GEOGRAPHICAL SEPARATION OF OLIGOPOLISTS CAN BE VERY COMPETITIVE*

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Abstract

A given number of single-product oligopolists locate in one of two separate market-places, which consumers access at a cost. Firms set prices and the CES consumers choose purchases at one or both market-places. Firm agglomeration in one market-place produces positive profits because of product differentiation. But under various assumptions, geographical separation of firms produces prices analogous to homogeneous product Bertrand, and is "very competitive", the reverse of textbook Hotelling. Agglomeration equilibria ensue, in a way that is dual to existing arguments for "head-to-head" competition in product lines. Quantity competition and search model alternatives are also investigated.

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1. INTRODUCTION

The paper studies a situation where a given number of single-product oligopolists locate in one of two geographically separated market-places or centres. Consumers face fixed costs of accessing the market-places (e.g. transport costs) and have a taste for variety over the products available; so they must choose whether to buy at one or both of the market-places and how much of the goods available there to buy. Given the resulting consumer demands each oligopolist sets a price for the sale of their product (a la Bertrand), generating a 2-stage location-price game which can have various types of subgame perfect equilibria. In particular, at some parameters there is an equilibrium in which firms agglomerate in one centre because separation across the two centres induces consumers to buy from just one centre and is "much more competitive", in a precise sense and in a way which is the opposite of the standard textbook, and many other Hotelling (1929) models[1]. Our main objective is to explain, and elaborate on, this result.

A natural application is to the location of shops at out-of-town shopping centres. Indeed our main model of consumer behaviour (section 2) is broadly that used by Stahl (1982, 1987) in studies relating to the location of retail stores. We add CES preferences and a complete analysis of the resulting consumer problem, including the decision to buy at one or both centres. This allows (section 3) a full characterisation of pure strategy equilibria of the duopoly location-price game when consumers are homogeneous in all aspects, including access costs. Agglomeration of the two firms in the same centre is then the only equilibrium possibility, but of two types. Our main interest is type A agglomeration which occurs when goods are relatively substitutable, and because separation is "much more competitive". Type B agglomeration emerges when goods are more complementary, because consumers now buy at both centres when firms are separate, leaving prices unchanged but increasing access cost expenditures and reducing profits compared to agglomeration. Although consumers prefer type B agglomeration of firms to the alternative of geographical separation, the same is (at some parameters) not true of the type A agglomeration – a planner interested solely in consumer welfare and with powers to direct firm location may want to veto agglomeration in one centre to allow consumers to benefit from the extra competition of geographical separation.

The rest of the paper explores the type A equilibrium in various different contexts, starting with oligopoly (Section 4) and heterogeneous consumers (Section 5), and finishing (section 7) with variations on Gehrig's (1998) model of competing financial centres and Dudey's (1992) Cournot model; section 6 explains how our main model achieves the inversion of the standard Hotelling logic.

There are several oligopolistic location-price models which also provide explanations of agglomeration of firms. Many of these (e.g. Ben-Akiva et al (1989), Gehrig (1998), Stahl (1982,1987)[2] rely on sufficient complementarity between goods, and have features more in common with our type B agglomeration. And in the standard monopolistic competition model agglomeration is accompanied by prices which are independent of firm locations (see Fujita et al. (1999), again akin to type B agglomeration. Klemperer (1992) seems to be the closest analogue for our main focus, the type A agglomeration. In Klemperer's duopoly, the geographical separation (1 firm at each centre) is given and firms choose whether or not to differentiate their physical product or "product line". Think of consumers as heterogeneously and bimodally located near the centres, with transport as the access cost. With identical product lines consumers buy from their local centre and prices may be high (no-one buys from both centres so firms have considerable local market power). With different product lines, the emergence of consumers who buy from both centres toughens competition, lowering prices and causing firms to "agglomerate", that is, to choose the same product line; they compete "head-to-head". In a sense our type A explanation of geographical agglomeration is dual to Klemperer's explanation of head-to-head competition. With homogenous consumers and given differentiated physical products or product lines our firms choose the same geographical location. With appropriate consumer heterogeneity and given separate geographical locations, Klemperer's firms choose the same product line.

2. CONSUMER BEHAVIOUR

There are two centres which can be thought of as located at the opposite ends (0 and 1) of the interval [0, 1]. N is the total set of firms, with n elements; subset N_0 (n_0 elements) is at 0 and N_1 (n_1 elements) is at 1 $\left(N_0 \cup N_1 = N,\, n_0 + n_1 = n\right)$. These firm locations are given here, to be endogenized in the following sections. Each firm sells (at constant marginal production cost c) a single good with a given specification whose price is denoted $p_i, i \in N_0 (i = 1,..., n_0)$ and q_i, $i \in N_1 (i = n_0 + 1,..., n)$. There is a unit mass of consumers who have CES preferences over the n goods available represented by the utility function;

$$u(x_h) = \left(\sum_{i=1}^{n} x_{ih}^{r} \right)^{\frac{1}{r}}, 0 < r < 1, x_h = (x_{1h},..., x_{nh})$$

So goods are always gross (r > 0) but imperfect (r < 1) substitutes. Each consumer has income y to be spent on the n goods[3] and on the fixed costs of access to the centre(s). For consumer h, t_{0h}, t_{1h} and t_{01h} (> t_{0h}, t_{1h}) are the costs of access to centre 0, centre 1 and both centres, respectively, leaving residual income (to be spent on the goods available at the accessed centre(s)) of $m_{0h} = y - t_{0h}$, $m_{1h} = y - t_{1h}$ and $m_{01h} = y - t_{01h}$, respectively.

Using the standard definitions of CES price indices for the two centres

separately, $P = \left(\sum_{i \in N_0} p_i^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}}$ and $Q = \left(\sum_{i \in N_1} q_i^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}}$, and $C = \left(\sum_{i \in N_0} p_i^{\frac{r}{r-1}} + \sum_{i \in N_1} q_i^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}}$

for the whole set of goods, consumer demands (x_{ih}) and indirect utilities (n_h) after buying at centre 0 only (2.1), at centre 1 only (2.2) and at both centres (2.3) are;

$$x_{ih} = \frac{m_{0h}}{P} \left(\frac{P}{p_i} \right)^{\frac{1}{1-r}}, i \in N_0; x_{ih} = 0, \; i \in N_1; \; n_{0h} = m_{0h} / P \qquad (2.1)$$

$$x_{ih} = 0, i \in N_0; x_{ih} = \frac{m_{1h}}{Q} \left(\frac{Q}{q_i} \right)^{\frac{1}{1-r}}, i \in N_1; \; n_{1h} = m_{1h} / Q \qquad (2.2)$$

$$x_{ih} = \frac{\hat{m}_{01h}}{C} \left(\frac{C}{p_i} \right)^{\frac{1}{1-r}}, i \in N_0; x_{ih} = \frac{m_{01h}}{C} \left(\frac{C}{q_i} \right)^{\frac{1}{1-r}}, \; i \in N_1; \; n_{01h} = m_{01h} / C \qquad (2.3)$$

In the special case where all firms are located in one centre ($N_0 = N$, $N_1 =$ Æ say) the consumer problem is entirely textbook standard, generating the CES solutions of (2.1). Otherwise (n_0, $n_1 \geq 1$) the consumer problem becomes "non-convex" as they must decide whether to buy at centre 0, centre 1 or both. The optimum will be given by (2.1), (2.2) or (2.3) depending on which of n_{0h}, n_{1h} or n_{01h} is the largest. Using the function

$$f(r,w) = \left(w^{\frac{r}{1-r}} - 1 \right)^{\frac{1-r}{r}}$$

and notation $w_{ih} = m_{ih} / m_{01h}, i = 0{,}1,$ lemma 1 provides an exact statement.

<u>Lemma 1</u> For the consumer problem when N_0, $N_1 \neq \varnothing$, there is a function g: $(1, \infty)^2 \rightarrow (0, 1)$ such that:

(a) if $r > g\,(w_{0h}, w_{1h})$ then there are no prices at which consumer h buys from both centres, the unique optimal demands being given by (2.1) if $P/Q < m_{0h}/m_{1h}$ and (2.2) if $P/Q > m_{0h}/m_{1h}$, with indifference between (2.1) and (2.2) if $P/Q = m_{0h}/m_{1h}$;

(b) if $r < g\,(w_{0h}, w_{1h})$ then there are prices at which consumer h buys from both centres, the unique optimal demands being given by (2.3) if P/Q Î $(f(r, w_{0h}), f(r, w_{1h})^{-1})$, (2.1) if $P/Q < f(r, w_{0h})$ and (2.2) if $P/Q > f(r, w_{1h})^{-1}$, with indifference between (2.1) and (2.3) if $P/Q = f(r, w_{0h})$ and between (2.2) and (2.3) if $P/Q = f(r, w_{1h})^{-1}$;

(c) the signs of $g(w_{0h}, w_{1h}) - r$, $f(r, w_{1h})^{-1} - f(r, w_{0h})$ and $w_{0h}^{\frac{r}{r-1}} + w_{1h}^{\frac{r}{r-1}} - 1$ are the same;

(d) if $w_{0h} = w_{1h} = w_h$ then $g\,(w_h, w_h) = \ell n\, 2/\ell n\,(2w_h)$.

<u>Proof</u> See Appendix.

The first part of Lemma 1 shows that if goods are sufficiently good substitutes $(r > g\,(w_{0h}, w_{1h}))$ then the consumer never buys from both centres – the taste for variety is never strong enough to compensate for the extra access cost of two-centre purchase. In this case the consumer spends all (residual) income at the centre offering the lower price index, if $m_{0h} = m_{1h}$; when $m_{0h} > m_{1h}$ (say) access costs are lower for centre 0, consumer expenditure is still "all-or-nothing", but now centre 0 wins this expenditure whenever $P/Q < m_{0h}/m_{1h}$ (>1) because of the lower access cost, and vice versa if $P/Q > m_{0h}/m_{1h}$. Conversely in Lemma 1(b) where $r < g\,(w_{0h}, w_{1h})$ there is a nonempty cone of price indices (P/Q Î $(f\,(r, w_{0h}), f\,(r, w_{1h})^{-1})$) at which the consumer would purchase from both centres, with reversion to single-centre purchasing at prices outside this cone.

Lemma 1 is concerned with individual consumer behaviour. In the sequel we focus on the case of "homogeneous consumers" in which access costs and residual incomes are the same for all consumers and for both centres - for all h, $m_{0h} = m_{1h} = m$, $m_{01h} = M$ so $w_{0h} = w_{1h} = w = m/M$, say. Aggregate demands are then given by (2.4) below if N_1 =Æ, (2.5) if N_0 = Æ and the statement in Lemma 2 if N_0, N_1 ¹ Æ.

$$x_i = \frac{m}{P}\left(\frac{P}{p_i}\right)^{\frac{1}{1-r}}, i \in N_0; \quad x_i = 0, \quad i \in N_1 \qquad (2.4)$$

$$x_i = 0, \quad i \in N_0; \quad x_i = \frac{m}{Q}\left(\frac{Q}{q_i}\right)^{\frac{1}{1-r}}, i \in N_1 \qquad (2.5)$$

$$x_i = \frac{M}{C}\left(\frac{C}{p_i}\right)^{\frac{1}{1-r}}, i \in N_0 \text{ ; } x_i = \frac{M}{C}\left(\frac{C}{q_i}\right)^{\frac{1}{1-r}}, i \in N_1 \qquad (2.6)$$

<u>Lemma 2</u> In the homogeneous consumer case where, for all h, $m_{0h} = m_{1h} = m$, $m_{01h} = M$ so $w_{0h} = w_{1h} = m/M = w$ and where N_0, N_1 ¹ Æ;

(a) if $r > g(w)$ then the aggregate demand for good i is given by (2.4) if $P < Q$, (2.5) if $P > Q$ and any convex combination of (2.4) and (2.5) if $P = Q$.

(b) if $r < g(w)$ then the aggregate demand for good i is given by (2.6) if P/Q Î $(f(r, w), f(r, w)^{-1})$, (2.4) if $P/Q < f(r, w)$, (2.5) if $P/Q > f(r, w)^{-1}$, any convex combination of (2.4) and (2.6) if $P/Q = f(r, w)$ and any convex combination of (2.5) and (2.6) if $P/Q = f(r, w)^{-1}$.

<u>Proof</u> The claims in (a) for P/Q ¹ 1 and in (b) for P/Q ¹ $f(r, w), f(r, w)^{-1}$ follow immediately from Lemma 1 and the consumer homogeneity assumption. The claims in (a) and (b) for the remaining borderline cases follow similarly since all consumers are then indifferent to the extremes of the convex combinations. ■

3. DUOPOLY WITH HOMOGENEOUS CONSUMERS

There are $n = 2$ firms and access costs are the same for both centres and for all consumers, so $m_{0h} = m_{1h} = m$, $m_{01h} = M$ and $w_{0h} = w_{1h} = m/M = w$ say, for all h. Firms choose the centre (0 or 1) in which they locate at stage I, and prices for the sale of their good at stage II. Ultimately we provide a full description of the pure strategy subgame perfect equilibria.

First suppose firms agglomerate at stage I, both locating in centre 0 (without loss of generality), so $N_0 = \{1, 2\}$ and N_1 = Æ; (2.4) then describes consumer demands and payoffs in the ensuing price subgame are $p_i(p_1, p_2) = (p_i - c)x_i, i = 1,2$. The equilibrium of this subgame is as follows.

<u>Proposition 1</u> In the duopoly model with homogenous consumers, the unique Nash equilibrium prices, profits and utilities in the price subgame following agglomeration of the two firms at the same centre are;

$$p_1^* = p_2^* = (2-r)r^{-1}c \quad ; \quad p_1^* = p_2^* = m(1-r)(2-r)^{-1}$$

$$n^* = mc^{-1}r(2-r)^{-1}\, 2^{\frac{1-r}{r}}$$

<u>Proof</u> The special case (n = 2) of Proposition 5 later. ■

Proposition 1 reveals the standard positive profit outcome of Bertrand competition when products are differentiated. Notice also that profits shrink towards zero as the differentiation disappears.

When the firms further differentiate the products by choosing separate locations (firm1 at 0, firm 2 at 1, without loss of generality) the price subgame is quite different and depends critically on the degree of substitutability (r). If $r > g(w)(=g(w,w)$ for short) , we know from lemma 2(a) that consumers will buy from just one centre/firm, that offering the lower price, with indifference at equal prices. Assuming an equal split of the market when prices are equal, the price subgame is now equivalent to the classic, homogeneous product Bertrand game with the usual “Bertrand paradox” zero-profit outcome:

<u>Proposition 2</u> In the duopoly model with homogeneous consumers where $r > g(w)$, the unique Nash equilibrium prices, profits and utilities in the price subgame when firms are geographically separated, are;

$$\tilde{p}_1 = \tilde{q}_2 = c \quad ; \quad \tilde{p}_1 = \tilde{p}_2 = 0 \quad ; \quad \tilde{n} = mc^{-1}$$

The immediate consequence of Propositions 1 and 2 is;

Theorem 1 In the duopoly model with homogeneous consumers where $r > g(w)$, the unique subgame perfect equilibrium of the location-price game has the 2 firms agglomerating at the same centre (leading to prices, profits and utilities of Proposition 1) because geographical separation causes consumers to buy from just one firm and is as if there is homogeneous product Bertrand competition and so zero profits for both firms.

We elaborate further on Theorem 1 in succeeding sections. For the time being somewhat imprecisely, we refer to the Theorem 1 equilibrium as "Type A agglomeration". First, however, the consequences of geographical separation are quite different from Proposition 2 when $r < g(w)$. From lemma 2(b) when prices belong to the cone p_1/q_2 Î $(f(r, w), f(r, w)^{-1})$ consumers buy at both centres leading to aggregate demands x_i in (2.6) and corresponding payoffs. Apart from a change of m to M these payoffs are the same as in the agglomeration subgame, and the argument of Proposition 1 produces the following candidate Nash equilibrium (the unique candidate in the price cone) for the current subgame, which has the property that neither firm will wish to Nash deviate to prices that remain in the above cone;

$$\left.\begin{array}{c} \hat{p}_1 = \hat{q}_2 = (2-r)r^{-1}c \quad ; \quad \hat{p}_1 = \hat{p}_2 = M(1-r)(2-r)^{-1} \\ \\ \hat{n} = Mc^{-1} r (2-r)^{\frac{1-r}{r}} \end{array}\right\} \quad (3.1)$$

Clearly neither firm wishes to raise its price from this candidate value so that prices are outside the cone, since this firm then loses the whole market and gets zero profits. However firm 1 (say), by lowering its price from the candidate equilibrium to $p_1 < \hat{q}_2 f(r, w)$ will capture the whole market and can attain profits arbitrarily close to:

$$\hat{\hat{p}}_1 = m\left(1 - c[\hat{q}_2 f(p, w)]^{-1}\right) \quad (3.2)$$

If follows that the candidate in (3.1) is indeed a Nash equilibrium of the current subgame iff $\hat{p}_1 \geq \hat{\hat{p}}_1$, an inequality which becomes:

$$w[2 - r - r f(r, w)^{-1}] \leq 1 - r \quad (3.3)$$

We have the following.

Lemma 3 There is a decreasing function $h:(1,\infty)\to(0,1)$ such that for all $w > 1$, $0 < h(w) < g(w)$ and (3.3) holds if and only if $r \le h(w)$; as $w \to 1$, $h(w) \to 1$ and as $w \to \infty$, $h(w) \to 0$.

Proof   See the appendix ■

Hence we have;

Proposition 3 In the duopoly model with homogeneous consumers where $r \le h(w)(< g(w))$, the unique Nash equilibrium prices profits and utilities in the price subgame when firms are geographically separated are given by (3.1).

Since $m > M$, $\hat{p}_i^* > \hat{p}_i$, $i = 1,2$, and geographical separation when goods are relatively weak substitutes $(r < h(m/M))$ lowers profits (without any affect on prices) since consumers still buy from both the separated firms, the extra access cost expenditure lowering profits.

Theorem 2 In the duopoly model with homogeneous consumers where $r < h(w)$ $(< g(w))$ the unique subgame perfect equilibrium of the location-price game has the 2 firms agglomerating at the same centre (leading to prices, profits and utilities of Proposition 1) because geographical separation leads consumers to buy from both firms, with no effect on prices but lowering profits because of additional consumer expenditure on access costs.

The equilibrium in Theorem 2 is quite different from that in Theorem 1; we refer to it as "Type B agglomeration". Thus, so far, for any $w > 1$, if goods are strong substitutes (r large enough) we have a type A agglomeration equilibrium whilst if they are weak substitutes (r small enough) we have a type B agglomeration equilibrium. Conversely, the properties of g and h in Lemmas 1 and 3 show that they both possess inverses with domain (1, ¥), so for each r Î (0, 1) there will be Type A agglomeration if w is large enough (so the cost of accessing two centres is much larger than that for one) and Type B agglomeration if w is small enough. Either way there is a parameter gap between Types A and B, in which there is no pure strategy equilibrium.

Proposition 4 In the duopoly model with homogeneous consumers where $h(w) < r < g(w)$ or $h^{-1}(r) < w < g^{-1}(w)$, there is no pure strategy equilibrium in the price subgame when firms are geographically separated.

Proof   We know from Lemma 3 that the candidate equilibrium of (3.1) is not an equilibrium when $h(w) < r$, and that there is no other equilibrium in the cone p_1/q_2 Î $(f(r,w), f(r,w)^{-1})$. We now show there is no equilibrium outside this cone. Notice first that $p_1, p_2 \geq 0$ in any such equilibrium (so $p_i \geq c$, i – 1,2) since each firm i can ensure zero profits by raising price enough ($pi > f(r, w)^{-1} p_j$). Suppose ($w$log) that firm 1 is the low price firm in an equilibrium outside the cone. If $p_1 = c$, raising p_1 to slightly more than $f(r, w)p_2$ takes us into the cone and produces positive profit for firm 1. If $f(r, w)\, p_2 > p_1 > c$ then firm 1 (still) gets the whole market and higher profits by raising p_1 a little. If $f(r, w)\, p_2 = p_1 > c$ then firm 1 would capture the whole market and higher profits by a sufficiently small price decrease. ■

The type A and type B equilibria have different welfare consequences. It is obvious that both consumers and firms prefer the agglomeration in type B equilibrium to the alternative of geographical separation – the extra expenditure on access costs lowers utility as well as profits ($\hat{n} < n^*$ since $M<m$). However consumers prefer geographical separation to type A agglomeration if $\tilde{n} > n^*$, which is possible.

Theorem 3 In the duopoly model with homogeneous consumers where $r > g(w)$, consumers would prefer geographical separation of firms to the equilibrium type A agglomeration if $r > r \cong 0.2737$.

Proof $\tilde{n} > n^*$ iff;

$$1 > r\,(2-r)^{-1}\, 2^{\frac{1-r}{r}} = F(r) \text{ say}$$

It is easy to check that $F(1) = 1$, $F'(r) = r(2 - r)^{-1}\, 2^{\frac{1-r}{r}} \left[p^{-1} + (2-r)^{-1} - r^{-2}\ell n2\right]$, $F'(1) = 2 - \ell n2 > 0$, $F'(r) = 0$ at $\hat{r} = 2\ell n2(2+\ell n2)^{-1}$, F is increasing (resp. decreasing) to the right (resp.left) of $\hat{r}$ and $F(r) \to +$ ¥ as $r \to 0$. It follows that there is a unique r^* Î (0, 1) where $F(r^*) = 1$ and so (4.2) is satisfied iff r Î $(r^*, 1)$; computation reveals $r^* \cong 0.2737$. ■

The rest of the paper explores type A agglomeration in various different contexts.

4. OLIGOPOLY WITH HOMOGENOUS CONSUMERS

We start with oligopoly. To shorten exposition we assume an even number of firms[4], so we have $n \geq 4$ firms choosing to locate at one of the two centres. If all firms agglomerate at the same centre, the following generalisation of proposition 1 emerges;

Proposition 5 In the oligopoly model with homogeneous consumers, the unique Nash equilibrium prices, profits and utilities in the price subgame following agglomeration of all firms at the same centre are;

$p_i^* = (n-r)(n-1)^{-1} r^{-1} c$, $i = 1,\dots,n$; $p_i^* = m(1-r)(n-r)^{-1}$, $i = 1,\dots, n$

$$n^* = m r (n-r)^{-1} (n-1) n^{\frac{1-r}{r}} c^{-1}$$

Proof. From (2.4) the payoff to firm 1 may be written;

$$p_1 = m(p_1 - c) p_1^{\frac{1}{r-1}} \left(\Sigma p_j^{\frac{r}{p-1}} \right)^{-1}$$

Differentiating and rearranging gives:

$$\frac{\partial p_1}{\partial p_1} = -m(1-r)^{-1}(p_1 - c) p_1^{\frac{1}{r-1}-1} a \left(\Sigma p_j^{\frac{r}{p-1}} \right)^{-1}, \text{ where}$$

$$a = 1 - (1-r)(p_1 - c)^{-1} p_1 - r \; p_1^{\frac{r}{r-1}} \left(\Sigma p_j^{\frac{r}{r-1}} \right)^{-1}$$

Differentiating again and using the stationary point condition $a = 0$ gives;

$$\frac{\partial^2 p_1}{\partial p^2_1} = -m(1-r)^{-1}(p_1 - c) p_1^{\frac{1}{r-1}-1} b \left(\Sigma p_j^{\frac{r}{r-1}} \right)^{-1}, \text{ where}$$

$$b = -(1-r)(p_1 - c)^{-1} + (1-r) p_1 (p_1 - c)^{-2} + r(1-r)^{-1} p_1^{-1}[1 - (1-r)(p_1 - c)^{-1} p_1]$$
$$- (1-r)^{-1} p_1^{-1} [1 - (1-r)(p_1 - c)^{-1} p_1]^2$$

Expanding the quadratic bracket and rearranging leads to

$b = (p_1 - c)^{-1} - p_1^{-1} > 0$ if $p_1 > c$

Hence $\partial^2 \pi_1 / \partial p_1^2 < 0$ at a stationary point of π_1 where $p_1 > c$ (i.e. profits are positive). So any positive profit stationary point of π_1 must (by continuity of π_1) be unique and the global maximum of π_1. So $a = 0$ defines firm 1's best responses in this price subgame, and the analogous condition for firm i produces the following Nash equilibrium conditions;

$$p_i^{\frac{r}{1-r}} \left[1 - (1-r)(p_i - c)^{-1} p_i\right] = r\left(\Sigma p_j^{\frac{r}{r-1}}\right)^{-1}, i = 1,\ldots,n$$

Since the left hand side is increasing in p_i any Nash equilibrium must be symmetric. Equating prices produces the required p_i* which gives π_i* when substituted into π_1 above, and v* when substituted into v_{oh} in (2.1). ■

Again because of the product differentiation, agglomeration produces positive profits, dissipating as $r \to 1$.

Geographical separation of firms can now occur in many different ways, ranging from the case where all firms but one are in the same centre to the case where firm numbers are the same at each centre. If $r > g(m/M)$ we know (lemmas 1(a), 2(a)) that in all these cases consumers never buy from both centres, the chosen centre being that with the lower price index which now in general depends on the size of the centres (n_0 and n_1) as well as prices. For instance suppose that $n_0 > n_1 \geq 1$ and that p (q) is the uniform price at centre 0 (1). Then consumers are indifferent between which centre to buy from if $P = Q$ or $p = aq = (n_o / n_1)^{\frac{1-r}{r}} q > q$; to the larger centre has an advantage because of its greater variety – its prices can be higher than at the rival centre up to the multiple $a = (n_0 / n_1)^{\frac{1-r}{r}}$ and yet it retains the whole market. Because of this, and analogous to the usual treatment of homogeneous product Bertrand games with asymmetric marginal costs (to avoid non-existence problems), we assume that centre 0 gets the whole market when $P = Q$ and $n_0 > n_1$.

And with the same analogy, we exclude prices below their marginal cost from firms' strategy sets when $n_0 > n_1$ (to avoid implausible equilibria otherwise). The consequences for price subgames are as follows.

Proposition 6 In the oligopoly model with homogeneous consumers where $r > g\,(w)$, the set of Nash equilibrium prices, profits and utility in the price subgame with geographically separate firms ($n_0 \geq n_1 \geq 1$) is;

(a) If $n_0 > n_1$ and assuming that centre 0 gets the whole market if $P = Q$ and that firms strategy sets exclude prices below margined cost,

(i) $\tilde{r}_i = (n_0 - r)(n_0 - 1)^{-1} r^{-1} c,\ i \in N_0$ and $\tilde{q}_i \geq c,\ i \in N_1$

$\tilde{p}_i = m\,(l - r)(n_0 - r)^{-1},\ i$ Î N_0 and $p_i = 0,\ i$ Î N_1

$\tilde{v} = mr\,(n_0 - r)^{-1}(n_0 - 1)\, n_0^{\frac{1-r}{r}}\, c^{-1}$

if ac ³ $\tilde{p}_i$

(ii) and if ac £ $\tilde{p}_i$ (as defined in (a) (i)) then

$\tilde{\tilde{p}}_i = ac, i \in N_0$ and $\tilde{\tilde{q}}_i = c, i \in N_1$

$\tilde{\tilde{p}}_i = (1 - a)mn_0^{-1}, i \in N_0$ and $\tilde{\tilde{p}}_i = 0, i \in N_1$

$\tilde{\tilde{v}} = mn_0^{\frac{1-r}{r}} c^{-1}$

(b) If $n_0 = n_1$, $\tilde{p}_i = c, i \in N_0, \tilde{q}_i = c, i \in N_0, \tilde{p}_i = 0, i \in N$ and $\tilde{v} = mn_0^{\frac{1-r}{r}} c^{-1}$

Proof

(a) (i) Notice from Proposition 5 that $\tilde{p}_i$ are the prices that would emerge if centre 0 captures the whole market at any P (as if $N = N_0$ or $N_1 = \varnothing$). The prices defined in the statement imply $P = Q$, so centre 0 does get the whole market at these prices, and the argument of Proposition 5 ensures that no centre 0 firm wishes to deviate to any other price (since they do not wish to even when centre 0 always gets the whole market). Thus for any feasible prices for centre 1 $(\tilde{q}_i \geq c, i \in N_1)$, centre 0 firms choose $\tilde{p}_i, i \in N_0$ and centre 0 firms get the whole market, ensuring the price claims in the statement. The profit and utility claims follow.

(a) (ii) The prices defined in the statement imply $P = Q$, and centre 0 gets the whole market, and positive profits. No centre 0 firm wishes to deviate to a higher price since centre 0 then loses the whole market, and they will neither wish to deviate to a lower price if (taking i =1, without loss of generality), $\partial p_1 / \partial p_1 \geq 0$ for $p_1 \leq \tilde{\tilde{p}}_1$ where

$p_j = \tilde{\tilde{p}}_j$, $j \neq 1$ and $p_1 = m(p_1 - c)p_1^{\frac{1}{r-1}}\left(\Sigma p_j^{\frac{r}{r-1}}\right)^{-1}$.

From the proof of Proposition 5, the required derivative sign requires $a \leq 0$, a condition which becomes:

$$rp_i \leq c + r(p_1 - c)\left[1 + (n_0 - 1)\left(\tilde{\tilde{p}}_1 / p_1\right)^{\frac{r}{r-1}}\right]^{-1}$$

A sufficient condition for this to hold for $p_1 \leq \tilde{\tilde{p}}_1$ is that $rp_i \leq c + p(p_1 - c)n_0^{-1}$ when $p_1 \leq \tilde{\tilde{p}}_1$ which becomes $p_1 \leq \tilde{p}_1$ when $p_1 \leq \tilde{\tilde{p}}_1$. This follows since $\tilde{p}_1 \geq \tilde{\tilde{p}}_1$. Thus no centre 0 firm wishes to Nash deviate from the prices in the statement. And clearly no centre 1 firm can benefit from such a deviation (raising prices fails to gain any market and lowering prices below c is not feasible). So the prices in the statement are Nash equilibrium prices. Moreover there is no other equilibrium. If so $Q > c$ and the centre 0 firms best response would be either $p_i = a\, Q$ or $p_i = \tilde{p}_i$ (as in (a)(i)), taking the whole market in both cases. But then a small Nash reduction in price by some centre 1 firm will capture the market for centre 1 and be profitable.

(b) When $n_0 = n_1$ the entire market goes to centre 0 if $P < Q$, centre 1 if $P > Q$ with equal shares if $P = Q$. The usual Bertrand reasoning ensures that the statement is the unique Nash equilibrium. ■

Part (a) (i) prevails if n_1 is sufficiently small relative to n_0. In this case the smaller centre is so uncompetitive that it has no effect on the larger centre – the prices at the larger centre are those that would also emerge with n_0 firms at 0 and zero firms at 1. The "within centre" competition at 0 produces prices at which the smaller centre cannot compete – even with $q_i = c$, $i \in N_1$, centre 0 captures the whole market. As n_1 increases relative to n_0, (a) (i) eventually gives way to (a) (ii) and margined cost prices at centre 1 do now restrain prices at the larger centre, to the level at which it just keeps the whole market. And when $n_1 = n_0$ we are back to the Bertrand paradox outcome, as in Proposition 2 for duopoly.

There is a homogenous product Bertrand parallel for all these subgames, not just when $n_0 = n_1$. Define a 2 "firm" (0 and 1) homogenous but asymmetric product Bertrand game, as follows. Let $p(q)$ be the price chosen by firm 0 (1), and c be the constant, symmetric marginal production cost. Let D denote the market demand function, let $p^m > c$ denote the monopoly price and assume monopoly profit is increasing on $[c, p^m]$ and decreasing on $[p^m, \infty)$. The 2 goods are perfect (linear indifference curves) but asymmetric (indifference curve slope ≠ 1) substitutes for all consumers, so that firm 0 captures the whole market if $p < aq$ where $a > 1$, and firm 1 has the whole market if $p > aq$. As in asymmetric cost Bertrand games assume that the firm offering the better product (0) takes the whole market if $p = aq$, and exclude prices below marginal cost. Thus strategy sets are $p \in [c, \infty), q \in [c, \infty)$ and payoffs are $p_0(p, q) = (p - c)D_0(p, q), p_1(p, q) = (q - c)D_1(p, q)$ where:

$$D_0(p,q) = \begin{cases} D(p) & \text{if } p \le aq \\ 0 & \text{if } p > aq \end{cases} \quad \text{and} \quad D_1(p,q) = \begin{cases} D(q) & \text{if } p > aq \\ 0 & \text{if } p \le aq \end{cases}$$

Employing similar reasoning that used for part (a) of Proposition 6, one finds the equilibria;

(i) if $ac \ge p^m$ then $p = p^m, q \ge c$

(ii) if $ac \le p^m$ then $p = ac, q = c$

The parallel is that if $a = (n_0 / n_1)^{\frac{1-r}{r}}$ and one interprets p^m as the price that would emerge at centre 0 if there were no firms at centre 1, then the equilibrium p and q in the 2 firm homogeneous, asymmetric product Bertrand game are exactly the prices that emerge in our oligopoly game, p for all firms at centre 0 and q at centre 1.

If $n_0 \ge n_1$ firms at 1 get zero profits whereas a switch to centre 0 would allow positive profits. Thus;

<u>Theorem 4</u> In the oligopoly model with homogenous consumers, where $r > g(w)$, the unique subgame perfect equilibrium of the location-price game has all the firms agglomerating at the same centre because any geographical separation causes consumers to buy from just one centre, and is as if there is (symmetric or asymmetric) homogenous product Bertrand competition and zero profits at one centre at least.

Type A agglomeration is now defined as an agglomeration subgame perfect equilibrium in which any geographical separation of firms is as if there is symmetric or asymmetric homogenous product Bertrand competition. So, tautologically, r > $g(w)$ implies Type A agglomeration under oligopoly with homogenous consumers. Turning to consumer preferences over alternative sized centres, it is easy to check that $\partial\tilde{v}/\partial n_0 > 0$ in proposition 6 so $\partial\tilde{v}/\partial n_1 < 0$ where $n_1 = n - n_0$. As n_1 increases from 0 in the range of (a)(i) n_0 declines, and the single-centre behaviour of centre 0 means there is declining variety and declining within centre competition; thus $\tilde{v}$ falls as n_1 increases. For large enough values of $n_1 (\leq n_0)$ Proposition 6 (a)(ii) and (b) take over and now $\partial\tilde{v}/\partial n_1 > 0$; centre 1 now restrains prices at centre 0 to the extent that consumers prefer a larger centre 1 in this range. Thus over the entire range $n_1 \in \left[0, \frac{n}{2}\right]$ consumer utility is a "u-shaped" function of n_1, attaining a maximum either at $n_1 = 0$ (full agglomeration) or $n_1 = \frac{n}{2}$ (equal sized centres). As with duopoly, for r large enough the latter is preferable.

Theorem 5 In the oligopoly model with homogenous consumers where $r > g(w)$ the consumers' most preferred location of firms is two equal-sized centres, and not the equilibrium type A agglomeration, if $r > r^*(n)$ where $r^*(n)$ is increasing in n and $r^*(n) \to \dfrac{1}{2}$ as $n \to \infty$.

Proof

From the preamble the consumers' most preferred location of firms is either $n_0 = n$, $n_1 = 0$ or $n_0 = n_1 = n/2$. The latter is preferred (using v^* in Proposition 5 and $\tilde{v}$ in Proposition 6(b)) if:

$$r(n-r)^{-1}(n-1)n^{\frac{1-r}{r}} < (n/2)^{\frac{1-r}{r}}$$

Alternatively: $$F(r,n) = r(n-r)^{-1}(n-1)2^{\frac{1-r}{r}} < 1$$

Now $F(1,n) = 1$ and $\partial F/\partial r = 2^{\frac{1-r}{r}} r(n-1)(n-r)^{-1}\left[r^{-1} + (n-r)^{-1} - r^{-2}\ell n2\right]$

So, at $r = 1$, $\partial F/\partial r = 1 - \ell n2 + (n-1)^{-1} > 0$. Moreover for r Î (0,1) the sign of $\partial F/\partial r$ is the same as that of $r - n\ell n2/(n + \ell n2)$, and $F(r, n) \to \infty$ as $r \to 0$. This ensures that for each n there is a unique r Î (0,1), $r^*(n)$ say, such that $F(r, n) < 1$ iff $r \in (r^*(n), 1)$. Since $\partial F/\partial n > 0$, $r^*(n)$ increases with n. Since $r = r^*(n)$ iff $r < 1$

and $F\,(r, n) -1 = 0$, and since as $n \to \infty$, $F\,(r, n) -1 \to \; r\,2^{\frac{1-r}{r}}$ - 1 = 0 at $r = \frac{1}{2}$, it follows that $r^*(n) \to \frac{1}{2}$ as $n \to \infty$ ■

In terms of the shopping centre example, a town planner with interest only in the townsfolk welfare and with powers over the location of shops at two out-of-town shopping centres, may wish to exercise this power and force shops to spread across the two centres, against their wishes. Although each consumer will shop at only one centre, and lose some variety, the competition between centres will reduce prices to such an extent that consumers benefit.

5. CONSUMER HETEROGENEITY

In this section we argue that the type A agglomeration emanating from the severe competition of geographical separation is robust to small relaxations of the consumer homogeneity assumptions used in Sections 3 and 4, prior to relating our model to those of the traditional Hotelling literature in the next section; with the latter in mind we adopt some Hotelling jargon. The interval $[0,1]$ is now Main Street, but unlike Hotelling, the only feasible locations for firms are at the extremes 0 and 1. Our homogenous consumer model would emerge if the unit mass of consumers were all located at the midpoint of Main Street[5]. We consider 2 perturbations of this homogenous specification.

5.1 Oligopoly with Side Street Heterogeneity

Think of Main Street (*M*) as joining the points (0,0) and (0,1) in R^2 and let Side Street (*S*) be any other compact straight line in R^2 such that $M \cap S$ = (0, ½). Consumers are distributed over *S* with some continuous distribution. Notice that *S* intersects *M* only at the midpoint of *M*. Consumers can travel within *S* and *M* and hear the transport costs. For each consumer $h \in (0,1)$ there will be a shortest return route length $s(h)$ from the consumer's location to the midpoint of *M* and so shortest route lengths to access centre 0 *(s(h)+ 1)*, centre 1 *(s(h)+1)* and both centres *(s(h)+2)*. Let consumer transport costs be some continuous increasing function *t* of shortest return route lengths, so that residual incomes of consumer *h* (assumed positive) are

$m_{0h} = y\text{-}t\left[s(h)+\tfrac{1}{2}\right]$, $m_{1h} = y\text{-}t\left[s(h)+\tfrac{1}{2}\right]$ and $m_{01h} = y - t\left[s(h)+1\right]$. Clearly for all h, $m_{0h} = m_{1h} = m_h$ say and for all h, $w_{0h} = w_{1h} = w_h = m_h/m_{01h}$ (>1).

If firms agglomerate in one centre (0, say) then Proposition 5 continues to describe the price subgame equilibrium, with m replaced by $\int_0^1 m_h dh$ in π_i* and by m_h in v*. When there is some geographical separation $(n_0 \geq n_1 \geq 1)$ Proposition 6 is also maintained, as follows. Given the compactness and continuity assumptions, w_h attains a minimum value as h varies over [0,1], w^* say where $w^*>1$. Since g is decreasing (see Lemma 1(d)), it follows that $r>g\ (w_h)$ for all $h \in (0,1)$ and no consumer buys from both centres if we assume $r>g\ (w^*)$. Then, since $m_{0h} = m_{1h}$ for all h, from Lemma 1(a) all consumers buy from the centre offering the lower price index and Proposition 6 carries through with $r>g\ (w^*)$ replacing $r>g\ (w)$ and m replaced by $\int_0^1 m_h dh$ in π_i and by m_h in $\tilde{n}$. Theorems 4 and 5 then follow also, with w replaced by w^*. In short, the Type A agglomeration story of Sections 3 and 4 carries through more or less unchanged with Side Street heterogeneity.

5.2 Duopoly With Main Street Heterogeneity

Consumers are now uniformly distributed over the middle subset $\left[\tfrac{1}{2}-e, \tfrac{1}{2}+e\right]$ of Main Street (there is no Side Street) and the centres are again at the ends (0 and 1). With $e = \tfrac{1}{2}$ this is the standard Hotelling assumption on consumer locations. Here we are interested in what happens when e is "small", and in the limit $e \to 0$ in particular. For consumer h at $\ell \in \left[\tfrac{1}{2}-e, \tfrac{1}{2}+e\right]$, where $\ell = \tfrac{1}{2} - e + 2eh$, shortest return route lengths are $2l$ for centre 0, $2(1 - l)$ for centre 1 and 2 for both centres. Assuming linear transport costs, h's residual incomes are $m_{0h} = y - 2t\ell$, $m_{1h} = y - 2t\,(1\text{-}\ell)$ and $m_{0h} = y - 2t$ (all assumed positive). As usual, if firms agglomerate then positive profits emerge.

<u>Proposition</u> 7 In the duopoly model with Main Street locations, the unique Nash equilibrium prices, profits and utilities in the price subgame following agglomeration of the two firms at centre 0 (without loss of generality) are:

$$p_1^* = p_2^* = (2-r)r^{-1}c; p_1^* = p_2^* = (y-t)(1-r)(2-r)^{-1}$$

$$n_h^* = m_{0h}c^{-1}r(2-r)^{-1}2^{\frac{1-r}{r}}, h \in [0,1]$$

Proof. After agglomeration at 0 (1 is symmetric) the duopoly game is the same as in the homogeneous case except individual residual incomes are m_{0h} and aggregate income is now $\int_0^1 m_h dh = y - t$, instead of m in Proposition 1. Amending Proposition 1 formulae produces Proposition 7. ■

Notice that the heterogeneity has no effect on these profits – varying ε does not affect aggregate residual income and so does not affect aggregate demands. But these profits shrink to 0 as $r \to 1$, as usual.

Under geographical separation (firm 1 at 0, 2 at 1), and as in the last section, there are no prices at which h buys from both centres if $w_{0h}^{\frac{r}{r-1}} + w_{1h}^{\frac{r}{1-r}} < 1$. Now it is easy to see that $w_{0h}^{\frac{r}{r-1}} + w_{1h}^{\frac{r}{1-r}}$ attains its maximum over $h \in [0,1]$ at $h = 0$ or at $h = 1$. Thus the condition $w_{00}^{\frac{r}{r-1}} + w_{10}^{\frac{r}{1-r}} < 1$, now assumed, again ensures that there are no prices at which any consumer buys from both centres, giving individual demands:

$$x_{1h} = m_{0h} / p_1, x_{2h} = 0 \text{ if } p_1/q_2 < m_{0h}/m_{1h}$$

$$x_{1h} = 0, x_{2h} = m_{1h} / q_2 \text{ if } p_1/q_2 > m_{0h}/m_{1h}$$

with indifference if $p_1/q_2 = m_{0h}/m_{1h}$. Letting $f(h) = m_{0h}/m_{1h}$ we have that $f(0)>1$, $f(1)<1$ and $f'(h) < 0$, $0<h<1$, producing aggregate demands:

$$x_1 = \int_0^1 (m_{0h} / p_1) dh, \quad x_2 = 0 \text{ if } p_1/q_2 \le f(1)$$

$$x_1 = \int_0^{f^{-1}(p_1/q_2)} (m_{0h} / p_1) dh, \quad x_2 = \int_{f^{-1}(p_1/q_2)}^1 (m_{1h} / q_2) dh \text{ if } p_1 / q_2 \in [f(1), f(0)]$$

$$x_1 = 0, x_2 = \int_0^1 (m_{1h} / q_2) dh \text{ if } p_1 / q_2 \ge f(0)$$

Proposition 8 In the duopoly model with Main Street locations, assume $w_{00}^{\frac{r}{r-1}} + w_{10}^{\frac{r}{r-1}} < 1$ where $w_{00} = \left[y - 2t\left(\tfrac{1}{2} - e\right)\right]/(y - 2t)$ and $w_{10} = \left[y - 2t\left(\tfrac{1}{2} + e\right)\right]/(y - 2t)$. Then the unique Nash equilibrium prices, profits and utilities in the price subgame when firms are geographically separated (firm 1 at 0, firm 2 at 1) are:

$$\tilde{r}_1 = \tilde{q}_2 = c(y - t + 2et)^2 / (y - t)^2$$

$$\tilde{p}_1 = \tilde{p}_2 = 2et(y - t + et)^2 / (y - t + 2et)^2$$

$$\tilde{v}(h) = \begin{cases} m_{0h} p_1^{-1} \text{ if } 0 \le h \le \frac{1}{2} \\ m_{1h} q_2^{-1} \text{ if } \frac{1}{2} \le h \le 1 \end{cases}$$

Proof

From the preamble, if $p_1/q_2 \in [f(1), f(0)]$ then

$$p_1(p_1, q_2) = (1 - c/p_1)\int_0^{f^{-1}} [y - 2t(\tfrac{1}{2} - e + 2eh)]dh$$

$$= (1 - c/p_1)[f^{-1}(y - t + 2et) - 2et(f^{-1})^2]$$

where $f^{-1} = f^{-1}(p_1/q_2) = \frac{1}{2} + (y - t)(1 - p_1/q_2)/4te(1 + p_1/q_2)$.

$$\frac{\partial p_1}{\partial p_1} = \frac{c}{p_1^2}\left[f^{-1}(y - t + 2et) - 2et(f^{-1})^2\right] + (1 - c/p_1)[y - t + 2et - 4etf^{-1}]/q_2 f'$$

where $f' = f'(f^{-1}(p_1/q_2)) = -8et(y - t)/[y - 2t(\frac{1}{2} + e - 2ef^{-1})]^2$

There will be a symmetric equilibrium $(p_1 = q_2, f^{-1} = \frac{1}{2}, f' = -8et/(y - t))$ if $\partial p_1/\partial p_1 = 0$ so:

$$\frac{c}{p_1^2}[\tfrac{1}{2}(y - t + 2et) - \tfrac{1}{2}et] - \left(1 - \frac{c}{p_1}\right)(y - t)^2/8etp_1 = 0$$

Rearranging, $p_1(= q_2) = c(y - t + 2et)^2/(y - t)^2$ and produces also the profit and utility formulae. ■

Thus the Main Street heterogeneity does not produce the previous zero profit Bertrand consequence of geographical separation. But profits do shrink to zero (monotonically; $\partial\tilde{p}_i/\partial e < 0$ everywhere) as the heterogeneity disappears ($e \to 0$). This suggests that the type A agglomeration equilibrium of the homogeneous duopoly model remains in an approximate sense for small amounts of Main Street heterogeneity. To be precise, consider the model of this section when $e = 0$. This is the homogeneous duopoly model with $m = y - t$ and $M = y - 2t$.

Fix r, y and t so that $r > g\,[(y-t)/(y-2t)]$ or (lemma 1) $2\,[(y - t)/(y - 2t)]^{\frac{r}{r-1}} < 1$. From theorem 1 there is then type A agglomeration when ε = 0. Now allow $e > 0$. The condition in the last Proposition that $w_{00}^{\frac{r}{r-1}} + w_{10}^{\frac{r}{1-r}} < 1$ is such that the left hand side increases with e; it follows from our fixing of r, y and t that there is $e^*>0$ such that $w_{00}^{\frac{r}{r-1}} + w_{10}^{\frac{r}{1-r}} < 1$ remains true for e Î $(0, e^*)$. On the other hand profits after agglomeration (proposition 7) exceed those under geographical separation for small enough, positive e, $e \in (0, e^{**})$ say where $e^{**} > 0$. Then for $e \in (0, \min(e^{*}, e^{**}))$ we have an agglomeration subgame perfect equilibrium in which profits after geographical separation shrink to 0 as $e \to 0$. Thus for small enough heterogeneity the type A agglomeration remains in this approximate sense, exhibiting a required robustness.

The consumer welfare result (Theorem 3) is similarly robust. Fix r, y and t so that $r > g\,[(y-t)/(y-2t)]$ and $r > r^{*}\ (\cong 0.2737)$. Then (Theorem 3), with $e = 0$ there is type A agglomeration but consumers would prefer geographical separation of the firms. Now allow $e > 0$. we know from the last paragraph that type A agglomeration continues in an approximate sense for $e \in (0, \min(e^{*}, e^{**}))$. All consumers would prefer geographical separation to this agglomeration if $\tilde{v}_h > v_h^{*}$ for all $h \in [0,1]$ where $\tilde{v}_h^{*}$ is given by proposition 7 and $\tilde{v}_h$ by proposition 8. Some manipulation[6] shows that all these inequalities hold if

$$(y-t)^2 (y-t+2et)^2 > r(2-r)^{-1} 2^{\frac{1-r}{r}}$$

However when $\varepsilon = 0$ this inequality is equivalent to our assumed $r > r^{*}$ (see the proof of Theorem 3) and so continues to hold for positive e which are small enough.

6. THE HOTELLING MODEL

There are essentially 3 differences between our duopoly model and the standard Hotelling duopoly model, as follows:

a) Consumer homogeneity or near homogeneity. Here consumer locations are homogeneous. In Hotelling, consumers are spread uniformly along the whole of Main Street.

b) Tastes for variety. Here the goods sold by firms are (physically) differentiated and consumers may wish to buy from both firms, demands emanating from CES preferences. In Hotelling the goods are physically perfect substitutes, differentiated only by location, with consumers buying inelastically from one of the firms.

c) Outside firm locations. The feasible locations for firms are strictly outside the set of consumer locations. In Hotelling firms may locate anywhere within the set of consumer locations (i.e. Main Street).

The consequences of these differences are extreme. In particular, the Hotelling 2-stage location-price game never has any kind of equilibrium in which firms agglomerate, since agglomeration now produces the Bertrand paradox zero profits, whilst geographical separation of the firms produces positive profits as the goods are then location differentiated for the heterogeneous consumers. The best-known positive result (D'Aspremont et al (1977)) is that under quadratic transport costs the subgame perfect equilibrium has maximum geographical separation of the firms (one at 0, one at 1). In the remainder of this section we argue that each of the (a), (b) and (c) differences is needed for this turnaround.

The role of consumer homogeneity is that it is needed for the exact Bertrand parallels of type A agglomeration, creating the zero profits under geographical separation of sections 3, 4 and 5.1. The small Main Street heterogeneity of section 5.2 leads to small separation profits that shrink to zero as the heterogeneity disappears, leading to approximate type A agglomeration. The strength of our overall claims thus depends tautologically on the proximity to homogeneity, and weakens as Main Street heterogeneity expands.

Tastes for variety, or physically differentiated products, are also essential since they allow positive profits for agglomerated firms. Changing our assumption here to that of Hotelling would entail $r = 1$, zero profits under agglomeration, and so agglomeration could never be an equilibrium in section 5.2.

Finally consider replacing our outside firm locations assumption with Hotelling's inside locations, in the model of Section 5.2. Now consumers located to the left of the leftmost firm in $\left[\frac{1}{2}-e, \frac{1}{2}+e\right]$ will never buy only from the rightmost firm and vice versa. Clearly the most likely inside firm locations to induce a price subgame equilibrium in which consumers buy from only one firm are where one firm is at $\frac{1}{2}-e$ and one at $\frac{1}{2}+e$. But, for given $r \in (0,1)$, this cannot happen for positive, sufficiently small e, as follows.

From lemma 1, the consumer at $\frac{1}{2}-e$ will buy from only one firm (the firm at $\frac{1}{2}-e$)

if and only if $P/Q < \left[\left(\frac{y}{y-4et}\right)^{\frac{r}{1-r}} - 1\right]^{\frac{r}{1-r}}$ and the consumer at $\frac{1}{2}+e$ will buy from

only one firm (at $\frac{1}{2}+e$) if and only if $P/Q < \left[\left(\frac{y}{y-4et}\right)^{\frac{r}{1-r}} - 1\right]^{\frac{r-1}{r}}$. Thus there are no

prices at which all consumers buy from only one firm if

$$\left[\left(\frac{y}{y-4et}\right)^{\frac{r}{1-r}} - 1\right]^{\frac{r}{1-r}} > \left[\left(\frac{y}{y-4et}\right)^{\frac{r}{1-r}} - 1\right]^{\frac{r-1}{r}}$$

or, $2^{\frac{1-r}{r}} > \frac{y}{y-4e_t}$

For given $r \in (0,1)$ this inequality holds for all e sufficiently small, which is sufficient to preclude type A agglomeration with inside locations and near homogenous consumers[7].

7. TWO FURTHER VARIATIONS

Our theme has been articulated in a model where firms compete in prices after locating in one of two centres, and where consumers buy a variety of goods, with full information. We have argued that competition between the centres can force firms into type A agglomeration. In this section we show that the argument carries over to two somewhat different settings. In section 7.1 consumers search a centre for their most desired good of which they wish to buy one unit (inelastically), and in section 7.2 firms compete in quantities at stage 2, generating a Cournot version of the previous Bertrand stories.

7.1 A search model

The model is Gehrig (1998) except that we replace his conventional assumptions (firms can locate anywhere on Main St. along which consumers are uniformly distributed) with the geography of section 4 (n firms to locate at 0 or 1, all consumers located at $\frac{1}{2}$). Alternatively the model is that of section 4, changing the consumer aspect as follows. Consumers wish to buy inelastically just one unit of one (differentiated) good. With n_0 firms at 0 and n_1 at 1, it is assumed that the products available at each centre are symmetrically distributed around a "characteristics" circle of perimeter 1. Always now, consumers buy from only one centre, and realise their most preferred characteristic after arriving at the chosen centre. They then search the chosen centre for the firm offering the product closest to their ideal, and buy the one unit from that firm. Assuming that the realized ideal good is equally likely to be anywhere on the circle, and that the consumer cost of buying the best alternative available is composed of the travel cost ($\frac{1}{2}t$ for each centre, following Gehrig), the price paid and a search cost proportional (with a factor m) to the circular distance between the ideal and the best available good, the full expected cost of buying from centre 0 (EC_0) and 1 (EC_1) is as follows, when p is the price at all firms in centre 0 and q is the uniform price at 1;

$$EC_0 = \tfrac{1}{2}t + p + m/4n_0,\ EC_1 = \tfrac{1}{2}t + q + m/4n_1 \qquad (7.1)$$

In each case the final term is the search cost. Clearly with uniform prices across both centres (p=q) consumers buy at the centre offering the greater variety. If a firm $d \in N_0$ (say) deviated to p_d then EC_0 changes to (see Gehrig (1998) for derivations of (7.1)-(7.3)),

$$EC_0 = \frac{1}{2}t + \left(\frac{n_0 - 1}{n_0}p + \frac{1}{n_0}p_d\right) + \frac{m}{4m_0} - \frac{1}{2v}(p - p_d)^2 \qquad (7.2)$$

And the deviant's profit is $\pi_d = (p_d - c)x_d$ where demand is,

$$x_d = 1/n_0 + \quad (p - p_d)/m \qquad (7.3)$$

To investigate price consequences of alternative locations, start with the agglomeration case (without loss of generality $N_0 = N$, $N_1 = \varnothing$). Using π_d above it is easy to derive the following subgame equilibrium prices, profits and consumer costs:

$$p_i = m/n + c, \quad \pi_i = m/n^2,\ i = 1, \ldots, n;\ EC_0 = \tfrac{1}{2}t + c + 5m/4n$$

If $n_0 \geq n_1 \geq 1$ we would expect (following section 4) that, when n_0 is not too much bigger than n_1, marginal cost prices would emerge at the smaller centre 1, and price at 0 would be the highest price at which $EC_0 \leq EC_1$. The appendix shows that this is the case when $5/6 \geq n_0/n(\geq 1/2)$, in which case;

$$\text{for } i \in N_0,\ p_i = \frac{m}{4}\left(\frac{1}{n_1} - \frac{1}{n_0}\right) + c;\ \text{for } i \in N_1,\ q_i = c \qquad (7.4a)$$

$$\text{for } i \in N_0,\ p_i = \frac{m}{4n_0}\left(\frac{1}{n_1} - \frac{1}{n_0}\right);\ \text{for } i \in N_1,\ p_i = 0 \qquad (7.4b)$$

$$EC_0 = EC_1 = \tfrac{1}{2}t + c + m/4n_1 \qquad (7.4c)$$

When $n_0/n > 5/6$, centre 1 is too small to affect centre 0 and the outcome is (see appendix for proof):

$$\text{for } i \in N_0,\ p_i = m/n_0 + c;\ \text{for } i \in N_1,\ q_i = c \qquad (7.5a)$$

$$\text{for } i \in N_0,\ p_i = m/n_0^2;\ \text{for } i \in N_1,\ p_i = 0 \qquad (7.5b)$$

$$EC_0 = \tfrac{1}{2}t + c + 5m/4n_0 \qquad (7.5c)$$

Everything is exactly analogous to section 4. Agglomeration at one centre is the unique subgame perfect equilibrium. As firms disperse to separate centres the smaller centre is completely ineffective if it is very small $(n_0/n > 5/6)$, and prices and profits at the larger centre, and consumer costs, increase as n_0 falls in this range. When the smaller centre is not too small $(5/6 \geq n_0/n \geq 1/2)$, it does restrain prices and profits at the larger centre, which decrease, as do consumer costs, as n_0 falls towards $\tfrac{1}{2}n$. Consumer costs are minimized (so utility is maximized) with equal sized centres, again.

The intuition of section 4 carries over also. Competition between centres is fierce because as soon as consumers face a "one or the other" choice between centres, the centres behave like single firms in a homogeneous Bertrand duopoly. Gehrig (1998) invokes an externality argument instead for his heterogeneous consumer model - if a firm in one centre lowers price it attracts business from rivals in the same centre but also attracts business to the centre from the other centre. This externality is of course present here and in section 4, but cannot provide a satisfactory account of type A agglomeration in our model. Indeed when $n = 2$ the externality disappears completely, but the agglomeration remains.

7.2 A Cournot model

In our Bertrand models, firms choose locations at stage 1 and prices at stage 2, and consumers have full information on prices prior to accessing a centre and buying goods, which we can helpfully now think of as at "stage 3". In our Cournot variation, firms choose locations at stage 1 and quantities at stage 2. Firms commit to quantities before consumers access the centres at stage 3. Prices which clear markets at the two centres are assumed to emerge at stage 3, a la Cournot. For simplicity we assume that consumers access just one centre, that there are 2 firms, that consumers are homogeneous, and that transport costs are linear.

If the firms locate together (at centre 0, without loss of generality) and choose quantities x_1, x_2, all consumers access centre 0 and the market clearing prices (CES inverse demands) are:

$$p_i = \left(y-t\right) x_i^r \left(x_1^r + x_2^r\right)^{-1},\ i = 1,2$$

It is straightforward to define payoffs in the stage 2 quantity subgame and show that they generate the unique equilibrium $x_1 = x_2 = r\left(y-1t\right)/4c$. So prices, profits and utilities are;

$$p_i = 2c/r,\ i = 1,2;\ p_i = \tfrac{1}{4}(y - t)(2 - r);\ n = 2^{1/r} r(y - t)/4c$$

If firm 1 is at 0 and firm 2 at 1 and a fraction λ of consumers access centre 0, the rest at 1, then market clearing prices given quantities x_1 (>0) and x_2(>0) are defined by

$$l(\mathrm{y} - t)/p_1 = x_1 \text{ and } (1 - l)(y - t)/q_2 = x_2$$

A consumer at centre 1 gets utility $(\mathrm{y} - t)/p_1$ which must be the same as for centre 2, $(\mathrm{y}-t)/q_2$ – otherwise all consumers would shop at the same centre. Hence $l = x_1/(x_1 + x_2)$ and the market clearing prices become $p_1=q_2=(y-t)/(x_1+x_2)$. Again it is simple to derive the unique quantity subgame equilibrium as $x_1=x_2=(y-t)/4c$, producing the prices, profits and utilities;

$$p_1 = q_2 = 2c;\ p_i = \tfrac{1}{4}(y - t);\ n = (y - t)/2c$$

Hence, as in section 3, prices and profits are lower under separation than under agglomeration, and agglomeration is the subgame perfect equilibrium. When firms agglomerate the product differentiation produces relatively high prices and relatively large profits. When they separate it is as if the goods become perfect substitutes, lowering prices and profits, analogous to Section 3 but not as extreme, as usual with Cournot versus Bertrand. The behaviour of consumer utility is also completely analogous to the Bertrand case: if r is large enough consumers would prefer geographical separation of the firms. This indicates that there is a qualitatively similar Cournot argument to the earlier Bertrand, and so the earlier arguments are not an artefact due to the Bertrand specification. The Cournot agglomeration is quite different from that of Dudey (1990), however, who essentially interchanges our stages 2 and 3. In that case a one-firm centre will behave monopolistically to its consumers (since they commit to the centre before quantities are chosen), so it gets no consumers and agglomeration equilibrium is again the result. Separation is then not an equilibrium since consumers expect small centres to be less competitive and do not use them.

8. CONCLUSIONS

The paper has shown how geographical separation of product differentiated oligopolists across two market-places can lead to fierce price competition, analogous to that of homogeneous product Bertrand models, whereas agglomeration of the oligopolists in one market-place allows the product differentiation to produce a more profitable outcome. This happens if consumers are relatively homogenous in their costs of accessing market-places and the differentiated goods are reasonably good substitutes so that consumers buy at only one market-place when firms are separated. And when it does happen firms will choose to agglomerate in one market-place (type A agglomeration), although consumers may indeed prefer the fiercer competition and lower prices of two active competing market-places.

The type A agglomerative forces reverse those of the textbook Hotelling model, a reversal due jointly to the consumer homogeneity and tastes for variety (product differentiation), and to the “outside” firm locations. In the type A equilibrium, firms choose to co-locate geographically with given, different product lines and with, in particular, consumer homogeneity, whereas in Klemperer (1992), firms choose the same product line given separate geographical locations and with now appropriate consumer heterogeneity.

Our main model has price competition, fully informed consumers and the out-of-town shopping example. However the main ideas emerge also with quantity competition, producing a Cournot agglomeration argument quite different from Dudey (1990), and with consumer search, producing an alternative argument for financial centre agglomeration to Gehrig (1998). In general terms the paper shows how competition between markets can be much more severe than competition within markets.

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FOOTNOTES

1. Tirole (1989) provides an excellent textbook exposition.

2. The Stahl references are explicitly focussed on location of shops and provide our main model of consumer behaviour, but not all provide full evaluations of location-price equilibria; Stahl (1982) assumes fixed prices, for instance.

3. Alternatively consumers have a Cobb-Douglas function over various CES sub-utility functions, one of which is defined over the set of goods *N*. *y* is then the (constant) budget share allocated to *N*.

4. Qualitatively this makes no difference.

5. In the terminology of Gabszewicz and Thisse (1986, see also 1992) the model is an "outside" location model - firms can locate only outside the given residential district. Our focus is completely different from their vertical differentiation analysis, where both firms are outside but on the same side of the consumer locations.

6. The graphs of $\tilde{v}_h$ (piecewise linear) and v_h^* (linear) against *h* show that $\tilde{v}_h > v_h^*$ for all *h* Î [0,1] if and only if either (a) $\tilde{v}_0 > v_0 *$ and $r(2-r)^{-1} 2^{\frac{1-p}{p}} c^{-1} > \tilde{p}_1^{-1}$ or (b) $\tilde{v}_{0.5} > v_{0.5}^*$ and $r(2-p)^{-1} 2^{\frac{1-r}{r}} c^{-1} < \tilde{p}_i^{-1}$. Substitution of formulae for $\tilde{v}, v*$ and $\tilde{p}_1$ from Propositions 7 and 8 shows that (a) is impossible and (b) becomes the inequality claimed in the text.

7. In fact (details omitted) for given ρ ∈ (0,1) and positive sufficiently small ε, agglomeration of the firms at any point of $\left[\frac{1}{2}-e, \frac{1}{2}+e\right]$ is a type B equilibrium.

APPENDIX

Proof of Lemma 1

(a), (c) Using (2.1), (2.3) and the definitions of P,Q,C, f and w_{0h}, the inequality $n_{01h} < n_{0h}$ becomes;

$$P/Q < f(r, w_{0h}) \qquad \text{(i)}$$

Similarly $n_{01h} < n_{1h}$ becomes;

$$f(r, w_{1h})^{-1} < P/Q \qquad \text{(ii)}$$

If $f(r, w_{1h})^{-1} < f(r, w_{0h})$ then clearly there are no prices at which h buys from both centres, and in this case comparing n_{0h} and n_{1h} reveals that h buys from a single centre as claimed.

After some rearrangement, the sign of $f(r, w_{1h})^{-1} - f(r, w_{0h})$ is, for all $r \in (0, 1)$ and $(w_{0h}, w_{1h}) \in (1, \infty)^2$, the same as that of $G(r, w_{0h}, w_{1h}) = w_{0h}^{\frac{r}{r-1}} + w_{1h}^{\frac{r}{r-1}} - 1$. Now $G \to 1$ as $r \to 0$, $G \to -1$ *as* $r \to 1$ and $\partial G/\partial r < 0$. So for each $(w_{0h}, w_{1h}) \in (1, \infty)^2$ there is a unique value for $r \in (0, 1)$ where $G = 0$, thus defining a function $g : (1, \infty)^2 \to (0, 1)$ where $g(w_{0h}, w_{1h}) - r$ has the sign of $f(r, w_{1h})^{-1} - f(r, w_{0h})$, completing the proof of (c). In particular, if $r > g(w_{0h}, w_{1h})$ then $f(r, w_{1h})^{-1} < f(r, w_{0h})$, which also completes the proof of (a).

(b) With g as defined above and $r < g(w_{0h}, w_{1h})$, it follows that $f(r, w_{1h})^{-1} > f(r, w_{0h})$. Reversing (i) and (ii) above it also follows that $n_{01h} > \max(n_{0h}, n_{1h})$ if $P/Q \in (f(r, w_{0h}), f(r, w_{1h})^{-1})$ which is nonempty since $r < g(w_{0h}, w_{1h})$; at such price indices the uniquely optimal consumer demands are thus given by (2.3). If $P/Q < f(r, w_{0h})$ (resp., $P/Q > f(r, w_{1h})^{-1}$) it is easy to check that $n_{0h} > \max(n_{1h}, n_{01h})$ $(\text{resp.,} n_{1h} > \max(n_{0h}, n_{01h}))$, so the uniquely optimal demands are those of (2.1) (resp., (2.2)), with the borderline indifference claimed.

(d) With the definition of g above, $r = g(w, w)$ if and only if $f(r, w) = f(r, w)^{-1}$, or $w^{\frac{r}{1-r}} = 2$ which is equivalent to $r = \ell n2/\ell n(2w)$. ■

Proof of Lemma 3

We change variables to $x = r\,(1\text{-}r)^{-1}$ which defines an increasing function mapping (0, 1) onto (0, ∞). (3.3) is then equivalent to (A1) which is equivalent to (A2):

$$x^{-1}\left(w^{x}-1\right)^{\frac{1}{x}}\left(2+x-w^{-1}\right)\le 1 \qquad \text{(A1)}$$

$$f(x,w)=\ell n\left(w^{x}-1\right)+x\,\ell n\left(2+x-w^{-1}\right)-x\,\ell n\,x\le 0 \qquad \text{(A2)}$$

As $w \to 1, f \to -\infty$; as $w \to \infty, f \to +\infty$; also $\partial f/\partial w > 0$ everywhere. So for each *x Î* (0, ¥) there is a unique *w* such that *j (x, w)* = 0, defining a function *w: (*0, *¥) →* *(*1, *¥)* where *w = w(x)* iff *f (x, w)* = 0. Moreover we show in Lemma A.1 below that $\partial f/\partial x > 0$ everywhere, so $w'(x) < 0$ everywhere and *w £ w(x)* is equivalent to (A2) (or (A1), or (3.3)). The inverse of *w(x)* defines the function *h: (*1, *¥) ® (0,* 1*)* where $h(w) = w^{-1}(w)/\left[1+w^{-1}(w)\right]$, and so $h'(w) < 0$ everywhere and $r \le h(w)$ iff (3.3) holds. For the claimed boundary behaviour of h to hold we must show:

(a) $\lim\limits_{x\to\infty} w(x) = 1$; this is lemma A.2 below

(b) $\lim\limits_{x\to 0} w(x) = +\infty$; this is lemma A.3 below.

Proof of lemma A.1

$$\frac{\partial f}{\partial x} = \frac{w^{x}\,\ell n w}{w^{x}-1} + \ell n\left(2+x-\frac{1}{w}\right) + \frac{x}{2+x-\frac{1}{w}} - \ell n x - 1$$

At the lower limit *w* = 1, $\frac{\partial f}{\partial x} = \ell n\left(\frac{1+x}{x}\right) + \frac{x}{1+x} - 1 > 0$ for all *x* > 0 since $\ell n\, u + \frac{1}{u}$ has a global lower bound value of 1 attained at *u* = 1.

Also, $$\frac{\partial}{\partial w}\frac{\partial f}{\partial x} = \frac{\partial}{\partial w}\left[w^{x}\ \ell n w\left(\frac{2+x-\frac{1}{w}}{x}\right)^{\frac{1}{x}}\right] + \frac{1}{w^{2}}\cdot\frac{1}{2+x-\frac{1}{w}} - \frac{1}{w^{2}}\cdot\frac{x}{\left(2+x-\frac{1}{w}\right)^{2}},$$

where the first term is clearly positive and the second and third terms become $w^{-2}\left(2+x-\frac{1}{w}\right)^{-2}\left(2-\frac{1}{w}\right) > 0$ also. It follows that $\frac{\partial f}{\partial x} > 0$ everywhere. ■

Proof of lemma A.2 $\left(w(x) \to 1 \text{ as } x \to \infty\right)$

Since w is decreasing it suffices to show that $w(x)$ cannot converge to any $\hat{w} \in (1, \infty)$ as $x \to \infty$. Let $(x_n, w_n)_{n=1}^{\infty}$ be a sequence where $w_n = w(x_n)$ for all n, $x_n \to \infty$ and $w_n \to \hat{w} \in (1, \infty)$. Reverting to the (A1) border definition of the function w, for all n;

$$\left[\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} - 1\right] x_n + \left(2 - \frac{1}{w_n}\right)\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} = 0 \tag{A2}$$

But (see lemma A.4 below), $\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} \to \hat{w}$ as $n \to \infty$, contradicting (A2) for n large enough. ■

<u>Proof of lemma A.3</u> $\quad (w(x) \to \infty$ as $x \to 0)$

Again it suffices to show that $w(x)$ cannot converge to any $\hat{w} \in (1, \infty)$ as $x \to \infty$. Define the sequence $(x_n, w_n)_{n=1}^{\infty}$ as in the proof of lemma A.2. The (A1) border requirement is equivalently, for all n;

$$\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} \left(\frac{2 - w_n^{-1}}{x_n} + 1\right) - 1 = 0 \tag{A3}$$

But now (see lemma A.5 below), both $\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}}$ and $x_n^{-1}\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} \to 0$ *as* $n \to \infty$, contradicting (A3) for n large enough. ■

<u>Proof of lemma A.4</u> $\quad \left(\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} \to \hat{w} \text{ if } w_n \to \hat{w} \in (1, \infty) \text{ and } x_n \to \infty\right)$

Note $\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} = w_n \left(1 - w_n^{-x_n}\right)^{\frac{1}{x_n}}$ and $\ell n \left(1 - w_n^{-x_n}\right)^{\frac{1}{x_n}} = \dfrac{1}{x_n} \ell n\left(1 - w_n^{-x_n}\right) \to 0$ as $n \to \infty$.

So $\left(1 - w_n^{-x_n}\right)^{\frac{1}{x_n}} \to 1$ and $\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} \to \hat{w}$, as $n \to \infty$. ■

<u>Proof of lemma A.5</u> (for any $k \geq 0$, $x_n^{-k}\left(w_n^{x_n} - 1\right)^{\frac{1}{x_n}} \to 0$ *as* $w_n \to \hat{w} \in (1, \infty)$ and $x_n \to 0$)

It suffices to show that, as $n \to \infty$, $-k \ell n x_n + \dfrac{1}{x_n} \ell n\left(w_n^{x_n} - 1\right) \to -\infty$. Now consider the function of x_n, $w_n^{x_n}$ (for fixed $w_n \in (1, \infty)$) at the 2 points x_n and 0; the slope of the chord joining the two corresponding points on the graph must equal the derivative at some intermediate point $q x_n$ where $q = q(x_n) \in (0, 1)$. That is, $w_n^{x_n} - 1 = x_n . \ell n w_n . w_n^{q x_n}$. Hence;

$$-k\ell n x_n + \frac{1}{x_n}\ell n\left(w_n^{x_n} - 1\right) = -k\,\ell n x_n + \frac{1}{x_n}\ell n\left(x_n\ \ell n\, w_n\right) + \ell n\left(w_n^{q}\right) \qquad \text{(A4)}$$

The last term on the right of (A4) belongs to $[0,\ \ell n\, w_n] \to [0,\ \ell n\, \hat{w}]$ as $n \to \infty$, and is bounded. Writing $r_n^{-1} = x_n\, \ell n\, w_n$, so $r_n \to \infty$ as $n \to \infty$, the first 2 terms on the right of (A4) are

$k\ \ell n\left(r_n\ \ell n\, w_n\right) - \left(r_n\ \ell n\, w_n\right)\ell n\, r_n = k\ \ell n\left(\ell n\, w_n\right) + \left(k - r_n\ \ell n\, w_n\right)\ell n\, r_n \to -\infty$ as $n \to \infty$, as required. ■

It remains to show that *h(w)* < *g(w)* for all *w* > 1. *r* ³ *g(w)* is equivalent (using Lemma 1(*d*)) to w^x ³ 2. But if w^x ³ 2, $\left(w^x - 1\right)^{\frac{1}{x}} \geq 1$, and $x^{-1}\left(2 + x - w^{-1}\right) > 1$ (since $w > 1$), so (A1) is violated and *r* > *h(w).* This ensures *g(w)* > *h(w)* as required. ■

Proofs of (7.4) and (7.5) If $q = c$ the price *p* which makes $EC_0 = EC_1$ in (7.1) is

$p = m(1/n_1 - 1/n_0)/4 + c$. No firm at centre 0 wants to raise price as the whole market then goes to 1. No firm *d* at 0 wants to lower price provided $\partial p_d / \partial p_d \geq 0$ at $p_d = p$;

$1/n_0 + p/m + c/m - 2p_d/m$ at $p_d = p$ if and only if $5n_1 \geq n_0$.

Thus when $n_0/n_1 \leq 5$, *or* $n_0/n \leq 5/6$, the above price *p*, with $q = c$, is a Nash equilibrium. The formula for profits (since each firm at centre 0 sells $1/n_0$ units) and consumer costs (substituting into (7.1)) follow, completing the proof of (7.4). When $n_0/n > 5/6$ the claimed prices and profits at centre 0 are as if centre 1 was absent. With these prices $EC_0 = \frac{1}{2}t + 5m/4n_0 + c$. With $q = c$, $EC_1 = \frac{1}{2}t + c + m/4n_1$ and $EC_1 > EC_0$, so firms at centre 1 cannot do better than $q = c$ and zero profits. Thus (7.5) follows. ■