_0 \in \mathcal{V}_0$, the cone $\mathcal{A}$ contains $(v_0, v_T) \in \mathcal{A}$, where v_T is defined by (39). According to Proposition 3.1, the availability of such bank accounts leads to relations (40) between the discount factors q_0^j and q_T^j.

Assume $m = 1$, i.e., initial endowments and contingent claims are measured in terms of one currency (cash). Suppose that cash is a traded riskless asset with non-random interest rate r over each time period $t-1, t$. Then an amount v_0^1 of cash deposited with the bank account at time 0 will yield the amount $B_T^1 v_0^1$ at time T, where $B_T^1 = (1+r)^T$. In this case, formula (40) coincides with formula (18) established in the classical case of a frictionless market with a riskless asset.

4 Consistent price systems

4.1 Consistent price systems: the definition

Let q_0, q_T be discount factors, that is, $(q_0, q_T) \in \mathcal{M}_0^* \times \mathcal{M}_T^*$. Let p_0 be a vector in $R_+^{N_0}$ and $p_t(\omega^t)$, $t = 1, 2, ..., T$, vector functions of ω^t with values in $R_+^{N_t}$. We will denote by $\mathcal{P}_t$ the set of nonnegative elements in $\mathcal{X}_t$ (understood in the sense of the standard partial ordering in $\mathcal{X}_t$). Thus $p_t \in \mathcal{P}_t$, $t = 0, ..., T$. The ith coordinate $p_t^i(\omega^t)$ of the vector $p_t = p_t(\omega^t)$ will be interpreted as a price of asset $i = 1, 2, ..., N_t$ at time t. The prices at time t, depend on the market history ω^t up to time t. A sequence

$$(q_0, p_0, p_1, p_2, ..., p_T, q_T)$$

will be called a *consistent price system* if the following requirements are fulfilled:

$$q_0 v_0 \geq p_0 h_0 \text{ for each } (v_0, h_0) \in W_0; \tag{42}$$

$$E p_{t-1} h_{t-1} \geq E p_t h_t \text{ for each } (h_{t-1}, h_t) \in Z_t; \tag{43}$$

$$E p_T h_T \geq E q_T v_T \text{ for each } (h_T, v_T) \in W_T. \tag{44}$$

In (43), t ranges through $1, ..., T$. It follows from (42) - (44) that

$$q_0 v_0 \geq p_0 h_0 \geq E p_1 h_1 \geq ... \geq E p_{T-1} h_{T-1} \geq E p_T h_T \geq E q_T v_T \tag{45}$$

for all sequences $(v_0, h_0, ..., h_T, v_T)$ satisfying

$$(v_0, h_0) \in W_0, (h_0, h_1) \in Z_1, ..., (h_{T-1}, h_T) \in Z_T, \ (h_T, v_T) \in W_T. \tag{46}$$

According to (42), if a portfolio h_0 can be constructed based on the initial endowment w_0, then the value $q_0 v_0$ of the initial endowment v_0 is not less that the value $p_0 h_0$ of the portfolio h_0. It follows from (43) that, for each feasible trading strategy $(h_0, ..., h_T)$, the expected value $E p_t h_t$ of the portfolio h_t does not increase in time. Requirement (44) states that the expected value $E q_T v_T$ of a contingent claim v_T does not exceed the expected value $E q_T h_T$ of each portfolio h_T that allows to superhedge this contingent claim.

The portfolio values involved in the above definition are measured in terms of the price vectors p_t. If conditions (43) hold for each $t = 1, ..., T$, we say that $p_0 \in \mathcal{P}_0$, ..., $p_T \in \mathcal{P}_T$ are *consistent asset prices*. The initial endowment v_0 and the contingent claim v_T are evaluated by using the discount factors q_0 and q_T, respectively. If there are several currencies $i = 1, 2, ..., m_0$ and $j = 1, ..., m_T$ in terms of which the coordinates v_0^i and v_T^j of the vectors v_0 and v_T are expressed, then the discount factors q_0^i and q_T^j may be viewed as "relative prices" on these currencies at time 0 and time T (which justifies the term "consistent price system").

Any sequence $(v_0, h_0, ..., h_T, v_T)$ satisfying (46) will be called a *superhedging programme.* Such a sequence specifies an initial endowment v_0 a contingent claim v_T and a feasible trading strategy $(h_0, ..., h_T)$ allowing to superhedge v_T starting from v_0. According to (45), a consistent price system does not increase (in the sense of expectations) along any superhedging programme.

It is clear from the above definitions that if $(q_0, p_0, p_1, p_2, ..., p_T, q_T)$ is a consistent price system, then (q_0, q_T) is a pair of consistent discount factors. Under general assumptions, one can prove the following converse statement: if (q_0, q_T) is a pair of consistent discount factors, then there exist $p_0, p_1, p_2, ..., p_T$ such that $(q_0, p_0, p_1, p_2, ..., p_T, q_T)$ is a consistent price system. This assertion will be proved in the next subsection.

4.2 Consistent discount factors and consistent price systems

Let us introduce the following assumptions.

(III) (a) If $(h_{t-1}, h_t) \in Z_t$, $h'_{t-1} \in \mathcal{X}_{t-1}$ and $h'_{t-1} \geq h_{t-1}$, then $(h'_{t-1}, h_t) \in Z_t$. (b) If $(h_T, v_T) \in W_T$, $h'_T \in \mathcal{X}_T$ and $h'_T \geq h_T$, then $(h'_T, v_T) \in W_T$.

(IV) (a) For each t, the cone Z_t contains an element $(\tilde{h}_{t-1}, \tilde{g}_t) \in Z_t$ such that $\tilde{g}_t > 0$. (b) There is a pair $(\tilde{v}_0, \tilde{g}_0) \in W_0$ such that $\tilde{g}_0 > 0$.

(We recall that strict inequalities, as well as non-strict ones, are understood coordinatewise and for each ω.)

The condition imposed on the cones Z_t in part (a) of hypothesis (III) is supposed to hold for each $t = 1, ..., T$. It means that if we can obtain a portfolio h_t by a feasible rebalancing of a portfolio h_{t-1}, we can also obtain h_t by a feasible rebalancing of any portfolio $h'_{t-1} \geq h_{t-1}$. Part (b) of assumption (III) expresses the analogous property of the cone W_T describing the possibilities of hedging a contingent claim v_T by liquidating a portfolio h_T. Hypothesis (IV) contains mild assumptions of non-degeneracy of Z_t and W_T.

Theorem 4.1 below holds under the assumptions (III) and (IV) (hypotheses (I) and (II) are not used in its proof).

Theorem 4.1 *Let q_0, q_T be consistent discount factors. Then there exist $p_0, p_1, ..., p_T$ such that $(q_0, p_0, p_1, ..., p_T, q_T)$ is a consistent price system.*

Proof. By virtue of the definition of consistent discount factors (q_0, q_T), we have $q_0 \in M_0^*$, $q_T \in M_T^*$ and $Eq_Tv_T - q_0v_0 \leq 0$ for all $(v_0, v_T) \in \mathcal{A}$. The last inequality means that the functional

$$Eq_Tv_T - q_0v_0 \tag{47}$$

of $v = (v_0, v_T)$ attains its maximum on the set $\mathcal{A}$ at $(0, 0)$ (and the maximum value of the functional is 0). Consequently, by virtue of the definition of $\mathcal{A}$ (see (35)), the functional

$F(\xi) = Eq_T v_T - q_0 v_0$ attains its maximum over the set of sequences

$$\xi = (v_0, g_0, g_1, ..., g_T, v_T) \tag{48}$$

satisfying the constraints

$$(v_0, g_0) \in W_0, \ (g_0, g_1) \in Z_1, \ ..., \ (g_{T-1}, g_T) \in Z_T, \ (g_T, v_T) \in W_T, \tag{49}$$

at the sequence $(0, 0, ..., 0)$, and the maximum value of $F(\xi)$ is zero. Note that sequences ξ of the form (48) are vectors in the space $\mathcal{V}_0 \times \mathcal{X}_0 \times \mathcal{X}_1 \times ... \times \mathcal{X}_T \times \mathcal{V}_T$.

Consider the set Θ of sequences θ of the form

$$\theta = ((v_0, g_0), (h_0, g_1), ..., (h_{T-1}, g_T), (h_T, v_T)), \tag{50}$$

where

$$(v_0, g_0) \in W_0, \ (h_0, g_1) \in Z_1, ..., (h_{T-1}, g_T) \in Z_T, \ (h_T, v_T) \in W_T. \tag{51}$$

Denote by Θ^0 the set of sequences $\theta \in \Theta$ satisfying

$$g_0 \geq h_0, \ g_1 \geq h_1, ..., g_T \geq h_T. \tag{52}$$

A sequence $\theta \in \Theta$ may be regarded as a plan for hedging v_T which starts from the initial endowment v_0 at time 0, and involves feasible portfolio rebalancing, as well as consumption, in the time periods $1, 2, ..., T$. A possibility of consumption is reflected by the fact that we allow inequalities $g_t \geq h_t$, rather than requiring equalities $g_t = h_t$.

For $\theta \in \Theta$, define

$$f(\theta) = Eq_T v_T - q_0 v_0$$

and

$$G(\theta) = (g_0 - h_0, g_1 - h_1, ..., g_T - h_T) \in \mathcal{X}_0 \times \mathcal{X}_1 \times ... \times \mathcal{X}_T.$$

Clearly

$$\Theta^0 = \{\theta \in \Theta : G(\theta) \geq 0\},$$

where the inequality is understood coordinatewise and for each ω. Equivalently, we can write

$$\Theta^0 = \{\theta \in \Theta : G(\theta) \in \mathcal{G}\}, \tag{53}$$

where $\mathcal{G}$ is the standard cone in the space $\mathcal{X}_0 \times \mathcal{X}_1 \times ... \times \mathcal{X}_T$.

Observe that the maximum of $f(\theta)$ on Θ^0 is equal to zero. Indeed, if a sequence θ of the form (50) satisfies (51) and (52), then the sequence

$$\theta' = ((v_0, g_0), (g_0, g_1), (g_1, g_2), ..., (g_{T-1}, g_T), (g_T, v_T))$$

also satisfies (52) (since the inequalities in (52) hold as equalities) and it also satisfies (51) by virtue of hypothesis (III). But $f(\theta') = F(\xi)$, where the sequence $\xi = (v_0, g_0, g_1, ..., g_T, v_T)$ satisfies constraints (49), and so

$$f(\theta) = f(\theta') = F(\xi) \leq 0.$$

We would like to apply the Kuhn-Tucker theorem (see the Appendix, Theorem A.7) to the problem of maximization of $f(\theta)$ on Θ subject to the constraint $G(\theta) \in \mathcal{G}$. To this end we have to check Slater's constraint qualification, stating that $G(\mathring{\theta}) \in \text{int}\, \mathcal{G}$ for some $\mathring{\theta} \in \Theta$. The interior of $\mathcal{G}$ consists of vector functions that are strictly positive in each coordinate and for each ω. Thus it is sufficient to construct a sequence

$$\mathring{\theta} = ((\mathring{v}_0, \mathring{g}_0), (\mathring{h}_0, \mathring{g}_1), ..., (\mathring{h}_T, \mathring{v}_T)) \in \Theta$$

for which

$$\mathring{g}_0 > \mathring{h}_0, \mathring{g}_1 > \mathring{h}_1, ..., \mathring{g}_T > \mathring{h}_T. \tag{54}$$

Define $(\mathring{v}_0, \mathring{g}_0) = (\tilde{v}_0, \tilde{g}_0)$ (see (IV), (b)). Then define $(\mathring{h}_0, \mathring{g}_1)$ as $(\lambda_1 \tilde{h}_0, \lambda_1 \tilde{g}_1) \in Z_1$(see (IV), (a)), where $\lambda_1 > 0$ is a sufficiently small number, such that $\mathring{g}_0 > \mathring{h}_0$. Then define $(\mathring{h}_1, \mathring{g}_2)$ as $(\lambda_2 \tilde{h}_1, \lambda_2 \tilde{g}_2) \in Z_2$ where $\lambda_2 > 0$ is a sufficiently small number for which $\mathring{g}_1 > \mathring{h}_1$. By continuing this procedure, we construct step by step a sequence $\mathring{\theta} \in \Theta$ with properties (54). At the last step, we define $(\mathring{h}_T, \mathring{v}_T) = (0, 0)$.

By virtue of Theorem A.7, there exists a linear functional $p \in \mathcal{G}^*$ such that

$$f(\theta) + p(G(\theta)) \leq 0 \tag{55}$$

for all $\theta \in \Theta$. A linear functional $p(g)$ on the space of $g = (g_0, g_1, ..., g_T) \in \mathcal{X}_0 \times \mathcal{X}_1 \times ... \times \mathcal{X}_T$ is given by

$$p(g) = p_0 g_0 + E p_1 g_1 + E p_2 g_2 + ... + E p_T g_T,$$

where $p_0 \in R^{n_0}$ and $p_t = p_t(\omega^t)$ are N_t-dimensional vector functions of ω^t. We have $p \in \mathcal{G}^*$ if and only if $p_t \geq 0$ for all $t = 0, ..., T$. Inequality (55) can be written

$$-q_0 v_0 + p_0(g_0 - h_0) + E p_1 (g_1 - h_1) + ... + E p_T (g_T - h_T) + E q_T v_T \leq 0, \tag{56}$$

where $(v_0, g_0) \in W_0$, $(h_0, g_1) \in Z_1$, ..., $(h_{T-1}, g_T) \in Z_T$, $(h_T, v_T) \in W_T$. Fix some $t = 1, ..., T$, consider any $(h_{t-1}, g_t) \in Z_t$ and define $(v_0, g_0) = 0$, $(h_{l-1}, g_l) = 0$ for all $l \neq t$ and $(h_T, v_T) = 0$. Then (56) yields $-Ep_{t-1}h_{t-1} + Ep_t g_t \leq 0$, which implies (43). Analogously, we obtain (42) and (44). This proves that $(q_0, p_0, p_1, ..., p_T, q_T)$ is a consistent price system. $\square$

Remark 4.1. Sequences θ possessing properties (51) and (52) can be called *superhedging programmes with consumption.* In the proof of Theorem 4.1, we used hypotheses (III) and (IV) in order to derive the following properties of the set Θ of such sequences.

(Θ.1) For each $\theta = ((v_0, g_0), (h_0, g_1), ..., (h_{T-1}, g_T), (h_T, v_T)) \in \Theta$, there exists a superhedging programme $(v_0, h'_0, h'_1, ..., h'_T, v_T)$ with the same initial endowment v_0 and the same contingent claim v_T.

(Θ.2) There exists a sequence $((\mathring{v}_0, \mathring{g}_0), (\mathring{h}_0, \mathring{g}_1), ..., (\mathring{h}_T, \mathring{v}_T)) \in \Theta$ satisfying (54).

Condition (Θ.1) expresses the assumption that if an initial endowment v_0 allows to superhedge a contingent claim v_T with consumption, then v_0 allows to superhedge v_T without it. According to (Θ.2), there exists a superhedging programme with strictly positive consumption in each time period. We can see from the proof of Theorem 4.1 that the theorem remains valid if hypotheses (III) and (IV) are replaced by more general conditions (Θ.1) and (Θ.2).

Finally, we note that condition (Θ.2) is obsolete when the cones W_t ($t = 0, T$) and Z_t ($t = 1, ..., T$) are polyhedral. Indeed, (Θ.2) is used to guarantee the validity of Slater's condition in the optimization problem considered in the course of the proof of Theorem 4.1. If all the above cones are polyhedral, then we obtain a linear programming problem, where no Slater's condition is needed (see the Appendix, Theorem A.7).

4.3 Criteria for no-arbitrage and hedging in terms of consistent price systems

Let us say that a consistent price system $(q_0, p_0, p_1, ..., p_T, q_T)$ is *strictly consistent* if $q_0 \in M_0^+$ and $q_T \in M_T^+$. By combining Theorems 3.1 and 4.1, we obtain the following result.

Theorem 4.2. *The validity of the no-arbitrage hypothesis ($\mathcal{NA}$) is equivalent to the existence of a strictly consistent price system.*

By using Theorems 3.2 and 4.1, we arrive at a superhedging criterion stated in terms of consistent price systems.

Theorem 4.3. *Let hypothesis ($\mathcal{NA}$) hold. Then for any $(v_0, v_T) \in \mathcal{V}_0 \times \mathcal{V}_T$ the following assertions are equivalent:*

($\mathcal{H}$) The contingent claim v_T can be superhedged starting from the initial endowment v_0, i.e. $(v_0, v_T) \in \mathcal{A}$.

($\mathcal{C}$) We have $Eq_T v_T \leq q_0 v_0$ for each strictly consistent price system $(q_0, p_0, p_1, ..., p_T, q_T)$.

It should be noted that the main advantage of the consideration of consistent price systems in place of (or in addition to) consistent discount factors lies in the following.

It might be difficult to verify directly that some (q_0, q_T) is a pair of consistent discount factors. For this aim one has to check inequality (37) involving the cone $\mathcal{A}$ which might have, generally, a quite complex structure. The analysis of consistent price systems allows to simplify the problem. It allows to "decompose" it – to reduce it to a family of simpler problems over each of the time periods $t-1, t$. Indeed, to check that $(q_0, p_0, p_1, ..., p_T, q_T)$ is a consistent price system, we have to verify separately every inequality in the chain (45). The analysis of each of these inequalities requires a separate consideration of each of the given cones $W_0, Z_0, ..., Z_T, W_T$.

4.4 Consistent price systems and supermartingales

In most of the specific examples, the cones Z_t $(t = 2, 3, ..., T)$ and W_T satisfy additional conditions included into hypothesis (V) below.

(V) (a) If $(h_{t-1}, h_t) \in Z_t$, then for any non-negative function $\gamma(\omega^{t-1})$ of ω^{t-1}, we have $(\gamma h_{t-1}, \gamma h_t) \in Z_t$. (b) The cone W_T contains with each pair of vector functions (h_T, v_T) the pair $(\gamma h_T, \gamma v_T)$, where $\gamma(\omega)$ is any non-negative function of ω.

Note that if $(h_{t-1}, h_t) \in Z_t$, then $(\gamma h_{t-1}, \gamma h_t) \in Z_t$ for each non-negative constant γ because Z_t is a cone. Condition (a) in (V) requires more. It states that $(\gamma h_{t-1}, \gamma h_t) \in Z_t$ for each *function* $\gamma(\omega^{t-1})$. This condition imposes additional restrictions on Z_t only if $t \geq 2$; therefore in (V) we assume that t takes on the values $2, 3, ..., T$. If assumption (Va) holds, we say that the cone Z_t is *decomposable with respect to* ω^{t-1}. Analogously, if (Vb) holds, we say that the cone W_T is *decomposable with respect to* ω.

For a random variable ζ, we will denote by $E(\zeta \mid \omega^{t-1})$ the conditional expectation of ζ given ω^{t-1}. For $t = 1$ (when ω^{t-1} does not make sense), the above notation will stand for the unconditional expectation $E\zeta$. Under hypothesis (V), we can give an equivalent definition of a consistent price system.

Proposition 4.1. *A sequence $(q_0, p_0, p_1, ..., p_T, q_T)$, where $q_0 \in M_0^*$, $q_T \in M_T^*$ and $p_t \in \mathcal{P}_t$ $(t = 0, ..., T)$, forms a consistent price system if and only if the following relations hold:*

$$q_0 v_0 \geq p_0 h_0 \text{ for each } (v_0, h_0) \in W_0; \tag{57}$$

$$p_{t-1} h_{t-1} \geq E(p_t h_t \mid \omega^{t-1}) \text{ for each } (h_{t-1}, h_t) \in Z_t \text{ and } t = 1, 2, ..., T; \tag{58}$$

$$p_T h_T \geq q_T v_T \text{ for each } (h_T, v_T) \in W_T. \tag{59}$$

Proof. Condition (57) simply repeats condition (42). Let us show that (59) is equivalent to (43). By taking the expectations of both sides of inequality (58) we obtain (43). Conversely, assume (43) holds. Consider any $(h_{t-1}, h_t) \in Z_t$ and take any function $\gamma(\omega^{t-1}) \geq 0$.

Then $(\gamma h_{t-1}, \gamma h_t) \in Z_t$ by virtue of part (a) of hypothesis (V), and so $E p_t \gamma h_t \leq E p_{t-1} \gamma h_{t-1}$ in view of (43). The last inequality implies $E\gamma[E(p_t h_t \mid \omega^{t-1})] \leq E\gamma p_{t-1} h_{t-1}$ for each $\gamma(\omega^{t-1}) \geq 0$, which byields (58). The equivalence of (59) and (44) is straightforward. □

From Proposition 4.1, we obtain the following result.

Proposition 4.2. *Let $(q_0, p_0, p_1, ..., p_T, q_T)$ be a consistent price system. Then for any superhedging programme $(v_0, h_0, h_1, ..., h_T, v_T)$, the random sequence $(p_0 h_0, p_1 h_1, ..., p_T h_T)$ is a supermartingale and inequalities (57) and (59) hold.*

It is natural to ask when the sequence $p_0, p_1, ..., p_T$ is a supermartingale or martingale itself. The answer is given in the following proposition.

Proposition 4.3. *Let condition (Va) hold. If N_t does not depend on t ($N_t = N$) and if Z_t contains (h, h) for each $h \in R^N$ (resp. for each $h \in R^N_+$), then for any consistent price system $(q_0, p_0, p_1, ..., p_T, q_T)$, the sequence of random vectors $p_0, ..., p_T$ is a martingale (resp. supermartingale).*

The assumption $(h, h) \in Z_t$ for a non-random portfolio h implies that a *buy-and-hold strategy* $(h, h, ..., h)$ is feasible. According to Proposition 4.3, if this is true for any h (resp. any nonnegative h), then the vectors $p_0, p_1, ..., p_T$ of asset prices involved in a consistent price system form a martingale (resp. supermartingale).

Proof of Proposition 4.3. When proving Proposition 4.2, we established the equivalence of (59) and (44) under assumption (Va). By substituting $h_t = h_{t-1} = h \in R^N_+$ into (59), we get $p_{t-1} \geq E(p_t \mid \omega^{t-1})$, which means that $p_0, p_1, ..., p_T$ is a supermartingale. If any $h \in R^N$ can be substituted into (59), we obtain that $p_{t-1} = E(p_t \mid \omega^{t-1})$, and so $p_0, p_1, ..., p_T$ is a supermartingale. □

5 Von Neumann-Gale model and set-valued dynamical systems

5.1 Von Neumann-Gale model

The *von Neumann-Gale model* [17], [38] is specified by a sequence of cones $\mathcal{Z}_t \subseteq R_+^{N_t} \times R_+^{N_t}$, $t = 1, 2, ...$ The model describes an economy in which, at time $t = 0, 1, ...$, there are N_t commodities $i = 1, 2, ..., N_t$. The state of the economy at time t is characterized by a commodity vector $x_t = (x_t^1, ..., x_t^{N_t}) \in R_+^{N_t}$. A *path (trajectory)* of the economic system is a finite or infinite sequence $x_0, x_1, x_2, ...$ such that

$$(x_{t-1}, x_t) \in \mathcal{Z}_t,\ t = 1, 2, ...$$

Elements $(x, y) \in \mathcal{Z}_t$ are called *input-output* pairs or *technological processes.* The sets $\mathcal{Z}_t$ are called *technology sets.* In the original [38] model, the cones $\mathcal{Z}_t$ were supposed to be polyhedral. Gale [17] generalized von Neumann's framework allowing general (not necessarily polyhedral) cones.

The main focus in the von Neumann-Gale model is on the analysis of paths that maximize growth rates over each time period $t-1, t$. Such paths are called *efficient.* The precise definition is as follows. A path $x_0, x_1, x_2, ...$ is called efficient if there exists a sequence of price vectors $p_0, p_1, ...$ ($p_t \in R_+^{N_t}$) such that $p_t x_t = 1$ and

$$p_t y \leq p_{t-1} x \text{ for all } (x, y) \in \mathcal{Z}_t. \tag{60}$$

By virtue of this definition, the *growth rate*

$$\frac{p_t y_t}{p_{t-1} y_{t-1}}$$

over the time period $t-1, t$ attains its maximum among all paths $y_0, y_1, ...$ on the path $x_0, x_1, ...$ Since $p_t x_t = 1$, the growth rate on $x_0, x_1, ...$ is constant and equal to one. (In the above definition, the assumption $p_t x_t = 1$ is not essential, what matters is that $p_t x_t$ is a strictly positive constant.)

5.2 Homogeneous convex dynamical systems

A (discrete-time) dynamical system is given by a sequence of sets Y_t, $t = 0, 1, ...$, and mappings

$$F_t : \ Y_{t-1} \to Y_t,\ t = 1, 2, ...,$$

of Y_{t-1} into Y_t. Points y in Y_t represent possible *states* of the dynamical system at time t. The mappings F_t specify the *law of dynamics.* If x_t is a state at time t, the state at time $t+1$ will be

$$x_{t+1} = F_{t+1}(x_t). \tag{61}$$

Sequences $x_0, x_1, x_2, ...$ satisfying (61) are called *paths* (*trajectories*) of the dynamical system under consideration.

A *set-valued (multivalued)* dynamical system is defined by a sequence of *multivalued mappings* $\Phi_t(x)$ assigning a set $\Phi_t(x) \subseteq Y_t$ (rather than a singleton) to each $x \in Y_{t-1}$. A path of the multivalued dynamical system is a sequence of states $x_0, x_1, ...$ such that

$$x_{t+1} \in \Phi_{t+1}(x_t).$$

If for each t, Y_t is a convex set in R^{N_t} and if the graph

$$Gr(\Phi_t) = \{(x, y) \in Y_{t-1} \times Y_t : \; y \in \Phi_t(x)\}$$

of the multivalued mapping $\Phi_t(\cdot)$ is a convex set in $R^{N_{t-1}} \times R^{N_t}$, then the multivalued dynamical system is called *convex*. The dynamical system is said to be *homogeneous and convex* if $Gr(\Phi_t)$ is a cone in $R^{N_{t-1}} \times R^{N_t}$.

5.3 The homogenous convex dynamical system generated by a von Neumann-Gale model

Given a von Neumann-Gale model specified by a sequence of cones $\mathcal{Z}_t \subseteq R^{N_{t-1}} \times R^{N_t}$, $t = 1, 2, ...$, we define a multivalued dynamical system by setting

$$\mathcal{Z}_t(x) = \{y \in R_+^{N_t} : (x, y) \in \mathcal{Z}_t\}, \; x \in R^{N_{t-1}}. \tag{62}$$

Then the cone $\mathcal{Z}_t$ is the graph of the multivalued mapping $\mathcal{Z}_t(x)$. Consequently, the multivalued dynamical system (62) is convex and homogeneous. Paths of the von Neumann-Gale model are nothing but paths of the multivalued system (62).

The *dual* dynamical system (to the system (62)) is defined as follows:

$$\mathcal{Z}_t^{\times}(p) = \{q \in R_+^{N_t} : qy \leq px \text{ for all } x, y \text{ satisfying } y \in \mathcal{Z}_t(x)\}, \; p \in R_+^{N_{t-1}}. \tag{63}$$

According to this definition, $p_0, p_1, ...$ is a path of the dual dynamical system (63) (or briefly a *dual path* or a *dual trajectory*) if $p_t y \leq p_{t-1} x$ for all $(x, y) \in \mathcal{Z}_t$. The last inequality coincides with (60), and so we can reformulate the definition of an efficient path, saying that *a path $x_0, x_1, x_2, ...$ is efficient if there exists a dual path $p_0, p_1, p_2, ...$ such that $p_t x_t = 1$.* Clearly, if $p_0, p_1, p_2, ...$ is a dual trajectory, then

$$p_0 x_0 \geq p_1 x_1 \geq p_2 x_2 \geq .... \tag{64}$$

for each trajectory $x_0, x_1, x_2, ...$ of the dynamical system (62).

5.4 Analogies between the von Neumann-Gale model and the securities market model

In the above considerations, we dealt with *deterministic* models. Paths $x_0, x_1, x_2, ...$ and dual paths $p_0, p_1, p_2...$ were sequences of non-random vectors. We can easily extend all the above considerations to the case where x_t and p_t are functions of $\omega^t = (a_1, a_2, ..., a_t)$, where $a_1, a_2, ...$ is a sequence of random states of the world taking values in a finite set $A = \{a^1, ..., a^L\}$. The scalar products of the form $p_t x_t$ can be replaced in the above analysis by the expectations $Ep_t x_t$. Clearly functions $p_t(\omega^t)$ and $x_t(\omega^t)$ may be regarded as vectors in R^{n_t}, where $n_t = N_t \cdot L^t$ (if A contains L elements, then there are L^t different sequences $\omega^t = (a_1, a_2, ..., a_t)$). The stochastic analogue of the von Neumann-Gale model is specified by a family of cones $\mathcal{Z}_t \subseteq R^{n_{t-1}} \times R^{n_t}$, $t = 1, 2, ...$, whose elements are pairs of non-negative functions $x = x(\omega^{t-1})$, $y = y(\omega^t)$. Paths in the stochastic model are sequences $x_0, x_1, ...$ such that $(x_{t-1}, x_t) \in \mathcal{Z}_t$. A dual path $p_0, p_1, ...$ is a sequence of nonnegative functions $p_t = p_t(\omega^t)$ such that

$$Ep_{t-1}x \geq Ep_t y \text{ for all } (x, y) \in \mathcal{Z}_t.$$

The above discussion suggests clear analogies between concepts related to the von Neumann-Gale model and notions associated with the dynamic securities market model. In the latter, we deal with a sequence of cones $W_0, Z_1, ..., Z_T, W_T$ describing possibilities of converting an initial endowment v_0 into the portfolio h_0 (purchasing the portfolio h_0), then trading over the time period $1, 2, ..., T$ according to a feasible strategy $(h_0, ..., h_T)$, and finally liquidating the portfolio h_T and superhedging the contingent claim v_T. Superhedging programmes $(v_0, h_0, ..., h_T, v_T)$ are nothing but paths in the model defined by the cones

$$\mathcal{Z}_1 = W_0,\ \mathcal{Z}_2 = Z_1,\ ..., \mathcal{Z}_{T+1} = Z_T,\ \mathcal{Z}_{T+2} = W_T.$$

Note, however, that in the financial context, vectors in the given cones $W_0, Z_1, ..., Z_T, W_T$ are not necessarily nonnegative, is the case in the conventional von Neumann-Gale setting. All the other analogies are straightforward. These analogies are summarized in the following table:

Von Neumann-Gale model	**Securities Market Model**
commodities	assets
commodity vectors	portfolios of assets
paths of economic dynamics	superhedging programmes
dual paths	consistent price systems
technology constraints	trading constraints

6 Two classes of models

6.1 Models described in terms of "value operators"

We will begin with a description of two classes of models. In the former class, the cones W_0 and W_T characterizing the possibilities of constructing the initial portfolio and liquidating the final one, have a special structure defined in terms of the "value operators". In the latter class (considered in the next subsection), the cones Z_t specifying the trading constraints have a special structure described by difference inclusions. A number of concrete examples analyzed in the remainder of the paper can be included into these two frameworks.

Let us assume that the cones W_t $(t = 0, T)$ are defined as follows. Suppose mappings (operators)

$$V_t : \mathcal{X}_t \to \mathcal{V}_t \ (t = 0, T)$$

are given, and the cones W_t are defined by

$$W_0 = \{(v_0, h_0) \in \mathcal{V}_0 \times \mathcal{X}_0 : v_0 \geq_0 V_0(h_0)\}, \tag{65}$$

$$W_T = \{(h_T, v_T) \in \mathcal{X}_T \times \mathcal{V}_T : v_T \leq_T V_T(h_T)\}. \tag{66}$$

The mappings V_t are called the *value operators.*

The operator V_0 associates with each initial portfolio h_0 its *purchase value,* indicating what is the minimum initial endowment needed to construct the portfolio h_0. The minimum is understood with respect to the partial ordering $\geq_0$. An important special case is where $V_0(h_0)$ is a scalar (i.e., the dimension m_0 of the space $\mathcal{V}_0$ of initial endowments is equal to one), and the partial ordering $\geq_0$ is standard. Then $V_0(h_0)$ is the amount of cash needed to purchase, taking into account the transaction costs, the assets contained in the portfolio h_0. The purchase is supposed to be made at the prices prevailing at time 0. Another important case is where $m_0 = N_0$ and $V_0(h_0) = h_0$. In this case the "purchase value" of the portfolio does not admit aggregation and can only be specified by the portfolio itself. This situation is characteristic for currency markets where positions of portfolios h_t represent holdings of N_t currencies (cf. Kabanov [23] and Kabanov and Stricker [25]).

The operator V_T associates with each final portfolio h_0 its *liquidation value.* Again, there are two important special cases. In the former, $m_T = 1$, and the scalar $V_T(\omega, h_T(\omega))$, depending on ω, represents the amount of money that can be obtained by liquidating the portfolio h_T with transaction costs. In the latter case, typical for models of currency markets, $m_T = N_T$ and $V_T(h_T) = h_T$.

Models described in terms of the value operators were considered by Evstigneev and Taksar [15]. The framework adopted in this paper, in which the cones W_0 and W_T are given, is more general. It is applicable, in particular, to those cases where $1 < m_T < N_T$ and there is no well-defined maximum with respect to the partial ordering $\leq_T$ in the set

$\{v_T : (h_T, v_T) \in W_T\}$. The latter case arises, for example, when there is a mixed asset market where N_t assets $i = 1, 2, ..., N_t$, including currencies and common stock, are traded at each time period $t = 1, 2, ..., T$, and where contingent claims are portfolios of several currencies $j = 1, 2, ..., m_T$. In a typical example, $m_T = 3$, i.e., a contingent claim has three components: the first component is measured in terms of dollar, the second in terms of euro, and the third in terms of the local currency (provided it is not dollar or euro).

Let us impose the following assumptions on the operators V_0 and V_T .Assume

$$V_0(\alpha h_0 + \alpha' h_0') \leq_0 \alpha V_0(h_0) + \alpha' V_0(h_0') \tag{67}$$

for all vectors $h_0, h_0' \in \mathcal{X}_0$ and all numbers $\alpha, \alpha' \geq 0$, and

$$V_T(\alpha h_T + \alpha' h_T) \geq_T \alpha V_T(h_T) + \alpha' V_T(h_T') \tag{68}$$

for all vector functions $h_T, h_T' \in \mathcal{X}_T$ and all numbers $\alpha, \alpha' \geq 0$. Further, suppose

$$h_t' \geq h_t \Rightarrow V_t(h_t') \geq_t V_t(h_t) \ (t = 1, 2). \tag{69}$$

Note that (67) and (68) contain assumptions of convexity and concavity similar to the conventional ones, but stated in terms of the general, not necessarily standard, partial orderings $\leq_0$ and $\geq_T$. Also, we emphasize that, in the monotonicity assumption (69), the inequality between h_t' and h_t is understood in the standard sense, while $V_t(h_t')$ and $V_t(h_t)$ are compared by using the partial ordering $\geq_t$.

Proposition 6.1. *If conditions (67), (68) and (69) hold, then the sets W_0 and W_T defined by (65) and (66) are cones satisfying all the requirements contained in hypotheses (II)-(IV).*

Proof. The sets W_0 and W_T are cones in view of inequalities (67) and (68). Conditions (IIa) and (IIb) follow immediately from (65) and (66). Property (IIIb) is a consequence of (69). To verify (IVb) consider any $\tilde{g}_0 > 0$ and put $\tilde{v}_0 = V_0(\tilde{g}_0)$. □

If the cones W_t $(t = 0, T)$ are described in terms of the value operators according to formulas (65) and (66), then one can express conditions (42) and (44), involved in the definition of a consistent price system, directly through the operators V_t $(t = 0, T)$. This is shown in the following proposition.

Proposition 6.2. *Let $q_0 \in \mathcal{M}_0^*$ and $p_0 \in R_+^{N_T}$. Then inequality (42) holds if and only if*

$$q_0 V_0(h_0) \geq p_0 h_0 \text{ for all } h_0 \in \mathcal{X}_0. \tag{70}$$

Let $q_T \in \mathcal{M}_T^*$ and let p_T be a function of ω with values in $R_+^{N_T}$. Then inequality (44) holds if and only if

$$E q_T V_T(h_T) \leq E p_T h_T \text{ for all } h_T \in \mathcal{X}_T. \tag{71}$$

Proof. Let us prove the latter of the two assertions. The former is proved similarly. Suppose (44) holds. Take any $h_T \in \mathcal{X}_T$ and define $v_T = V_T(h_T)$. Then $(h_T, v_T) \in W_T$, and so (71) follows from (44). Conversely, suppose (71) is valid. Consider any $(h_T, v_T) \in W_T$. By virtue of (66), $v_T \le_T V_T(h_T)$. Consequently, $Eq_T v_T \le Eq_T V_T(h_T)$ because $q_T \in \mathcal{M}_T^*$. By combining the last inequality and (71), we obtain (44). □

We formulate a list of assumptions on the value operators V_0 and V_T that imply the conditions already introduced and, in addition, guarantee that the cone W_T defined by (66) is decomposable with respect to ω. Recall that the last property means (see condition (Vb)) that W_T contains with each pair of vector functions (h_T, v_T) the pair $(\gamma h_T, \gamma v_T)$, where $\gamma(\omega)$ is any nonnegative function of ω. Let us say that *the operator V_T is decomposable* (with respect to ω) if

$$V_T(h_T) = V_T(\omega, h_T(\omega)), \tag{72}$$

where $V_T(\omega, \cdot)$ is a mapping of R^{N_T} into R^{m_T} given for each ω. Slightly abusing notation, we write V_T both for the operator $V_T(\cdot)$ and the mapping $V_T(\omega, \cdot)$. The representation (66) means that the value of $V_T(h_T)$ is defined "for each ω separately".

(VI) (a) The operator V_0 is *homogeneous and convex:*

$$V_0(\alpha_0 h_0) = \alpha_0 V_0(h_0) \tag{73}$$

for all scalars $\alpha \ge 0$ and vectors $h_0 \in \mathcal{V}_0$;

$$V_0(h_0 + h_0') \le_0 V_0(h_0) + V_0(h_0') \text{ for all } h_0, h_0' \in \mathcal{X}_0. \tag{74}$$

(b) The operator V_T is decomposable (i.e., it admits a representation (72)), and the mapping $V_T(\omega, \cdot)$ is *homogeneous and concave for each ω:*

$$V_T(\alpha_T h_T) = \alpha_T V_T(h_T) \tag{75}$$

for all scalar functions $\alpha_T = \alpha_T(\omega) \ge 0$ and vector functions $h_T \in \mathcal{X}_T$;

$$V_T(h_T + h_T') \le_T V_T(h_T) + V_T(h_T') \text{ for all } h_T, h_T' \in \mathcal{X}_T. \tag{76}$$

(c) The operators V_t $(t = 0, T)$ are *monotone*, i.e., requirement (69) is satisfied.

Clearly, conditions (73) and (74) imply (67), and conditions (75) and (76) imply (68). Moreover, we have

$$V_T(\omega, \alpha_T h_T + \alpha_T' h_T) \ge_T \alpha_T V_T(\omega, h_T) + \alpha_T' V_T(\omega, h_T') \tag{77}$$

for all $h_T, h_T' \in \mathcal{X}_T$ and all non-negative functions $\alpha_T = \alpha_T(\omega)$ and $\alpha_T' = \alpha_T'(\omega)$.

Proposition 6.3. *If the cone $\mathcal{M}_T$ is decomposable, then the cone W_T defined by (66) is decomposable (and so condition (Vb) holds).*

Proof. Let $(h_T, v_T) \in W_T$ and let $\gamma(\omega)$ be a nonnegative scalar function. Then $v_T \leq_T V_T(h_T)$, i.e., $V_T(h_T) - v_T \in \mathcal{M}_T$. Consequently, $\gamma V_T(h_T) - \gamma v_T \in \mathcal{M}_T$, since $\mathcal{M}_T$ is decomposable. Thus $\gamma v_T \leq_T \gamma V_T(h_T)$, and since the mapping $V_T(\omega, \cdot)$ is homogeneous, we get $\gamma v_T \leq_T V_T(\gamma h_T)$, which means that $(\gamma h_T, \gamma v_T) \in W_T$. □

Proposition 6.4. *Inequality (44) holds if and only if*

$$q_T(\omega) V_T(\omega, b) \leq p_T(\omega) b \text{ for all } b \in R^{N_T}. \tag{78}$$

Proof. By virtue of Proposition 6.2, (44) is equivalent to (71). We can see that (78) implies (71). To prove the converse assume (71) holds and consider some $h_T \in \mathcal{X}_T$. For any function $\gamma(\omega)$, we have $V_T(\gamma h_T) = \gamma V_T(h_T)$, and so $E\gamma q_T V_T(h_T) \leq E\gamma p_T h_T$ by virtue of (75). This implies

$$q_T V_T(h_T) \leq p_T h_T \text{ for all } h_T \in \mathcal{X}_T. \tag{79}$$

Suppose (78) fails to hold for some $\omega = \bar{\omega}$ and $b = \bar{b}$. Then, by setting $h_T(\omega) = \bar{b}$ for $\omega = \bar{\omega}$ and $h_T(\omega) = 0$ for $\omega \neq \bar{\omega}$, we arrive at a contradiction with (79). This completes the proof of the proposition. □

6.2 Models described in terms of difference inclusions

Let us assume that the number N_t of assets traded at time t does not depend on t ($N_t = N$). Suppose the cones Z_t describing trading constraints are of the form

$$Z_t = \{(h_{t-1}, h_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : h_t - h_{t-1} \in M_t\}, \tag{80}$$

where $M_t \subseteq \mathcal{X}_t$ $(t = 1, ..., T)$ are some given cones. In models of this kind, feasible trading strategies are sequences $\{h_0, ..., h_T\}$ satisfying

$$h_t \in \mathcal{X}_t,\ t = 0, ..., T;\ h_t - h_{t-1} \in M_t,\ t = 1, ..., T.$$

Note that in (80) possible transitions from one portfolio, h_{t-1}, to another, h_t, are defined in terms of the *difference* between the vectors h_t and h_{t-1}. One can obtain a portfolio h_t by rebalancing a portfolio h_{t-1} if and only if $h_t - h_{t-1} \in M_t$. Therefore we say that models with constraint sets Z_t of the form (80) are defined in terms of *difference inclusions.*

We first examine conditions on M_t guaranteeing the validity of the assumptions imposed on Z_t in Section 4.

Proposition 6.5. *Let the cone Z_t be defined by (80) $(t = 1, ..., T)$. Then condition (IVa) holds. If $M_t \supseteq -\mathcal{P}_{t-1}$ $(t = 1, ..., T)$, then Z_t satisfies condition (IIIa). If for each $t = 1, ..., T$, the cone M_t is decomposable with respect to ω^{t-1}, then Z_t satisfies (Va).*

Proof. Fix any strictly positive vector $\tilde{h}_{t-1} \in \mathcal{X}_{t-1}$ and define $\tilde{g}_t = \tilde{h}_{t-1}$. Then $\tilde{h}_{t-1} = \tilde{g}_t \in \mathcal{X}_{t-1} \subseteq \mathcal{X}_t$ and $\tilde{g}_t - \tilde{h}_{t-1} = 0 \in M_t$. Hence Z_t contains $(\tilde{h}_{t-1}, \tilde{g}_t)$ such that $\tilde{g}_t > 0$, which yields (IVa).

Consider any $(h_{t-1}, h_t) \in Z_t$ and $\mathcal{X}_{t-1} \ni h'_{t-1} \geq h_{t-1}$. Then $g_{t-1} := h'_{t-1} - h_{t-1} \in \mathcal{P}_{t-1}$. Consequently, $h_t - h'_{t-1} = h_t - h_{t-1} - g_{t-1} \in M_t$. Thus $\left(h'_{t-1}, h_t\right) \in Z_t$, and so Z_t satisfies (IIIa).

Suppose, for each $t = 2, 3, ..., T$, the cone M_t is decomposable with respect to ω^{t-1}. Consider any element (h_{t-1}, h_t) of Z_t and any function $\gamma(\omega^{t-1}) \geq 0$. We have $\gamma h_{t-1} \in \mathcal{X}_{t-1}$ and $\gamma h_t \in \mathcal{X}_t$. Since M_t is decomposable with respect to ω^{t-1}, we obtain $\gamma h_t - \gamma h_{t-1} = \gamma(h_t - h_{t-1}) \in M_t$. □

Let us examine the structure of consistent asset prices in models defined in terms of difference inclusions. Recall that such prices are given by sequences of vectors $p_0 \in \mathcal{P}_0,\ ..., p_T \in \mathcal{P}_T$ satisfying (43).

Proposition 6.6. *Vectors $p_t \in \mathcal{P}_t$, $t = 0, 1, ..., T$, form a sequence of consistent asset prices if and only if the sequence $p_0, ..., p_T$ is a martingale and*

$$Ep_t y \leq 0,\ y \in M_t, \tag{81}$$

for all $t = 1, 2, ..., T$. If the cone M_t is decomposable with respect to ω^t, i.e.,

$$M_t = \{x \in \mathcal{X}_t :\ x(\omega) \in M_t(\omega)\}, \tag{82}$$

where $M_t(\omega)$ is a cone in R^N, then (81) is equivalent to the assertion that

$$p_t(\omega^t)b \leq 0,\ \textit{for all } \omega^t \textit{and all } b \in M_t(\omega). \tag{83}$$

Proof. Observe that $(h_{t-1}, h_t) \in Z_t$ if and only if

$$h_{t-1} \in \mathcal{X}_t \text{ and } h_t = h_{t-1} + y_t, \text{ where } y_t \in M_t.$$

Therefore condition (43), defining consistent asset prices, holds if and only if

$$Ep_t y + Ep_t h_{t-1} \leq Ep_{t-1} h_{t-1},\ h_{t-1} \in \mathcal{X}_{t-1},\ y \in M_t.$$

In turn, this inequality is equivalent to the following two relations: (81) and $Ep_t h_{t-1} \leq Ep_{t-1} h_{t-1}$ $(h_{t-1} \in \mathcal{X}_{t-1})$. Since $\mathcal{X}_{t-1}$ is a linear space, the last inequality holds if and only if $Ep_t h_{t-1} = Ep_{t-1} h_{t-1}$, which implies that $p_0, ..., p_T$ is a martingale. □

7 Examples and applications

7.1 The case of a frictionless market

In this section we consider four specific models that can be included into the general framework we developed. All these models are described in terms of value operators; all except the last one are defined in terms of different inclusions. All the cones involved are polyhedral, so that condition (I) (see Section 3) holds automatically. The verification of all the assumptions in (II) - (V) used in each specific case is straightforward.

We begin with reconsidering the case of a frictionless market. The model we deal with is essentially the same as in Section 2. However, the existence of a riskless asset is not assumed. Also, in the definition of a self-financing strategy, we assume inequality $S_t h_t \leq S_t h_{t-1}$ between the portfolio values, rather than equality. The main conclusions we will obtain will be fully analogous to those derived in Section 2, but they will be expressed in the general terms of consistent price systems.

Suppose for each $t = 0, ..., T$, we are given a vector $S_t(\omega^t) = (S_t^1(\omega^t), ..., S_t^N(\omega^t)) \geq 0$ specifying the market prices of N assets $i = 1, 2, ..., N$ at time t. Assume that initial endowments and contingent claims are measured in terms of a single currency, so that $m_0 = m_T = 1$. Thus $\mathcal{V}_0 = R^1$, and $\mathcal{V}_T$ is the space of scalar-valued functions $v(\omega^T)$. Assume that $\mathcal{M}_0$ and $\mathcal{M}_T$ are the standard cones of non-negative elements, consequently, the partial orderings $\leq_0$ and $\leq_T$ are standard. Let the cones Z_t, $t = 1, ..., T$, and W_t $(t = 0, T)$ be defined by

$$Z_t = \{(h_{t-1}, h_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \; S_t h_t \leq S_t h_{t-1}\}, \tag{84}$$

$$W_0 = \{(v_0, h_0) \in \mathcal{V}_0 \times \mathcal{X}_0 : \; v_0 \geq S_0 h_0\}, \tag{85}$$

$$W_T = \{(h_T, v_T) \in \mathcal{X}_T \times \mathcal{V}_T : \; S_T h_T \geq v_T\}. \tag{86}$$

The inequality $S_t h_t \leq S_t h_{t-1}$ in (84) expresses the self-financing condition. According to (85), an initial endowment v_0 allows to purchase a portfolio h_0 if the value $S_0 h_0$ of this portfolio expressed in terms of the prices S_0 (without transaction costs) does not exceed v_0.

Note that the model under consideration is defined in terms of difference inclusions, since we can represent Z_t in the form

$$Z_t = \{(h_{t-1}, h_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \; h_t - h_{t-1} \in M_t\},$$

where

$$M_t = \{y \in \mathcal{X}_t : S_t y \leq 0\}. \tag{87}$$

The cones W_0 and W_T are defined in terms of the value operators

$$V_0(h_0) = S_0 h_0, \; V_T(h_T) = S_T h_T.$$

In each of the examples we analyze in this section, we are primarily interested in the description of the class of strictly consistent price systems. Recall that such price systems exist if and only if the no arbitrage hypothesis holds (see Theorem 4.2). Furthermore, they are used for the characterization of the set of initial endowments needed to superhedge a contingent claim (see Theorem 4.3).

Suppose asset $i = 1$ is cash and put $B_t(\omega^t) = S_t^1(\omega^t)$. There is a bank account such that one unit of cash deposited with the account at time 0 yields B_t units of cash at time t. Assuming that $B_t(\omega^t) > 0$, consider the *discounted prices*

$$s_t^i = S_t^i / B_t$$

of assets $i = 1, 2, ..., N$. Clearly $s_t^1 = 1$. Denote by $\mathcal{Q}_T$ the set of those real-valued functions $q(\omega) > 0$ for which

$$E(q s_t \mid \omega^{t-1}) = E(q \mid \omega^{t-1}) s_{t-1}, \; t = 1, 2, ..., T. \tag{88a}$$

Since $E(q s_t \mid \omega^{t-1}) = E[E(q \mid \omega^t) s_t \mid \omega^{t-1}]$, equality (88a) means that the sequence $E(q \mid \omega^t) s_t$, $t = 0, ..., T$, is a martingale (with respect to the original measure P on Ω). Define

$$Q(\omega) = \frac{q(\omega)}{Eq} P(\omega), \; \omega \in \Omega. \tag{89}$$

Then Q is a probability measure on Ω, *equivalent to P with density* $\dfrac{q}{Eq} > 0$. Conditional expectations with respect to Q and P are related to each other by the formula

$$E^Q(\xi \mid \omega^{t-1}) = \frac{E(q\xi \mid \omega^{t-1})}{E(q \mid \omega^{t-1})}, \tag{90}$$

holding for any random variable $\xi(\omega)$. By using (90), we can write (88a) in the form

$$E^Q(s_t \mid \omega^{t-1}) = s_{t-1}, \tag{91}$$

which means that the discounted price process $s_0, ..., s_T$ is a *martingale with respect to Q*. Such measures Q are called *equivalent martingale measures.* If asset $i = 1$ is riskless, i.e. $B_t = (1+r)^t$, then this notion coincides with the notion of a risk-neutral measure examined in Sections 1 and 2.

Theorem 7.1. *Let $(q_0, p_0, ..., p_T, q_T)$ be a sequence such that $p_t \in \mathcal{P}_t$ $(t = 1, 2, ..., T)$, $q_0 \geq 0$ is a number and $q_T(\omega) \geq 0$ is a real-valued function of ω. Then the following three assertions are equivalent.*

(a) The sequence $(q_0, p_0, ..., p_T, q_T)$ is a strictly consistent price system.

(b) The random process $p_0, ..., p_T$ is a martingale, $q_T > 0$, and there exist nonnegative real-valued functions $q_1(\omega^1), ..., q_{T-1}(\omega^{T-1})$ such that

$$p_t = q_t S_t, \; t = 0, 1, ..., T. \tag{92}$$

(c) We have

$$q_T B_T \in \mathcal{Q}_T;$$

the discount factors q_0 and q_T satisfy

$$q_0 = E\frac{q_T B_T}{B_0}; \tag{93}$$

and the vectors of asset prices $p_0, ..., p_T$ are expressed through $S_0, ..., S_T$ by the formula:

$$p_t = q_t S_t, \textit{ where } q_t = E(\frac{q_T B_T}{B_t}|\omega^t), \; t = 0, ..., T. \tag{94}$$

Proof. By virtue of Propositions 6.2, 6.4 and 6.6, a sequence $(q_0, p_0, ..., p_T, q_T)$ is a consistent price system if and only if

$$q_0 S_0 = p_0, \; q_T S_T = p_T, \tag{95}$$

$p_0, ..., p_T$ is a martingale and

$$p_t(\omega^t)b \leq 0 \text{ for all } b \text{ satisfying } S_t(\omega^t)b \leq 0, \; t = 1, 2, ..., T \tag{96}$$

(see (78) and (83)). Observe that condition (96) holds if and only if

$$p_t(\omega^t) = l_t(\omega^t) S_t(\omega^t) \tag{97}$$

for some real-valued function $l_t(\omega^t) \geq 0$ $(t = 1, ..., T)$. To prove this statement observe that if p_t is of the form (97), then the second inequality in (96) implies the first one. Conversely, assume (96) is valid. Fix some ω^t and consider the linear programming problem: maximize $p_t(\omega^t)b$ subject to $-S_t(\omega^t)b \geq 0$. According to (96), $b = 0$ is a solution to this problem. By virtue of the Kuhn-Tucker theorem (see the Appendix), there exists $l = l_t(\omega) \geq 0$ such that $p_t(\omega^t)b - lS_t(\omega^t)b \leq 0$ $(b \in R^N)$ which yields (97).

(a)⇒(b) Suppose $(q_0, p_0, ..., p_T, q_T)$ is a strictly consistent price system. Then $q_T > 0$. Consider the functions l_t in (97) and define $q_t = l_t$, $t = 1, 2, ..., T-1$. Then $p_0, ..., p_T$ is a martingale satisfying (92).

(b)⇒(c) If the vector process $q_t S_t$, $t = 0, ..., T$, is a martingale, its first coordinate $q_t S_t^1 = q_t B_t$ is a martingale, which yields $q_t B_t = E(q_T B_T \mid \omega^t)$, which implies formula (94)

and the expression for q_t in (93). It remains to show that $q_T B_T \in \mathcal{Q}_T$. This follows from the relations $q_T > 0$ and

$$E(q_T B_T s_t \mid \omega^{t-1}) = E[E(q_T B_T \mid \omega^t)\frac{S_t}{B_t} \mid \omega^{t-1}] =$$

$$E(q_t S_t \mid \omega^{t-1}) = q_{t-1} S_{t-1} = q_{t-1} B_{t-1} s_{t-1} = E(q_T B_T \mid \omega^{t-1}) s_{t-1}.$$

(The second of these equalities holds because $p_t = q_t S_t,\ t = 0, ..., T$, is a martingale.)

(c)⇒(a) Since $q_T > 0$, we obtain that $q_0 > 0$ by virtue of (93). Relations (95) and (96) hold because $p_t = q_t S_t$. To show that p_t is a martingale, we write

$$E(p_t \mid \omega^{t-1}) = E(q_t S_t \mid \omega^{t-1}) = E(q_t B_t s_t \mid \omega^{t-1}) =$$

$$E[E(q_T B_T \mid \omega^t) s_t \mid \omega^{t-1}] = E[q_T B_T s_t \mid \omega^{t-1}] =$$

$$E[q_T B_T \mid \omega^{t-1}] s_{t-1} = E[\frac{q_T B_T}{B_{t-1}} \mid \omega^{t-1}] S_{t-1} = q_{t-1} S_{t-1} = p_{t-1},$$

where the third equality follows from (94) and the fifth from (88a) with $q = q_T B_T \in \mathcal{Q}_T$. □

Assertion (c) of Theorem 7.1 demonstrates the relation between consistent price systems and equivalent martingale measures. If $(q_0, p_0, ..., p_T, q_T)$ is a consistent price system, then $q_T B_T \in \mathcal{Q}_T$, and so $q_T B_T/(E q_T B_T)$ is the density of an equivalent martingale measure (see (88a), (89) and (91)). Conversely, if q is the density of an equivalent martingale measure, then $q \in \mathcal{Q}_T$, and by setting

$$q_T = \frac{q}{B_T},\ q_0 = E\frac{q_T B_T}{B_0},\ p_t = q_t S_t,$$

where $q_t = E(\frac{q_T B_T}{B_t}|\omega^t)$ $(t = 0, ..., T)$, we obtain a consistent price system. By virtue of Theorem 4.2, a consistent price system exists if and only if the no arbitrage hypothesis holds. This we conclude that the latter hypothesis holds if and only if there exists an equivalent martingale measure. This leads to a proof of the Fundamental Theorem of Asset Pricing in the model under consideration.

We can see from Theorem 7.1 that (q_0, q_T) is a pair of strictly consistent discount factors (in symbols, $(q_0, q_T) \in \mathcal{Q}$) if and only if $q_0 = E(q_T \frac{B_T}{B_0})$ and $q_T B_T / E(q_T B_T)$ is the density of an equivalent martingale measure. By virtue of Theorem 4.3, an initial endowment v_0 is sufficient to superhedge a contingent claim v_T if and only if $q_0 v_0 \geq E q_T v_T$ for all $(q_0, q_T) \in \mathcal{Q}$, in other words, if and only if

$$v_0 \geq \sup_{(q_0,q_T)\in\mathcal{Q}} \frac{E q_T v_T}{q_0} = \sup_{(q_0,q_T)\in\mathcal{Q}} \frac{E q_T B_T \dfrac{v_T}{B_T}}{E(\dfrac{q_T B_T}{B_0})} =$$

$$\sup_{q_T B_T \in \mathcal{Q}_T} E[\frac{q_T B_T}{E(q_T B_T)} \frac{B_0 v_T}{B_T}] = \sup_{q \in \mathcal{Q}_T} E(\frac{q}{Eq} \frac{B_0 v_T}{B_T}) = \sup_{Q\in\mathbb{Q}} E^Q \frac{B_0 v_T}{B_T},$$

where the last supremum is taken over the set $\mathbb{Q}$ of all equivalent martingale measures Q and the last but one with respect to their densities q. We formulate the result obtained in the following theorem.

Theorem 7.2. *An initial endowment v_0 is sufficient to superhedge a contingent claim v_T if and only if*

$$v_0 \geq \sup_{Q\in\mathbb{Q}} E^Q \frac{B_0 v_T}{B_T}.$$

If $i = 1$ is a riskless asset, i.e. $B_t = (1 + r)^t$, where $r > 0$ is a non-random number, then we arrive at the formula

$$v_0 \geq \sup_{Q} E^Q \frac{v_T}{(1 + r)^T}$$

fully analogous to that we obtained in Theorem 2.1.

7.2 The conventional model with proportional transaction costs

The material of the present subsection is based on the work of Jouini and Kallal [20] (see also Pham and Touzi [30]). Suppose that, for each $t = 0, ..., T$, we are given a vector $S_t(\omega^t) = (S_t^1(\omega^t), ..., S_t^N(\omega^t)) \geq 0$ specifying the market prices of N assets $i = 1, 2, ..., N$ at time t. Assume that $m_0 = m_T = 1$, and so elements of the space $\mathcal{V}_0$ of initial endowments are real numbers (amounts of cash) and elements of the space $\mathcal{V}_T$ of contingent claims scalar-valued functions $v(\omega)$ (amounts depending on the random situation ω at the terminal moment of time T). Let $\mathcal{M}_0$ and $\mathcal{M}_T$ be the standard cones of non-negative elements in $\mathcal{V}_0$ and $\mathcal{V}_T$, generating the standard partial orderings.

Fix some numbers $\lambda^i \geq 0$ and $1 > \mu^i \geq 0$ – *transaction cost rates.* By selling one unit of asset i at time t, one gets $(1 - \mu^i)S_t^i$, and in order to buy one unit of asset i, one has to pay $(1 + \lambda^i)S_t^i$. As in the classical model considered in the previous subsection, assume that asset $i = 1$ is cash. Suppose $B_t := S_t^1 > 0$ and

$$\lambda^1 = \mu^1 = 0$$

(there are no transaction costs for operating the bank account).

For any vector $a = (a^1, ..., a^N) \in R^N$ and any $i = 1, 2, ..., N$, define

$$\tau^i(a) = (1+\lambda^i)a^i_+ + (1-\mu^i)a^i_-, \tag{98}$$

where $a^i_+ = \max\{a^i, 0\}$ and $a^i_- = \min\{a^i, 0\}$. The functions $\tau^i(a)$ are convex and homogeneous. Consider the mapping $\tau : R^N \to R^N$ acting by the formula

$$\tau(a) = (\tau^1(a), ..., \tau^l(a)). \tag{99}$$

Define the cones Z_t, $t = 1, ..., T$, and W_t $(t = 0, T)$ by

$$Z_t = \{(h_{t-1}, h_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \; S_t\tau(h_t - h_{t-1}) \leq 0\}, \tag{100}$$

$$W_0 = \{(v_0, h_0) \in \mathcal{V}_0 \times \mathcal{X}_0 : \; v_0 \geq S_0\tau(h_0)\}, \tag{101}$$

$$W_T = \{(h_T, v_T) \in \mathcal{X}_T \times \mathcal{V}_T : \; -S_T\tau(-h_T) \geq v_T\}. \tag{102}$$

The inequality $S_t\tau(h_t - h_{t-1}) \leq 0$ in involved in (100) can be written

$$\sum_{i=1}^N (1+\lambda^i)S^i_t(h^i_t - h^i_{t-1})_+ \leq -\sum_{i=1}^N (1-\mu^i)S^i_t(h^i_t - h^i_{t-1})_- \, .$$

The last relation expresses a self-financing condition: assets are purchased only at the expense of sales of other assets. In order to construct a portfolio h_0 at time 0, one needs the amount

$$S_0\tau(h_0) = \sum_{i=1}^N (1+\lambda^i)S^i_0(h^i_0)_+ + \sum_{i=1}^N (1-\mu^i)S^i_0(h^i_0)_-$$

and when liquidating a portfolio h_T at time T, one gets

$$-S_T\tau(-h_T) = -\sum_{i=1}^N (1+\lambda^i)S^i_T(-h^i_T)_+ - \sum_{i=1}^N (1-\mu^i)S^i_T(-h^i_T)_- =$$

$$\sum_{i=1}^N (1+\lambda^i)S^i_T(h^i_T)_- + \sum_{i=1}^N (1-\mu^i)S^i_T(h^i_T)_+.$$

This leads to the definitions of the cones W_0 and W_T in (101) and (102).

The model under consideration is defined in terms of difference inclusions, since we can represent Z_t in the form

$$Z_t = \{(h_{t-1}, h_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : \; h_t - h_{t-1} \in M_t\}, \tag{103}$$

where

$$M_t = \{y \in \mathcal{X}_t : S_t \tau(y) \le 0\}. \tag{104}$$

The cones W_0 and W_T are defined in terms of the value operators

$$V_0(h_0) = S_0 \tau(h_0), \; V_T(h_T) = -S_T \tau(-h_T). \tag{105}$$

Consider the *discounted prices* $s_t^i = S_t^i / B_t^i$ of assets $i = 1, 2, ..., N$. Denote by $\bar{\mathcal{Q}}_T$ the set of those real-valued functions $q(\omega) > 0$ for which there exists a sequence $\bar{s}_0, \bar{s}_1(\omega^1), ..., \bar{s}_T(\omega^T)$ of vector functions with values in R^N such that

$$(1 - \mu^i) s_t^i \le \bar{s}_t^i \le (1 + \lambda^i) s_t^i, \; t = 0, 2, ..., T, \tag{106}$$

and

$$E(q\bar{s}_t \mid \omega^{t-1}) = E(q \mid \omega^{t-1}) \bar{s}_{t-1}, \; t = 1, 2, ..., T. \tag{107}$$

The set $\bar{\mathcal{Q}}_T$ consists of functions $q(\omega) > 0$ for which some sequence $\bar{s}_t$ satisfying (106) is a martingale with respect to the measure $Q(\omega) = \dfrac{q(\omega)}{Eq} P(\omega)$. If $\lambda^i = \mu^i = 0$ (there are no transaction costs), then $\bar{\mathcal{Q}}_T$ coincides with the set $\mathcal{Q}_T$ introduced in the previous subsection.

Theorem 7.3. *Let $(q_0, p_0, ..., p_T, q_T)$ be a sequence such that $p_t \in \mathcal{P}_t$ $(t = 1, 2, ..., T)$, $q_0 \ge 0$ is a number and $q_T(\omega) \ge 0$ is a real-valued function of ω. Then the following assertions are equivalent.*

(a) The sequence $(q_0, p_0, ..., p_T, q_T)$ is a strictly consistent price system.

(b) The random process $p_0, ..., p_T$ is a martingale, $q_T > 0$, and there exist nonnegative real-valued functions $q_1(\omega^1), ..., q_{T-1}(\omega^{T-1})$ such that

$$q_t S_t^i (1 - \mu^i) \le p_t^i \le q_t S_t^i (1 + \lambda^i), \; t = 0, ..., T. \tag{108}$$

(c) The function $q_T B_T$ belongs to the class $\bar{\mathcal{Q}}_T$, the discount factors q_0 and q_T are related to each other by formula $q_0 = E(\dfrac{q_T B_T}{B_0})$, and the sequence of asset prices $p_0, ..., p_T$ is a martingale satisfying (108), where $q_1, ..., q_{T-1}$ are defined by $q_t = E(\dfrac{q_T B_T}{B_t} | \omega^t)$.

Proof. By virtue of Propositions 6.2, 6.4 and 6.6, a sequence $(q_0, p_0, ..., p_T, q_T)$ is a consistent price system if and only if

$$q_0 S_0 \tau(b) \ge p_0 b, \; p_T(\omega) b \ge -q_T(\omega) S_T(\omega) \tau(-b) \text{ for all } b \in R^N. \tag{109}$$

$p_0, ..., p_T$ is a martingale, and

$$p_t(\omega^t)b \leq 0 \text{ for all } b \in R^N \text{ satisfying } S_t(\omega^t)\tau(b) \leq 0, \tag{110}$$

$(t = 1, 2, ..., T)$. In the model we deal with, the inequalities involved in (109) are equivalent to those in (70) and (78)). Property (110) is derived from (83) under the assumption that M_t defined by (104).

Let us show that condition (110) holds if and only if

$$l_t S_t^i(1 - \mu^i) \leq p_t^i \leq l_t S_t^i(1 + \lambda^i), \; i = 1, ..., N, \tag{111}$$

for some real-valued functions $l_t = l_t(\omega^t) \geq 0$ $(t = 1, ..., T)$. Consider the following optimization problem: maximize $p_t(\omega^t)b$ over $b \in R^N$ subject to $-S_t(\omega^t)\tau(b) \geq 0$. Property (110) is equivalent to the assertion that $b = 0$ is a solution to this problem. By virtue of the Kuhn-Tucker theorem (see the Appendix), this assertion holds if and only if there exists $l = l_t(\omega) \geq 0$ such that

$$p_t(\omega^t)b - l_t(\omega)S_t(\omega^t)\tau(b) \leq 0 \text{ for all } b \in R^N. \tag{112}$$

(The Kuhn-Tucker theorem can be applied because the function $-S_t(\omega^t)\tau(b)$ is concave and the Slater condition $-S_t(\omega^t)\tau(b) > 0$ is fulfilled for $b = (-1, 0, 0, ..., 0)$.) Since $S_t(\omega^t)\tau(b) = \sum_{i=1}^N S_t^i(\omega^t)\tau^i(b)$, inequality (109) is valid if and only if the analogous "coordinatewise" inequality

$$p_t^i(\omega^t)r - l_t(\omega)S_t^i(\omega^t)\tau^i(r) \leq 0, \; i = 1, ..., N, \; r \in R^1, \tag{113}$$

is valid. It remains to observe that, for $r < 0$, (113) is equivalent to the first inequality in (111), and for $r > 0$, (113) is equivalent to the second inequality in (111).

We note that relations (109) hold if and only if, for $t = 0, T$, we have

$$q_t S_t^i(1 - \mu^i) \leq p_t^i \leq q_t S_t^i(1 + \lambda^i), \; i = 1, ..., N, \; t = 0, T. \tag{114}$$

This follows from the assertion that 113) is equivalent to (111). To use this assertion it is sufficient to set $l_t = q_t$ $(t = 0, T)$.

(a)⇒(b) Suppose $(q_0, p_0, ..., p_T, q_T)$ is a strictly consistent price system. Then $q_T > 0$. Consider the functions l_t in (111) and define $q_t = l_t$, $t = 1, 2, ..., T - 1$. Then $p_0, ..., p_T$ is a martingale satisfying (108).

(b)⇒(c) If the vector process p_t, $t = 0, ..., T$, is a martingale, its first coordinate p_t^1 is a martingale. From (108), by using the relations $S_t^1 = B_t$ and $\lambda^1 = \mu^1 = 0$, we get $p_t^1 = q_t B_t$. The fact that the process $q_t B_t$, $t = 0, ..., T$, is a martingale implies (94) and (93). It remains to show that $q_T B_T \in \bar{Q}_T$. Define $q = q_T B_T$ (> 0) and $\bar{s}_t = p_t / q_t B_t$. Then $\bar{s}_t$ satisfies (106) by virtue of (108), and we have

$$E(q\bar{s}_t \mid \omega^{t-1}) = E(\frac{q_T B_T}{q_t B_t} p_t \mid \omega^{t-1}) = E[(\frac{q_T B_T}{q_t B_t} \mid \omega^t) p_t \mid \omega^{t-1}] =$$

$$E[p_t \mid \omega^{t-1}] = p_{t-1} = q_{t-1}B_{t-1}\bar{s}_{t-1} = E(q \mid \omega^{t-1})\bar{s}_{t-1},\ t = 1, 2, ..., T,$$

which means that $q = q_T B_T \in \bar{\mathcal{Q}}_T$.

(c)⇒(a) Since $q_T > 0$ (which is true because $q_T B_T \in \bar{\mathcal{Q}}_T$), we obtain that $q_0 > 0$ by virtue of (93). Inequalities (114) follow from (108) with $t = 0, T$. By setting $l_t = q_t,\ t = 1, ..., T-1$, we obtain (111) as a consequence of (108). Since $p_t,\ t = 0, ..., T$, is a martingale, we find that $(q_0, p_0, ..., p_T, q_T)$ is a consistent price system. □

Denote by $\bar{\mathbb{Q}}$ the class of probability measures Q equivalent to P with density q/Eq, where $q \in \bar{\mathcal{Q}}_T$.

Theorem 7.4. *An initial endowment v_0 is sufficient to superhedge a contingent claim v_T if and only if*

$$v_0 \geq \sup_{Q \in \bar{\mathbb{Q}}} E^Q \frac{B_0 v_T}{B_T}.$$

Proof. By virtue of Theorem 4.3, v_0 is sufficient to superhedge v_T if and only if

$$q_0 v_0 \geq E q_T v_T \tag{115}$$

for all strictly consistent price systems $(q_0, p_0, ..., p_T, q_T)$. In view of assertion (c) of Theorem 7.3, for any such price system, we have

$$B_T q_T \in \bar{\mathcal{Q}}_T \text{ and } q_0 = E\frac{q_T B_T}{B_0}. \tag{116}$$

Conversely, suppose discount factors q_0, q_T satisfy (116). Consider the process $\bar{s}_t$ satisfying (106) and (107) with $q = q_T B_T \in \bar{\mathcal{Q}}_T$. Define $q_t = E(q_T B_T B_t^{-1} \mid \omega^t)$, $t = 1, ..., T-1$, and $p_t = q_t B_t \bar{s}_t$. Then (108) follows from (106) and formulas (93), (94) follow from the definition of q_t and (116). The sequence $p_0, ..., p_T$ is a martingale because

$$E(p_t \mid \omega^{t-1}) = E(q_t B_t \bar{s}_t \mid \omega^{t-1}) == E(E(q_T B_T \mid \omega^t)\bar{s}_t \mid \omega^{t-1}) =$$

$$E(q_T B_T \bar{s}_t \mid \omega^{t-1}) = E(q_T B_T \mid \omega^{t-1})\bar{s}_{t-1} = E(q_T B_T \mid \omega^{t-1})(q_t B_t)^{-1} p_{t-1} = p_{t-1}$$

by virtue of (107) with $q = q_T B_T$. Thus $(q_0, p_0, ..., p_T, q_T)$ is a consistent price system in view of assertion (c) of Theorem 7.3.

By using (115) and (116), we obtain that v_0 is sufficient to superhedge v_T if and only if

$$v_0 \geq \frac{E q_T v_T B_0}{E q_T B_T} \text{ for all } q_T \text{ with } q_T B_T \in \bar{\mathcal{Q}}_T,$$

or in other words, if and only if

$$v_0 \geq \sup_{q \in \bar{\mathcal{Q}}_T} E\frac{q v_T B_0}{B_T E q} = \sup_{Q \in \bar{\mathbb{Q}}} E^Q \frac{v_T B_0}{B_T},$$

which completes the proof. □

7.3 A multicurrency model with short sales

The model we discuss here is a version of that studied in a series of papers by Kabanov and coauthors (see, e.g., [23], [25], [22], [9]). Consider a financial market where N currencies $i = 1, 2, ..., N$ are traded. Admissible portfolios h_t at time t are any vector functions $h_t(\omega^t)$ with values in R^N, so that short sales are allowed. For each $t = 1, 2, ..., T$, we are given an $N \times N$ matrix $(\mu_t^{ij}(\omega^t))$ with $\mu_t^{ij} > 0$ and $\mu_t^{ii} = 1$. The numbers μ_t^{ij} represent the *exchange rates* of the currencies (including transaction costs). For one unit of currency j, at time t, one can get μ_t^{ij} units of currency i. A portfolio of currencies $h_{t-1} = (h_{t-1}^1, ..., h_{t-1}^N)$ can be exchanged to a portfolio $h_t = (h_t^1, ..., h_t^N)$ at time t in a random situation ω^t if and only if there exists a nonnegative $N \times N$ matrix $(b_t^{ij}(\omega^t))$ such that

$$h_t^i \le h_{t-1}^i + \sum_{j \ne i} \mu_t^{ij} b_t^{ij} - \sum_{j \ne i} b_t^{ji}, \ i = 1, ..., N. \tag{117}$$

Here, b_t^{ji} stands for the amount of currency i exchanged into currency j. Therefore the sum $\sum_{j \ne i} b^{ji}$ is subtracted from the ith position of the portfolio. In the course of the exchange, one gets the amount $\sum_{j \ne i} \mu_t^{ij} b_t^{ij}$ of currency i. Hence the sum $\sum_{j \ne i} \mu_t^{ij} b_t^{ij}$ is added to the ith position of the portfolio. The inequality (rather than equality) in (117) points to a possibility of consumption.

To embed the above model into the general framework studied in this paper, we define

$$M_t = \{y \in \mathcal{X}_t : \ y^i \le \sum_{j \ne i} \mu_t^{ij} b_t^{ij} - \sum_{j \ne i} b_t^{ji} \text{ for some matrix } (b_t^{ji}(\omega^t)) \ge 0\}$$

and $Z_t = \{(h_{t-1}, h_t) : \ h_t - h_{t-1} \in M_t\}$. Thus the cone Z_t is defined in terms of difference inclusions. Note that the cone M_t is decomposable with respect to ω^t: we have $y \in M_t$ if and only if $y(\omega^t) \in M_t(\omega^t)$, where

$$M_t(\omega^t) = \{b \in R^N : b^i \le \sum_{j \ne i} \mu_t^{ij} b_t^{ij} - \sum_{j \ne i} b_t^{ji} \text{ for some matrix } (b_t^{ji}) \ge 0\}.$$

Further, we assume that $m_0 = m_T = N$, $\mathcal{V}_0 = \mathcal{X}_0$ and $\mathcal{V}_T = \mathcal{X}_T$, and so both initial endowments and contingent claims are represented by portfolios of currencies. We define the value operators as the identity mappings

$$V_0(h_0) = h_0 \ (h_0 \in \mathcal{X}_0), \ V_0(h_0) = h_0 \ (h_0 \in \mathcal{X}_0). \tag{118}$$

These operators characterize wealth contained in a portfolio h by the vector h itself: there are no natural aggregate indicators of value in the multicurrency context. To complete the model description, we define the cones$\mathcal{M}_0$ and $\mathcal{M}_T$ (and the partial orderings associated with them) in the spaces $\mathcal{V}_0$ and $\mathcal{V}_T$ to be standard.

The main results related to the model at hand are presented in the following theorem.

Theorem 7.5 (cf. [25], Theorems 3.2 and 4.1). *In the model under consideration, strictly consistent price systems are sequences* $(q_0, p_0, ..., p_T, q_T)$ *such that* $q_0 = p_0$, $p_T = q_T$ *and* $p_0, ..., p_T$ $(p_t \in \mathcal{P}_t)$ *is a martingale satisfying the following conditions:*

$$p_t > 0,\ t = 0, ..., T \tag{119}$$

and

$$\mu_t^{ij} p_t^i \le p_t^j,\ t = 1, 2, ..., T,\ i, j = 1, ..., N, \tag{120}$$

where p_t^i *is the ith coordinate of the vector* p_t. *The no-arbitrage hypothesis* $(\mathcal{NA})$ *is equivalent to the existence such a martingale. A contingent claim* $v_T \in \mathcal{V}_T$ *can be superhedged starting from an initial endowment* $v_0 \in \mathcal{V}_0$ *if and only if* $Ep_0 v \ge Ep_T w$ *for any martingale* $p_0, ..., p_T$ *satisfying (119) and (120).*

Proof. By virtue of Propositions 6.2, 6.4 and 6.6, a sequence $(q_0, p_0, ..., p_T, q_T)$ is a strictly consistent price system if and only if $q_0 > 0$, $q_T > 0$,

$$q_0 h_0 \ge p_0 h_0\ (h_0 \in \mathcal{X}_0),\ p_T h_T \ge q_T h_T\ (h_T \in \mathcal{X}_T), \tag{121}$$

$p_0, ..., p_T$ is a martingale and

$$p_t(\omega^t) b \le 0 \text{ for all } b \le \sum_{j \ne i} \mu_t^{ij}(\omega^t) b^{ij} - \sum_{j \ne i} b^{ji}, \tag{122}$$

where (b^{ji}) is any nonnegative matrix (see (70), (78) and (83)). Clearly the inequalities in (121) hold if and only if

$$q_0 = p_0 \text{ and } p_T = q_T. \tag{123}$$

Property (122) can be equivalently stated as

$$\sum_{i=1}^N p_t^i(\omega^t)[\sum_{j \ne i} \mu_t^{ij}(\omega^t) b^{ij} - \sum_{j \ne i} b^{ji}] \le 0 \tag{124}$$

for all $(b^{ji}) \ge 0$. Since $\mu_t^{ii} = 1$, we can replace in (124) the sums over $j \ne i$ by the analogous sums over $j = 1, ..., N$. This yields

$$\sum_{i,j=1}^N p_t^i \mu_t^{ij} b^{ij} \le \sum_{i,j=1}^N p_t^j b^{ij} \text{ for all } (b^{ji}) \ge 0,$$

which is equivalent to (120). As long as $p_0, ..., p_T$ is a martingale and the equalities in (123) are valid, condition (119) holds if and only if $q_0 > 0$ and $q_T > 0$ because $p_t = E(q_T \mid \omega^t)$. This completes the proof of the first assertion of Theorem 7.5. To prove the second and the third assertions it is sufficient to refer to Theorems 4.2 and 4.3. □

7.4 Currency exchange without borrowing and short sales

We consider another multicurrency model, similar to that proposed in [15]. As in the foregoing subsection, we are given, for each $t = 1, ..., T$, a matrix $\{\mu_t^{ij}(\omega^t)\}$ with $\mu_t^{ij} > 0$ and $\mu_t^{ii} = 1$, specifying the exchange rates of N currencies $i = 1, 2, ..., N$. In the present model, portfolios $h_t(\omega^t)$ are supposed to be nonnegative vectors with values in R^N, which excludes possibilities of short sales. A portfolio $h_{t-1} = (h_{t-1}^1, ..., h_{t-1}^N) \in \mathcal{P}_{t-1}$ can be transformed into portfolio $h_t = (h_t^1, ..., h_t^N) \in \mathcal{P}_t$ at time t if and only if there exists a nonnegative matrix $(d_t^{ji}(\omega^t))$ (the *exchange matrix*) such that

$$h_{t-1}^i(\omega^{t-1}) \geq \sum_{j=1}^N d_t^{ji}(\omega^t),\ 0 \leq h_t^i(\omega^t) \leq \sum_{j=1}^N \mu_t^{ij}(\omega^t) d_t^{ij}(\omega^t). \tag{125}$$

The set of all such portfolio pairs will be denoted by Z_t. Here, d_t^{ji} $(i \neq j)$ stands for the amount of currency i converted into currency j. The amount d_{t-1}^{ii} of currency i is left unexchanged. The first inequality in (125) is a balance constraint for the currency i: one cannot exchange more of it than is available at time $t-1$ (no borrowing is allowed). The second inequality in (125) says that, at time t, the ith position of the portfolio cannot be greater than the sum $\sum_{j=1}^l \mu_t^{ij} d_{t-1}^{ij}$ obtained as a result of the exchange. We define $m_0 = m_T = N$, put $\mathcal{V}_t = \mathcal{X}_t$, $t = 0, T$, and denote by $\mathcal{M}_t$ the standard cones in the spaces $\mathcal{V}_t$ $(t = 0, T)$. Finally, we set

$$W_0 = \{(v_0, h_0) \in \mathcal{X}_0 \times \mathcal{P}_0 : v_0 \geq h_0\}, \tag{126}$$

$$W_T = \{(h_T, v_T) \in \mathcal{P}_T \times \mathcal{X}_T : h_T \geq v_T\}. \tag{127}$$

The model we have just described will be denoted by $\mathbf{M}$.

Observe that, in the deterministic case (when the space of states of the world consists of one element), the cone Z_t can be represented in the form

$$Z_t = \{(x, y) :\ x \geq \mathbf{A}d,\ 0 \leq y \leq \mathbf{B}_t d \text{ for some } d \in R_+^{N^2}\}.$$

Here $\mathbf{A} : R^{N^2} \to R^N$ and $\mathbf{B}_t : R^{N^2} \to R^N$ are nonnegative linear operators transforming an element $d = (d^{ij}) \in R^{N^2}$ into the vectors $\mathbf{A}d$ and $\mathbf{B}_t d$ whose coordinates are defined by

$$(\mathbf{A}d)^i = \sum_{j=1}^N d^{ji} \text{ and } (\mathbf{B}_t d)^i = \sum_{j=1}^N \mu_t^{ij} d^{ij}\ (i = 1, 2, ..., N).$$

Thus the model $\mathbf{M}$ is a direct stochastic analogue of the von Neumann [38] model of economic growth ($\mathbf{A}$ and $\mathbf{B}_t$ being the counterparts of the "technology matrices").

The following theorem contains results regarding the model $\mathbf{M}$.

Theorem 7.6. *In the model* $\mathbf{M}$*, strictly consistent price systems are sequences* $(q_0, p_0, ..., p_T, q_T)$ *such that* $q_T > 0$

$$q_0 \geq p_0, \ p_T \geq q_T, \tag{128}$$

and $p_0, ..., p_T$ $(p_t \in \mathcal{P}_t)$ *is a strictly positive supermartingale satisfying the following condition:*

(π) *For every* $t = 1, 2, ..., N$*, there exists a strictly positive vector function* $\pi_t(\omega^t)$ *such that*

$$\mu_t^{ij} p_t^i \leq \pi_t^j, \ t = 1, 2, ..., T, \ i, j = 1, ..., N, \tag{129}$$

and

$$E(\pi_t \mid \omega^{t-1}) \leq p_{t-1}. \tag{130}$$

The no-arbitrage hypothesis holds always. A contingent claim $v_T \in \mathcal{V}_T$ *can be superhedged starting from an initial endowment* $v_0 \in \mathcal{V}_0$ *if and only if* $Ep_0 v \geq Ep_T w$ *for any strictly positive supermartingale* $p_0, ..., p_T$ *satisfying condition* (π)*.*

Proof. To characterize consistent price systems, we examine conditions (42) - (44). The first and the third of these hold if and only if $q_0 \geq p_0$ and $p_T \geq q_T$. Requirement (43) can be written in the following equivalent form:

$$E \sum_{i=1}^{N} p_t^i(\omega^t) \sum_{j=1}^{N} \mu_t^{ij}(\omega^t) d_t^{ij}(\omega^t) - E \sum_{i=1}^{N} p_{t-1}^i(\omega^{t-1}) h_{t-1}^i(\omega^{t-1}) \leq 0 \tag{131}$$

for each vector function $h_{t-1} = h_{t-1}(\omega^{t-1})$ and each matrix function $d_t = (d_t^{ji}(\omega^t))$ satisfying

$$h_{t-1}^i(\omega^{t-1}) \geq \sum_{j=1}^{N} d_t^{ji}(\omega^t). \tag{132}$$

This property means that $(h_{t-1}, d_t) = (0, 0)$ maximizes the expression on the left-hand side of (43) among all $(h_{t-1}, d_t) = (h_{t-1}(\omega^{t-1}), d_t(\omega^t)) \geq 0$ subject to constraint (43). By applying the Kuhn-Tucker theorem to the linear programming problem at hand, we obtain that the above property holds if and only if there exists a vector function $\pi_t = \pi_t(\omega^t) \geq 0$ such that

$$E \sum_{i=1}^{N} p_t^i \sum_{j=1}^{N} \mu_t^{ij} d_t^{ij} - E \sum_{i=1}^{N} p_{t-1}^i h_{t-1}^i + E\pi_t [h_{t-1}^i - \sum_{j=1}^{N} d_t^{ji}] \leq 0 \tag{133}$$

for all $(h_{t-1}, d_t) = (h_{t-1}(\omega^{t-1}), d_t(\omega^t)) \geq 0$. Replacing the term $E\pi_t h_{t-1}^i$ in (133) by $E[E(\pi_t \mid \omega^{t-1}) h_{t-1}^i]$ and maximizing the expression in (133) separately with respect to h_{t-1} and d_t, we obtain that (133) is equivalent to (129) and (130). Observe that conditions (129) and (130) imply

$$E(\mu_t^{ij} p_t^i \mid \omega^{t-1}) \leq p_{t-1}^j,\ t = 1, 2, ..., T,\ i, j = 1, ..., N. \tag{134}$$

By setting $i = j$ and using the fact that $\mu_t^{ii} = 1$, we obtain the inequality $E(p_t \mid \omega^{t-1}) \leq p_{t-1}$, and so any sequence $p_0, ..., p_T$ $(p_t \in \mathcal{P}_t)$ satisfying (π) is a supermartingale. Thus we have shown that $(q_0, p_0, ..., p_T, q_T)$ is a consistent price system if and only if the relations in (128) hold and $p_0, ..., p_T$ $(p_t \in \mathcal{P}_t)$ is a supermartingale satisfying condition (π).

Suppose $(q_0, p_0, ..., p_T, q_T)$ is a strictly consistent price system. Then $p_T \geq q_T > 0$, which implies that the supermartingale $p_0, ..., p_T$ is strictly positive (indeed, $p_t \geq E(p_T \mid \omega^t) > 0$). Conversely, if $q_T > 0$, the relations in (128) hold and $p_0, ..., p_T$ is a supermartingale, then we have $q_0 \geq p_0 = Ep_T \geq Eq_T > 0$. Thus $q_0 > 0$, and so $(q_0, p_0, ..., p_T, q_T)$ is a strictly consistent price system. This completes the proof of the first assertion of Theorem 7.6.

To verify $(\mathcal{NA})$ observe that if $(h_{t-1}, h_t) \in Z_t$ and $h_{t-1} = 0$, then $h_t = 0$. This is immediate from (128). Thus if $(v_0, h_0, ..., h_T, v_T)$ is a superhedging programme with $v_0 \leq 0$ and $v_T \geq 0$, we have $v_0 \geq h_0 \geq 0$, hence $h_0 = h_1 = ... = h_T = 0$, which implies $v_T = 0$ because $v_T \leq h_T \leq 0$.

The last assertion of Theorem 7.6 is a direct consequence of Theorem 4.3. □

Consider a modification of the model $\mathbf{M}$ (which will be denoted by $\mathbf{M}'$) described like $\mathbf{M}$, with the only difference that the cones Z_t are defined as follows. A pair of portfolios $(h_{t-1}, h_t) \in \mathcal{P}_{t-1} \times \mathcal{P}_t$ belongs to Z_t if there exists a matrix function $d_{t-1}^{ji} = d_{t-1}^{ji}(\omega^{t-1}) \geq 0$ such that

$$h_{t-1}^i(\omega^{t-1}) \geq \sum_{j=1}^N d_{t-1}^{ji}(\omega^{t-1}),\ 0 \leq h_t^i(\omega^t) \leq \sum_{j=1}^N \mu_t^{ij}(\omega^t) d_{t-1}^{ij}(\omega^{t-1}). \tag{135}$$

The difference between (125) and (135) lies in the fact that, in the former formula, the exchange matrix d_t depends on the market history ω^t (which is known at time t), whereas in the latter, the matrix d_{t-1} depends on ω^{t-1} (which is known one time period earlier). Although this assumption might seem restrictive, it is fulfilled for an important class of exchange strategies. This is the class of *fixed-mix strategies* (see, e.g., [10]), for which

$$d_{t-1}^{ji}(\omega^{t-1}) = h_{t-1}^i(\omega^{t-1}) \theta^{ji},$$

where $(\theta^{ji}) \geq 0$ is a fixed (non-random) matrix with $\sum_{j=1}^N \theta^{ji} = 1$. The number θ^{ji} represents the *proportion* of the available amount of currency j that is exchanged into currency i.

Theorem 7.7. *For the model $\mathbf{M}'$, all the assertions of Theorem 7.6 remain valid with the only difference that condition (π) involved in the characterization of consistent price systems should be replaced by condition (134).*

Proof. The arguments basically repeat those in the proof of Theorem 7.6. Only when analyzing requirement (43), we observe that this requirement can be written in the following equivalent form:

$$E\sum_{i=1}^{N} p_t^i(\omega^t)\sum_{j=1}^{N}\mu_t^{ij}(\omega^t)d_{t-1}^{ij}(\omega^{t-1}) - E\sum_{i=1}^{N} p_{t-1}^i(\omega^{t-1})\sum_{j=1}^{N} d_{t-1}^{ji}(\omega^{t-1}) \le 0 \tag{136}$$

for each matrix function $d_{t-1} = (d_{t-1}^{ji}(\omega^{t-1})) \geq 0$. Clearly (136) holds if and only if

$$E[E(p_t^i\mu_t^{ij} \mid \omega^{t-1})d_{t-1}^{ij}] \le Ep_{t-1}^i d_{t-1}^{ji}$$

for each i, j and $d_{t-1}^{ji} = d_{t-1}^{ji}(\omega^{t-1}) \geq 0$, which is equivalent to (134). $\square$

Appendix: Some facts of convex analysis

A.1. Separation theorems.

Theorem A.1. *Let X and Y be convex sets in R^n such that*

$$X \cap Y = \emptyset.$$

Then there exists a linear function qx $(x \in X)$ such that

$$qx \leq qy$$

for all $x \in X$, $y \in Y$.

Theorem A.2. *If X and Y are convex sets in R^n such that the distance between X and Y is strictly positive, then there is a linear function qx such that*

$$\sup_{x \in X} qx < \inf_{y \in Y} qy.$$

A.2. Separation of cones by strictly positive functionals. Let K be a closed proper cone[4] in R^n. The cone K defines a partial ordering on R^n

$$x \leq_K y \;\Leftrightarrow\; y - x \in K.$$

For example, if $K = R^n_+$ (standard cone in R^n), then the inequality $x \leq_K y$ means that $x^i \leq y^i$ for all $i = 1, ...n$. We write K^* for the set of all those linear functionals q on R^n for which $qx \geq 0$, $x \in K$. We denote by K^+ the set of those q in K^* that satisfy $qx > 0$ for each non-zero element $x \in K$. Functionals in K^* (resp. in K^+) are called *positive* (resp. *strictly positive*). The cone K^* is called the *dual* to the cone K.

Let A be a closed cone in R^n.

Theorem A.3. *The following two assertions are equivalent:*

(**NA**) $A \cap K = \{0\}$.

(**Q**) *There exists $q \in K^+$ such that $qa \leq 0$ for all $a \in A$.*

Define

$$\mathcal{Q} = \{q \in K^+ : \; qa \leq 0, \; a \in A\}.$$

Theorem A.3 says that

$$(\mathbf{NA}) \Leftrightarrow \mathcal{Q} \neq \emptyset.$$

To prove Theorem A.3, fix some convex compact set $\Sigma \subseteq K$ such that $0 \notin \Sigma$ and $K = \{\lambda x : \lambda \geq 0, \, x \in \Sigma\}$. The set Σ exists since the cone K is proper and closed (one can take as Σ the convex hull of the intersection of K with the Euclidean sphere in R^n).

[4] A cone K is called *proper* if the relations $x \in K$ and $-x \in K$ imply $x = 0$.

Proof of Theorem A.3. The implication (**Q**)$\Rightarrow$(**NA**) is straightforward. Let (**NA**) hold. Since the closed convex set A does not intersect the convex compact set Σ, the distance between A and Σ is strictly positive, and so, by virtue of Theorem A.2, there exists a linear functional q on R^n which strongly separates A and Σ, i.e.,

$$\sup_{a\in A} qa < \inf_{b\in\Sigma} qb$$

This implies $\sup_{a\in A} qa = 0$, as long as A is a cone. Furthermore, we have $qx > 0$ for each $x \in K$, $x \neq 0$, because any such x is of the form $x = \lambda b$, where $\lambda > 0$ and $b \in \sum$. $\square$

Remark A.1. Theorem A.3 reflects the content of various versions of the FTAP (in the finite-dimensional case). In this work, we derive from it some specific versions of FTAP in several models. Condition (**NA**) (interpreted in the applications as the *no-arbitrage hypothesis*) states that if $a \in A$ and $0 \leq_K a$, then $a = 0$. Assertion (**Q**) deals with a strictly positive functional q whose maximum on A is zero. According to Theorem A.3, the existence of this functional is equivalent to (**NA**). We may interpret (**NA**) as the hypothesis that the point 0 is a *Pareto-optimal element* of the set A, the optimality being defined in terms of the partial ordering $\leq_K$. Theorem A.3 expresses the idea that an element is Pareto-optimal if and only if it maximizes a strictly positive linear functional (for related results see, e.g., [2], Section 3.5).

When A is a linear space, Theorem A.3 admits the following refinement.

Theorem A.4. *If A is a linear space, then*

$$Q = \{q \in K^+ : \ qa = 0, \ a \in A\},$$

and (**NA**) *holds if and only if the following assertion is valid:*

($\mathbf{Q}_0$) *There exists $q \in K^+$ such that $qa = 0$ for all $a \in A$.*

Proof. If A is a linear space, then the inequality $qa \leq 0$ holds for all $a \in A$ if and only if $qa = 0$ for each $a \in A$. $\square$

Remark A.2. Theorem A.4 corresponds to the classical case of a frictionless market. In that case, assertion ($\mathbf{Q}_0$) turns into the statement regarding the existence of an equivalent martingale measure.

A.3. A duality theorem. Theorem A.5. below characterizes elements u in the cone $A - K$ in terms of *dual variables* - linear functionals q in Q. It may be thought of as an abstract version of various hedging results.

Theorem A.5. *Let* (**NA**) *hold. Then, for any $u \in R^n$, the following assertions are equivalent:*

(**H**) *There exists an element $a \in A$ such that $u \leq_K a$ (i.e., $u \in A - K$).*

(**U**) *We have*

$$qu \leq 0 \text{ for all } q \in Q.$$

Note that $u \in A - K \iff u = a - k$ for some $a \in A$ and $k \in K \Rightarrow a - u = k \in K \iff u \leq_K a$.

Before proving Theorem A.5, we consider the following consequence of it.

Theorem A.6. *Let* (**NA**) *hold. Suppose A is a linear space. Then, for any $u \in R^n$, the following assertions are equivalent:*

(**H**$_0$) *The vector u belongs to A.*

(**U**$_0$) *We have*

$$qu = 0 \text{ for all } q \in Q.$$

Proof. Suppose (**H**$_0$) holds. Then by virtue of Theorem A.5 (the implication (**H**)⇒(**U**)), we have $qu \leq 0$ for all $q \in$Q. Since A is a linear space, this means that $qu = 0$ for all $q \in$Q, which is stated in (**U**$_0$).

Let (**U**$_0$) hold. In view of Theorem A.5, we have $u \leq_K a$, i.e., $a - u \in K$ for some $a \in A$. We have to show that $u = a$ for some $a \in A$. To this end let us apply Theorem A.5 to a modified model in which the cone K is replaced by $-K$. For this model, condition (**NA**) is fulfilled as well. Indeed, if $a \in -K$ for some $a \in A$, then $-a \in K \cap A$ (since A is a linear space), and so $a = 0$. Furthermore, (**U**$_0$) holds for the modified model too. Indeed, if $q \in (-K)^+$, and $qa = 0$, $a \in A$, then $-q \in K^+$ and $-qa = 0$, $a \in A$. Consequently, $-qu = 0$, and so $qu = 0$. By applying Theorem A.5 in this situation, we find an $a' \in A$ such that $u \leq_{(-K)} a'$, which means $a' - u \in -K$, or $u - a' \in K$. By adding up this relation and the relation $a - u \in K$, we get $a - a' \in K$. Furthermore, $a - a' \in A$. This implies $a = a'$ in view of (**NA**). Hence $a - u \in K$ and $u - a \in K$, which yields $u = a$ because K is a proper cone. □

A.4. A lemma for Theorem A.5. The proof of Theorem A.5 is based on Lemma A.1 below. Denote by B_ϵ the closed ball $\{x \in R^n : |x| \leq \epsilon\}$, where $|\cdot|$ is the Euclidean norm in R^n, and consider the closed ϵ-neighborhood of Σ:

$$\Sigma_\epsilon = \Sigma + B_\epsilon.$$

Put

$$K_\epsilon = \{\lambda y : \ \lambda \geq 0, \ y \in \Sigma_\epsilon\}.$$

If $\epsilon > 0$ is small enough (which will be assumed in what follows), the convex compact set Σ_ϵ does not contain 0, and so the cone K_ϵ spanned on Σ_ϵ is proper and closed.

Lemma A.1. *Let condition* (**NA**) *be fulfilled. Let u be a vector in R^n such that $(A - u) \cap K = \emptyset$. Then there exists a number $\epsilon > 0$ for which the distance between the sets $A - u$ and K_ϵ is strictly positive.*

Proof. Suppose the contrary. Then there exist sequences $a_k \in A$, $\lambda_k \geq 0$, $s_k \in \Sigma$, $b_k \in R^n$ such that $|b_k| \leq k^{-1}$ and

$$|a_k - u - \lambda_k s_k - \lambda_k b_k| \leq k^{-1}$$

$(k = 1, 2, ...)$. We may assume without loss of generality that $s_k \to s \in \Sigma$, since Σ is compact. If λ_k is bounded, then, along a subsequence, $\lambda_k \to \lambda \geq 0$. Consequently, $a_k - u \to$

$\lambda s \in K$, since $\lambda_k b_k \to 0$. Thus $a_k \to a \in A$, where $a - u \in K$. This is a contradiction, since $(A - u) \cap K = \emptyset$. If λ_k is unbounded, then by passing to a subsequence, we obtain $0 < \lambda_k \to \infty$,

$$|\frac{a_k}{\lambda_k} - b_k - \frac{u}{\lambda_k} - s_k| \to 0,$$

and $a_k \lambda_k^{-1} \to s \in \Sigma$ (since $b_k \to 0$ and $u\lambda_k^{-1} \to 0$). Thus $s \in A$, because $a_k \lambda_k^{-1} \in A$ and A is closed. This contradicts hypothesis (**NA**), according to which $A \cap K = \{0\}$. $\square$

A.5. Proof of Theorem A.5. Let (**H**) hold. Then $a - u \in K$ for some $a \in A$, and if $q \in$Q, we have

$$0 \leq q(a - u) = qa - qu \leq -qu,$$

which yields (**U**). Conversely, suppose (**U**) is true, while (**H**) is not. The latter means that $(A - u) \cap K = \emptyset$. Indeed, if $g \in K$ and $g \in A - u$, then $g = a - u$, $a \in A$, and so $u \leq_K a$. By Lemma A.1, the distance between $A - u$ and K_ϵ is strictly positive for some $\epsilon > 0$. Consequently (see Theorem A.2), there exists a linear functional q strongly separating the sets $A - u$ and K_ϵ:

$$\sup_{a \in A}(qa - qu) < \inf_{b \in K_\epsilon} qb. \tag{137}$$

Since K_ϵ is a cone, the infimum on the right-hand side is equal to zero. Consequently, $qb \geq 0$ for $b \in K_\epsilon$, which yields $qs > 0$, $s \in \Sigma$. Therefore $q \in K^+$. By using (137) and the fact that $0 \in A$, we obtain the inequality $-qu < 0$. The functional q is bounded above on A, and hence it is nonpositive on A. Thus $q \in$Q and $qu > 0$. A contradiction. $\square$

A.6. The Kuhn-Tucker theorem. Let Θ be a convex subset in R^n and $f(\theta)$, $\theta \in \Theta$, a concave real-valued function defined on Θ. Let $\mathcal{G}$ be a cone in R^k and $G(\theta)$ a vector function on Θ with values in R^k. Assume that G is concave in the following sense

$$G(c\theta_1 + (1 - c)\theta_2 - cG(\theta_1) - (1 - c)G(\theta_2) \in \mathcal{G} \tag{138}$$

for all θ_1, $\theta_2 \in \Theta$ and $c \in [0, 1]$. (Clearly (138) holds if G is affine, i.e., the expression in (138) is equal to zero.) Consider the following optimization problem.

(**M**) Maximize $f(\theta)$ on the set Θ subject to the constraint

$$G(\theta) \in \mathcal{G}. \tag{139}$$

Suppose that one of the conditions (**SL**) or (**LP**) below holds.

(**SL**) (Slater's constraint qualification). There is a vector $\theta \in \Theta$ such that $G(\theta) \in \text{int}\,\mathcal{G}$.

(**LP**) The set Θ is polyhedral and f, G are affine.

(If (**LP**) holds, (**M**) is a linear programming problem.)

Theorem A.7. *Let $\bar{\theta}$ be an element of Θ satisfying the constraint $G(\theta) \in \mathcal{G}$. Then the following assertions are equivalent.*

(i) The vector $\bar{\theta}$ is a solution to optimization problem (**M**).
(ii) There exists a linear functional such that

$$f(\theta) + p(G(\theta)) \le f(\bar{\theta}) + p(G(\bar{\theta})),\ \theta \in \Theta, \tag{140}$$

and

$$p(G(\bar{\theta})) = 0. \tag{141}$$

Condition (140) states that $\bar{\theta}$ maximizes the Lagrangian $L(p, \theta) = f(\theta) + p(G(\theta))$ over all $\theta \in \Theta$ (not necessarily satisfying (141)). In this sense, p is said to *relax* the constraint (139). Equality (141) is called the *complementary slackness condition.* If $p \in \mathcal{G}^*$, then two relations (140) and (141) are equivalent to one:

$$f(\theta) + pG(\theta) \le f(\bar{\theta}),\ \theta \in \Theta. \tag{142}$$

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