Natural Agglomeration*

Pierre M. Picard[†] and Takatoshi Tabuchi[‡]

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Abstract

This paper considers the racetrack economic approach, where manufacturing activities are distributed continuously. We seek constant-access equilibria and show that smooth equilibrium distributions are always unstable for almost all transport cost functions, whereas agglomeration in 1 or 2 atomic cities is stable for any economic parameters given regular transport costs, such as linear transport costs.

Keywords: agglomeration, continuous distribution, asymptotic stability, Fourier series.

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1 Introduction

The number of cities that naturally emerge from market interactions in continuous space is an issue that worries spatial economists and economic geographers. Indeed, given the bunch of literature based on the special case of two regions, such as Krugman (1991), it is important to check the foundation of spatial configurations with large number of cities or regions.

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[†]University of Manchester, pierre.picard@man.ac.uk.

[‡]University of Tokyo. ttabuchi@e.u-tokyo.ac.jp.

Krugman's (1993) develops the racetrack economic approach and attempts to support the view that agglomeration processes lead to few cities. He demonstrates and simulates that a configuration with 12 symmetric cities of equal size is often unstable against small perturbations. Brakman, Garretsen and Marrewijk (2001) provide simulations supporting the same kind of result and they suggest that 2-city agglomeration is likely to emerge. Fujita, Krugman and Venables (1999, chapters 6 and 17) and Mossay (2003) prove the instability of flat earth, where workers distribute uniformly over space, in Krugman's (1991) model. On the other hand, Tabuchi, Thisse and Zeng (2002) and Anas (2003) provide more analytical light on the topic given identical transport costs between any two regions.

This paper considers both discrete distributions of atomic cities and continuous distributions of cities in a unified framework. It attempts to fill the gap in the economic geography literature by investigating the nature and the number of stable equilibria. Extending Salop's (1979) model, we assume that economic space is a continuous circumference on which firms and also workers-consumers are perfectly mobile. The economic space is symmetric in that there exists no "first nature" locational advantage.

The results obtained in this paper contrast with those obtained in traditional theories in spatial competition. In particular, spatial competition models based on a circular space yield equilibria where the economic activity is dispersed. In particular, it is shown that firms disperse equidistantly around a circumference when they choose their location before their prices (Anderson, de Palma and Thisse 1992) and that similar configurations are obtained when firms choose their location before their output levels (Pal, 1998; Matsushima, 2001; Matsumura and Shimizu, 2002). This literature does not give support the existence of agglomerations in a single location unless some geographical constrained are considered. For instance, agglomeration in a single location can occur when space is limited to a line segment (de Palma, Ginsburgh, Papageorgiou and Thisse, 1985; Anderson and Neven, 1991). The existence of borders indeed provides a locational advantage to the central place. By contrast, the present paper shows that agglomeration in a single location exists when some workers-consumers may choose to locate close to firms.

Spatial distributions of manufacturing activities are sensitive to the shapes of transport costs. The impact of transportation costs on economic agglomeration is not well documented in the economic geography literature which generally restricts attention to iceberg and linear transportation costs. In this paper, we therefore do not assume particular transport costs, but consider a class of transport costs including linear, exponential, sinusoidal, and so on.

We firstly study the existence and the stability of continuous spatial distributions of workers. Such spatial distributions are characterized by no empty location and continuous distributions of manufacturing activities. Many equilibria can exist according to the shape of transportation costs and to the spatial frequency contents of both transport cost functions and spatial distribution. We also show that equilibrium exists with non-uniform distributions of manufacturing activities. However, any continuous equilibrium distribution is unstable in the sense that there always exist a spatial perturbation of workers' migration that does not converge back to the equilibrium. Stable equilibria therefore includes no continuous distributions of economic activities.

We secondly investigate the equilibrium conditions of configurations in which economic activity is concentrated on a finite number of atomic cities. It is shown that the nature of equilibria is very similar to that for continuous distributions. Many equilibria may exist according the shape of transportation costs and to the spatial frequency contents of both transport cost function and spatial distribution of workers. However, we show that equilibrium distributions with large number of cities are unstable as it is the case for continuous distributions. By contrast, there always exists a configuration with either 1 or 2 atomic cities that is stable if transport costs have the regular properties used in the literature, including those of linear transport costs. Such a finiteness property in the number of cities might be comparable to the properties of the limited number of firms in natural oligopoly in the literature of vertical differentiation (Shaked and Sutton, 1983).

The remainder of the paper is organized as follows. The model is presented in Section 2. Existence of continuous equilibria are analyzed in Section 3, while its stability is dealt with in Section 4. On the other hand, Section 5 characterizes existence of atomic city equilibrium, and Sections 6-8 consider its stability properties mainly in the cases of sinusoidal and linear transport costs. Section 9 concludes.

2 The Model

We assume that immobile farmers and perfectly mobile workers are located on a circumference with perimeter equal to 1. Farmers are uniformly distributed around the circumference with a distribution density equal to A. Workers are located according to the density $\lambda(y)L$ with $\int_0^1 \lambda(y)dx = 1$. Each firm produces a single variety $i \in [0, M]$ and requires ϕ workers to operate its plant. There are thus $\lambda(y)M$ varieties produced at location y. Equilibrium in the whole labor market implies that $L = \phi M$.

Let y and $x \in [0, 1]$ be the coordinates of a producer and a consumer on the circumference. The consumer demand of a variety produced at location y and consumed at location x is given by the function $q(y, x)$. As in Ottaviano, Tabuchi and Thisse (2002), consumers' preferences are identical across individuals and are described by the following quasi-linear utility with quadratic sub-utility functions:

$$\begin{aligned} U(q_0, q(\cdot, x)) \quad &= \quad \alpha \int_0^1 q(y, x)\lambda(y)\, M\ dy \ - \frac{\beta - \gamma}{2} \int_0^1 [q(y, x)]^2 \lambda(y)\, M\ dy \\ &\quad - \frac{\gamma}{2} \left[\int_0^1 q(y, x)\lambda(y)\, M\ dy \right]^2 + q_0 \end{aligned}$$

where $\alpha > 0$, $\beta > \gamma > 0$ and q_0 is the numéraire. The budget constraint of a consumer located at x is equal to

$$\int_0^1 p(y, x)q(y, x)\lambda(y)\, M\ dy + q_0 \leq w(x) + \bar{q_0}$$

where $p(y, x)$ is the price of a variety produced at location y and sold at location x, $w(x)$ is his/her income residing at location x, and $\bar{q_0}$ is the consumer's initial endowment. Whereas mobile workers' income depends on their location, the immobile farmers' income does not depend on location. Indeed, it assumed that immobile farmers produce the same constant-returns-to-scale good that can be transported at zero cost. Therefore, the market for this agricultural good clears at the same price in every location and yield the same income to farmers. We can normalize this income to 1 without loss of generality. Each consumer maximizes his/her utility, which leads to the following demand

$$q(y, x) = a - (b + cM)p(y, x) + cP(x)$$

where $a \equiv \alpha/[(\beta + (M - 1)\gamma] > 0$, $b \equiv 1/[\beta + (M - 1)\gamma] > 0$, $c \equiv \gamma/(\beta - \gamma)[\beta + (M - 1)\gamma] > 0$, and $P(x) = \int_0^1 p(y, x)\lambda(y)\, M dy$ is the price index at location x.

Each price-discriminating firm at location y maximizes its profits

$$\Pi(y) = \int_0^1 [p(y,x) - \tau(y,x)]\, q(y,x)\, [\lambda(x) L + A]\, dx - \phi w(y)$$

where $\tau(y,x)$ is the unit transport costs from locations y to x incurred by the firm, and $w(y)$ is the wage paid to workers employed by the firm at location y. The first-order condition yields the optimal price of variety produced and consumed at the same location x:

$$p(x,x) = \frac{2a\phi + cL \int_0^1 \tau(x,y)\, \lambda(y) dy}{2(2b\phi + cL)}$$

and the optimal price of variety produced at y and consumed at x:

$$p(y,x) = p(x,x) + \frac{1}{2}\tau(y,x)$$

For the sake of clarity, we define the firms' prices net of transport costs on a variety produced in x and sold in y as

$$m(x,y) \equiv p(x,y) - \tau(x,y) = p(y,y) - \frac{1}{2}\tau(x,y)$$

To avoid corner solutions, we impose that the prices net of transport costs are always positive. That is, we require that $m(x,y) > 0$ for any x, y and $\lambda(x)$, which is equivalent to $\min p(y,y) > \max \frac{1}{2}\tau(x,y)$.

In the long run, entry occurs until firms earn zero profit. Entry determines the workers' wage at location x as

$$w(x) = \frac{b\phi + cL}{\phi^2} \int_0^1 [m(x,y)]^2 [\lambda(y) L + A]\, dy$$

As in Ottaviano *et al.* (2002), the consumer surplus of an individual located at x is given by

$$\begin{aligned} S(x) &= \frac{a^2 L}{2b\phi} - \frac{aL}{\phi}\int_0^1 p(y,x)\,\lambda(y)\, dy - \frac{cL^2}{2\phi^2}\left[\int_0^1 p(y,x)\,\lambda(y)\, dy\right]^2 \\ &\quad + \frac{b\phi + cL}{2\phi^2} L \int_0^1 [p(y,x)]^2 \lambda(y)\, dy \end{aligned}$$

The worker's indirect utility is therefore given by $V(x) = S(x) + w(x)$.

Before turning to the study of the equilibria for several types of spatial distributions, we need to be more explicit about the transport cost function. We decompose unit transport costs from locations x to y as

$$\tau(x,y) \equiv \frac{\tau}{2}[1 - C(x-y)]$$

where τ is the *amplitude* of transportation costs and where $C(x)$ captures the *shape* of transportation costs. The function $C(x) : \mathbb{R} \to [-1, 1]$ is a periodic function such that $C(x) = C(l + x) = C(l - x)\ \forall l \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, and such that $C(0) = 1$, $C(1/2) = -1$, and $C'(x) \leq 0$ $\forall x \in [0, 1/2]$. This function shares many similarities with the cosine function; indeed $C(x) = \cos 2\pi x$ fulfills these conditions. Also the crenellated periodic function $C(x) = C_1(x) \equiv 1 - 4x$ for all $x \in [0, 1/2]$ fulfills these conditions; it captures linear transport cost. Figure 1 shows examples of shapes of transport cost functions.

INSERT FIGURE 1 HERE

Given this definition, the condition to avoid corner solutions becomes:

$$\tau < \frac{2a\phi}{2b\phi + cL} \tag{1}$$

which is obtained from $\min p(y, y) > \max \frac{1}{2}\tau(x, y)$.

Finally, we will show in the sequel that the spatial frequency content of shape of transport costs drives the results about the existence and stability of equilibria. For this purpose, we define the Fourier decomposition of $C(x)$ and its square as

$$C(x) = \sum_{m=-\infty}^{\infty} c_m \exp(2\pi Imx) \qquad \text{and} \qquad [C(x)]^2 = \sum_{m=-\infty}^{\infty} d_m \exp(2\pi Imx)$$

where $I^2 = -1$ and $\exp 2\pi Ikx \equiv \cos 2\pi kx + I \sin 2\pi kx$. We readily have

$$\begin{aligned} c_m &= \int_0^1 C(x) \exp(-2\pi Imx)\ dx \\ d_m &= \int_0^1 [C(x)]^2 \exp(-2\pi Imx)\ dx \end{aligned}$$

Because the transport costs are even functions and return real values, the Fourier coefficients are real and symmetric with respect to m: $c_m = c_{-m} \in \mathbb{R}$ and $d_m = d_{-m} \in \mathbb{R}$. In other words, transport cost functions are approached by Fourier series with cosine components only. Moreover, one easily gets the following relationship: $d_0 > 0$ and

$$d_m = \sum_{k=-\infty}^{\infty} c_k c_{m-k} \tag{2}$$

3 Quasi-Smooth and Dense Equilibria

In this section, we consider quasi-smooth and dense equilibrium distributions of firms and workers, where manufacturing is active in any location.

Definition 1 *A spatial distribution $\lambda(x)$ is said to be quasi-smooth if $\lambda(x)$ is continuous and $\lambda'(x)$ is piecewise continuous $\forall x$. It is dense if $\lambda(x) > 0$ $\forall x$.*

In the sequel, we seek the conditions under which an equilibrium exists for quasi-smooth and dense distributions. Collecting the results of section 2, the worker's indirect utility can be rewritten as a function of $\lambda(\cdot)$ and x:

$$\begin{aligned} V(x) &= W_0 + W_1 \int_0^1 [f_1(y)]^2\, dy + W_2 \int_0^1 f_1(y)\lambda(y)\, dy \\ &\quad + W_3 \int_0^1 [f_1(y)]^2 \lambda(y)\, dy + V_1 f_1(x) + V_2 f_2(x) - V_3 [f_1(x)]^2 \\ &\quad - V_4 \int_0^1 C(x-y) f_1(y) dy - V_5 \int_0^1 C(x-y)\lambda(y) f_1(y) dy \end{aligned} \tag{3}$$

where the accessibility measures are given by

$$f_1(x) \equiv \int_0^1 C(x-z)\lambda(z)\, dz \qquad f_2(x) \equiv \int_0^1 [C(x-z)]^2 \lambda(z)\, dz$$

and where all constants W_j's and V_j's are all positive and 'generically' different from zero (see Appendix 1 for these expressions). Whereas the constant V_j's are the coefficients of terms that depends on x, the constant W_j's are coefficients of terms independent from x.

It is known that Fourier series of quasi-smooth functions converge (Iorio and de Magalhães Iorio, 2002, p.102). Hence, a quasi-smooth spatial distribution can be decomposed by its Fourier series:

$$\lambda(x) \equiv \sum_{k=-\infty}^{\infty} \lambda_k \exp(2\pi I k x) \tag{4}$$

where $\lambda_k \in \mathbb{C}$, the set of complex numbers. Because $\lambda(x)$ returns real values and because $\int_0^1 \lambda(x)\, dx = 1$, we have that $\lambda_{-k} = \overline{\lambda_k}$ where $\overline{\lambda_k}$ is the complex conjugate of λ_k and $\mathrm{Re}(\lambda_0) = 1$.

The Fourier series of $\lambda(x)$ and $C(x)$ make explicit the spatial frequency contents of those functions. Expanding the terms of the utility yields

$$
\begin{aligned}
f_1(x) &= \sum_{k=-\infty}^{\infty} \lambda_k c_k \exp(2\pi I k x) \\
f_2(x) &= \sum_{k=-\infty}^{\infty} \lambda_k d_k \exp(2\pi I k x) \\
[f_1(x)]^2 &= \sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} (\lambda_m c_m \lambda_{k-m} c_{k-m}) \right] \exp(2\pi I k x) \\
\int_0^1 C(x-y) f_1(y) dy &= \sum_{k=-\infty}^{\infty} \lambda_k (c_k)^2 \exp(2\pi I k x) \\
\int_0^1 C(x-y) \lambda(y) f_1(y) dy &= \sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} (\lambda_m c_m \lambda_{k-m} c_k) \right] \exp(2\pi I k x)
\end{aligned}
$$

Thus, $V(x) - V(0) = \sum_{k=-\infty}^{\infty} V_k \exp(2\pi I k x)$, where

$$
V_k = \lambda_k \left[V_1 c_k + V_2 d_k - V_4 (c_k)^2 \right] - \sum_{m=-\infty}^{\infty} \lambda_m c_m \lambda_{k-m} \left[V_3 c_{k-m} + V_5 c_k \right] \quad (5)
$$

Equilibrium is attained when $V_k = 0$ for all $k \neq 0$.

In this paper, we will focus on a particular type of equilibria: the constant-access equilibria. Because constant-access provides additional symmetry to the locational problem, the nature and the stability of constant-access equilibria can be determined in a general way. In many cases, non-constant access equilibria exist but their stability is difficult to assess.

3.1 Constant-access Equilibria

Definition 2 *A distribution $\lambda(x)$ is said to be constant-access if*

$$
c_k \lambda_k = c_k c_m \lambda_{k-m} = 0 \qquad \forall k \neq m \in \mathbb{Z} \quad (6)
$$

We immediately know from (5) that any constant-access distribution yields equilibria. In this case, the functions $f_1(x)$ and $f_2(x)$ are constant for all x.[1] The function $f_1(x)$ corresponds to the additional transport costs that workers located at x incur when they rise their consumption by one unit of each variety.

[1] It should be noted that the constant-access is endogenously determined here, while it is often exogenously specified in the literature, such as Tabuchi *et al.* (2002).

The function $f_2(x)$ corresponds to a higher moment of this additional transport costs. Constant-access brings additional symmetry: workers have the same access to varieties wherever they locate. Equivalently, firms have the same access to consumers.

By definition of equilibrium, a dense distribution yields is an equilibrium if the worker's utility is the same everywhere. It can be readily shown that any constant-access distribution is an equilibrium. Functions $f_1(x)$ and $f_2(x)$ can be decomposed with their frequency components as

$$f_1(x) = \sum_{k=-\infty}^{\infty} c_k \lambda_k \exp(2\pi I k x) \quad \text{and} \quad f_2(x) = \sum_{k=-\infty}^{\infty} d_k \lambda_k \exp(2\pi I k x)$$

An equilibrium exists if these expressions are constant, which is satisfied if $c_k \lambda_k = c_k c_m \lambda_{k-m} = 0$ for all $k \neq m \in \mathbb{Z}$. In this case, by (2), we have that $d_k \lambda_k = 0$, and thus $f_1(x) = f_1^* = c_0/2$ and $f_2(x) = f_2^* = d_0/2$, where $-1 \leq c_0 \leq 1$ and $0 \leq d_0 \leq 1$. This yields the following lemma.

Lemma 1 *Any constant-access, quasi-smooth and dense distribution of workers is an equilibrium.*

This lemma means the following. Let $Y = Y_c \cup Y_d$, where Y_c is a set of frequencies $m \neq 0$ such that $c_m \neq 0$ and where Y_d is a set of frequencies $m \neq 0$ such that $d_m \neq 0$. Let also Z be a set of frequencies k such that $\lambda_k \neq 0$. The set Z includes the frequency $k = 0$. Then, any distribution with spatial frequencies in Z is an equilibrium if $Z \cap Y = \varnothing$.

It is easy to infer that flat earth $\lambda^*(x) = 1$ (i.e., $Z = \{0\}$ because $\lambda_k = 0$ $\forall k \neq 1$) is a quasi-smooth and dense equilibrium distribution of workers that is compatible with all shapes of transport costs (i.e. for any c_m). From equality (6), it can be seen that *the set of quasi-smooth and dense equilibria is smaller, the richer the spatial frequency content of transport cost functions.* To illustrate this, it is worth looking at specific examples.

Linear transport costs: Assume that $C_1(x) = 1 - 4x \; \forall x \in [0, 1/2]$. Then, $c_0 = 0$, $c_m = 4\left[1 - (-1)^m\right] / (\pi m)^2 > 0$ and $d_m = 8\,(-1)^m / (\pi m)^2 > 0$. Thus, we readily get that either $c_m \neq 0$ or $d_m \neq 0$ for any $m \geq 1$. Hence, $Y = \mathbb{Z}^0$, the set of natural numbers. It must be that flat earth ($Z = \{0\}$) is the unique quasi-smooth equilibrium distribution.

Step transport costs: Assume that transport is costless within some distance ξ from production location and is constant above this distance: $C(x) = 1$ for $0 \leq x < \xi$ and $C(x) = -1$ if $\xi < x \leq 1/2$, where $0 < \xi < 1/2$,. Then, one can compute that $c_0 = 8\xi - 2$ and $c_m = \frac{4\sin 2\xi m\pi}{m\pi}$ for $m \geq 1$. If ξ is not a rational number so that $2\xi m \neq k$ with $m, k \in \mathbb{Z}$, then $c_m \neq 0$ for all $m \neq 0$, and hence $Y = \mathbb{Z}^0$. It must be that flat earth ($Z = \{0\}$) is the unique quasi-smooth equilibrium distribution. Nevertheless, there exists shapes of transport costs that yield equilibria with other spatial distributions than flat earth.

Sinusoidal transport costs: Assume that $C(x) = \cos 2\pi x$. Then $c_1 = 1$, $d_0 = 1$, $d_2 = 1/2$ and $c_m = d_m = 0$ otherwise. Hence, $Y = \{-2, -1, 1, 2\}$. Any spatial distribution with the shape $\lambda^*(x) = 1 + \sum_{m \geq 3} \lambda_m \exp(2\pi I m x) > 0$ is an equilibrium. For instance, the three-peaked distribution $\lambda^*(x) = 1 + 0.5\cos 6\pi x$ as well as flat earth is an equilibrium.

Non-constant-access equilibria: In many cases, constant-access equilibria exist at the same time as non-constant-access equilibria. Although the former are independent of the economic parameters $(a, b, c, ...)$, the latter generally depends on the economic conditions. Indeed, it is difficult to obtain a general characterization of those non-constant-access equilibria, but it is possible to find examples for some particular shapes of transport costs. For instance, when the transport cost function is a Fourier series truncated to the K-th component,

$$C(x) = \sum_{k=-K}^{K} c_k \exp(2\pi I k x)$$

it can be shown that non-constant-access distributions $\lambda^*(x)$ with the coefficients:

$$\begin{aligned}
\lambda_k &= 0 \qquad \text{for all } k \notin \{0, \pm K, \pm 2K\} \\
\|\lambda_K\|^2 &= \frac{V_2 d_{2K}}{V_3 V_5 (c_K)^4} \left[V_1 c_K + V_2 d_K - (c_K)^2 (V_4 + V_5)\right] \\
\lambda_{2K} &= (\lambda_K)^2 \frac{V_3}{V_1} \frac{(c_K)^2}{d_{2K}}
\end{aligned}$$

are equilibria.[2] In the case of sinusoidal transport costs (where $K = 1$), this is shown to be the unique non-constant-access equilibrium distribution. In contrast to constant-access equilibrium distributions, this distribution changes in

[2] A proof can be obtained upon request.

accordance with the economic parameters since it is determined by the economic parameters. Because the nature and stability of non-constant-access equilibria are too complicated to characterize in general, we will focus on constant-access equilibria in the sequel.

4 Stability of Quasi-Smooth and Dense Equilibria

In this section, we extend Krugman's (1993) racetrack economic approach for any (non-flat) equilibrium and study its stability. To this aim we analyze the stability against small cyclical perturbations on the quasi-smooth and dense equilibrium distribution $\lambda^*(x)$. We first present the dynamics of workers' migration, derive the equilibrium conditions of workers' distribution around the space, and finally study whether those perturbations attenuate or amplify due to (infinitesimally) small perturbations.

4.1 Dynamic Behavior of Workers

Introducing the time variable t, the variables $\lambda(x,t)$ and $V(x,t)$ are now time dependent. We assume myopic workers in the time and space dimensions: workers consider only the current period and the utility differential with respect to their neighboring locations in their migration decisions. More specifically, we assume that the number of migrating workers is proportional to the difference of their instantaneous utility between their current location x and their neighboring location $x \pm \varepsilon$ where $\varepsilon > 0$ is small enough. If $V(x-\varepsilon,t) > V(x,t)$, then we assume that $\nu_0 [V(x-\varepsilon,t) - V(x,t)]$ workers move from locations x to $x-\varepsilon$, otherwise $\nu_0 [V(x,t) - V(x-\varepsilon,t)]$ workers move from locations $x-\varepsilon$ to x, where coefficient ν_0 measures the speed of adjustment. Similarly, if $V(x+\varepsilon,t) > V(x,t)$, then $\nu_0 [V(x+\varepsilon,t) - V(x,t)]$ workers move from location x to location $x+\varepsilon$, otherwise $\nu_0 [V(x,t) - V(x+\varepsilon,t)]$ workers move from location $x+\varepsilon$ to location x. The resulting flows yield the following local motion equation:

$$\frac{\partial}{\partial t}\lambda(x,t) = \nu_0 [2V(x,t) - V(x-\varepsilon,t) - V(x+\varepsilon,t)]$$

This motion process respect the law of conservation of the total mass of workers. Indeed, the total number of workers remains fixed since $\int_0^1 \partial\lambda(x,t)/\partial t$

$dx = 0$. Still, the motion process will be well defined provided that the number of workers at each location remains positive. That is, we must choose sufficiently small ν and ε such that $\nu_0 |V(x,t) - V(x \pm \varepsilon, t)| \leq L\lambda(x,t)$. For dense distribution of workers, we have that $\min_{x,t} \lambda(x,t) > 0$ and it suffices that $\nu_0 \max_{x,t} |V(x,t) - V(x \pm \varepsilon, t)| \leq L \min_{x,t} \lambda(x,t)$.

For sufficiently small ε, the above expression can be approximated to the following motion equation:

$$\frac{\partial}{\partial t}\lambda(x,t) = -\nu \frac{\partial^2}{\partial x^2} V(x,t) \tag{7}$$

where $\nu = \nu_0 \varepsilon^2/2$, which is set equal to 1 without loss of generality.

We finally note that as in Mossay (2003), stability results do not depend on this local motion process where workers consider neighboring locations only. In the Appendix 2, we prove the validity of our results for a global motion process where workers consider all locations in their migration choice.

4.2 Perturbed Equilibrium Distributions

Let us consider a quasi-smooth and dense equilibrium distribution

$$\lambda^*(x) = \sum_{m \in Z} \lambda_m \exp(2\pi I m x),$$

where $Z \cap Y = \varnothing$ and $\lambda_0 = 1$. In order to check stability, we now allow for (infinitesimally) small temporal variations of the variables. Small perturbations are defined as $\widetilde{\lambda}(x,t) = \lambda(x,t) - \lambda^*(x)$ and $\widetilde{V}(x,t) = V(x,t) - V^*$, where a tilde refers to the perturbed values of these variables, and where $\lambda^*(x)$ and V^* are the equilibrium values of these variables. Also, using the definitions $\widetilde{f}_1(x,t) = f_1(x,t) - f_1^* = \int_0^1 C(x-y)\widetilde{\lambda}(y,t)\,dy$ and $\widetilde{f}_2(x,t) = f_2(x,t) - f_2^* = \int_0^1 [C(x-y)]^2 \widetilde{\lambda}(y,t)\,dy$, the utility function and the motion equation (7) can be linearized by dropping terms with perturbations of order strictly higher than one. This yields

$$\begin{aligned} \widetilde{V}(x,t) &= V_1 \widetilde{f}_1(x,t) + V_2 \widetilde{f}_2(x,t) - V_4 \int_0^1 C(x-y)\,\widetilde{f}_1(y,t)dy \\ &\quad - V_5 \int_0^1 C(x-y)\,\lambda^*(y)\,\widetilde{f}_1(y,t)dy + \text{constant} \end{aligned} \tag{8}$$

where we used the obvious facts that $\int_0^1 \widetilde{\lambda}(y,t)\,dy = 0$ and $\int_0^1 \lambda^*(y)\,dy = 1$.

The motion equation (7) can be linearized as

$$\frac{\partial \widetilde{\lambda}(x,t)}{\partial t} = -\frac{\partial^2 \widetilde{V}(x,t)}{\partial x^2} \tag{9}$$

Note that equations (8) and (9) constitute a homogenous system of linear partial differential equations. It can be studied by its 'normal modes' below.

4.3 Normal Modes and Instability

Let the system be perturbed by initial normal mode functions of the form $\exp(2\pi Ikx)$, where k is the normal mode frequency. Then, the solution of the system is given by $\exp(2\pi Ikx)\exp(s_k t)$, where s_k are amplifying parameters. Since the system of equations is linear, any linear combination of normal mode solutions is also a solution of the system. Because we focus on quasi-smooth and dense spatial distributions, any initial perturbation can be decomposed by its Fourier series $\widetilde{\lambda}(x,0) = \sum_{k=-\infty}^{\infty} \widetilde{\lambda}_k \exp(2\pi Ikx)$, where $\widetilde{\lambda}_k$ are normal mode amplitudes. Since the perturbation does not alter the total size of the workers' population, it must be that $\int_0^1 \widetilde{\lambda}(x,0)\, dx = 0$ and thus $\widetilde{\lambda}_0 = 0$. As a result, the response of the system to such an initial perturbation is equal to

$$\widetilde{\lambda}(x,t) = \sum_{k=-\infty}^{\infty} \widetilde{\lambda}_k \exp(2\pi Ikx + s_k t)$$

Stability is related to the normal modes by the following definition:

Definition 3 *A quasi-smooth and dense equilibrium $\lambda^*(x)$ is asymptotically stable if any sufficiently small change in the distribution results in a movement back toward the equilibrium.*

Therefore, an equilibrium is unstable if there exists a (infinitesimally small) perturbation of the equilibrium distribution that does not attenuate. That is, there exists a normal mode $k(\geq 1)$ that does not vanish: $s_k \geq 0$ or $r_k \geq 0$.

Krugman *et al.* (1999) study the stability of flat earth ($\lambda^*(x) = 1$, $Z = \{0\}$, $f_1^* = c_0$). We here provide the condition for stability of any constant-access equilibrium distribution $\lambda^*(x)$. Plugging this function into (8) and (9) yields the following lemma:

Lemma 2 *A constant-access, quasi-smooth and dense equilibrium distribution is unstable if and only if there exists one k such that $s_k > 0$ where*

$$\begin{aligned} s_k &= (2\pi k)^2 \left[\left[V_1 - (2V_3 + V_5)\, c_0 \right] c_k + V_2 d_k - (V_4 + V_5)\,(c_k)^2 \right] \\ &= \frac{\pi^2 k^2 \tau L\,(b\phi + cL)}{4\phi^2\,(2b\phi + cL)^2} \left[\begin{array}{l} \left[8\phi\,(2a - \tau b)\,(3b\phi + 2cL) - 2\tau cL\,(4b\phi + 3cL)\, c_0 \right] c_k \\ +3\tau\,(2b\phi + cL)^2\, d_k - 4\tau c\,(2b\phi + cL)\,(A + L)\,(c_k)^2 \end{array} \right] \end{aligned} \tag{10}$$

Proof. See Appendix 3. ■

For general transport cost functions, a constant-access equilibrium will be unstable if, for some k, the squared bracket term in the above expression is positive. Note that small transport costs τ and large manufacturing demand a decrease the likelihood of flat earth stability, whereas large farming population A always increases the likelihood of flat earth stability. However, the impact of the farming population is rather weak for perturbations with high frequencies because, as c_k is a decreasing series, the term in $(c_k)^2$ becomes much smaller than the terms in c_k and d_k at high frequencies. For high frequencies, the shapes of transport cost have then a dramatic impact on stability.

For high frequencies, the amplifying parameter s_k has the same sign as $\overline{V_1} c_k + V_2 d_k$, where $\overline{V_1} \equiv V_1 - (2V_3 + V_5)\, c_0$. The question of stability and instability is then related to whether this expression is always negative or whether it can sometimes be positive. When the transport cost function $C(x)$ is differentiable on the interval $[0, 1/2]$, the values of the Fourier coefficients c_k and d_k are closely related to the properties of the transport cost function at the points $x = 0$ and $1/2$ (see Appendix 4). That is, they are related to the derivatives of $C(x)$ evaluated at $x = 0$ and $1/2$.

Let us define

$$Q_h \equiv (-1)^h \left[\overline{V_1} C^{(2h-1)}(0) + V_2 D^{(2h-1)}(0) \right] + \left| \overline{V_1} C^{(2h-1)}(1/2) + V_2 D^{(2h-1)}(1/2) \right|$$

where $C^{(2h-1)}$ and $D^{(2h-1)}$ are the derivatives of $C(x)$ and $[C(x)]^2$ to the order $2h-1, h \in \mathbb{N}$. In some cases, Q_h can be equal to zero. Thus, we need to define $H \in \mathbb{N}$ such that

$$Q_H \neq 0 \qquad \text{and} \qquad Q_h = 0 \text{ for } h = 1, \ldots, H-1 \tag{11}$$

In Appendix 4, we show that s_k has the same sign as Q_H for sufficiently large k. Therefore, there exists a $\bar{k}$ such that (i) $s_k < 0$ for all $k > \bar{k}$ if $Q_H < 0$ and (ii) $s_k > 0$ for some $k > \bar{k}$ if $Q_H > 0$. Part (i) implies that all high frequencies attenuate. It can be readily checked that lower frequencies also attenuate if the farming population A is sufficiently larger. Part (ii) implies that some high frequencies amplify and that instability always prevails. These arguments lead to the next proposition and corollary.

Proposition 1 *Assume that $C(x)$ is differentiable and H exists. Then, any constant-access, quasi-smooth and dense distribution $\lambda^*(x)$ is a stable (resp. unstable) equilibrium for sufficiently large A if $Q_H < 0$ (resp. $Q_H > 0$).*

The literature has provided support to the view that flat earth is unstable in the case of very specific shape of transport costs (Fujita *et al.*, 1999; Mossay, 2003). The proposition suggests that *flat-earth instability is a property for a large class of transport cost functions.*

Nevertheless, the proposition also states that *flat earth can be stable* under some class of shape of transport costs. Take for instance the following polynomial transport cost function: $C(x) = 1 - 2x^2 - 96x^3 + 248x^4 - 160x^5$ (which is shown in Figure 1 with a dotted curve) This cost function is such that $C'(0) = C'(1/2) = D'(0) = D'(1/2) = C'''(1/2) = D'''(1/2) = 0$ and that $C'''(0) = -576$, $D'''(0) = -1152$. Thus, $Q_1 = 0$ and $Q_2 = -576\,[Q_A + 2Q_B] < 0$, which implies stability of flat earth when A is sufficiently large.

The intuition behind Proposition 1 and the example is as follows. First, note that since the farming population is the dispersion force in the model, a large number of farmers rises the likelihood of dispersion. The effect of the farming population is particularly important against low frequencies of perturbations. For instance, if workers are forced to locate in the North of the earth (i.e. a perturbation with frequency equal to one), they have incentives to relocate to the South because of the weaker competition and close access to Southern farmers. By contrast, if workers locate in close, repeated and dense areas (i.e. a perturbation with high frequency), they do not have incentives to relocate far away because access and competition conditions are rather similar everywhere. So, the dispersion role of the farming sector is likely to be important for perturbation with low frequencies. Second, suppose again that the perturbation has a high spatial frequency. Since $C'(0) = C'(1/2) = 0$, workers can always slightly relocate to a close location without altering its average cost and its utility. Therefore, the advantage of agglomeration disappears. All in all, dispersion becomes stable.

It must be noted that H does not exist when the transport cost function has a finite number of Fourier coefficients. For example, in the case of $C(x) = \cos 2\pi x$, we have $Q_h = 0$ for all h. In this case, any constant-access, quasi-smooth and dense equilibrium distribution is unstable for such a shape of transport cost with a limited frequency content. To see this, suppose that the frequency content of the transport cost function is limited to the frequency $\bar{k}$: that is, $c_k = 0$ for any $k > \bar{k}$. By (2), it is easily checked that the frequency content of the square of transport cost goes to the frequency $2\bar{k}$. Hence, the term in $d_{2\bar{k}}$ is shown to be positive, while $c_{2\bar{k}}$ is zero, which implies $s_{2\bar{k}} > 0$. As a result, *any Fourier approximation of transport cost function with a finite number of terms implies instability.*

It is reasonable to argue that transport costs should strictly increase with distance: $C'(x) < 0$. More generally, if we consider the weaker restriction of $C'(0) < 0$ or $C'(1/2) < 0$, we get $H = 1$ and $Q_1 < 0$, which establishes the following corollary.

Corollary 1 *Assume that $C(x)$ is three-times differentiable with $C'(0) < 0$ or $C'(1/2) < 0$. Then, any constant-access, quasi-smooth and dense equilibrium distribution $\lambda^*(x)$ is unstable.*

This corollary states that the instability property of any constant-access, quasi-smooth and dense equilibrium distribution holds for 'acceptable' transport cost functions. It also says that flat earth, which is the limit case of symmetric atomic cities when the number of cities goes to infinity, is always unstable. Therefore, the racetrack economic approach developed by Fujita *et al.* (1999) turns out to contain no stable constant-access equilibrium. Its intuition is that *workers and firms always have an incentive to move and form agglomerations in which some transport costs can be saved.*

To sum up, many shapes of transport costs yield to multiplicity of equilibria with quasi-smooth and dense distributions of workers. However, flat earth is the unique constant-access equilibrium for all shapes of transport costs. Still, flat earth is unstable under acceptable conditions on the shape of transport costs. Because of this negative result on quasi-smooth and dense distributions, it is worth studying alternative spatial distributions of workers. In the next section, we explore equilibria with atomic cities.

5 Equilibrium with Many Atomic Cities

Suppose that there are n atomic cities located equidistantly on the circumference with perimeter equal to 1. Whereas the density of farmers is uniform across the circumference and equal to A, workers are now distributed over n atomic cities and located at $x_j \equiv j/n$, $j = 0, 1, \ldots, n-1$. We focus on equilibria, where workers and firms locate only in cities. The spatial distribution of workers is

$$\lambda(x) = \begin{cases} \frac{1}{n}\sum_{k=0}^{\infty} \lambda_k \exp(2\pi I k x) & \text{if } x = x_j,\ j = 0, 1, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases}$$

with $\lambda_0 = 1$, $\lambda_{-k} = \overline{\lambda_k}$ and λ_k $(k \geq 1)$ are small enough to respect $\lambda(x) > 0$ for all x. The spatial distribution is *symmetric* if $\lambda(x_j) = 1/n$ for all j.

Before studying discrete equilibrium distributions, it is natural to introduce the discrete Fourier series associated to transport cost functions as

$$C(x_j) = \sum_{m=0}^{n-1} c_m^n \exp(2\pi I m x_j) \qquad \text{and} \qquad [C(x_j)]^2 = \sum_{m=0}^{n-1} d_m^n \exp(2\pi I m x_j)$$

where the coefficients c_m^n and d_m^n are defined as

$$c_m^n \equiv \frac{1}{n}\sum_{j=0}^{n-1} C(x_j) \exp(-2\pi I m x_j) \qquad \text{and} \qquad d_m^n \equiv \frac{1}{n}\sum_{j=0}^{n-1} [C(x_j)]^2 \exp(-2\pi I m x_j)$$

These coefficients are the discrete counterpart of the Fourier transform coefficients in the racetrack economic model where workers' location choice is continuous. More specifically, using the identities

$$\frac{1}{n}\sum_{j=0}^{n-1} \exp(2\pi I k x_j) = \begin{cases} 1 & \text{if } k = nl,\ l \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

we get the following relationships:

$$\begin{aligned} d_m^n &= 2\sum_{k=0}^{n-1} c_k^n c_{m-k}^n \\ c_m^n &= c_{n-m}^n = \frac{1}{n}\sum_{l=-\infty}^{\infty} c_{m-nl} \qquad \text{and} \qquad d_m^n = d_{n-m}^n = \frac{1}{n}\sum_{l=-\infty}^{\infty} d_{m-nl} \end{aligned}$$

for all $n \geq m$. Note that these discrete Fourier coefficients converge to the continuous Fourier coefficients when n becomes very large: $\lim_{n\to\infty} c_m^n = c_m$ and $\lim_{n\to\infty} d_m^n = d_m$.

The workers indirect utility with n cities becomes

$$\begin{aligned} V(x) &= W_0 + W_1 \int_0^1 [g_1(y)]^2 dy + W_2 \sum_{j=0}^{n-1} g_1(x_j)\lambda(x_j) \\ &\quad + W_3 \sum_{j=0}^{n-1} [g_1(x_j)]^2 \lambda(x_j) + V_1 g_1(x) + V_2 g_2(x) - V_3 [g_1(x)]^2 \\ &\quad - V_4 \int_0^1 C(x-y) g_1(y) dy - V_5 \sum_{j=0}^{n-1} C(x-x_j)\lambda(x_j) g_1(x_j) \quad (12) \end{aligned}$$

where

$$g_1(x) \equiv \sum_{i=0}^{n-1} C(x-x_i)\lambda(x_i) \qquad \text{and} \qquad g_2(x) \equiv \sum_{i=0}^{n-1} [C(x-x_i)]^2 \lambda(x_i)$$

When there are $n(\geq 2)$ cities located at $x = x_j$, $j = 0, 1, \ldots, n-1$, it must be that $V(x)$ is the maximum at each city location x_j. Noting that $c_k^n = c_{k+n}^n$ and $d_k^n = d_{k+n}^n$, we get

$$\begin{aligned} g_1(x_j) &\equiv \sum_{i=0}^{n-1} C(x_j - x_i)\lambda(x_i) = \sum_{k=0}^{n-1} c_k^n \boldsymbol{\lambda}_k^n \exp(2\pi I k x_j) \\ g_2(x_j) &\equiv \sum_{i=0}^{n-1} [C(x_j - x_i)]^2 \lambda(x_i) = \sum_{k=0}^{n-1} d_k^n \boldsymbol{\lambda}_k^n \exp(2\pi I k x_j) \end{aligned}$$

where

$$\boldsymbol{\lambda}_k^n \equiv \sum_{l=-\infty}^{\infty} \lambda_{k+nl}$$

and where $k = 0, 1, \ldots, n-1$. These two functions are constant for all $j = 0, 1, \ldots, n-1$ at an equilibrium with constant-access, whose definition is given by the following.

Definition 4 *A discrete distribution $\lambda(x_j)$ is said to be constant-access if*

$$c_k^n \boldsymbol{\lambda}_k^n = c_k^n c_m^n \boldsymbol{\lambda}_{k-m}^n = 0 \qquad \forall k \neq m \in \mathbb{N} \quad (13)$$

The constant-access equilibrium conditions (13) are the discrete counterpart of (6). According to Ginsburgh, Papageorgiou and Thisse (1985), spatial equilibrium is defined as follows.

Definition 5 *A spatial equilibrium is a distribution $\lambda(x)$ in the space $[0, 1]$ such that either $V(x) = \overline{V}$ for $\lambda(x) > 0$, or $V(x) \leq \overline{V}$ for $\lambda(x) = 0$.*

Similar to the quasi-smooth and dense equilibria, we can say the following.

Lemma 3 *Any constant-access discrete distribution of $n(\geq 2)$ atomic cities is a spatial equilibrium if and only if $V(x) \leq V(x_j)$ for all x and for $j = 0, 1, \ldots, n-1$.*

This is the discrete counterpart of Lemma 1. The additional condition $V(x) \leq V(x_j)$ requires that the utility in the hinterland does not exceed that in the cities because workers are not restricted to locate in cities. Obviously, when there are few cities, farmers in the hinterland are badly served by firms charging high prices, and hence firms would find it profitable to locate there and workers would increase their utility by following the firms. As a result, symmetric distribution $\lambda^*(x_j) = 1/n$, $\forall x_j$ is not necessarily a spatial equilibrium in the case of atomic cities.

For a specific shape of transport costs, many non-uniform spatial distributions are likely to exist provided that transport costs include few spatial frequencies. Let $Y^n = Y_c^n \cup Y_d^n$, where Y_c^n is a set of integers $m \neq 0$ such that $c_m^n \neq 0$ and Y_d^n is a set of integers $m - j$ such that $m, j \in Y_c^n$. Let also Z be a set of positive integers k such that $\boldsymbol{\lambda}_k^n \neq 0$. Then, any distribution with spatial frequencies in Z is a spatial equilibrium if $Z \cap Y^n = \varnothing$.

6 Stability of Many Atomic City Equilibria

While spatial equilibrium is defined by the possible deviation of a single worker/firm to any location including the hinterland, asymptotic stability is defined here by the possibility of deviations in multiple directions to cities only. At the equilibrium, workers' utility is higher in cities than in the hinterland. Because the perturbations studied in the asymptotic stability analysis are infinitesimal, workers' utility remains higher in cities.[3] For expositional purposes, let $\Lambda \equiv (\lambda(x_0), \lambda(x_1), ..., \lambda(x_{n-1}))$ and $\Lambda^* \equiv (\lambda^*(x_0), \lambda^*(x_1), ..., \lambda^*(x_{n-1}))$. As defined before, an equilibrium Λ^* is asymptotically stable if any sufficiently small change in the distribution results in a movement back toward the equilibrium.

Analogous to the case of quasi-smooth and dense distribution, we assume that workers compare utilities of neighboring cities. Dynamics of n cities is

[3] Note that we do not consider configurations of parameters such that workers' utility is the same as within cities as at some locations outside cities. In this case, the stability criteria should be defined with respect to the hinterland too, which is beyond the scope of this paper.

therefore depicted by

$$\frac{d\lambda^*(x_j)}{dt} = 2V(x_j, \Lambda) - V(x_{j-1}, \Lambda) - V(x_{j+1}, \Lambda) \qquad j = 0, 1, \ldots, n-1 \quad (14)$$

where we normalize the speed of adjustment ν to 1 as before. Note that dynamics (14) satisfies the principle of a constant total mass since $\sum_{j=0}^{n-1} d\lambda^*(x_j)/dt = 0$.

Differentiating the RHS of (14) by $\lambda^*(x_i)$ and evaluating it at x_j, we get the Jacobian matrix J whose elements are defined by

$$J_{j,i} \equiv \frac{\partial}{\partial\lambda^*(x_i)} \left[2V(x_j, \Lambda) - V(x_{j-1}, \Lambda) - V(x_{j+1}, \Lambda)\right]$$

The constant-access equilibrium condition is now written as $g_1(x_j, \Lambda^*)) = g_1^*$ and. $g_2(y, \Lambda^*)) = g_2^*$ where g_1^* and g_2^* are constants. Using these properties and the fact that $\sum_{j=1}^{n-1} \lambda^*(x_j) = 1$, we compute

$$\begin{aligned} \frac{\partial V(x_m, \Lambda^*)}{\partial\lambda^*(x_i)} &= V_1 C(x_i - x_m) + V_2 \left[C(x_i - x_m)\right]^2 \\ &\quad -2V_3 g_1^* C(x_i - x_m) - V_4 \int_0^1 C(x_i - y) C(y - x_m)\, dy \\ &\quad -V_5 \sum_{j=0}^{n-1} \left[C(x_i - x_m) + C(x_j - x_m)\right] C(x_i - x_j) \lambda^*(x_j) \end{aligned}$$

For any equilibrium distribution $\lambda^*(x_j)$ satisfying conditions (13), this expression has the following symmetry property: $\partial V(x_0)/\partial\lambda^*(x_i) = \partial V(x_k)/\partial\lambda^*(x_{i+k})$ for all $k \in \mathbb{Z}$. Let us define

$$v_i \equiv \frac{\partial V(x_0)}{\partial\lambda^*(x_j)} \qquad \text{for } j = \bmod |i|$$

where $\bmod |i|$ is the modulo n function: $\bmod |i| = i - nl$ if $l \le i < nl$, $l \in \mathbb{Z}$. Because of the symmetry, we note that $v_i = \partial V(x_k)/\partial\lambda^*(x_{i+k})$ for all $k = 0, 1, \ldots, n-1$. This yields a circulant Jacobian matrix $J_{j,i} = J_{|i-j|,0}$ where $J_{k,0} = 2v_k - v_{k-1} - v_{k+1}$. According to Bellman (1970, pp.242-243), the n eigenvalues of this matrix are known as

$$s_k^n = \sum_{j=0}^{n-1} J_{j,0} \exp(-2\pi I k x_j) \qquad \text{for } k = 0, 1, \ldots, n-1$$

which is rewritten as

$$\begin{aligned}
s_k^n &= \sum_{j=0}^{n-1} (2v_j - v_{j-1} - v_{j+1}) \exp(-2\pi I k x_j) \\
&= \sum_{j=0}^{n-1} v_j \left[2\exp\frac{-2\pi Ikj}{n} - \exp\frac{-2\pi Ik(j+1)}{n} - \exp\frac{-2\pi Ik(j-1)}{n} \right] \\
&= \sum_{j=0}^{n-1} v_j \left(2\exp\frac{-2\pi Ikj}{n} - 2\exp\frac{-2\pi Ikj}{n}\cos\frac{2\pi Ik}{n} \right) \\
&= 2\left(1 - \cos\frac{2\pi k}{n}\right) \sum_{j=0}^{n-1} v_j \exp(-2\pi Ikx_j) \qquad (15)
\end{aligned}$$

Since $1-\cos\frac{2\pi k}{n} > 0$, the eigenvalue s_k^n is proportional to the discrete Fourier transform of v_j. Because $s_0^n = 0$, we pay attention to s_k^n for $k = 1, \dots, n-1$. Before characterizing stability, we finally need to define

$$e_k^n \equiv \sum_{l=-\infty}^{\infty} (c_{k-nl})^2 \geq 0$$

Note that $\lim_{n\to\infty} e_k^n = (c_k)^2$. Using the above, we obtain the following lemma:

Lemma 4 *An equilibrium with atomic cities satisfying conditions (13) is unstable if and only if there exists $k \in \{1, 2, \dots, n-1\}$ such that $s_k^n > 0$ where*

$$\begin{aligned}
s_k^n &= n\left[[V_1 - (2V_3 + V_5)c_0^n] c_k^n + V_2 d_k^n - \left(V_4 e_k^n + V_5 (c_k^n)^2\right)\right] \qquad (16) \\
&= \frac{n\tau L(b\phi + cL)}{32\phi^2(2b\phi + cL)^2} \left[\begin{array}{c} [8\phi(2a - b\tau)(3b\phi + 2cL) - 2\tau cL(4b\phi + 3cL)c_0^n] c_k^n \\ +3\tau(2b\phi + cL)^2 d_k^n - 4\tau c(2b\phi + cL)\left(Ae_k^n + L(c_k^n)^2\right) \end{array} \right]
\end{aligned}$$

Moreover,

$$\lim_{n\to\infty} \frac{s_k^n}{n} = \frac{s_k}{(2\pi k)^2}$$

Proof. Appendix 5. ■

This lemma is very similar to Lemma 2; it is equivalent to the latter for infinitely many atomic cities. Hence, stability of uniform distributions in atomic cities share similar properties. As in Proposition 1, *equilibrium distributions with atomic cities are unstable when n is large enough.* Inspection of expression (16) also allows us to establish the following properties.

Proposition 2 *Equilibria with atomic cities satisfying conditions (13) are stable if there are sufficiently many immobile farmers (large A). They are unstable if the manufacturing demand is high (large a), if the transport costs are low (small τ) and if goods are very bad substitutes (small c).*

Proof. For large A, we obviously get $s_k^n > 0$ for all k. Since c_k^n multiplies a in (16), large enough a yields instability if there exists a positive c_k^n with $k \in \{1, 2, \ldots, n-1\}$. This is true because one can check that $C(0) = \sum_{k=0}^{n-1} c_k^n = 1$, $c_0^n = (1/n) \sum_{k=0}^{n-1} C(x_k) \leq 1$, and thus,

$$\sum_{k=1}^{n-1} c_k^n \geq 0 \qquad (17)$$

Note that decreasing transport costs $C(x_j)$ cannot yield $c_k^n = 0$ for all $k \neq 0$. Hence, there always exists a strictly positive c_k^n with $k \neq 0$. The same argument applies to τ. Finally, when $c \to 0$, we get $s_k^n = 2(2a - b\tau)\, c_k^n + b\tau d_k^n$. Applying (17) for d_k^n, we have $\sum_{k=1}^{n-1} d_k^n > 0$, and hence $\sum_{k=1}^{n-1} s_k^n > 0$. There exists at least one k such that $s_k^n > 0$ when $c \to 0$. ■

We are nevertheless unable to provide a general condition for instability of equilibria with small numbers of cities except if we put some additional symmetry in the shape of transport costs.

Definition 6 *The transport cost function $C_2(x) : \mathbb{R} \to [0, 1]$ is doubly symmetric if*

$$C_2(x) = -C_2(1/2 - x) \qquad \forall x \in [0, 1/2]$$

This means that $C_2(x)$ is not only symmetric around $l/2 \in \mathbb{N}$ but it is also *anti-symmetric* around $l/2 + 1/4$, $l \in \mathbb{N}$. Examples of such functions are the linear transport cost function, $1 - 4x$, the composition of sinusoidal cost functions like $\sum_m c_{2m-1} \cos(2m - 1)\pi x$, some cubic transport cost functions like $-8x^3 + 6x^2 - 5x + 1$, and so on. One can check that the associated Fourier coefficients c_k is zero for even k, which leads to $c_2^n = e_2^n = 0$ and $d_2^n > 0$, and hence $s_2^n > 0$ holds for all even $n (\geq 4)$.

Proposition 3 *Suppose doubly symmetric transport costs. Then, equilibrium distributions with even numbers ($n \geq 4$) of atomic cities satisfying conditions (13) is unstable.*

This proposition suggests that stable equilibria include few atomic cities. We now focus on the stability of 2-city configurations.

7 Stable Equilibrium with Few Atomic Cities

In this section, we consider equilibria with 1 or 2 atomic cities. When there is one city with $\lambda^*(0) = 1$, we have

$$\begin{aligned} V(x)|_{\lambda^*(0)=1} &= W_0 + W_1 + W_2 + W_3 + (V_1 - V_5)\, C(x) + (V_2 - V_3)\, [C(x)]^2 \\ &\quad - V_4 \int_0^1 C(x-y)\, C(y) dy \end{aligned}$$

By definition of the spatial equilibrium, a one-city equilibrium with $\lambda^*(0) = 1$ exists if and only if $V(0) \geq V(x)$ for all $x \in [0, 1]$. This condition is rewritten as

$$(V_1 - V_5)\,[1 - C(x)] + (V_2 - V_3)\left[1 - [C(x)]^2\right] - V_4 \left[\frac{d_0}{2} - \int_0^1 C(x-y)\, C(y) dy\right] \geq 0 \tag{18}$$

for all x, where $V_1 - V_5 > 0$, $V_2 - V_3 > 0$ and $V_4 > 0$.

Suppose that we have two atomic cities located at $x = 0$ and $x = x_1$, where x_1 needs not be 1/2. Let the workers' distribution be $\lambda(0) = \lambda$ and $\lambda(x_1) = 1 - \lambda$. We have

$$g_1(x) = C(x)\lambda + C(x - x_1)(1 - \lambda) \qquad \text{and} \qquad g_2(x) = (C(x))^2 \lambda + (C(x - x_1))^2 (1 - \lambda)$$

Then, the inter-city utility differential is

$$\begin{aligned} & V(0) - V(x_1)|_{\lambda(0)=\lambda, \lambda(x_1)=1-\lambda} \\ = \;& (2\lambda - 1) \left\{ \begin{array}{c} (V_1 - V_5)\,[1 - C(x_1)] + (V_2 - V_3)\left[1 - (C(x_1))^2\right] \\ -V_4 \left[\frac{d_0}{2} - \int_0^1 C(y - x_1)\, C(y) dy\right] \end{array} \right\} \end{aligned}$$

Hence, the two-city symmetric equilibrium is stable if

$$(V_1 - V_5)\,[1 - C(x_1)] + (V_2 - V_3)\left[1 - (C(x_1))^2\right] - V_4 \left[\frac{d_0}{2} - \int_0^1 C(y - x_1)\, C(y) dy\right] < 0 \tag{19}$$

One can check that this condition is consistent with (16) for $n = 2$, $k = 1$ and $x_1 = 1/2$. The two-city stability condition (19) is similar to the one-city equilibrium condition (18) except for the presence of variables x and x_1. This implies that if there exists a 1-city equilibrium, then there exists no configuration with 2 cities that is stable. By contrast, if there does not exist a 1-city equilibrium, but there exists a 2-city equilibrium, then a 2-city equilibrium is stable. As a result, we can state that *one-city equilibrium and two-city stable equilibrium cannot simultaneously exist* for any shape of transport costs and for any distance between cities. Still, this does not provide information about the existence of a 2-city stable equilibrium. Further properties of stability can be characterized for equidistant cities ($x_1 = 1/2$), symmetric spatial distribution ($\lambda^*(0) = \lambda^*(1/2) = 1/2$), for transport cost functions with double symmetry: $C(x) = C_2(x)$. The worker's utility reduces to

$$V(x)|_{\lambda^*(0)=\lambda^*(1/2)=1/2} = W_0 + W_1 + W_2 + W_3 + \frac{V_2}{2}\left[(C_2(x))^2 + (C_2(x-1/2))^2\right]$$

which has maxima at $x = 0, 1/2$. We thus establish the following proposition.

Proposition 4 *Suppose doubly symmetric transport costs. Then, there always exists a stable a equilibrium with either* 1 *or* 2 *cities.*

Proposition 4 confirms two-region models, such as Krugman (1991) and Ottaviano *et al.* (2002). The proposition sharply contrasts with Proposition 1: *for any doubly symmetric transport costs, there is always a stable equilibrium of* 1 *or* 2 *atomic cities, whereas there is no stable equilibrium of quasi-smooth and dense spatial distributions.* Thus, Proposition 4 exhibits *natural agglomeration* in economic geography. Note however that if the transport costs are not doubly symmetric (e.g. exponential transport costs), there may be no stable equilibrium with 1 or 2 cities for certain parameters. In such a case, stable equilibria with 3 or more cities would exist.

8 Stable Equilibria with Specific Transport Costs

The previous sections are informative about equilibria with very small and very large number of atomic cities. We can nevertheless obtain clearer results for specific shapes of transport costs functions. In this section, we study the nature of stable equilibria of atomic cities in the cases of sinusoidal and linear transport costs. In both cases, the property of double symmetry applies.

8.1 Sinusoidal Transport Costs

Suppose that $C(x) = \cos 2\pi x$. Then, we get $g_1(x) = \cos 2\pi x$ if $n = 1$ and $g_1(x) = 0$ otherwise; $g_2(x) = \frac{1}{2} + \frac{1}{2}\cos 4\pi x$ for $n = 1, 2$ and $g_2(x) = \frac{1}{2}$ otherwise.

Also, we have $c_1 = d_0 = 1$, $d_2 = 1/2$ and $c_m = d_m = 0$ for any other m. Thus, $c_1^2 = 2$ and $c_1^n = 1$ for $n \geq 3$.

For $n = 1, 2$, Proposition 4 applies: there always exists a stable equilibrium with either 1 or 2 cities.

For $n = 3$, the indirect utility writes as $V(x) = W_0 + \frac{1}{2}W_1 + \frac{1}{2}V_2$, which is constant and which is equal to the utility under flat earth. Because $c_0^3 = 0$ and $c_1^3 = c_2^3 = 1$, there exists no constant-access equilibrium other than symmetric distribution of atomic cities. It is also readily shown that the symmetric distribution equilibrium with 3 cities yields a lower utility than those with 2 cities. Interestingly, the utility level is the same not only across cities but also across the hinterland as those of flat earth. Workers may relocate to outside cities due to constant $V(x)$ for all x as it is the case in flat earth. Such a 'weak' equilibrium is possible for combinations of any shape of transport costs that has a bounded frequency content and that has a large number of cities. This result has a very similar structure to the stability result of flat earth. The stability of a 3-city equilibrium is given by the sign of s_2^3. Because $c_2^3 = e_2^3 = 2d_2^3 = 1$, we can state that the 3 symmetric atomic cities is unstable if and only if $8\phi(2a - b\tau)(3b\phi + 2cL) + 3\tau(2b\phi + cL)^2/2 - 2\tau c(2b\phi + cL)(A + L) > 0$. Instability will occur provided that a and L are sufficiently large, while c, τ and A are sufficiently small. Otherwise, the symmetric equilibrium is stable.

For more than 3 cities, many constant-access equilibria may occur. For instance, when there are 6 cities located equidistantly, then there exist non-uniform equilibrium distributions exhibiting hierarchical system of cities. We have $c_k^6 \neq 0$ for $k = 1, 5$, $d_k^6 \neq 0$ for $k = 2, 4$, and $c_3^6 = d_3^6 = 0$. This implies that the non-uniform distribution $\lambda^*(x_j) = 1 + \lambda_3 \sin 6\pi x_j + \mu_3 \cos 6\pi x_j$ is a spatial equilibrium. This yields the following distribution: $\lambda^*\left(\frac{0}{6}\right) = \lambda^*\left(\frac{2}{6}\right) = \lambda^*\left(\frac{4}{6}\right) = \frac{1}{n} + \Delta$ and $\lambda^*\left(\frac{1}{6}\right) = \lambda^*\left(\frac{3}{6}\right) = \lambda^*\left(\frac{5}{6}\right) = \frac{1}{n} - \Delta$ for any $\Delta \in [0, 1/n]$. However, for more than 3 cities, constant-access equilibria are always unstable. It can indeed be checked that $c_0^n = c_2^n = e_2^n = 0 < d_2^n$ and thus $s_2^n > 0$ for any $n \geq 4$.

8.2 Linear Transport Costs

Under linear transport costs $C(x) = C_1(x)$, equilibrium conditions differ according to even or odd numbers of cities. The following lemma is useful (the proof is contained in Appendix 6).

Lemma 5 *For linear transport costs, we have*

$$\begin{aligned} g_1(x) &= \frac{1-(-1)^n}{2n^2} C_1(nx) \\ g_2(x) &= \begin{cases} \frac{n^2-4}{3n^2} + \frac{1}{n^2}\left[(C_1(nx))^2 + 1\right] & \textit{if } n \textit{ is odd} \\ \frac{n^2-4}{3n^2} + \frac{1}{n^2}\left[2C_1\left(\frac{n}{2}x\right)\right]^2 & \textit{if } n \textit{ is even} \end{cases} \\ G(x) &\equiv \int_0^1 C_1(y-x)\, C_1(ny)\, dy = \frac{1-(-1)^n}{12n^4}\left[3C_1(nx) - C_1(nx)^3\right] \end{aligned}$$

Suppose even number of cities. The mill price $p(x,x)$ is shown to be constant across locations for even number of cities. Using Lemma 5, we can show that the indirect utility is given by

$$V(x) = W_0 + V_2\left[\frac{n^2-4}{3n^2} + \frac{1}{n^2}\left[2C_1\left(\frac{n}{2}x\right)\right]^2\right]$$

which attains a maximum at city locations $x_j = 0, 1/n, \ldots, (n-1)/n$ because $V_2 > 0$. This implies that *once a symmetric equilibrium with even number of cities is reached, workers do not move to the hinterland, and hence new cities never emerge for any marginal changes in parameter values.* However, as shown in Proposition 3, *only the 2-city configuration can be stable.*

The utility level at the symmetric equilibrium decreases with even n and converges to the level obtained under flat earth as n goes to infinity.

Suppose odd number of cities. The mill price $p(x,x)$ is not constant in the case of odd number of cities. Again using Lemma 5, the indirect utility is written as

$$\begin{aligned} V(x) &= W_0 + W_1\frac{1}{n^3} + W_2\frac{1}{n^2} + W_3\frac{1}{n^4} + V_1\frac{1}{n^2}C_1(nx) \\ &\quad + V_2\left[\frac{n^2-4}{3n^2} + \frac{1}{n^2}\left[(C_1(nx))^2 + 1\right]\right] - V_3\frac{1}{n^4}\left[C_1(nx)\right]^2 \\ &\quad - V_4\frac{1}{6n^4}\left[3C_1(nx) - \left[C_1(nx)\right]^3\right] - V_5\frac{1}{n^4}C_1(nx) \end{aligned}$$

This function is a linear combination of a periodic functions with n periods and with period $2n$ on the interval $[0,1]$. We can thus restrict the analysis to the interval $[0, 1/2n]$. Since it can be shown that $V'(0) < 0 < V'(1/2n)$ and $V'''(x) < 0$ for $x \in [0, 1/2n]$, $V(x)$ is quasi-convex in the interval of $[0, 1/2n]$. This suggests that there can exist maxima in the utility function at $x = 0, 1/2n$. Unlike the case with even number of cities, this utility may have the extrema not only at the city locations $x_j = 0, 1/n, \ldots, (n-1)/n$, but also at the midpoints of city locations $x = 1/2n, 3/2n, \ldots, (2n-1)/2n$. Therefore, $V(0) \geq V(1/2n)$ is a necessary and sufficient condition for any odd number of symmetric cities to be a spatial equilibrium under linear transport costs. For the sake of conciseness, we do not present the explicit form of this condition. We nevertheless stress that it is likely to be broken for some feasible sets of parameters.

Stability is difficult to check too. Since $c_k^n = 2n^{-2}\left[1 + (-1)^k \cos\frac{k\pi}{n}\right]^{-1}$, we also have $d_k^n > 0$ and $e_k^n > 0$ for all odd n. As a result, the negative term containing A in (16) does not vanish for any finite n. This means that a spatial equilibrium with an odd number of cities must be stable for sufficiently large A, which contrasts with any even number of cities. In fact, when $A = 100000$, $a = 3$, $b = 2$, $c = 1$, $\phi = 1$, $L = 100$, there are four stable equilibria with symmetric configurations of $n = 1, 2, 3, 5$.[4] This example shows that odd numbers of stable equilibrium cities can be larger the even numbers of cities. It is easy to build examples with still larger odd numbers of cities.

Finally, the utility level at the symmetric equilibrium is shown to be decreasing in odd n and equal to the flat earth utility for $n \to \infty$.

To sum up, our analysis suggests that existence and stability of equilibria depend on the degree of symmetry in the configuration of cities (even n versus odd n) and also on the degree in the shape of transport costs. The richer the symmetries in the transport cost functions, the larger the set of equilibria with atomic cities but the smaller the set of stable equilibria.

9 Conclusion

We have considered the racetrack economic approach, where manufacturing activities are distributed continuously and discretely around a circumference of a circle, and where the economic interactions were identical to those in Ottaviano *et al.*'s (2002) model. Investigating the nature and the number of stable equilibria, we have shown that constant-access, quasi-smooth and dense equilibrium

[4] They are

$$\begin{aligned} n &= 1 \quad \text{for } \tau \in [0, 0.000238) \\ n &= 2 \quad \text{for } \tau \in (0.000238, 0.019608) \\ n &= 3 \quad \text{for } \tau \in (0.000215, 0.002141) \\ n &= 5 \quad \text{for } \tau \in (0.002511, 0.019608) \end{aligned}$$

where the upper bound $0.019608 = \frac{2a\phi}{2b\phi + cL}$ is given by the tradability condition (1).

distributions are unstable for almost all transport cost functions, whereas agglomeration in 1 or 2 atomic cities is stable for any economic parameters given some symmetry properties in the transport cost functions.

Our finding vindicates that the assumption of two regions in most of the literature is neither noxious nor a mathematical convenience. When compared with Hotelling or Cournot competition, agglomeration is a distinct property in economic geography accruing from the existence of home market effects which results in forming a limited number of atomic cities instead of dispersed and continuous urban configurations.

Appendix 1: The weights V_i and W_i

We compute the weights V_i. Remember that

$$\tau(x,y) \equiv \frac{\tau}{2}\left[1 - C(x-y)\right]$$

$$f_1(x) \equiv \int_0^1 C(x-z)\,\lambda(z)\,dz \qquad f_2(z) \equiv \int_0^1 \left[C(x-z)\right]^2 \lambda(z)\,dz$$

We have that $p(y,x) = \alpha_1 + \alpha_2 f_1(x) - \frac{\tau}{4}C(x-y)$ and $m(x,y) = \alpha_3 + \alpha_2 f_1(y) + \frac{\tau}{4}C(x-y)$, where

$$\alpha_1 \equiv \frac{2a\phi + cL\frac{\tau}{2}}{2(2b\phi + cL)} + \frac{\tau}{4} \qquad \alpha_2 \equiv -\frac{cL\tau}{4(2b\phi + cL)} \qquad \alpha_3 \equiv \frac{2a\phi + cL\frac{\tau}{2}}{2(2b\phi + cL)} - \frac{\tau}{4}$$

We can compute

$$\begin{aligned}
\int_0^1 p(y,x)\,\lambda(y)\,dy &= \alpha_1 + \left(\alpha_2 - \frac{\tau}{4}\right) f_1(x) \\
\int_0^1 \left[p(y,x)\right]^2 \lambda(y)\,dy &= (\alpha_1)^2 + (\alpha_2)^2 \left[f_1(x)\right]^2 + \left(\frac{\tau}{4}\right)^2 f_2(x) \\
&\quad + 2\alpha_1\alpha_2 f_1(x) - \frac{\tau}{4} 2\alpha_1 f_1(x) - \frac{\tau}{4} 2\alpha_2 \left[f_1(x)\right]^2
\end{aligned}$$

and thus

$$\begin{aligned}
S(x) &= \frac{a^2 L}{2b\phi} - \frac{aL}{\phi}\alpha_1 - \frac{cL^2}{2\phi^2}(\alpha_1)^2 + \frac{b\phi + cL}{2\phi^2} L (\alpha_1)^2 \\
&\quad + \left[\left(-\frac{aL}{\phi} - \frac{cL^2}{2\phi^2} 2\alpha_1\right)\left(\alpha_2 - \frac{\tau}{4}\right) + \frac{b\phi + cL}{2\phi^2}\frac{1}{2}\alpha_1 L (4\alpha_2 - \tau)\right] f_1(x) \\
&\quad + \left[-\frac{cL^2}{2\phi^2}\left(\alpha_2 - \frac{\tau}{4}\right)^2 + \frac{b\phi + cL}{2\phi^2}\frac{1}{2}\alpha_2 L (2\alpha_2 - \tau)\right] \left[f_1(x)\right]^2 \\
&\quad + \frac{b\phi + cL}{2\phi^2} L \left(\frac{\tau}{4}\right)^2 f_2(x)
\end{aligned}$$

$$
\begin{aligned}
&\int_0^1 (m(x,y))^2 A dy \\
= \; & (\alpha_3)^2 A + (\alpha_2)^2 A \int_0^1 [f_1(y)]^2 dy + \left(\frac{\tau}{4}\right)^2 A \int_0^1 [C(x-y)]^2 dy \\
&+ 2\alpha_3\alpha_2 A \int_0^1 f_1(y)dy + 2\alpha_3 \frac{\tau}{4} A \int_0^1 C(x-y)\, dy + 2\alpha_2 \frac{\tau}{4} A \int_0^1 C(x-y) f_1(y) dy
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 [m(x,y)]^2 L\lambda(y)\, dy \\
= \; & (\alpha_3)^2 L + (\alpha_2)^2 L \int_0^1 [f_1(y)]^2 \lambda(y)\, dy + \left(\frac{\tau}{4}\right)^2 L \int_0^1 [C(x-y)]^2 \lambda(y)\, dy \\
&+ 2\alpha_3\alpha_2 L \int_0^1 f_1(y)\lambda(y)\, dy + 2\alpha_3 \frac{\tau}{4} L \int_0^1 C(x-y)\lambda(y)\, dy \\
&+ 2\alpha_2 \frac{\tau}{4} L \int_0^1 C(x-y) f_1(y)\lambda(y)\, dy
\end{aligned}
$$

Note that $\int_0^1 C(x-y)\, dy = 0$ and $\int_0^1 f_1(y)dy = \int_0^1 \int_0^1 C(x-y)dy\lambda(x)dx = 0$. Thus, grouping terms and substituting for α_1, α_2 and α_3, we get

$$
\begin{aligned}
V(x) = \; & W_0 + W_1 \int_0^1 [f_1(y)]^2 dy + W_2 \int_0^1 f_1(y)\lambda(y)\, dy \\
&+ W_3 \int_0^1 [f_1(y)]^2 \lambda(y)\, dy + V_1 f_1(x) + V_2 f_2(x) - V_3 [f_1(x)]^2 \\
&- V_4 \int_0^1 C(x-y) f_1(y) dy - V_5 \int_0^1 C(x-y) f_1(y)\lambda(y)\, dy
\end{aligned}
$$

where

$$
\begin{aligned}
W_0 &= \frac{(b\phi + cL)\left((2A+3L)\, b\phi + cL^2\right)}{8b\phi\,(2b\phi + cL)^2} (2a - b\tau)^2 + \frac{\tau^2 A\,(b\phi + cL)}{32\phi^2} d_0 > 0 \\
W_1 &= \frac{\tau^2 c^2 A L^2 (b\phi + cL)}{16\phi^2 (2b\phi + cL)^2} > 0 \qquad W_2 = \frac{\tau^2 c^2 L^3 (b\phi + cL)}{16\phi^2 (2b\phi + cL)^2} > 0 \\
W_3 &= \frac{\tau^2 c^2 L^3 (b\phi + cL)}{16\phi^2 (2b\phi + cL)^2} > 0 \\
V_1 &= \frac{\tau L\,(3b\phi + 2cL)(b\phi + cL)(2a - b\tau)}{4\phi\,(2b\phi + cL)^2} > 0 \\
V_2 &= \frac{3\tau^2 L\,(b\phi + cL)}{32\phi^2} > 0 \qquad V_3 = \frac{\tau^2 c^2 L^3 (b\phi + cL)}{32\phi^2 (2b\phi + cL)^2} > 0 \\
V_4 &= \frac{\tau^2 c A L\,(b\phi + cL)}{8\phi^2 (2b\phi + cL)} > 0 \qquad V_5 = \frac{\tau^2 c L^2 (b\phi + cL)}{8\phi^2 (2b\phi + cL)} > 0
\end{aligned}
$$

which is (3). Note that condition (1) guarantees $V_1 > 0$.

Appendix 2: Global motion process

This paper studies stability under a *local motion* process. We here show that stability under the local motion process is equivalent to stability under a global motion process.

For quasi-smooth and dense distributions, a *global motion* process may be depicted by

$$\frac{d\lambda(x,t)}{dt} = V(x,t) - \int_0^1 V(z,t)\, dz$$

by which workers moves to any location that has a higher utility than the average utility on the circumference. Obviously, the total number of workers remains fixed: $\int_0^1 \frac{d\lambda(x,t)}{dt} dx = 0 \ \forall t$. For small perturbations, we get

$$\frac{d\widetilde{\lambda}(x,t)}{dt} = \widetilde{V}(x,t)$$

and for normal modes with spatial frequency k, we get

$$s_k \widetilde{\lambda}_k = \widetilde{V}_k$$

This must be compared with the local motion and its resulting normal mode equality: $s_k \widetilde{\lambda}_k = (2\pi k)^2 \widetilde{V}_k$. Obviously, the sign of s_k does not depend on whether motion is local or global.

For equilibria with atomic cities, we propose a similar global motion equation:

$$\frac{d\lambda_k}{dt} = V(x_k, \Lambda) - \frac{1}{n} \sum_{j=0}^{n-1} V(x_j, \Lambda) \qquad k = 0, 1, \ldots, n-1$$

Net migration flow in a city x_k is proportional to the difference between that region's utility and the average utility. Differentiating the RHS of this equation by λ_l and evaluating it at Λ^*, we get the Jacobian matrix J, whose elements are given by

$$J_{m,l} = \frac{\partial}{\partial \lambda_l} V(x_m, \Lambda^*) - \frac{1}{n} \sum_{p=0}^{n-1} \frac{\partial}{\partial \lambda_l} V(x_p, \Lambda^*)$$

Taking advantage of the symmetry property, we note that partial derivatives are equal to

$$v_j \equiv \frac{\partial}{\partial \lambda_0} V(x_j, \Lambda_0) = \frac{\partial}{\partial \lambda_l} V(x_m, \Lambda^*).$$

for any l and m such that $j = \bmod |l-m|$. This also yields a circulant Jacobian matrix $J_{n,l} = v_j - \overline{v}$ where $j = |l-n|$ and where $\overline{v} = \frac{1}{k}\sum_{j=0}^{k-1} v_j$. The n eigenvalues of this matrix are known as

$$s_k^n = \sum_{j=0}^{n-1} (v_j - \overline{v}) \exp(2\pi I k x_j) = \sum_{j=0}^{n-1} v_j \exp(2\pi I k x_j) \qquad k = 0, 1, \ldots, n-1$$

Each eigenvalue has the same sign as that in (15) for the local motion process since $1 - \cos\frac{2\pi k}{n} > 0$ holds for $1 \le k < n$ in (15).

Appendix 3: Proof of Lemma 2

At a constant-access equilibrium $\lambda(x) = \lambda^*(x)$, we must have

$$f_1(x) = c_0 \qquad \text{and} \qquad f_2(x) = d_0$$

In the dynamic setting, small perturbations are defined as $\widetilde{\lambda}(x,t) \equiv \lambda(x,t) - \lambda^*(x)$, $\widetilde{f}_1(x,t) \equiv f_1(x,t) - c_0$ and $\widetilde{f}_2(x,t) \equiv f_2(x,t) - d_0$. Dropping terms in perturbations with order higher than one, we can write the perturbation in the worker's utility as

$$\begin{aligned}\widetilde{V}(x) \quad &= \quad V_1\widetilde{f}_1(x,t) + V_2\widetilde{f}_2(x,t) - 2V_3\widetilde{f}_1(x,t)f_1^* - V_4\int_0^1 C(x-y)\,\widetilde{f}_1(y,t)dy \\ &\quad -V_5\int_0^1 C(x-y)\,\widetilde{\lambda}(y)\,f_1^* dy - V_5\int_0^1 C(x-y)\,\lambda^*(y)\,\widetilde{f}_1(y,t)dy\end{aligned}$$

Since $\widetilde{\lambda}(x,t) = \sum_{k=-\infty}^{\infty}\widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$ and since $\int_0^1 \exp(2\pi I(k-m)y)\,dy$ is equal to 1 if $k = m$ and to 0 otherwise, we have

$$\widetilde{f}_1(x,t) = \int_0^1 C(x-y)\,\widetilde{\lambda}(y)\,dy = \sum_{k=-\infty}^{\infty} c_k\widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$$

$$\widetilde{f}_2(x,t) = \sum_{k=-\infty}^{\infty} d_k\widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$$

$$\int_0^1 C(x-y)\,\widetilde{f}_1(y,t)dy = \sum_{k=-\infty}^{\infty} (c_k)^2\,\widetilde{\lambda}_k \exp(2\pi I m x + s_k t)$$

$$\int_0^1 C(x-y)\,\widetilde{\lambda}(y)\,f_1^* dy = \widetilde{f}_1(x,t)f_1^* = \sum_{k=-\infty}^{\infty} c_0 c_k\widetilde{\lambda}_k \exp(2\pi I k x + s_k t)$$

$$\begin{aligned}\int_0^1 C(x-y)\,\lambda^*(y)\,\widetilde{f}_1(y,t)dy &= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_l\lambda_m^* c_k\widetilde{\lambda}_k \int_0^1 \exp(2\pi I(ky+my+lx-ly)+s_k t)\,dy \\ &= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_m c_k \lambda_{m-k}^*\widetilde{\lambda}_k \exp(2\pi I m x + s_k t) = \sum_{k=-\infty}^{\infty} (c_k)^2\,\widetilde{\lambda}_k \exp(2\pi I k x + s_k t)\end{aligned}$$

where the last equation is due to the condition of constant-access equilibrium: $c_m c_k \lambda^*_{m-k}$ for all $k \neq m$ and $\lambda^*_0 = 1$.

Using $\widetilde{V}(x) = \sum_{k=1}^{\infty} \widetilde{V}_k \widetilde{\lambda}_k \exp(2\pi Ikx + s_k t)$, we obtain

$$\widetilde{V}_k = V_1 c_k + V_2 d_k - (V_4 + V_5)(c_k)^2 - (2V_3 + V_5) c_0 c_k$$

implying that the eigenvalues s_k are the same for any quasi-smooth and dense distribution.

Appendix 4: Proof of Proposition 1

When $C(x)$ is differentiable, Fourier coefficient c_k with sufficiently large k is rewritten as

$$\begin{aligned}
c_k &= 4\int_0^{1/2} C(x)\cos 2\pi kx \, dx \\
&= -4\int_0^{1/2} \frac{C'(x)\sin 2\pi kx}{2\pi k} dx \\
&= 4\frac{(-1)^k C'(1/2) - C'(0)}{(2\pi k)^2} + 4\int_0^{1/2} \frac{C'''(x)\sin 2\pi kx}{(2\pi k)^3} dx \\
&= 4\frac{(-1)^k C'(1/2) - C'(0)}{(2\pi k)^2} - 4\frac{(-1)^k C'''(1/2) - C'''(0)}{(2\pi k)^4} \\
&\quad -4\int_0^{1/2} \frac{C^{(5)}(x)\sin 2\pi kx}{(2\pi k)^5} dx \\
&= 4\sum_{h=1}^{H} \frac{(-1)^k C^{(2h-1)}(1/2) - C^{(2h-1)}(0)}{(-1)^{h-1}(2\pi k)^{2h}} + 4\int_0^{1/2} \frac{C^{(2H+1)}(x)\sin 2\pi kx}{(-1)^{H-1}(2\pi k)^{2H+1}} dx \\
&\approx 4\sum_{h=1}^{H} \frac{(-1)^k C^{(2h-1)}(1/2) - C^{(2h-1)}(0)}{(-1)^{h-1}(2\pi k)^{2h}}
\end{aligned}$$

The last near equality holds because the last integral is bounded and its absolute value decreases faster with k than the other term due to the continuity of $C^{(2H+1)}(x)$.

Similarly, using $D(x) \equiv [C(x)]^2$, we have

$$\begin{aligned}
d_k &= 4\sum_{h=1}^{H} \frac{(-1)^k D^{(2h-1)}(1/2) - D^{(2h-1)}(0)}{(-1)^{h-1}(2\pi k)^{2h}} + 4\int_0^{1/2} \frac{D^{(2H+1)}(x)\sin 2\pi kx}{(-1)^{H-1}(2\pi k)^{2H+1}} dx \\
&\approx 4\sum_{h=1}^{H} \frac{(-1)^k D^{(2h-1)}(1/2) - D^{(2h-1)}(0)}{(-1)^{h-1}(2\pi k)^{2h}}
\end{aligned}$$

Third, since $(c_k)^2 \ll c_k$ holds for sufficiently large k, we have

$$s_k \approx 4(\pi k)^2 \frac{\tau L(b\phi + cL)}{8\phi^2(2b\phi + cL)^2}\left(\overline{V_1}c_k + V_2 d_k\right) \approx 16(\pi k)^2 \sum_{h=1}^{H} R_{h,k}$$

where

$$R_{h,k} \equiv \frac{(-1)^{h-1}}{(2\pi k)^{2h}}\left\{-\left[\overline{V_1}C^{(2h-1)}(0) + V_2 D^{(2h-1)}(0)\right] + (-1)^k\left[\overline{V_1}C^{(2h-1)}(1/2) + V_2 D^{(2h-1)}(1/2)\right]\right\}$$

Since

$$\operatorname{sgn}\{\max\{R_{h,k}, R_{h,k+1}\}\} = \operatorname{sgn}\{Q_h\}$$

holds, by using (11), we obtain

$$\operatorname{sgn}\{\max\{s_k, s_{k+1}\}\} = \operatorname{sgn}\{Q_H\}$$

(i) When $Q_H < 0$, from this expression, it can be seen that $\exists \bar{k}$ such that $s_k < 0\ \forall k > \bar{k}$. Therefore, if $Q_H < 0$ is satisfied, high frequencies will not amplify. Furthermore, low frequencies will also not amplify if A is sufficiently large so that the equilibrium will be stable.

(ii) When $Q_H > 0$, there always exist some high frequencies that amplify and the equilibrium is unstable.

Appendix 5: Proof of Lemma 4

We must compute the eigenvalues $\sum_{j=0}^{n-1} v_j \exp(-2\pi I k x_j)$ for $0 \le k < n$ where

$$\begin{aligned} v_j &= \frac{\partial V(0,\Lambda^*)}{\partial \lambda^*(x_j)} = V_1 C(x_j) + V_2 [C(x_j)]^2 - 2V_3 g_1^* C(x_j) \\ &\quad - V_4 \int_0^1 C(x_j - y) C(y)\, dy \\ &\quad - V_5 \sum_{p=0}^{n-1} [C(x_j) + C(x_p)] C(x_j - x_p) \lambda^*(x_p) \end{aligned}$$

We compute each line separately. The first line obviously yields

$$n\left(V_1 c_k^n + V_2 d_k^n - 2V_3 g_1^* c_k^n\right)$$

whereas the second line gives

$$
\begin{aligned}
& -V_4 \sum_{j=0}^{n-1} \int_0^1 C\left(x_j - y\right) C\left(y\right) dy \exp\left(-2\pi I k x_j\right) \\
= \; & -V_4 \sum_{j=0}^{n-1} \sum_{l=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} c_l c_p \int_0^1 \exp\left(2\pi I \left(l x_j - l y + p y - k x_j\right)\right) dy \\
= \; & -V_4 \sum_{j=0}^{n-1} \sum_{p=-\infty}^{\infty} \left(c_p\right)^2 \exp\left(2\pi I \left(p - k\right) x_j\right) \\
= \; & -n V_4 \sum_{l=-\infty}^{\infty} \left(c_{nl+k}\right)^2 = -n V_4 e_k^n
\end{aligned}
$$

To compute the last line, we firstly note that for $p = \{0, 1, ..., n-1\}$,

$$
\lambda^*(x_p) = \sum_{q=-\infty}^{\infty} \lambda_q^* \exp(2\pi I q p/n) = \sum_{r=0}^{n-1} \sum_{l=-\infty}^{\infty} \lambda_{r+nl}^* \exp(2\pi I p\left(r + nl\right)/n) = \sum_{r=0}^{n-1} \boldsymbol{\lambda}_r^n \exp(2\pi I r x_p)
$$

One can then show that $\sum_{r=0}^{n-1} \lambda^*(x_p) = n\boldsymbol{\lambda}_0^n = 1$, and thus $\boldsymbol{\lambda}_0^n = 1/n$. Note also that

$$
\begin{aligned}
\sum_{p=0}^{n-1} \sum_{t=0}^{n-1} \sum_{r=0}^{n-1} c_t^n \boldsymbol{\lambda}_r^n \exp\left(2\pi I \left(-t x_p + r x_p\right)\right) &= n \sum_{r=0}^{n-1} c_r^n \boldsymbol{\lambda}_r^n \\
\sum_{j=0}^{n-1} \sum_{s=0}^{n-1} \sum_{r=0}^{n-1} c_s^n c_r^n \boldsymbol{\lambda}_r^n \exp\left(2\pi I \left(s x_j + r x_j - k x_j\right)\right) &= n c_k^n c_0^n \boldsymbol{\lambda}_0^n + 2n \sum_{r=0}^{n-1} c_{k-r}^n c_r^n \boldsymbol{\lambda}_r^n
\end{aligned}
$$

As a consequence, the first term in the last line can be computed as

$$
\begin{aligned}
& \sum_{j=0}^{n-1} \left[-V_5 \sum_{p=0}^{n-1} C\left(x_j\right) C\left(x_j - x_p\right) \lambda^*\left(x_p\right) \right] \exp\left(-2\pi I k x_j\right) \\
= \; & -V_5 \sum_{j=0}^{n-1} \sum_{p=0}^{n-1} \sum_{s=0}^{n-1} \sum_{t=0}^{n-1} \sum_{r=0}^{n-1} c_s^n c_t^n \boldsymbol{\lambda}_r^n \exp\left(2\pi I \left(s x_j + t x_j - t x_p + r x_p - k x_j\right)\right) \\
= \; & -n V_5 \sum_{j=0}^{n-1} \sum_{s=0}^{n-1} \sum_{r=0}^{n-1} c_s^n c_r^n \boldsymbol{\lambda}_r^n \exp\left(2\pi I \left(s x_j + r x_j - k x_j\right)\right) \\
= \; & -n V_5 c_k^n c_0^n \boldsymbol{\lambda}_0^n + 2n V_5 \sum_{r=1}^{n-1} c_{k-r}^n c_r^n \boldsymbol{\lambda}_r^n \\
= \; & -n V_5 c_k^n c_0^n
\end{aligned}
$$

where the last equality is due to constant-access condition ($c_r^n \boldsymbol{\lambda}_r^n = 0$ for all $r \neq 0$).

Finally, the second term of the last line is equal to

$$
\begin{aligned}
& \sum_{j=0}^{n-1}\left[-V_5 \sum_{p=0}^{n-1} C\left(x_p\right) C\left(x_j-x_p\right) \lambda^*\left(x_p\right)\right] \exp\left(-2\pi I k x_j\right) \\
= & -V_5 \sum_{j=0}^{n-1} \sum_{p=0}^{n-1} \sum_{s=0}^{n-1} \sum_{t=0}^{n-1} \sum_{r=0}^{n-1} c_s^n c_t^n \boldsymbol{\lambda}_r^n \exp\left(2\pi I\left(s x_p+t x_j-t x_p+r x_p-k x_j\right)\right) \\
= & -n V_5 \sum_{p=0}^{n-1} \sum_{s=0}^{n-1} \sum_{r=0}^{n-1} c_s^n c_k^n \boldsymbol{\lambda}_r^n \exp\left(2\pi I\left(s x_p-k x_p+r x_p\right)\right) \\
= & -n^2 V_5 c_{-k}^n c_k^n \boldsymbol{\lambda}_0^n-2 n^2 V_5 \sum_{r=1}^{n-1} c_{r-k}^n c_k^n \boldsymbol{\lambda}_r^n \\
= & -n V_5\left(c_k^n\right)^2
\end{aligned}
$$

where the last equality is due to constant-access condition ($c_{r-k}^n c_k^n \boldsymbol{\lambda}_r^n$ for all $r \neq 0$).

Thus, using $g_1^*=n c_0^n \boldsymbol{\lambda}_0^n=c_0^n$ under constant-access condition, we get

$$
s_k^n=\sum_{j=0}^{n-1} v_j \exp\left(2\pi I k x_j\right)=n\left[V_1 c_k^n+V_2 d_k^n-2 V_3 c_0^n c_k^n-V_4 e_k^n-V_5 c_k^n c_0^n-V_5\left(c_k^n\right)^2\right]
$$

and

$$
\lim_{n \rightarrow \infty} s_k^n / n=\left[V_1 c_k+V_2 d_k-\left(V_4 e_k^n+V_5\left(c_k\right)^2\right)-\left(2 V_3+V_5\right) c_0 c_k\right]=\frac{s_k}{8\pi^2 k^2}
$$

Note that eigenvalues are real numbers.

Appendix 6: Proof of Lemma 5:

(i) We prove that $g_1(x)=\sum_{j=0}^{n-1} C_1\left(x_j-x\right) \lambda\left(x_j\right)=\frac{1-(-1)^n}{2n^2} C_1(n x)$. Symmetry imposes that $\sum_{j=0}^{n-1} C_1\left(x_j-x\right)=0$ when n is even. For $n=1$, we readily get $\sum_{j=0}^{n-1} C_1\left(x_j-x\right)=C_1(x)$. For $n=3,5, \ldots$, we get

$$
\begin{aligned}
\sum_{j=0}^{n-1} C_1\left(x_j-x\right) & = C_1(x)+\sum_{l=1}^{(n-1)/2}\left[C_1\left(\frac{l}{n}-x\right)+C_1\left(\frac{l}{n}+x\right)\right] \\
& = (1-4x)+\sum_{l=1}^{(n-1)/2}\left[\left(1-4\left(\frac{l}{n}-x\right)\right)+\left(1-4\left(\frac{l}{n}+x\right)\right)\right] \\
& = \frac{1-4xn}{n}=\frac{1}{n} C_1(n x) \text{ for } n=3,5, \ldots
\end{aligned}
$$

(ii) We prove that

$$g_2(x) = \sum_{j=0}^{n-1} [C_1(x_j - x)]^2 \lambda(x_j) = \begin{cases} \frac{n^2-4}{3n^2} + \frac{1}{n^2}\left[(C_1(nx))^2 + 1\right] & \text{if } n \text{ is odd} \\ \frac{n^2-4}{3n^2} + \frac{1}{n^2}\left[2C_1\left(\frac{n}{2}x\right)\right]^2 & \text{if } n \text{ is even} \end{cases}$$

We obviously get that $g_2(x) = n(C_1(nx))^2$ for $n = 1, 2$. In order to determine $g_2(x)$ for $n > 2$, let $\hat{h} : [0, 1/2] \to [0, 1]$, $\hat{h}(x) = (1 - 4x)^2$. One can check that $\hat{h}(x) = (C_1(x))^2$ $\forall x \in [0, 1/2]$. For $n = 3, 5, 7, ...$ and for $x \in [0, 1/2n]$, we can write

$$\begin{aligned} \sum_{j=0}^{n-1} [C_1(x_j - x)]^2 &= \hat{h}(x) + \sum_{l=1}^{\frac{n-1}{2}} \left(\hat{h}\left(\frac{l}{n} - x\right) + \hat{h}\left(\frac{2l-1}{2n} - x\right)\right) \\ &= (1-4x)^2 + \sum_{l=1}^{\frac{n-1}{2}} \left[\left(1 - 4\left(\frac{l}{n} - x\right)\right)^2 + \left(1 - 4\left(\frac{2l-1}{2n} - x\right)\right)^2\right] \\ &= \frac{1}{n}(1-4nx)^2 + \frac{n^2-1}{3n} \\ &= \frac{1}{n}\hat{h}(nx) + \frac{n^2-1}{3n} \end{aligned}$$

where we used the equality $\sum_{l=1}^{m} l^2 = \frac{1}{6}m(m+1)(2m+1)$. Hence,

$$g_2(x) = \frac{1}{n^2}[C_1(nx)]^2 + \frac{n^2-1}{3n^2}$$

For $n = 4, 6, 8, ...$ and for $x \in [0, 1/n]$, we can write

$$\begin{aligned} \sum_{j=0}^{n-1} [C_1(x_j - x)]^2 &= 2\hat{h}(x) + 2\sum_{l=1}^{\frac{n}{2}-1} \hat{h}\left(\frac{l}{n} - x\right) \\ &= 2(1-4x)^2 + 2\sum_{l=1}^{\frac{n}{2}-1} \left(1 - 4\left(\frac{l}{n} - x\right)\right)^2 \\ &= \frac{4}{n}(1-2nx)^2 + \frac{n^2-4}{3n} \\ &= \frac{4}{n}\hat{h}\left(\frac{n}{2}x\right) + \frac{n^2-4}{3n} \end{aligned}$$

Hence, we get

$$g_2(x) = \frac{n^2-4}{3n^2} + \frac{1}{n^2}\left[2C_1\left(\frac{n}{2}x\right)\right]^2$$

(iii) We prove that $G(x) \equiv \int_0^1 C_1(ny)\, C_1(y-x)\, dy = \frac{1-(-1)^n}{12n^2}\left[3C_1(nx) - C_1(nx)^3\right]$. Let $\hat{g} : [0, 1/2] \to [-1, 1]$, $\hat{g}(x) = 1 - 4x$. We have that $C_1(x) = \hat{g}(x - l)$ if $x \in [l, l + 1/2]$ with $l \in \mathbb{N}$ and $C_1(x) = \hat{g}(l + 1 - x)$ if $x \in [l + 1/2, l + 1]$.

For even n, we have $\int_0^1 C_1(ny)\,C_1(x-y)\,dy = 0$ for $x \in [0, 1/2n]$ since $\hat{g}_1(x) = 0$.

For odd $n \geq 3$, we can write

$$\begin{aligned}
& \int_0^1 C_1(ny)\,C_1(y-x)\,dy \\
= & \int_0^x \hat{g}(ny)\,\hat{g}(x-y)\,dy + \int_x^{\frac{1}{2n}} \hat{g}(ny)\,\hat{g}(y-x)\,dy \\
& + \sum_{l=1}^{\frac{n-1}{2}} \left(\int_{\frac{2l-1}{2n}}^{\frac{2l}{2n}} \hat{g}(l-ny)\,\hat{g}(y-x)\,dy + \int_{\frac{2l}{2n}}^{\frac{2l+1}{2n}} \hat{g}(ny-l)\,\hat{g}(y-x)\,dy \right) \\
& + \int_{\frac{1}{2}}^{\frac{1}{2}+x} \hat{g}\left(\frac{n+1}{2} - ny\right) \hat{g}(y-x)\,dy + \int_{\frac{1}{2}+x}^{\frac{1}{2}+\frac{1}{2n}} \hat{g}\left(\frac{n+1}{2} - ny\right) \hat{g}(x-y+1)\,dy \\
& + \sum_{l=\frac{n+1}{2}}^{n-1} \left(\int_{\frac{2l}{2n}}^{\frac{2l+1}{2n}} \hat{g}(ny-l)\,\hat{g}(x-y+1)\,dy + \int_{\frac{2l+1}{2n}}^{\frac{2l+2}{2n}} \hat{g}(l+1-ny)\,\hat{g}(x-y+1)\,dy \right) \\
= & \frac{1 - 24n^2x^2 + 32n^3x^3}{3n^2} = \frac{3\hat{g}(nx) - \hat{g}(nx)^3}{6n^2}
\end{aligned}$$

for $x \in [0, 1/2n]$. This function has a maximum at $x = 0$ and a minimum at $x = 1/2n$. By symmetry, the above convolution is equal to $3\hat{g}(-nx) - \hat{g}(-nx)^3/6n^2$ for any $x \in [-1/2n, 0]$. This pattern is repeated with frequency n. Therefore, for any $x \in \mathbb{R}$, we can write

$$\int_0^1 C_1(ny)\,C_1(y-x)\,dy = \frac{3C_1(nx) - C_1(nx)^3}{6n^2}$$

One can check that this result is also valid for $n = 1$.

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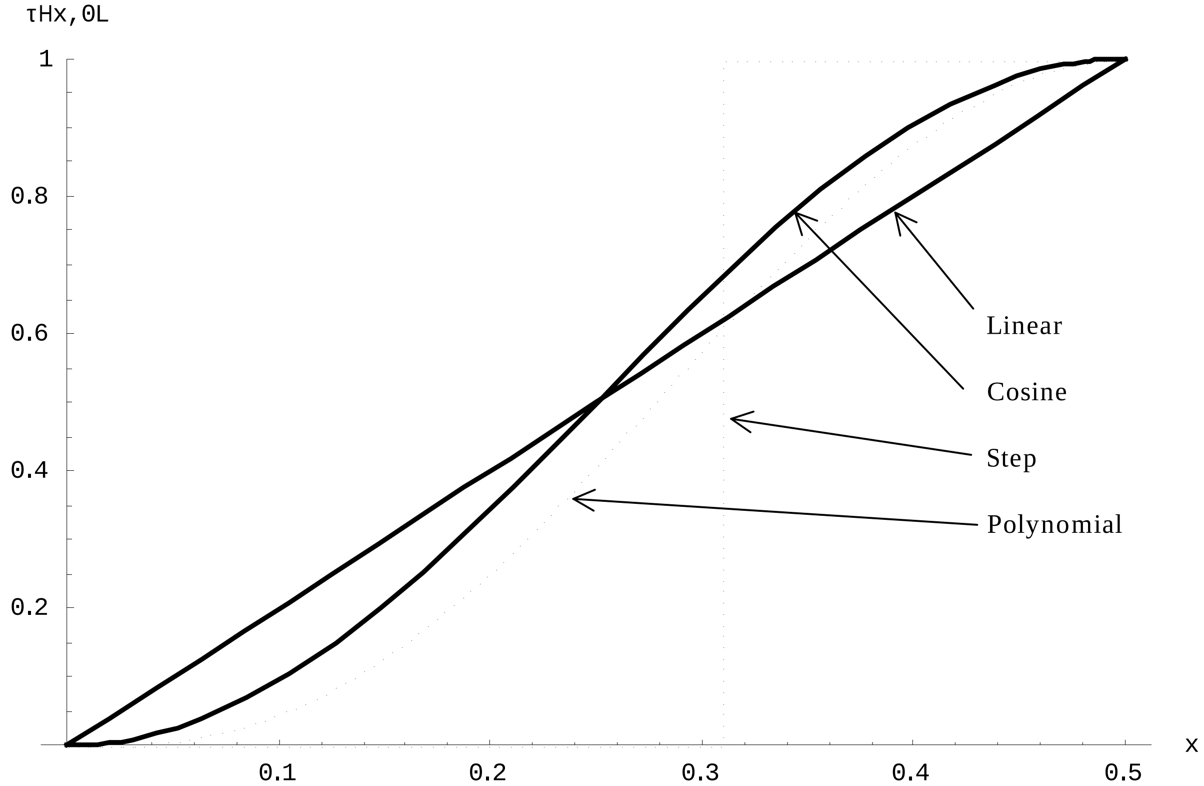

Figure 1 : Shapes of transportation costs
(Linear: C(x)=1-4x; Cosine: C(x)= cos2πx; Step: C(x)= 1 if x>0.31, C(x)= 0 otherwise; Polynomial: $C(x)=1-2x^2-96x^3+248x^4-160x^5$)