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MANCHESTER 1824

Economics
Discussion Paper Series
EDP-1328

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December 2013

Economics
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December 11, 2013

[1]The authors acknowledge the support from the ESRC (UK), under grant RES-062-23-1351.

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ABSTRACT

This paper makes two contributions in relation to the use of information criteria for inference on structural breaks when the coefficients of a linear model with endogenous regressors may experience multiple changes. Firstly, we show that suitably defined information criteria yield consistent estimators of the number of breaks, when employed in the second stage of a two-stage least squares (2SLS) procedure with breaks in the reduced form taken into account in the first stage. Secondly, a Monte Carlo analysis investigates the finite sample performance of a range of criteria based on BIC, HQIC and AIC for equations estimated by 2SLS. Versions of the consistent criteria BIC and HQIC perform well overall when the penalty term weights estimation of each break point more heavily than estimation of each coefficient, while AIC is inconsistent and badly over-estimates the number of true breaks.

Keywords: structural breaks, information criteria, instrumental variables estimation

JEL codes: C13, C26

1 Introduction

Information criteria are routinely used to select a specific model from a range of time-invariant linear specifications. It is not surprising, therefore, that a number of authors extend the approach by proposing versions of these criteria for the purpose of estimating the number of structural breaks in linear models. For example, Yao (1988) considers a version of the criterion of Schwarz (1978) [referred to as BIC] for this purpose while Ninomiya (2005) develops a version of the Akaike (1973) criterion [AIC].[1] Yao (1988) establishes that his criterion is consistent for estimation of the number of breaks in the mean of an *i.i.d.* Gaussian process, while the arguments of Bai (2000) indicate that a wider range of penalty functions - although not AIC - will similarly deliver a consistent estimate of this key parameter.[2]

A further important difference between Yao's (1988) and Ninomiya's (2005) criterion is the weight attached to the estimated break date in the penalty function: Yao (1988) effectively counts each break date estimated as equivalent to a single coefficient while the analytical results obtained by Ninomiya (2005) lead him to increase the weight on break date estimation to three times that of a coefficient. While Ninomiya's (2005) analysis applies only to a mean shift model, Hall, Osborn, and Sakkas (2012) show that his arguments extend to a variety of regression models estimated via least squares. Indeed, Hall, Osborn, and Sakkas (2013) provide a Monte Carlo analysis of the finite sample performance of a range of consistent information criteria for structural break inference in the linear OLS context, finding that a modified BIC-based penalty function and a version of the criterion of Hannan and Quinn (1979) [HQIC] both perform well when a relative weighting of three is applied for break date estimation. In particular, the use of this relative weighting scheme in the penalty function substantially reduces the problem of spurious break detection found in the study of Bai and Perron (2006) when using BIC.

However, economic models often include endogenous regressors, rendering OLS-based techniques inappropriate. Very recently, Hall, Han, and Boldea (2012) and Boldea, Hall, and Han (2012) have extended the OLS approach of Bai and Perron (1998) to develop a hypothesis testing methodology for structural break inference in the two stage least squares [2SLS] context. Although this methodology provides researchers with techniques that are (asymptotically) valid for 2SLS, nevertheless it has the practical disadvantage that the method involves dividing the

[1] Also see Liu, Wu, and Zidek (1997) and Zhang and Siegmund (2007).

[2] Bai's (2000) analysis is in the context of vector autoregressions.

sample into sub-samples over which the reduced form is judged stable. With the moderate sample sizes often available to practitioners, this sample splitting can lead to the partitions having relatively few observations over which testing can be conducted for the structural form equation.

Motivated by these considerations, the present paper extends the information criteria approach to structural break estimation in linear models with endogenous regressors estimated by 2SLS. More explicitly, we establish generic conditions under which information criteria methods yield consistent estimation of the number of breaks in the structural equation. These conditions cover penalty functions that both behave as a function of the sample size like either BIC or HQIC and also attach to each estimated break either the same or three times the weight as an estimated coefficient. However, in line with other results relating to model specification, including Shibata (1976), methods based on AIC are not consistent and may asymptotically over-estimate the number of true breaks. Although our approach requires breaks in the reduced form equation(s) to be appropriately taken into account, implementation does not require the sample used for break inference in the structural form to be split based on the reduced form partitions.

The paper also undertakes an extensive Monte Carlo analysis of the performance of information criteria for the estimation of the number of breaks in a structural equation estimated by 2SLS, examining versions of BIC, HQIC and AIC that count an estimated break as effectively equivalent to one and three individual coefficients, respectively. In line with our OLS analysis in Hall, Osborn, and Sakkas (2013), we find that BIC and HQIC perform well when combined with the higher relative weight of three for break estimation and this applies in cases with both *i.i.d.* and positively autocorrelated disturbances.

The outline of the paper is as follows. Section 2 discusses the assumptions made on the structural equation of interest to the researcher, with the consistency of the information criteria approach in the 2SLS context established in Section 3. The results of our Monte Carlo study are detailed in Section 4, with conclusions drawn in Section 5.

2 The Structural Equation

Consider the case in which the equation of interest is a structural relationship from a simultaneous system, with this equation exhibiting m breaks, such that

$$y_t \ = \ x_t'\beta_{x,i}^0 \ + \ z_{1,t}'\beta_{z_1,i}^0 \ + \ u_t, \qquad i=1,...,m+1, \qquad t=T_{i-1}^0+1,...,T_i^0 \tag{1}$$

where $T_0^0 = 0$ and $T_{m+1}^0 = T$, where T is the total sample size. Thus, y_t is the dependent variable, while x_t is a $p_1 \times 1$ vector of endogenous explanatory variables, $z_{1,t}$ is a $p_2 \times 1$ vector of exogenous variables including the intercept, and u_t is a mean zero error. We define $p = p_1 + p_2$. As usual in the literature, we require the break points to be asymptotically distinct.

Assumption 1 $T_i^0 = [T\lambda_i^0]$, *where* $0 < \lambda_1^0 < ... < \lambda_m^0 < 1$.[3]

As a structural equation, we allow the explanatory variables, x_t, to be correlated with the errors, u_t and x_t requires a reduced form representation to be estimated using appropriate instruments. This estimation is done *a priori* in the first stage of a Two Stage Least Squares (2SLS) procedure. Furthermore, we allow for this reduced form to be subject to discrete shifts in the sample period,

$$x_t^{'} \ = \ z_t^{'}\Delta_0^{(i)} \ + \ v_t^{'}, \qquad i=1,2,\ldots,h+1, \qquad t=T_{i-1}^*+1,\ldots,T_i^* \tag{2}$$

where $T_0^* = 0$ and $T_{h+1}^* = T$. The vector $z_t = (z_{1,t}', z_{2,t}')'$ is $q \times 1$ and contains variables that are uncorrelated with both u_t and v_t and are appropriate instruments for x_t in the first stage of the $2SLS$ estimation. The parameter matrices are $\Delta_0^{(i)} = (\delta_{1,0}^{(i)}, \delta_{2,0}^{(i)}, ..., \delta_{p_1,0}^{(i)})$, each with dimension $q \times p_1$, and each $\delta_{j,0}^{(i)}$ is dimension $q \times 1$, for $j = 1, ..., p_1$. The points $\{T_i^*\}$ are assumed to be generated as follows.

Assumption 2 $T_i^* = [T\pi_i^0]$, *where* $0 < \pi_1^0 < \ldots < \pi_h^0 < 1$.

Note that the break fractions in the reduced form, $\pi^0 = [\pi_1^0, \pi_2^0, \ldots, \pi_h^0]'$, may or may not coincide with the breaks in the structural equation, $\lambda^0 = [\lambda_1^0, \lambda_2^0, \ldots, \lambda_m^0]'$. Also note that (2) can be re-written as follows

$$x_t(\pi^0)^{'} \ = \ \tilde{z}_t(\pi^0)^{'}\Theta_0 \ + \ v_t^{'}, \qquad t=1,2,\ldots,T \tag{3}$$

[3] $[\cdot]$ denotes the integer part of the quantity in the brackets.

where $\Theta_0 = [\Delta_0^{(1)'}, \Delta_0^{(2)'}, \ldots \Delta_0^{(h+1)'}]'$. $\tilde{z}_t(\pi^0) = \iota(t,T) \otimes z_t$, $\iota(t,T)$ is a $(h+1) \times 1$ vector with first element $\mathcal{I}\{t/T \in (0, \pi_1^0]\}$, $h+1^{th}$ element $\mathcal{I}\{t/T \in (\pi_h^0, 1]\}$, k^{th} element $\mathcal{I}\{t/T \in (\pi_{k-1}^0, \pi_k^0]\}$ for $k = 1, 2, \ldots, h$ and $\mathcal{I}\{\cdot\}$ is an indicator variable that takes the value one if the event in the curly brackets occurs.

Let $\hat{\pi} = [\hat{\pi}_1, \hat{\pi}_2, \ldots, \hat{\pi}_h]'$ denote estimators of π^0. It is assumed these estimators satisfy the following condition.

Assumption 3 $\hat{\pi} \;=\; \pi^0 \,+\, O_p(T^{-1})$

This condition would be satisfied if, for example, the break dates in the reduced form are estimated by applying Bai and Perron's (1998) methodology or consistent information criteria equation by equation and then pooling the estimates of the break fractions. Let $\hat{x}_t(\hat{\pi})$ denote the resulting fitted values that is,

$$\hat{x}_t(\hat{\pi})' \;=\; \tilde{z}_t(\hat{\pi})'\hat{\Theta}_T(\hat{\pi}) \;=\; \tilde{z}_t(\hat{\pi})' \left(\sum_{t=1}^{T} \tilde{z}_t(\hat{\pi})\tilde{z}_t(\hat{\pi})' \right)^{-1} \sum_{t=1}^{T} \tilde{z}_t(\hat{\pi}) x_t' \tag{4}$$

where $\tilde{z}_t(\hat{\pi})$ is defined analogously to $\tilde{z}_t(\pi^0)$ based on the estimator of the true break points in the reduced form.

To facilitate our analysis we impose the following assumptions:

Assumption 4 *(i)* $h_t = (u_t, v_t')' \otimes z_t$ *is an array of real valued* $(p+1)q \times 1$ *random vectors defined on the probability space* $(\Omega, \mathcal{F}, P)$, $V_T = Var[\sum_{t=1}^{T} h_t]$ *is such that* $diag[\xi_{T,1}^{-1}, \ldots, \xi_{T,(p+1)q}^{-1}] = \Xi_T^{-1}$ *is* $O(T^{-1})$ *where* Ξ_T *is the* $(p+1)q \times (p+1)q$ *diagonal matrix with the eigenvalues* $(\xi_{T,1}, \ldots, \xi_{T,(p+1)q})$ *of* V_T *along the diagonal; (ii)* $E[h_{t,i}] = 0$ *and, for some* $d > 2$, $\|h_{t,i}\|_d < \Gamma < \infty$ *for* $t = 1, 2, \ldots$ *and* $i = 1, 2, \ldots n$ *where* $h_{t,i}$ *is the* i^{th} *element of* h_t*; (iii)* $\{h_{t,i}\}$ *is near epoch dependent with respect to* $\{g_t\}$ *such that* $\|h_t - E[h_t|\mathcal{G}_{t-\zeta}^{t+\zeta}]\|_2 \le \nu_\zeta$ *with* $\nu_\zeta = O(\zeta^{-1/2})$ *where* $\mathcal{G}_{t-\zeta}^{t+\zeta}$ *is a sigma- algebra based on* $(g_{t-\zeta}, \ldots, g_{t+\zeta})$*; (iii)* $\{g_t\}$ *is either* ϕ*-mixing of size* $\zeta^{-d/(2(d-1))}$ *or* α*-mixing of size* $\zeta^{-d/(d-2)}$*; (iv)* $V_T(r) = Var[T^{-1/2}\sum_{t=1}^{[Tr]} h_t]$ *satisfies* $V_T(r) \to rV$ *uniformly in* $r \in [0,1]$ *where* V *is a pd matrix.*

Assumption 5 $Var[u_t] = \sigma_u^2$, $Cov[u_t, v_t] = \Sigma_{uv}$*, and* $Var[v_t] = \Sigma_v$*, for all* t*.*

Assumption 6 $rank\{\Upsilon_i^0\} \;=\; p$ *where* $\Upsilon_i^0 \;=\; \left[\Delta_0^{(i)}, \Pi\right]$*, for* $i \;=\; 1, 2, \cdots, h+1$ *where* $\Pi' = [I_{p_2}, 0_{p_2 \times (q-p_2)}]$*,* I_a *denotes the* $a \times a$ *identity matrix and* $0_{a \times b}$ *is the* $a \times b$ *null matrix.*

Assumption 7 *For $\sharp = 0, *$, there exists an $l_\sharp > 0$ such that for all $l > l_\sharp$, the minimum eigenvalues of $A_{il} = (1/l)\sum_{t=T_i^\sharp+1}^{T_i^\sharp+l} z_t z_t'$ and of $\bar{A}_{il} = (1/l)\sum_{t=T_i^\sharp-l}^{T_i^\sharp} z_t z_t'$ are bounded away from zero for all $i = 1, ..., \nu^\sharp + 1$ where $\nu^0 = m$ and $\nu^* = h$.*

Assumption 8 *$T^{-1}\sum_{t=1}^{[Tr]} z_t z_t' \stackrel{p.u}{\to} Q_{ZZ}(r)$ uniformly in $r \in [0,1]$ where $Q_{ZZ}(r)$ is positive definite for any $r > 0$ and strictly increasing in r. $Q_{ZZ}(r) - Q_{ZZ}(s)$ is positive definite for any $r > s$.*

Assumption 4 allows substantial dependence and heterogeneity in $(u_t, v_t')' \otimes z_t$ but at the same time imposes sufficient restrictions to deduce a Functional Central Limit Theorem for $T^{-1/2}\sum_{t=1}^{[Tr]} h_t$; see Wooldridge and White (1988). This assumption also contains the restrictions that the implicit population moment condition in 2SLS is valid - that is $E[z_t u_t] = 0$ - and the conditional mean of the reduced form is correctly specified. Assumption 5 restricts the unconditional variance and covariances of the structural equation and reduced form errors to be constant over time. Assumption 6 implies the standard rank condition for identification in IV estimation in the linear regression model[4] because Assumptions 4(ii), 6 and 8 together imply that

$$T^{-1}\sum_{t=[sT]+1}^{[Tr]} z_t[x_t', z_{1,t}'] \stackrel{p}{\to} [Q_{ZZ}(r) - Q_{ZZ}(s)]\Upsilon_0 = Q_{Z,[X,Z_1]}(r,s) \text{ uniformly in } r > s+\epsilon, r, s \in [0,1] \quad (5)$$

where $Q_{Z,[X,Z_1]}(r,s)$ has rank equal to p for any r, s (satisfying the above conditions). Note this assumption implies $q \geq p$. Assumption 7 requires that there be enough observations near the true break points in either the structural equation or reduced form so that they can be identified and is analogous to the extension proposed in Bai and Perron (1998) to their Assumption A2.

3 Consistency of an Information Criterion

Suppose now that a researcher knows neither the number nor the location of the breaks in the structural equation. Consider the case where an arbitrary number n breaks are estimated at $\tau(n) = [\tau_1, \tau_2, \ldots, \tau_n]'$ with $0 < \tau_1 < \tau_2 < \ldots < \tau_n < 1$, $\tau_0 = 0$, and $\tau_{n+1} = 1$. Then, the second stage of 2SLS can begin with the estimation of (1) via OLS for each possible n-partition

[4] See *e.g.* Hall (2005)[p.35].

of the sample that is,

$$y_t \;=\; \hat{x}_t(\hat{\pi})^{'}\beta^*_{x,i} \;+\; z'_{1,t}\beta^*_{z_1,i} \;+\; \tilde{u}_t(\hat{\pi}), \quad i = 1, ..., n+1; \quad t = T_{i-1}+1, ..., T_i; \tag{6}$$

where $T_i = [\tau_i T]$, and the regressors x_t are estimated using the fitted values of the first stage of 2SLS, $\hat{x}_t(\hat{\pi})$. We further assume that

Assumption 9 *Equation (6) is estimated over all partitions* $(T_1, ..., T_n)$ *such that* $T_i - T_{i-1} > max\{q-1, \epsilon T\}$ *for some* $\epsilon > 0$ *and* $\epsilon < inf_i(\lambda^0_{i+1} - \lambda^0_i)$ *and* $\epsilon < inf_j(\pi^0_{j+1} - \pi^0_j)$.

Assumption 9 requires that each segment considered in the estimation contains a positive fraction of the sample asymptotically; in practice ϵ is chosen to be small in the hope that the last part of the assumption is valid. Letting $\beta^{*\prime}_i = ({\beta^*_{x,i}}', {\beta^*_{z_1,i}}')'$, for a given n-partition, the estimates of $\beta^* = (\beta^{*\prime}_1, \beta^{*\prime}_2, ..., {\beta^*_{n+1}}')'$ are obtained by minimizing the sum of squared residuals

$$S_T(T_1, ..., T_n;\, \beta) = \sum_{i=1}^{n+1} \sum_{t=T_{i-1}+1}^{T_i} \left\{y_t - \hat{x}_t(\hat{\pi})'\beta_{x,i} - z'_{1,t}\beta_{z_1,i}\right\}^2$$

with respect to $\beta = ({\beta_1}', {\beta_2}', ..., {\beta_{n+1}}')'$. We denote these estimators by $\hat{\beta}(\tau(n))$. The estimators of the break points, $(\hat{T}_1, ..., \hat{T}_n)$, are then defined as

$$\hat{\tau}(n) \;=\; (\hat{T}_1, ..., \hat{T}_n) = \arg\min_{T_1, ..., T_n} S_T\left(T_1, ..., T_n;\; \hat{\beta}(\tau(n))\right) \tag{7}$$

where the minimization is taken over all possible partitions, $(T_1, ..., T_n)$. The 2SLS estimates of the regression parameters, $\hat{\beta}(\hat{\tau}(n)) = (\hat{\beta}'_1, \hat{\beta}'_2, ..., \hat{\beta}'_{n+1})'$, are the regression parameter estimates associated with each of the estimated partitions.

The estimators $\hat{\tau}(n)$ and $\hat{\beta}(\hat{\tau}(n))$ are calculated conditional on n. While the above considers arbitrary n, we seek an estimator for the true number of structural breaks m, which is typically unknown *a priori*. Hall, Han, and Boldea (2012) - HHB hereafter - propose a method for estimation of m based on the sequential application of certain test statistics for parameter variation. Here we consider an alternative approach based on minimization of the following information criterion (IC),

$$IC\,(\tau(n); n, \hat{\pi}) \;=\; \ln\left[\hat{\sigma}^2(\tau(n); n, \hat{\pi})\right] + K(n, T), \tag{8}$$

where

$$\hat{\sigma}^2(\tau(n); n, \hat{\pi}) = (T-p)^{-1} RSS(\tau(n); n, \hat{\pi}), \tag{9}$$

$$RSS(\tau(n); n, \hat{\pi}) = \sum_{j=1}^{n} RSS_j(\tau(n); n, \hat{\pi}), \tag{10}$$

$$RSS_j(\tau(n); n, \hat{\pi}) = \sum_{t=[\tau_{j-1}T]+1}^{[\tau_j T]} \left\{ y_t - \hat{x}_t(\hat{\pi})' \hat{\beta}_{x,i} - z_{1,t}' \hat{\beta}_{z_1,i} \right\}^2, \tag{11}$$

and $K(n,T)$ is a deterministic penalty term governed by the following Assumption,

Assumption 10 *$K(n,T) = o(1)$ as $T \to \infty$, it is a strictly increasing function of n, and $TK(n,T) \to \infty$ as $T \to \infty$.*

Then, the estimated number of breaks, denoted $\hat{n}$, is the value that minimizes the IC, that is

$$\hat{n} = argmin_{n \in \mathcal{N}} IC(\tau(n); n, \hat{\pi}). \tag{12}$$

where $\mathcal{N} = \{0, 1, \ldots, N\}$. The associated estimators of the break locations are $\hat{\tau}(\hat{n})$. N is the maximum number of breaks considered and we assume this is large enough to ensure $m \in \mathcal{N}$:

Assumption 11 *$N \geq m$.*

The proof of consistency of our method rests on the limiting properties of $RSS(\tau(n); n, \hat{\pi})$. The following lemma presents the limiting behaviour of $RSS_j(\tau(n); n, \hat{\pi})$.

Lemma 1 *Let y_t be generated by (1), x_t be generated by (2), $\hat{x}_t(\hat{\pi})$ be generated by (4) and Assumptions 1-9 hold. Then, for segment j of the data, $t = [\tau_{j-1}T]+1, \ldots, [\tau_j T]$,*

(i) If $\lambda_{i-1}^0 \leq \tau_{j-1}, \tau_j \leq \lambda_i^0$ then

$$T^{-1} RSS_j(\tau(n); n, \hat{\pi}) \stackrel{p.u}{\to} (\tau_j - \tau_{j-1}) \Gamma_i.$$

(ii) If there exists i and $\kappa > 0$ such that $\lambda_i^0, \lambda_{i+1}^0, \ldots, \lambda_{i+\kappa}^0 \in [\tau_{j-1}, \tau_j]$ then

$$\begin{aligned} T^{-1} RSS_j(\tau(n); n, \hat{\pi}) \stackrel{p.u}{\to} & (\lambda_i^0 - \tau_{j-1}) \Gamma_i + (\lambda_{i+1}^0 - \lambda_i^0) \Gamma_{i+1} + \ldots \\ & + (\lambda_{i+k}^0 - \lambda_{i+\kappa-1}^0) \Gamma_{i+\kappa} + (\tau_j - \lambda_{i+\kappa}^0) \Gamma_{i+\kappa+1} + F \end{aligned}$$

where $\Gamma_i = \sigma_u^2 + 2\Sigma_{uv}\beta_{x,i}^0 + \beta_{x,i}^{0\prime}\Sigma_v\beta_{x,i}^0$. $\overset{p.u}{\to}$ *denotes limit in probability, that exists uniformly in a segment defined by* $\tau_{j-1} + \epsilon < \tau_j$*, for* $\epsilon > 0$ *and* $\tau_{j-1}, \tau_j \in [0,1]$*.* F *is a positive constant (that is defined in the proof) which depends on* τ_{j-1}*,* τ_j*, certain limit matrices and the parameters of the model.*

Lemma 1 demonstrates the impact of neglected breaks on the residual sum of squares in segment j. Part (i) states that if there are no neglected breaks then $T^{-1}RSS_j(\tau(n); n, \hat{\pi})$ converges to the (scaled) variance $(\tau_j - \tau_{j-1})\Gamma_i$; part (ii) shows that if there are neglected breaks then $T^{-1}RSS_j(\tau(n); n, \hat{\pi})$ converges to its scaled variance plus a positive constant. Notice that the scaled variance in question is that of $u_t + \beta_{x,i}^{0\prime}v_t$, and this reflects both the error u_t and the measurement error inherent in the substitution of $\hat{x}_t(\hat{\pi})$ for x_t.

Given the additivity of $RSS(\cdot)$ in $RSS_j(\cdot)$, the results in Lemma 1 can be used to deduce the limiting behaviour of $T^{-1}RSS(\cdot)$ for any partition. For any partition with no neglected breaks, $T^{-1}RSS(\cdot)$ converges to $\Gamma = \sum_{i=1}^{n}\Gamma_i$ and for any partition with at least one neglected break $T^{-1}RSS(\cdot)$ converges to $\Gamma + \xi$, $\xi > 0$. This behaviour, combined with Assumptions 10 and 11 implies the consistency of $\hat{n}$ for m, and this combined with HHB[Theorem 1] implies the consistency of $\hat{\tau}(\hat{n})]$. This is stated formally in the following theorem.

Theorem 1 *Under Assumptions 1-11,*

$$[\hat{n}, \hat{\tau}(\hat{n})] \overset{p.u}{\to} [m, \lambda^0]$$

where $\lambda^0 = [\lambda_1^0, \ldots, \lambda_m^0]'$ *is the collection of the true break fractions in (1).*

Remark 1: To implement the estimation procedure, it is necessary to pick a penalty term that satisfies Assumption 10. A natural choice that leads to a consistent IC is

$$K(n,T) = [(n+1)p + kn]\, ln(T)/T, \tag{13}$$

which is associated with BIC, because this choice has been found to work well in other settings. Applied in this 2SLS context, the proposal of Yao (1988) sets $k = 1$, and this penalty gives the criterion that we refer to simply as BIC. However, the analysis of Hall, Osborn, and Sakkas (2012) suggests that $k = 3$ may be appropriate, and we refer to the resulting criterion as SBBIC, indicating structural break SIC, since this treats the estimation of break dates as having a distinct weight from that of the individual coefficients of (6). Following Hall, Osborn, and Sakkas (2013) who consider the OLS case, we also employ two versions of HQIC, with

$$K(n,T) = 2[(n+1)p + kn]\, ln[ln(T)]/T \tag{14}$$

for $k = 1$ (referred to as HQIC) and $k = 3$ (SBHQIC). These criteria using the penalty (14) also satisfy Assumption 10. However, the choice associated with AIC (Akaike (1974)), where

$$K(n,T) = 2[(n+1)p + kn]\,/T \tag{15}$$

does not satisfy Assumption 10 and its yields an estimator that has a zero probability of choosing too few breaks but a non-zero probability of choosing too many breaks in the limit. Once again, we use $k = 1$ (labelled as AIC in the results) and $k = 3$ (SBAIC).

Remark 2: HHB propose a methodology for estimation of m based on the sequential application of tests for various forms of parameter variation. If these tests are performed with a fixed significance level then the resulting estimator of m has a zero probability of underfitting but a non-zero probability of overfitting in the limit due to the non-zero probability of type one errors inherent in the decision rules for the tests. Simulation results in HHB suggest that the tendency to overfit can be substantially reduced by using 1% significance levels; nevertheless, the resulting estimator of the number of breaks is not consistent. This may be seen as an advantage of the IC approach.

Remark 3: A further difference between HHB's approach and the IC approach is in terms of the assumptions about the limiting behaviour of the instrument cross-product matrix. The theory underlying certain tests employed in HHB's methodology requires the standardized partial sum instrument cross-product matrix to be linear in the sampling fraction within the assumed regimes under the appropriate null that is, $T^{-1}\sum_{t=T^0_{i-1}+1}^{T^0_{i-1}+[rT]} z_t z_t' \xrightarrow{p} rQ_i$, uniformly in $r \in (0, \lambda^0_i - \lambda^0_{i-1}]$, where Q_i is a pd matrix of constants. This rules out changes in the mean and variance of the instruments at different times from the changes in the structural parameters. This assumption is more restrictive than Assumption 8. Thus the IC approach is potentially more robust to such changes in the behaviour of z_t (in the limit).

4 Simulation Evidence

The first subsection outlines the set-up employed for our Monte Carlo analysis, with results discussed in the second subsection.

4.1 Methodology

We assess the performance of the aforementioned information criteria in a variety of cases with different numbers and locations of breaks in both the reduced form (RF) and structural form (SF) equations of (1) and (2). These nine cases are given in Table 1 and include models with no RF breaks and zero to two SF breaks; one RF break and zero to two SF breaks, including the case where one break is coincidental in both equations (Case 5) and non-coincidental (Case 7); two RF breaks and one or two SF breaks. For each case we investigate the effect of sample size ($T = 120$ and 240), break magnitude, autocorrelation, and the effect of explanatory power in first stage (RF) breaks estimation on the second stage (SF) breaks estimation. The focus is on the scenario most relevant in practice, where the number and locations of breaks are unknown in both equations and the same IC is applied for structural break inference in each of these. Tables 2 to 9 present the empirical probability of each IC to pick 0, 1, 2, or ≥ 3 breaks in the RF and SF for each case, based on a sample of 2000 replications of the data generating processes (DGPs) discussed below. The results shown present RF results once for all cases with the same RF since by keeping the seed of the pseudo-random number generator the same these estimations give identical results. The cases with the same RF are separated in the tables with horizontal lines.

We consider DGPs for which the SF equation includes a constant and one endogenous variable, so in our experiments (1) becomes

$$y_t = \beta_{1,i} + \beta_{2,i} x_t + u_t \quad i = 1, ..., m + 1.$$

All cases of no breaks use $\beta_1 = 0.5$, with $\beta_2 = 0.1$ in Tables 2 to 5 and $\beta_2 = 1$ in Tables 6 to 9. When breaks exist in the SF, we use the same coefficient values as in the no breaks case, but alternate the coefficients' sign between segments. These choices of coefficient values were made in order to present meaningful and comparable results, where ICs do not pick the true number of breaks 100% of the time, or have effectively no power. In the cases of one break this becomes

$$y_t = \begin{cases} \beta_1 + \beta_2 x_t + u_t \text{ if } t \leq [\lambda_1 T] \\ -\beta_1 - \beta_2 x_t + u_t \text{ if } t > [\lambda_1 T] \end{cases}$$

while for two breaks we set

$$y_t = \begin{cases} \beta_1 + \beta_2 x_t + u_t \text{ if } t \leq [\lambda_1 T] \\ -\beta_1 - \beta_2 x_t + u_t \text{ if } [\lambda_1 T] < t \leq [\lambda_2 T] \\ \beta_1 + \beta_2 x_t + u_t \text{ if } t > [\lambda_2 T]. \end{cases}$$

The simulated RF equations based on (2) are

$$x_t = z_t' \Delta_j + v_t \quad j = 1, ..., h+1,$$

where z_t contains $q = 4$ instruments and an intercept. The coefficient values for the intercept are the same as in the SF but the regressor coefficients, common across regressors, are determined so that they yield $R^2 = 0.3$ or $R^2 = 0.5$ that is ensured by using $\delta = \sqrt{R^2/(q - q \times R^2)}$ (see Hahn and Inoue (2002)). Performance of the IC under these different cases of explanatory power are presented as different rows of results. Across segments, the signs of the RF coefficients alternate as in the SF equations.

We use two different dynamic structures, each presented in different tables, to generate the u_t, v_t as well as the instruments z_t. In the case of $i.i.d$ errors we draw from a multivariate (six-dimensional) standard normal distribution, where the errors have $Cov[u_t, v_t] = 0.5$ and are uncorrelated with the instruments while the instruments have $Cov[z_{it}, z_{jt}] = 0 \;\; \forall i \neq j$. To explore the effect of autocorrelation in the behaviour of the IC we simulate each case with AR(1) processes for both the SF errors $u_t = \phi_u u_{t-1} + \varepsilon_t$, and the instruments $z_{i,t} = \phi_z z_{i,t-1} + \varepsilon_{it}$. We set the autoregressive parameter to 0.5 for both, and to ensure that $Var[u_t] = 1$ we set $Var[\varepsilon_t] = (1 - \phi_u^2) Var[u_t]$, while to retain $Cov[u_t, v_t] = Cov[(1 - \phi_u L)^{-1} \varepsilon_t, v_t] = 0.5$ we set $Cov[\varepsilon_t, v_t] = 0.5$. Similar considerations for the AR(1) in the instruments means setting $Var[\varepsilon_{it}] = (1 - \phi_z^2) Var[z_{it}]$. Finally, when searching for the break locations we allow for a maximum of five breaks, set a trimming parameter (ϵ in Assumption 9) of 0.10, that is, the minimum length of a segment can be 10% of the sample size) and use the efficient search algorithm developed in Bai and Perron (2003).

The presentation of the results is as follows. Tables 2 and 3 give the results for the small sample size ($T = 120$) and the "small" breaks ($\beta_2 = 0.1$) for the two different dynamic structures, i.i.d and AR(1) respectively. Tables 4 and 5 change to the larger sample size ($T = 240$) and the next four tables (6 to 9) repeat the models of tables 2 to 5 but for the larger magnitude of breaks given by $\beta_2 = 1$. To aid interpretation within these tables, the highest empirical probability of detecting the true number of breaks is shown in bold for each case considered.

4.2 Results

When no breaks occur in the DGP for either the SF or the RF equations ($h = 0$, $m = 0$), Case 1 of Tables 2, 4, 6 and 8 show a good performance of BIC when the disturbances are *i.i.d.*. More explicitly, the BIC criterion, employing $k = 1$ in (13), performs very well in not detecting spurious breaks in the RF, with good results consequently also seen in the SF when there is no autocorrelation. Even with the smaller sample size of $T = 120$, spurious breaks are infrequently detected by BIC (Tables 2 and 6). However, any such spurious breaks are removed by the use of the criterion SBBIC, which applies a higher weight of $k = 3$ to break date estimation. As may be anticipated, the use of HQIC leads to the estimation of some spurious breaks in both the reduced and structural form equations, with this feature being more marked for the SF. An increase in the sample size from $T = 120$ to $T = 240$ reduces spurious break detection in the SF from around 13%-15% to about 10%-11% (compare Table 2 with Table 4, and Table 6 with Table 8). Use of the modified criterion SBHQIC, however, eliminates the vast majority of these, resulting in less than 1% spurious SF breaks for Case 1 across Tables 2, 4, 6 and 8.

Compared to the performances of these criteria, the use of the inconsistent AIC yields poor inference on the number of breaks when none apply in the DGP. This is particularly marked when the penalty term (15) employs $k = 1$, which effectively counts each break date estimated as equivalent to a single coefficient, and leads to three or more spurious breaks being detected in the clear majority of replications with *i.i.d.* disturbances. While the number of spurious breaks is reduced by the use of SBAIC, these nevertheless occur in a substantial percentage of replications, standing at around 18% in the most favourable scenario of Table 8.

Turning to the DGPs with autocorrelation, notice firstly that autocorrelation in the RF regressors with $h = 0$ in Tables 3, 5, 7 and 9 leads to very similar break detection results for the RF compared to when the regressors are *i.i.d.* However, when breaks occur ($h = 1$ or 2) and the sample size is relatively small, autocorrelation reduces the accuracy of RF break detection by the BIC-based and HQIC-based methods in Tables 3 and 7 compared with Tables 2 and 6. The AIC-based methods are always poor and autocorrelated regressors have little effect on their RF performance.

Although the BIC-based and HQIC-based criteria remain consistent in the presence of stationary autocorrelation, it is clear that the positively autocorrelated $AR(1)$ disturbances lead to a deterioration of performance for all criteria applied to the SF when this experiences no breaks [5]. However, allocating the heavier weight to break date estimation in SBBIC and SBHQIC alleviates this feature. For example, for Case 1 in Table 3, BIC yields spurious breaks in more than 40% of the replications, which is reduced to less than 7% by SBBIC, with the corresponding percentages for HQIC and SBHQIC being 83% and 32%, respectively, with these performances improving marginally with $T = 240$ in Table 5. Not surprisingly, a stronger role for the SF regressors ($\beta_2 = 1$) also leads to improved performances for these consistent criteria in Tables 7 and 9, with the marked improvement shown by BIC, HQIC and SBHQIC particularly noteworthy. AIC and SBAIC also show an increased tendency to detect spurious breaks with $AR(1)$ rather than *i.i.d.* disturbances, but they remain poor in comparison to the consistent criteria. Indeed, this is always the case irrespective of the number of true breaks, and hence we do not explicitly discuss these criteria further.

In Case 2, where a single break applies in the SF but the RF is stable ($h = 0$, $m = 1$), SBBIC is the most accurate method in terms of the correct detection of the SF break with *i.i.d.* disturbances in Tables 2, 4, 6 and 8. This is followed in accuracy by SBHQIC, with these results corresponding with this corresponding to their good performances in our OLS study, in Hall, Osborn, and Sakkas (2013). While BIC does well in Table 2, when the magnitudes of the SF coefficients and the magnitudes of the changes are relatively small, it has a tendency to over-estimate the number of breaks for the larger magnitudes in Table 6. However, and not surprisingly, HQIC has a greater tendency than BIC to over-estimate the number of breaks.

The presence of any break is more difficult to detect when two reverting breaks apply with a stable RF (Case 3, with $h = 0$, $m = 2$) in these tables, so that BIC, SBBIC and SBHQIC often erroneously imply no breaks are present, with this being particularly a feature of Table 2.

[5] Since there is generally only modest deterioration in the detection of RF breaks with autocorrelated regressors, the deterioration in performance in the SF can be attributed primarily to autocorrelation in the second stage model itself.

However, this is largely eliminated for the larger sample size and/or larger coefficient magnitudes (Tables 4, 6 and 8), except sometimes for SBBIC. Note in Table 6 that the RF R^2 plays an important role for the performance of SBBIC, with the poor fitted values resulting from the DGP with relatively low explanatory power causing this criterion to often detect no breaks in the SF, whereas the higher R^2 value leads to much improved detection of the two breaks.

As in Case 1 where no breaks occur, the presence of unmodelled $AR(1)$ disturbances leads to an increased tendency for all criteria to detect spurious breaks in the SF (compare Cases 2 and 3 for Table 2 with Table 3, and similarly Tables 4 and 5, 6 and 7, 8 and 9). Although with $T = 240$ and $\beta_2 = 1$ in Table 9, SBBIC has very good accuracy for detection of the true number of breaks in the SF with positively autocorrelated errors, its performance is less impressive at the other extreme of $T = 120$ and $\beta_2 = 0.1$ in Table 3 where it often under-specifies the numbers of true breaks, especially in Case 3 when $m = 2$. On the other hand, BIC, SBHQIC and (especially) HQIC often over-estimate the number of breaks for Cases 2 and 3 in this latter table.

Since the consistent criteria BIC, SBBIC, HQIC and SBHQIC correctly detect the presence of a single RF break in the vast majority of replications across Cases 4 to 7, the characteristics just discussed largely continue to apply when $h = 1$. This is can be seen particularly in Tables 8 and 9, where $T = 240$ and $\beta_2 = \pm 1$, and the results for Cases 4 to 7 match the corresponding Cases 1 to 3 where $h = 0$. Other settings, however, show a greater influence from estimation of RF breaks.

For the same SF coefficients as in Tables 8 and 9, but with the smaller sample size of $T = 120$, Tables 6 and 7 illustrate the additional difficulties that apply when the DGP exhibits RF breaks. Compared with results for $h = 0$, SBBIC more often under-estimates the number of SF breaks for Cases 4 to 7 when the RF R^2 is low at 0.3, but the performance largely matches that with $h = 0$ when R^2 is 0.5. Further, the relative timing of breaks in the reduced and structural forms plays a role with this criterion. In Tables 6 and 7, for example, $m = 1$ is more often correctly specified using SBBIC for Case 5 (when $\pi_1 = \lambda_1 = 0.5$) than for Case 7 (where $\pi_1 = 0.3$, $\lambda_1 = 0.6$). On the other hand, for the smaller breaks in Tables 2 and 3, where $\beta_2 = 0.1$, SBBIC (and in general BIC) has better performance for Case 7 than Case 5. Overall, the performance of SBHQIC is more robust to the timing of these breaks.

When $h = 2$ in Cases 8 and 9 with $T = 120$, BIC and (to a greater extent) SBBIC often miss the presence of any RF break, particularly when the coefficients are of smaller magnitude and

$R^2 = 0.3$. Table 6, for example, shows how this leads to a deterioration of the performance of these criteria for the detection of SF breaks compared with the situation when $R^2 = 0.5$, with this being particularly clear for Case 9 with two SF breaks. This feature is also seen, but to a lesser extent, in the number of SF breaks detected by SBHQIC. The different performances of SBBIC for the two RF scenarios in Case 9 extends also to $T = 240$ in Tables 8 and 9. This applies despite the criterion correctly detecting two RF breaks in 91-95% of replications, suggesting a role for the estimation of the RF break dates themselves, and not simply the number of these.

Overall, these simulation results indicates that best performing criteria are SBBIC and SB-HQIC. The former works well for the detection of breaks in both the reduced and structural form equations across many of the cases considered, but can fail to detect any breaks when two breaks of the reverting form are present in the SF. With breaks of such reverting form, the use of SBHQIC more satisfactorily detects the presence of breaks, but at the cost of over-specifying the number of breaks in other cases. The heavier weighting of break date estimation implied by the use of $k = 3$ in (13) and (14) generally works better than $k = 1$, while the inconsistent AIC-based criteria do not appear to be useful if the correct detection of the number of breaks is an important consideration.

5 Conclusions

This paper makes two contributions in relation to the use of information criteria for inference on structural breaks when the coefficients of a linear model with endogenous regressors may experience multiple changes. Firstly, we show that suitably defined information criteria yield consistent estimators of the number of breaks, when employed in the second stage of a two-stage least squares (2SLS) procedure with breaks in the reduced form taken into account in the first stage. Secondly, a Monte Carlo analysis investigates the finite sample performance of a range of criteria based on BIC, HQIC and AIC for equations estimated by 2SLS. Versions of the consistent criteria BIC and HQIC perform well overall when the penalty term weights estimation of each break point more heavily than estimation of each coefficient, while AIC is inconsistent and badly over-estimates the number of true breaks.

Appendix

Mathematical Appendix

Proof of Lemma 1

Case(i) Assume that (1) is stable for $t = [\tau_{j-1}T] + 1, \ldots, [\tau_j T]$, where τ_i denotes the estimated break fraction, so that for some i,

$$y_t = x_t' \beta^0_{x,i} + z_{1,t}' \beta^0_{z_1,i} + u_t \;\; t = [\tau_{j-1}T] + 1, \ldots, [\tau_j T]. \tag{16}$$

Let $\hat{\beta}_j$ be the 2SLS estimator of $\beta^0_i = [\beta^{0\prime}_{x,i}, \beta^{0\prime}_{z_1,i}]'$ based on (16) using $\hat{x}_t(\hat{\pi})$ defined in (4), and define $w_t(\pi) = [\hat{x}_t(\pi)', z_{1,t}']'$. Then, we have

$$\hat{\beta}_j \;=\; \left(\sum_j w_t(\hat{\pi}) w_t(\hat{\pi})' \right)^{-1} \sum_j w_t(\hat{\pi}) y_t \;=\; \beta^0_i + \left(\sum_j w_t(\hat{\pi}) w_t(\hat{\pi})' \right)^{-1} \sum_j w_t(\hat{\pi}) \tilde{u}_t(\hat{\pi}),$$

where $\sum_j$ denotes $\sum_{[\tau_{j-1}T]+1}^{[\tau_j T]}$ and

$$\tilde{u}_t(\hat{\pi}) \;=\; y_t \;-\; w_t(\hat{\pi})' \beta^0_i. \tag{17}$$

To facilitate the analysis of $RSS_j(\tau(n); n, \hat{\pi})$ (henceforth RSS_j), we consider $y_t - w_t(\hat{\pi})' \hat{\beta}_j$. Defining $\tilde{u}_t(\pi^0)$ and $\hat{x}_t(\pi^0)$ analogously to $\tilde{u}_t(\hat{\pi})$ and $\hat{x}_t(\hat{\pi})$, it can shown that (17) implies

$$\begin{aligned} y_t - w_t(\hat{\pi})' \hat{\beta}_j \;&=\; \tilde{u}_t(\pi^0) + \left[\hat{x}_t(\pi^0)' - \hat{x}_t(\hat{\pi})' \right] \beta^0_{x,i} - w_t(\hat{\pi})' \left(\sum_j w_t(\hat{\pi}) w_t(\hat{\pi})' \right)^{-1} \\ &\quad \times \sum_j w_t(\hat{\pi}) \tilde{u}_t(\hat{\pi}). \end{aligned} \tag{18}$$

From (18), it follows that

$$T^{-1} RSS_j \;=\; T^{-1} \sum_j (A_t + B_t - C_t)^2, \tag{19}$$

where $A_t \;=\; \tilde{u}_t(\pi^0)$, $B_t \;=\; \left[\hat{x}_t(\pi^0)' - \hat{x}_t(\hat{\pi})' \right] \beta^0_{x,i}$, and

$$C_t \;=\; w_t(\hat{\pi})' \left(\sum_j w_t(\hat{\pi}) w_t(\hat{\pi})' \right)^{-1} \sum_{t=[Ts]+1}^{[Tr]} w_t(\hat{\pi}) \tilde{u}_t(\hat{\pi}).$$

We now consider in turn the terms obtained by multiplying out the quadratic in (19).

First for A_t^2: using (16) and substituting for $\hat{x}_t(\pi^0)$ from (3), we have

$$\begin{aligned} T^{-1}\sum_j \tilde{u}_t(\pi^0)^2 &= T^{-1}\sum_j \left\{y_t - w_t(\pi^0)'\beta_i^0\right\}^2 \\ &= T^{-1}\sum_j \left\{u_t + [x_t - \hat{x}_t(\pi^0)]'\beta_{x,i}^0\right\}^2 \\ &= T^{-1}\sum_j \left\{u_t + v_t'\beta_{x,i}^0 - \tilde{z}_t(\pi^0)'\left[\hat{\Theta}_T(\pi_0) - \Theta_0\right]\beta_{x,i}^0\right\}^2. \end{aligned} \tag{20}$$

$\hat{\Theta}_T(\pi_0)$ is the (infeasible) OLS estimator constructed using the true reduced form break fractions $\{\pi^0\}$ as the break dates, and as such, may be decomposed as

$$\hat{\Theta}_T(\pi_0) = \Theta_0 + \left(\sum_{t=1}^T \tilde{z}_t(\pi^0)\tilde{z}_t(\pi^0)'\right)^{-1}\sum_{t=1}^T \tilde{z}_t(\pi^0)v_t'.$$

Substituting this formula into (20) we obtain

$$T^{-1}\sum_j A_t^2 = T^{-1}\sum_j \{a_t + b_t - c_t\}^2, \tag{21}$$

where $a_t = u_t$, $b_t = v_t'\beta_{x,i}^0$, and $c_t = \tilde{z}_t(\pi^0)'\left(\sum_{t=1}^T \tilde{z}_t(\pi^0)\tilde{z}_t(\pi^0)'\right)^{-1}\sum_{t=1}^T \tilde{z}_t(\pi^0)v_t'\beta_{x,i}^0$.

By Assumptions 4 and 5, it follows that for the terms a_t^2, b_t^2, and $2a_tb_t$ in (21), respectively we have,

$$\begin{aligned} T^{-1}\sum_j u_t^2 &\overset{p.u}{\to} (\tau_j - \tau_{j-1})\sigma_u^2, \\ T^{-1}\sum_j \beta_{x,i}^{0'} v_t v_t'\beta_{x,i}^0 &\overset{p.u}{\to} (\tau_j - \tau_{j-1})\beta_{x,i}^0\Sigma_v\beta_{x,i}^0, \\ T^{-1}2\sum_j u_t v_t'\beta_{x,i}^0 &\overset{p.u}{\to} (\tau_j - \tau_{j-1})2\Sigma_{uv}\beta_{x,i}^0. \end{aligned}$$

For the remaining terms in (21), using Assumptions 2 and 8 we have $T^{-1}\sum_{t=[Ts]+1}^{[Tr]} z_t z_t' \overset{p.u}{\to} Q_{ZZ}(r) - Q_{ZZ}(s) = M_{ZZ}(s,r)$ for $r > s + \epsilon$ is also *pd* and monotonically increasing. Also by Assumptions 2 and 8, it follows that

$$T^{-1}\sum_{t=1}^T \tilde{z}_t(\pi^0)\tilde{z}_t(\pi^0)' \overset{p.u}{\to} \tilde{Q}_{ZZ}(1),$$

also *pd*, where $\tilde{Q}_{ZZ}(1)$ is the block diagonal matrix $diag(Q_1, Q_2, \ldots Q_{h+1})$ and $Q_i = Q_{ZZ}(\pi_i^0) - Q_{ZZ}(\pi_{i-1}^0)$ and we set $\pi_0^0 = 0$, $\pi_{h+1}^0 = 1$. Then, for a segment of the data $t = [\tau_{j-1}T] + 1, \ldots, [\tau_j T]$, it follows that

$$T^{-1}\sum_{t=[\tau_{j-1}T]+1}^{[\tau_j T]} \tilde{z}_t(\pi^0)\tilde{z}_t(\pi^0)' \overset{p.u}{\to} \tilde{Q}(\tau_{j-1}, \tau_j) \text{ in } \tau_{j-1}, \tau_j,\ (\tau_j > \tau_{j-1} + \epsilon) \text{ and } pd, \tag{22}$$

where - assuming $\pi_i^0 < \tau_{j-1} \leq \pi_{i+1}^0$ and $\pi_{i+\ell}^0 < \tau_j \leq \pi_{i+\ell+1}^0$ without loss of generality - $\tilde{Q}(s,r) = [0_{(h+1)q\times iq}, A(\tau_{j-1},\tau_j), 0_{(h+1)q\times(h-i-\ell-1)q}]$ and $A(\tau_{j-1},\tau_j)$ is the block diagonal matrix $diag\{Q_{ZZ}(\pi_{i+1}^0) - Q_{ZZ}(\tau_{j-1}), Q(i+2), \ldots, Q(i+\ell), Q_{ZZ}(\tau_j) - Q_{ZZ}(\pi_{i+\ell}^0)\}$.

Furthermore, it follows from Assumptions 2 and 4, that $T^{-1/2}\sum_{t=1}^{[Tr]} \tilde{z}_t(\pi^0) \otimes \{(u_t, v_t{}')'\}$ is $O_p(1)$ via a central limit theorem. The above suffice to show that for the remaining terms in (21) we have, $T^{-1}\sum_j a_t c_t \overset{p.u}{\to} 0$, $T^{-1}\sum_j b_t c_t \overset{p.u}{\to} 0$ and $T^{-1}\sum_j c_t^2 \overset{p.u}{\to} 0$.

Combining these results regarding the term A_t^2 in (19) it follows that

$$T^{-1}\sum_j A_t^2 \overset{p.u}{\to} (\tau_j - \tau_{j-1})\Gamma_i \tag{23}$$

with Γ_i defined in Lemma 1.

The term involving B_t^2 in (19) can be written as

$$\begin{aligned} T^{-1}\sum_j B_t^2 &= T^{-1}\sum_j \beta_{x,i}^{0\prime}\left[\hat{x}_t(\pi^0) - \hat{x}_t(\hat{\pi})\right]\left[\hat{x}_t(\pi^{0\prime}) - \hat{x}_t(\hat{\pi})'\right]\beta_{x,i}^0 \\ &= \beta_{x,i}^{0\prime}\left\{T^{-1}\sum_j\left[\hat{x}_t(\pi^0)\hat{x}_t(\pi^0)' + \hat{x}_t(\hat{\pi})\hat{x}_t(\hat{\pi})' - 2\hat{x}_t(\pi^0)\hat{x}_t(\hat{\pi})'\right]\right\}\beta_{x,i}^0. \end{aligned} \tag{24}$$

The following results will determine the probability limit of (24). From Assumptions 3 and 8 it follows that

$$T^{-1}\sum_{t=1}^T \tilde{z}_t(\hat{\pi})\tilde{z}_t(\hat{\pi})' = T^{-1}\sum_{t=1}^T \tilde{z}_t(\pi^0)\tilde{z}_t(\pi^0)' + o_p(1) \overset{p.u}{\to} \tilde{Q}_{ZZ}(1) \tag{25}$$

and also,

$$T^{-1}\sum_j \tilde{z}_t(\hat{\pi})\tilde{z}_t(\pi^0)' \overset{p.u}{\to} \tilde{Q}(\tau_{j-1},\tau_j). \tag{26}$$

From Assumptions 3, 4, and 8 it follows that

$$T^{-1}\sum_{t=1}^T \tilde{z}_t(\hat{\pi})x_t' = T^{-1}\sum_{t=1}^T \tilde{z}_t(\pi^0)x_t' + o_p(1) \overset{p.u}{\to} \tilde{Q}_{ZZ}(1)\Theta_0 \tag{27}$$

By (25), (27), and (4),

$$\begin{aligned} T^{-1}\sum_j \hat{x}_t(\hat{\pi})\hat{x}_t(\hat{\pi})' &= T^{-1}\sum_{t=1}^T x_t\tilde{z}_t(\hat{\pi})'\left(\sum_{t=1}^T \tilde{z}_t(\hat{\pi})\tilde{z}_t(\hat{\pi})'\right)^{-1}\sum_j \tilde{z}_t(\hat{\pi})\tilde{z}_t(\hat{\pi})' \\ &\quad\times\left(\sum_{t=1}^T \tilde{z}_t(\hat{\pi})\tilde{z}_t(\hat{\pi})'\right)^{-1}\sum_{t=1}^T \tilde{z}_t(\hat{\pi})x_t' \\ &\overset{p.u}{\to} \Theta_0'\tilde{Q}_{ZZ}'(1)\tilde{Q}_{ZZ}^{-1}(1)\tilde{Q}(\tau_{j-1},\tau_j)\tilde{Q}_{ZZ}^{-1}(1)\tilde{Q}_{ZZ}(1)\Theta_0 \\ &= \Theta_0'\tilde{Q}(\tau_{j-1},\tau_j)\Theta_0 \end{aligned} \tag{28}$$

which is *pd* by the construction of $\tilde{Q}(\tau_{j-1}, \tau_j)$. Similarly, we have

$$T^{-1} \sum_j \hat{x}_t(\pi^0) \hat{x}_t(\pi^0)' \overset{p.u}{\to} \Theta_0' \tilde{Q}(\tau_{j-1}, \tau_j) \Theta_0. \tag{29}$$

For the last term in (24), we combine (25), (26), (27), and (4), and use Assumption 2 to deduce that

$$\begin{aligned} 2T^{-1} \sum_j \hat{x}_t(\pi^0) \hat{x}_t(\hat{\pi})' &= 2T^{-1} \sum_{t=1}^{T} x_t \tilde{z}_t(\pi_0)' \left(\sum_{t=1}^{T} \tilde{z}_t(\pi_0) \tilde{z}_t(\pi_0)' \right)^{-1} \sum_j \tilde{z}_t(\pi_0) \tilde{z}_t(\hat{\pi})' \\ & \quad \times \left(\sum_{t=1}^{T} \tilde{z}_t(\hat{\pi}) \tilde{z}_t(\hat{\pi})' \right)^{-1} \sum_{t=1}^{T} \tilde{z}_t(\hat{\pi}) x_t' \\ &\overset{p.u}{\to} 2\Theta_0' \tilde{Q}(\tau_{j-1}, \tau_j) \Theta_0. \end{aligned} \tag{30}$$

Combining (28), (29), and (30), the probability limit of (24) is

$$T^{-1} \sum_j B_t^2 \overset{p.u}{\to} 0. \tag{31}$$

Now consider the terms involving C_t in (19). Start by considering $\sum_j w_t(\hat{\pi}) u_t(\hat{\pi})$. If we expand $\tilde{u}_t(\hat{\pi})$ similarly to (18), and substitute for $\hat{x}_t(\hat{\pi})$, then from (4) we obtain

$$\sum_j w_t(\hat{\pi}) \tilde{u}_t(\hat{\pi}) = \sum_j w_t(\hat{\pi}) \left\{ \tilde{u}_t(\pi^0) + \left[\hat{x}_t(\pi^0)' - \hat{x}_t(\hat{\pi})' \right] \beta_{x,i}^0 \right\}.$$

Thus, we have

$$\begin{aligned} \sum_j \hat{x}_t(\hat{\pi}) \tilde{u}_t(\hat{\pi}) &= \sum_{t=1}^{T} x_t \tilde{z}_t(\hat{\pi})' \left(\sum_{t=1}^{T} \tilde{z}_t(\hat{\pi}) \tilde{z}_t(\hat{\pi})' \right)^{-1} \left[\sum_j \tilde{z}_t(\hat{\pi}) \tilde{u}_t(\pi_0) \right. \\ &+ \sum_j \tilde{z}_t(\hat{\pi}) \tilde{z}_t(\pi_0)' \left(\sum_{t=1}^{T} \tilde{z}_t(\pi_0) \tilde{z}_t(\pi_0)' \right)^{-1} \sum_{t=1}^{T} \tilde{z}_t(\pi_0) x_t' \beta_{x,i}^0 \\ &- \left. \sum_j \tilde{z}_t(\hat{\pi}) \tilde{z}_t(\hat{\pi})' \left(\sum_{t=1}^{T} \tilde{z}_t(\hat{\pi}) \tilde{z}_t(\hat{\pi})' \right)^{-1} \sum_{t=1}^{T} \tilde{z}_t(\hat{\pi}) x_t' \beta_{x,i}^0 \right]. \end{aligned} \tag{32}$$

From (25), (26), and (27), the last two terms inside the brackets in (32) cancel out asymptotically. The same equations and also Assumption 4, after expanding $\tilde{u}_t(\pi_0)$ similarly to (21), give

$$\begin{aligned} T^{-1} \sum_j \tilde{z}_t(\hat{\pi}) \tilde{u}_t(\pi_0) &= T^{-1} \sum_j \tilde{z}_t(\hat{\pi}) u_t + T^{-1} \sum_j \tilde{z}_t(\hat{\pi}) v_t' \beta_{x,i}^0 \\ & \quad -T^{-1} \sum_j \tilde{z}_t(\hat{\pi}) \tilde{z}_t(\pi_0) \left(\sum_{t=1}^{T} \tilde{z}_t(\pi_0) \tilde{z}_t(\pi_0)' \right)^{-1} \sum_{t=1}^{T} \tilde{z}_t(\pi_0) v_t' \beta_{x,i}^0 \\ &\overset{p.u}{\to} 0 \end{aligned} \tag{33}$$

and therefore it follows from (32) that,

$$T^{-1}\sum_j \hat{x}_t(\hat{\pi})\tilde{u}_t(\hat{\pi}) \stackrel{p.u}{\rightarrow} 0. \tag{34}$$

It can also be shown via similar arguments that

$$T^{-1}\sum_j z_{1,t}\tilde{u}_t(\hat{\pi}) \stackrel{p.u}{\rightarrow} 0. \tag{35}$$

Using (34)-(35), together with (28) and (30) it follows that the limiting behaviour of the terms involving C_t in $T^{-1}RSS_j$ are:

$$T^{-1}\sum_j C_t^2 \stackrel{p.u}{\rightarrow} 0 \tag{36}$$

$$2T^{-1}\sum_j A_tC_t \stackrel{p.u}{\rightarrow} 0 \tag{37}$$

$$2T^{-1}\sum_j B_tC_t \stackrel{p.u}{\rightarrow} 0 \tag{38}$$

The last remaining term of $T^{-1}RSS_j$ involves $2A_tB_t$,

$$\begin{aligned} T^{-1}\sum_j A_tB_t &= T^{-1}\sum_j \tilde{u}_t(\pi^0)\left[\hat{x}_t(\pi^0)^{'} - \hat{x}_t(\hat{\pi})^{'}\right]\beta_{x,i}^0 \\ &= T^{-1}\sum_j \left[\tilde{u}_t(\pi^0)\hat{x}_t(\pi^0)^{'} - \tilde{u}_t(\pi^0)\hat{x}_t(\hat{\pi})^{'}\right]\beta_{x,i}^0 \end{aligned}$$

Using (21), and (4) the first term inside the summation can be expanded as

$$\begin{aligned} T^{-1}\sum_j \tilde{u}_t(\pi^0)\hat{x}_t(\pi^0)^{'} &= T^{-1}\sum_j \left[u_t + v_t^{'}\beta_{x,i}^0 - \tilde{z}_t(\pi^0)^{'}\left(\sum_{t=1}^{T}\tilde{z}_t(\pi^0)\tilde{z}_t(\pi^0)^{'}\right)^{-1}\sum_{t=1}^{T}\tilde{z}_t(\pi^0)v_t^{'}\beta_{x,i}^0\right] \\ &\quad \times\tilde{z}_t(\hat{\pi})'\left(\sum_{t=1}^{T}\tilde{z}_t(\hat{\pi})\tilde{z}_t(\hat{\pi})'\right)^{-1}\sum_{t=1}^{T}\tilde{z}_t(\hat{\pi})x_t' \\ &\stackrel{p.u}{\rightarrow} 0 \end{aligned}$$

since by Assumption 4, $T^{-1}\sum_{t=[Ts]+1}^{[Tr]} u_t\tilde{z}_t(\pi^0) \stackrel{p.u}{\rightarrow} 0$ and $T^{-1}\sum_{t=[Ts]+1}^{[Tr]} v_t^{'}\tilde{z}_t(\pi^0) \stackrel{p.u}{\rightarrow} 0$. Also using the same arguments and (25), and (27), $T^{-1}\sum_j \tilde{u}_t(\pi^0)\hat{x}_t(\hat{\pi})^{'} \stackrel{p.u}{\rightarrow} 0$ as well, resulting in

$$T^{-1}\sum_j A_tB_t \stackrel{p.u}{\rightarrow} 0. \tag{39}$$

Collecting the results regarding the limit of $T^{-1}RSS_j$ in (19), found in (23), (31), (36), (37), (38), and (39), it follows that

$$T^{-1}RSS_j \stackrel{p.u}{\rightarrow} (\tau_j - \tau_{j-1})\Gamma_i. \tag{40}$$

Case (ii): we first consider the case where segment j contains one neglected break and then discuss how the argument extends to more than one neglected break. Let the neglected break be at λ_i^0 so that the model is

$$\begin{aligned} y_t &= x_t^{'}\beta_{x,i}^0 + z_{1,t}^{'}\beta_{z_1,i} + u_t, \qquad t = [\tau_{j-1}T]+1,...,[\lambda_i^0 T] \\ y_t &= x_t^{'}\beta_{x,i+1}^0 + z_{1,t}^{'}\beta_{z_1,i} + u_t, \qquad t = [\lambda_i^0 T]+1,...,[\tau_j T]. \end{aligned} \tag{41}$$

Then the residual sum of squares in this segment, RSS_j, may be decomposed as

$$\begin{aligned} T^{-1}RSS_j &= T^{-1}\sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]}\left[y_t - w_t(\hat{\pi})^{'}\hat{\beta}_j\right]^2 + T^{-1}\sum_{t=[\lambda_i^0 T]+1}^{[\tau_j T]}\left[y_t - w_t(\hat{\pi})'\hat{\beta}_j\right]^2 \\ &= \xi_1 + \xi_2, \text{ say, respectively.} \end{aligned} \tag{42}$$

We focus on ξ_1. Substituting for y_t from (17), we have

$$\begin{aligned} \xi_1 &= T^{-1}\sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]}\left[w_t(\hat{\pi})^{'}\beta_i^0 + \tilde{u}_t(\hat{\pi}) - w_t(\hat{\pi})^{'}\hat{\beta}_j\right]^2 \\ &= T^{-1}\sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]}\left[\tilde{u}_t(\hat{\pi}) - w_t(\hat{\pi})^{'}\left(\hat{\beta}_j - \beta_i^0\right)\right]^2 \\ &= T^{-1}\sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]}\left[\tilde{u}_t(\hat{\pi})^2 - 2\tilde{u}_t(\hat{\pi})w_t(\hat{\pi})^{'}\left(\hat{\beta}_j - \beta_i^0\right)\right. \\ &\qquad \left. + \left(\hat{\beta}_j - \beta_i^0\right)^{'} w_t(\hat{\pi})w_t(\hat{\pi})^{'}\left(\hat{\beta}_j - \beta_i^0\right)\right]. \end{aligned} \tag{43}$$

The first term in this sum can be written as, $\tilde{u}_t(\hat{\pi})^2 = \left[\tilde{u}_t(\pi^0) + \left(\hat{x}_t(\hat{\pi})^{'} - \hat{x}_t(\pi^0)^{'}\right)\beta_{x,i}^0\right]^2$. Using the results in the proof of Case (i) above for the limits of the terms involving sums of A_t^2, B_t^2, and A_tB_t, found in (23), (31), and (39), it can be shown that,

$$T^{-1}\sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]}\tilde{u}_t(\hat{\pi})^2 \overset{p.u}{\to} (\lambda_i^0 - \tau_{j-1})\Gamma_i. \tag{44}$$

To proceed we need to derive $plim\left(\hat{\beta}_j - \beta_i^0\right)$ where

$$\hat{\beta}_j = \left(\sum_j w_t(\hat{\pi})w_t(\hat{\pi})'\right)^{-1}\sum_j w_t(\hat{\pi})y_t. \tag{45}$$

Using similar arguments to (28) and (34) we have that

$$T^{-1} \sum_j \hat{x}_t(\hat{\pi}) \hat{x}_t(\hat{\pi})' \overset{p.u}{\rightarrow} \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \tag{46}$$

where $\bar{\Upsilon}_0 = [\Theta_0, \bar{\Pi}]$, $\bar{\Pi} = \imath_{h+1} \otimes \Pi$, and

$$\begin{aligned} \sum_j w_t(\hat{\pi}) y_t &= T^{-1} \sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]} w_t(\hat{\pi}) \left[w_t(\hat{\pi})' \beta_i^0 + \tilde{u}_t(\hat{\pi}) \right] \\ &\quad + T^{-1} \sum_{t=[\lambda_i^0 T]+1}^{[\tau_j T]} w_t(\hat{\pi}) \left[w_t(\hat{\pi})' \beta_{i+1}^0 + \tilde{u}_t(\hat{\pi}) \right] \\ &\overset{p.u}{\rightarrow} \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \lambda_i^0) \bar{\Upsilon}_0 \beta_i^0 + \bar{\Upsilon}_0' \tilde{Q}(\lambda_i^0, \tau_j) \bar{\Upsilon}_0 \beta_{i+1}^0. \end{aligned} \tag{47}$$

Combining (45), (46), and (47) results in

$$plim \left(\hat{\beta}_j - \beta_i^0 \right) \overset{p.u}{\rightarrow} \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\}^{-1} \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \lambda_i^0) \bar{\Upsilon}_0 \beta_i^0 + \bar{\Upsilon}_0' \tilde{Q}(\lambda_i^0, \tau_j) \bar{\Upsilon}_0 \beta_{i+1}^0 \right\} - \beta_i^0. \tag{48}$$

Furthermore, β_i^0 can be written as,

$$\begin{aligned} \beta_i^0 &= \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\}^{-1} \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\} \beta_i^0 \\ &= \left\{ \bar{\Upsilon}' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\}^{-1} \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \lambda_i^0) \bar{\Upsilon}_0 \beta_i^0 + \bar{\Upsilon}_0' \tilde{Q}(\lambda_i^0, \tau_j) \bar{\Upsilon}_0 \beta_i^0 \right\}. \end{aligned}$$

Substituting this into (48) and after some algebra it is shown that

$$plim \left(\hat{\beta}_j - \beta_i^0 \right) \overset{p.u}{\rightarrow} \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\}^{-1} \bar{\Upsilon}_0' \tilde{Q}(\lambda_i^0, \tau_j) \bar{\Upsilon}_0 \left(\beta_{i+1}^0 - \beta_i^0 \right) = P_1. \tag{49}$$

P_1 is ensured to be non-zero because $\tilde{Q}(r, s)$ is a block diagonal matrix and each block is positive definite via Assumption 9, and also $\beta_{i+1}^0 \neq \beta_i^0$.

Going back to ξ_1, it follows that from (34), (35), (49) and (46) that the last two terms in (43) have the following probability limits

$$T^{-1} \sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]} \tilde{u}_t(\hat{\pi}) w_t(\hat{\pi})' \left(\hat{\beta}_j - \beta_i^0 \right) \overset{p.u}{\rightarrow} 0 \tag{50}$$

and

$$\left(\hat{\beta}_j - \beta_i^0 \right)' \sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]} w_t(\hat{\pi}) w_t(\hat{\pi})' \left(\hat{\beta}_j - \beta_i^0 \right) \overset{p.u}{\rightarrow} P_1' \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \lambda_i^0) \bar{\Upsilon}_0 P_1 > 0, \tag{51}$$

since $\tilde{Q}(r, s)$ is positive definite and $P_1 \neq 0$.

Collecting the results from (44), (50), and (51),

$$\xi_1 \overset{p.u}{\to} (\lambda_i^0 - \tau_{j-1})\Gamma_i + P_1^{'}\bar{\Upsilon}_0'\tilde{Q}(\tau_{j-1}, \lambda_i^0)\bar{\Upsilon}_0 P_1 \ > \ (\lambda_i^0 - \tau_{j-1})\Gamma_i. \tag{52}$$

Analogously for ξ_2, we have

$$\xi_2 \overset{p.u}{\to} (\tau_j - \lambda_i^0)\Gamma_{i+1} + P_2^{'}\bar{\Upsilon}_0'\tilde{Q}(\lambda_i^0, \tau_j)\bar{\Upsilon}_0 P_2 \ > \ (\tau_j - \lambda_i^0)\Gamma_{i+1} \tag{53}$$

where

$$P_2 \ = \ \left\{\bar{\Upsilon}_0'\tilde{Q}(\tau_{j-1}, \tau_j)\bar{\Upsilon}_0\right\}^{-1} \bar{\Upsilon}_0'\tilde{Q}(\tau_{j-1}, \lambda_i^0)\bar{\Upsilon}_0 \left(\beta_i^0 - \beta_{i+1}^0\right) \ \neq \ 0.$$

$$T^{-1}RSS_j \ \overset{p.u}{\to} \ (\lambda_i^0 - \tau_{j-1})\Gamma_i + (\tau_j - \lambda_i^0)\Gamma_{i+1} \ + \ F_1 \tag{54}$$

where

$$F_1 \ = \ P_1^{'}\Theta_0^{'}\tilde{Q}(\tau_{j-1}, \lambda_i^0)\Theta_0 P_1 \ + \ P_2^{'}\Theta_0^{'}\tilde{Q}(\lambda_i^0, \tau_j)\Theta_0 P_2 \ > \ 0. \tag{55}$$

This line of argument extends to more than one neglected break. We now show how two neglected breaks in a segment of the structural equation extend the case discussed above. To do this, we must evaluate the limiting distribution of RSS_j in a segment where there are two neglected breaks, denoted λ_i^0, and λ_{i+1}^0. Therefore, the data generation process is

$$\begin{aligned}
y_t &= x_t'\beta_{x,i}^0 \ + \ z_{1,t}^{'}\beta_{z_1,i}^0 \ + \ u_t, \qquad t = [\tau_{j-1}T]+1, ..., [\lambda_i^0 T] \\
y_t &= x_t'\beta_{x,i+1}^0 \ + \ z_{1,t}^{'}\beta_{z_1,i}^0 \ + \ u_t, \qquad t = [\lambda_i^0 T]+1, ..., [\lambda_{i+1}^0 T] \\
y_t &= x_t'\beta_{x,i+2}^0 \ + \ z_{1,t}^{'}\beta_{z_1,i}^0 \ + \ u_t, \qquad t = [\lambda_{i+1}^0 T]+1, ..., [\tau_j T].
\end{aligned} \tag{56}$$

The RSS for this segment can be decomposed as,

$$\begin{aligned}
T^{-1}RSS_j &= T^{-1}\sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]}\left(y_t - w_t(\hat{\pi})^{'}\hat{\beta}_j\right)^2 + T^{-1}\sum_{t=[\lambda_i^0 T]+1}^{[\lambda_{i+1}^0 T]}\left(y_t - w_t(\hat{\pi})^{'}\hat{\beta}_j\right)^2 \\
&\quad + T^{-1}\sum_{t=[\lambda_{i+1}^0 T]+1}^{[\tau_j T]}\left(y_t - w_t(\hat{\pi})^{'}\hat{\beta}_j\right)^2 \\
&= \xi_1 \ + \ \xi_2 \ + \ \xi_3.
\end{aligned} \tag{57}$$

Focusing on ξ_1, as in (43), this term can be written as

$$\begin{aligned}
\xi_1 &= T^{-1}\sum_{t=[\tau_{j-1}T]+1}^{[\lambda^0 T]}\left[\tilde{u}_t(\hat{\pi})^2 - 2\tilde{u}_t(\hat{\pi})w_t(\hat{\pi})^{'}\left(\hat{\beta}_j - \beta_i^0\right)\right. \\
&\quad \left. + \ \left(\hat{\beta}_j - \beta_i^0\right)^{'} w_t(\hat{\pi})w_t(\hat{\pi})^{'}\left(\hat{\beta}_j - \beta_i^0\right)\right].
\end{aligned} \tag{58}$$

where $\hat{\beta}_j$ is defined as in (45). To analyse the limit of $\hat{\beta}_j$, note that (46) holds within the scenario considered here. However, this time we have but in this case,

$$
\begin{aligned}
\sum_j w_t(\hat{\pi}) y_t &= T^{-1} \sum_{t=[\tau_{j-1}T]+1}^{[\lambda_i^0 T]} w_t(\hat{\pi}) \left[w_t(\hat{\pi})' \beta_i^0 + \tilde{u}_t(\hat{\pi}) \right] \\
&+ T^{-1} \sum_{t=[\lambda_i^0 T]+1}^{[\lambda_{i+1}^0 T]} w_t(\hat{\pi}) \left[w_t(\hat{\pi})' \beta_{i+1}^0 + \tilde{u}_t(\hat{\pi}) \right] \\
&+ T^{-1} \sum_{t=[\lambda_{i+1}^0 T]+1}^{[\tau_j T]} w_t(\hat{\pi}) \left[w_t(\hat{\pi})' \beta_{i+2}^0 + \tilde{u}_t(\hat{\pi}) \right]
\end{aligned}
$$

and so

$$
\begin{aligned}
\sum_j w_t(\hat{\pi}) y_t \overset{p.u}{\to} & \ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \lambda_i^0) \bar{\Upsilon}_0 \beta_i^0 + \bar{\Upsilon}_0' \tilde{Q}(\lambda_i^0, \lambda_{i+1}) \bar{\Upsilon}_0 \beta_{i+1}^0 \\
& + \bar{\Upsilon}_0' \tilde{Q}(\lambda_{i+1}^0, \tau_j) \bar{\Upsilon}_0 \beta_{i+2}^0 .
\end{aligned}
$$

By a similar argument like the one that lead to (48), for $plim\left(\hat{\beta}_{x,j} - \beta_{x,i}^0\right)$ here it follows that

$$
\begin{aligned}
plim\left(\hat{\beta}_j - \beta_i^0\right) \overset{p.u}{\to} & \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\}^{-1} \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \lambda_i^0) \bar{\Upsilon}_0 \beta_i^0 + \bar{\Upsilon}_0' \tilde{Q}(\lambda_i^0, \lambda_{i+1}^0) \bar{\Upsilon}_0 \beta_{i+1}^0 \right. \\
& \left. + \bar{\Upsilon}_0' \tilde{Q}(\lambda_{i+1}^0, \tau_j) \bar{\Upsilon}_0 \beta_{i+2}^0 \right\} - \beta_i^0 . \quad (59)
\end{aligned}
$$

We can rewrite β_i^0 as

$$
\begin{aligned}
\beta_i^0 &= \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\}^{-1} \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\} \beta_i^0 \\
&= \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\}^{-1} \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \lambda_i^0) \bar{\Upsilon}_0 + \bar{\Upsilon}_0' \tilde{Q}(\lambda_i^0, \lambda_{i+1}^0) \bar{\Upsilon}_0 \right. \\
& \quad \left. + \bar{\Upsilon}_0' \tilde{Q}(\lambda_{i+1}^0, \tau_j) \bar{\Upsilon}_0 \right\} \beta_i^0
\end{aligned}
$$

Then, after substituting this equation into (59) and rearranging terms, we obtain

$$
\begin{aligned}
plim\left(\hat{\beta}_j - \beta_i^0\right) \overset{p.u}{\to} & \left\{ \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \tau_j) \bar{\Upsilon}_0 \right\}^{-1} \\
& \left\{ \bar{\Upsilon}_0' \tilde{Q}(\lambda_i^0, \lambda_{i+1}^0) \bar{\Upsilon}_0 (\beta_{i+1}^0 - \beta_i^0) \right. \\
& \left. + \bar{\Upsilon}_0' \tilde{Q}(\lambda_{i+1}^0, \tau_j) \bar{\Upsilon}_0 (\beta_{i+2}^0 - \beta_i^0) \right\} = K_1, \text{ say.} \quad (60)
\end{aligned}
$$

This expression can be used to construct an equivalent to (51), but with K_1, that together with (44) and (50) give that the limit of ξ_1 is

$$
\xi_1 \overset{p.u}{\to} (\lambda_i^0 - \tau_{j-1}) \Gamma_i + K_1' \bar{\Upsilon}_0' \tilde{Q}(\tau_{j-1}, \lambda_i^0) \bar{\Upsilon}_0 K_1 . \quad (61)
$$

Analogously, we have

$$\xi_2 \overset{p.u}{\to} (\lambda_{i+1}^0 - \lambda_i^0)\Gamma_{i+1} + K_2^{'}\bar{\Upsilon}_0^{\prime}\tilde{Q}(\lambda_i^0, \lambda_{i+1}^0)\bar{\Upsilon}_0 K_2 \tag{62}$$

with

$$K_2 = \left\{\bar{\Upsilon}_0^{\prime}\tilde{Q}(\tau_{j-1}, \tau_j)\bar{\Upsilon}_0\right\}^{-1}\left\{\bar{\Upsilon}_0^{\prime}\tilde{Q}(\tau_{j-1}, \lambda_i^0)\bar{\Upsilon}_0(\beta_i^0 - \beta_{i+1}^0) + \bar{\Upsilon}_0^{\prime}\tilde{Q}(\lambda_{i+1}^0, \tau_j)\bar{\Upsilon}_0(\beta_{i+2}^0 - \beta_{i+1}^0)\right\}$$

and,

$$\xi_3 \overset{p.u}{\to} (\tau_j - \lambda_{i+1}^0)\Gamma_{i+2} + K_3^{'}\bar{\Upsilon}_0^{\prime}\tilde{Q}(\lambda_{i+1}^0, \tau_j)\bar{\Upsilon}_0 K_3 \tag{63}$$

with

$$K_3 = \left\{\bar{\Upsilon}_0^{\prime}\tilde{Q}(\tau_{j-1}, \tau_j)\bar{\Upsilon}_0\right\}^{-1}\left\{\bar{\Upsilon}_0^{\prime}\tilde{Q}(\tau_{j-1}, \lambda_i^0)\bar{\Upsilon}_0(\beta_i^0 - \beta_{i+2}^0) + \bar{\Upsilon}_0^{\prime}\tilde{Q}(\lambda_i^0, \lambda_{i+1}^0)\bar{\Upsilon}_0(\beta_{i+1}^0 - \beta_{i+2}^0)\right\}.$$

Combining the above (61), (62), and (63) concludes that

$$T^{-1}RSS_j \overset{p.u}{\to} (\lambda_i^0 - \tau_{j-1})\Gamma_i + (\lambda_{i+1}^0 - \lambda_i^0)\Gamma_{i+1} + (\tau_j - \lambda_{i+1}^0)\Gamma_{i+2} + F_2 \tag{64}$$

where

$$\begin{aligned} F_2 \;\; = \;\; & K_1^{'}\bar{\Upsilon}_0^{\prime}\tilde{Q}(\tau_{j-1}, \lambda_i^0)\bar{\Upsilon}_0 K_1 \; + \; K_2^{'}\bar{\Upsilon}_0^{\prime}\tilde{Q}(\lambda_i^0, \lambda_{i+1}^0)\bar{\Upsilon}_0 K_2 \\ & + K_3^{'}\bar{\Upsilon}_0^{\prime}\tilde{Q}(\lambda_{i+1}^0, \tau_j)\bar{\Upsilon}_0 K_3 \; > \; 0. \end{aligned} \tag{65}$$

To see that F_2 is positive definite consider the following. Since $\tilde{Q}(r, s)$ is positive definite for any $r > s$ and $\bar{\Upsilon}_0$ is full rank, it suffices to show that not all K_1, K_2, and K_3 can be zero. By (60), K_1 is defined as the $plim\left(\hat{\beta}_j - \beta_i^0\right)$ and analogously K_2 and K_3 are $plim\left(\hat{\beta}_j - \beta_{i+1}^0\right)$, and $plim\left(\hat{\beta}_j - \beta_{i+2}^0\right)$ respectively. From the solution for, say K_1, given in (60), it can be deduced that there can exist a combination of break locations and parameter values for which K_1 is zero. The intuition for this is that $\hat{\beta}_j$, that is estimated over $t = [\tau_{j-1}T] + 1, \ldots, [\tau_j T]$, happens to converge to β_i^0. But if this is the case then at least one of K_2 and K_3 will be non zero since $\beta_i^0 \neq \beta_{i+1}^0 \neq \beta_{i+2}^0$ by the assumption that segment j has two neglected breaks. Therefore, the sum of terms involving those three in (65) will be strictly positive.

The same argument to the case of two neglected breaks in the segment extends to cases with more than two neglected breaks but the proofs are supressed here for brevity. Instead, we present the general form of RSS_j, for κ neglected breaks, that is

$$\begin{aligned} T^{-1}RSS_j(\tau(n); n, \hat{\pi}) \overset{p.u}{\to} & (\lambda_i^0 - \tau_{j-1})\Gamma_i \; + \; (\lambda_{i+1}^0 - \lambda_i^0)\Gamma_{i+1} \; + \ldots \\ & + (\lambda_{i+k}^0 - \lambda_{i+k-1}^0)\Gamma_{i+k} \; + \; (\tau_j - \lambda_{i+k}^0)\Gamma_{i+k+1} \; + \; F. \end{aligned}$$

where F is a positive definite matrix defined as,

$$F \;=\; K_1^{'}\bar{\Upsilon}_0^{'}\tilde{Q}(\tau_{j-1},\lambda_i^0)\bar{\Upsilon}_0 K_1 \;+K_2^{'}\bar{\Upsilon}_0^{'}\tilde{Q}(\lambda_i^0,\lambda_{i+1}^0)\bar{\Upsilon}_0 K_2 \;+\ldots+\; K_\kappa^{'}\bar{\Upsilon}_0^{'}\tilde{Q}(\lambda_i^0,\lambda_{i+1}^0)$$

where K_ς is defined as

$$\begin{aligned} K_\varsigma \;&=\; \left\{\bar{\Upsilon}_0^{'}\tilde{Q}(\tau_{j-1},\tau_j)\bar{\Upsilon}_0\right\}^{-1}\left\{\bar{\Upsilon}_0^{'}\tilde{Q}(\tau_{j-1},\lambda_i^0)\bar{\Upsilon}_0(\beta_i^0-\beta_{i+\varsigma-1}^0)\;+\;\bar{\Upsilon}_0^{'}\tilde{Q}(\lambda_i^0,\lambda_{i+1}^0)\bar{\Upsilon}_0(\beta_{i+1}^0-\beta_{i+\varsigma-1}^0)+\ldots\right.\\ &\quad \left.+\bar{\Upsilon}_0^{'}\tilde{Q}(\lambda_{i+\kappa-1}^0,\tau_j)\bar{\Upsilon}_0(\beta_{i+\kappa}^0-\beta_{i+\varsigma-1}^0)\right\} \end{aligned}$$

for $\varsigma = 1,2,\ldots,\kappa+1$.

Proof of Theorem 1

Lemma 1 can be used to establish a proof for the consistency of the information criterion $I(\tau(n);n,\hat{\pi})$ in selecting the true number of breaks. This can be achieved by considering the possible cases where the 2SLS procedure may over-fit, under-fit or correctly identify the true number of breaks (m) in the model. Firstly denote

$$\Gamma(\lambda^0,m,\beta^0) \;=\; \sum_{j=1}^{m+1}(\lambda_j-\lambda_{j-1})\left(\sigma_u^2 \;+\; 2\Sigma_{uv}\beta_j^0 \;+\; \beta_j^{0'}\Sigma_v\beta_j^0\right)$$

where with $\lambda^0 = (\lambda_1^0,\lambda_2^0,\ldots,\lambda_m^0)'$ and $\beta^0 = (\beta_1^{0'},\beta_2^{0'},\ldots,\beta_{m+1}^{0'})'$. $\Gamma(\lambda^0,m,\beta^0)$ is then the sum of the Γ_i (23) across all segments of the data. In the cases where there are neglected breaks in one or more segments, using the results of *Case* (*ii*) in (54) and (64) we can show that by adding the terms involving $\Gamma_i,\ i=1,2,\ldots,m+1$ across segments, the break fractions of the incorrectly estimated breaks (τ_j,τ_{j-1}) will cancel out and so the limit of all terms involving $Var[u_t,v_t^{'}|z_t]$ will be $\Gamma(\lambda^0,m,\beta^0)$. To illustrate this, consider the case where for only one segment j *s.t.* $[\tau_{j-1}T]+1,\ldots,[\tau_j T]$ there is one neglected break λ_j (as shown in (54)). Then,

$$\begin{aligned} \Gamma(\tau(n),n;\lambda^0,\beta^0) \quad &= \quad (\lambda_1^0-0)\Gamma_1 \;+\ldots+\; (\tau_{j-1}-\lambda_{j-1}^0)\Gamma_j \;+\; (\lambda_j^0-\tau_{j-1})\Gamma_j \;+(\tau_j-\lambda_j^0)\Gamma_{j+1} \;+\; F_1\\ &\qquad +(\lambda_{j+1}^0-\tau_j)\Gamma_{j+1} \;+\ldots+\; (1-\lambda_m^0)\Gamma_{m+1}\\ &= \quad (\lambda_1^0-\lambda_0^0)\Gamma_1 \;+\ldots+\; (\lambda_j^0-\lambda_{j-1}^0)\Gamma_j \;+\; (\lambda_{j+1}^0-\lambda_j^0)\Gamma_{j+1} \;+\ldots+\; (1-\lambda_m^0)\Gamma_{m+1} \;+\; F_1\\ &= \quad \Gamma(\lambda^0,m,\beta^0)+F_1 \end{aligned}$$

since the true vectors of coefficients β_j^0 are stable in each segment j.

Also, it follows directly from the analysis of $Case\ (ii)$ that a straight forward generalization to the case of a segment with more than two neglected breaks will result in a limit function with the basic characteristics of (64). Denote $F(\tau(n), \lambda^0)$ the collection of any terms of the form of F_1 (55), F_2 (65), or the equivalent of the general case of more than two neglected breaks, that will exist when any number of segments $[\tau_{j-1}T]+1, \ldots, [\tau_j T]$ include one, two, or more neglected breaks. As shown in Lemma 1(ii), all these terms will be strictly positive.

Then, the behaviour of the information criterion can be examined in the following cases:

(1) $if\ n = m$. The estimation procedure has identified the correct number of breaks. The following two scenarios are possible,

(1.1) $if\ \tau(n) = \lambda^0$ then there will not be neglected breaks in any segment and by $Case\ (i)$,

$$I\left(\tau(n); n, \hat{\pi}\right) \overset{p.u}{\to} \Gamma(\lambda^0, m, \beta^0)$$

(1.2) $if\ \tau(n) \neq \lambda^0$ then there must exist $j\ s.t.\ [\tau_{j-1}T]+1, \ldots, [\tau_j T]$ contains at least one neglected break, and therefore

$$I\left(\tau(n); n, \hat{\pi}\right) \overset{p.u}{\to} \Gamma(\lambda^0, m, \beta^0) \ + \ F(\tau(n), \lambda^0)$$

where $F(\tau(n), \lambda^0) \ > \ 0$

(2) $if\ n < m$. The estimation procedure has under-fitted the model. Then there must exist a segment $j\ s.t.\ [\tau_{j-1}T]+1, \ldots, [\tau_j T]$ contains at least one neglected break, and

$$I\left(\tau(n); n, \hat{\pi}\right) \overset{p.u}{\to} \Gamma(\lambda^0, m, \beta^0) \ + \ F(\tau(n), \lambda^0)$$

where $F(\tau(n), \lambda^0) \ > \ 0$

(3) $if\ n > m$. Then the following two scenarios are possible

(3.1) if $\tau(n)$ does not contain λ^0 then there must exist $j\ s.t.\ [\tau_{j-1}T]+1, \ldots, [\tau_j T]$ includes at least one λ_i^0 and

$$I\left(\tau(n); n, \hat{\pi}\right) \overset{p.u}{\to} \Gamma(\lambda^0, m, \beta^0) \ + \ F(\tau(n), \lambda^0)$$

where $F(\tau(n), m) \ > \ 0$

(3.2) if $\tau(n)$ contains λ^0 consider

$$D_T \ = \ \left\{I\left(\tau(n); n, \hat{\pi}\right) - I\left(\lambda^0; m\right)\right\}$$

and

$$D_T = Tln\left\{\hat{\Gamma}(\tau(n); n, \hat{\beta})/\hat{\Gamma}(\lambda^0; m, \beta^0)\right\} + T\{K(q,n,T) - K(q,m,T)\}$$
$$= -QLR_T + T\{K(q,n,T) - K(q,m,T)\}$$

where QLR_T is the quasi likelihood ratio test for $H_0 : \tau(n) = \lambda^0$ which is a nested test as $\tau(n) \in \lambda^0$. Under its H_0 by standard arguments $QLR_T = Op(1)$, and since $T\{K(q,n,T) - K(q,m,T)\} \overset{p.u}{\to} +\infty$ it follows that

$$D_T \to \infty.$$

Taken together, cases (1), (2), and (3) imply desired result.

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Table 1: Simulation cases

Case	h	m	π_1	π_2	λ_1	λ_2
1	0	0	-	-	-	-
2	0	1	-	-	0.5	-
3	0	2	-	-	0.3	0.6
4	1	0	0.5	-	-	-
5	1	1	0.5	-	0.5	-
6	1	2	0.5	-	0.3	0.6
7	1	1	0.3	-	0.6	-
8	2	1	0.3	0.6	0.5	-
9	2	2	0.3	0.6	0.2	0.4

Notes: h: number of breaks in the reduced form; m: number of breaks in the structural form; π_1, π_2: locations of reduced form breaks (as fractions of sample size); λ_1, λ_2: locations of structural form breaks;

Table 2: Unknown reduced form breaks, i.i.d. errors, T=120, $\beta_2 = 0.1$

			BIC				SBBIC				HQ				SBHQ				AIC				SBAIC			
Case		R^2	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3
1	RF	0.3	99.80	0.20	0.00	0.00	100.00	0.00	0.00	0.00	93.10	5.45	1.30	0.15	99.35	0.60	0.05	0.00	0.00	0.10	0.05	99.85	0.95	1.35	3.10	94.60
(h=0,m=0)		0.5	99.80	0.20	0.00	0.00	100.00	0.00	0.00	0.00	93.10	5.45	1.30	0.15	99.35	0.60	0.05	0.00	0.00	0.10	0.05	99.85	0.95	1.35	3.10	94.60
	SF	0.3	98.10	1.85	0.05	0.00	**100.00**	0.00	0.00	0.00	84.70	10.30	4.10	0.90	99.05	0.95	0.00	0.00	1.00	1.40	6.35	91.25	29.70	14.00	19.60	36.70
		0.5	98.00	1.95	0.05	0.00	**100.00**	0.00	0.00	0.00	84.75	10.05	4.25	0.95	99.15	0.85	0.00	0.00	0.75	1.05	6.30	91.90	28.15	13.20	20.90	37.75
2	SF	0.3	0.55	96.70	2.65	0.10	10.05	89.95	0.00	0.00	0.10	80.55	15.15	4.20	0.90	**97.10**	1.90	0.10	0.00	1.15	4.20	94.65	0.00	20.50	24.15	55.35
(h=0,m=1)		0.5	0.50	96.50	2.85	0.15	9.00	91.00	0.00	0.00	0.20	79.95	15.40	4.45	0.70	**97.35**	1.85	0.10	0.00	1.15	3.80	95.05	0.00	20.55	23.45	56.00
3	SF	0.3	26.75	4.75	66.40	2.10	89.55	1.90	8.50	0.05	2.05	2.25	**78.25**	17.45	37.65	4.55	56.80	1.00	0.00	0.00	2.65	97.35	0.00	0.05	25.05	74.90
(h=0,m=2)		0.5	25.60	4.55	67.60	2.25	89.00	1.85	9.10	0.05	1.90	2.00	**78.10**	18.00	36.20	4.65	58.00	1.15	0.00	0.00	2.20	97.80	0.00	0.05	24.85	75.10
4	RF	0.3	0.00	99.60	0.40	0.00	0.80	99.20	0.00	0.00	0.00	88.85	8.95	2.20	0.00	99.00	1.00	0.00	0.00	0.05	0.20	99.75	0.00	1.20	2.25	96.55
(n=1,m=0)		0.5	0.00	99.70	0.30	0.00	0.00	100.00	0.00	0.00	0.00	89.15	8.70	2.15	0.00	99.15	0.85	0.00	0.00	0.05	0.15	99.80	0.00	1.40	2.35	96.25
	SF	0.3	98.80	0.95	0.25	0.00	**100.00**	0.00	0.00	0.00	84.50	10.85	3.85	0.80	99.40	0.45	0.15	0.00	1.05	1.75	6.60	90.60	21.90	18.70	21.15	38.25
		0.5	98.75	1.05	0.20	0.00	**100.00**	0.00	0.00	0.00	85.50	9.70	3.75	1.05	99.25	0.65	0.10	0.00	0.90	1.55	6.50	91.05	25.50	15.10	20.20	39.20
5	SF	0.3	20.60	**76.90**	2.35	0.15	70.65	29.35	0.00	0.00	3.90	76.75	15.40	3.95	27.50	71.00	1.45	0.05	0.00	1.25	4.10	94.65	1.05	22.00	24.85	52.10
(h=1,m=1)		0.5	6.90	**90.35**	2.55	0.20	44.40	55.50	0.10	0.00	0.95	79.20	15.45	4.40	9.80	88.50	1.65	0.05	0.00	1.45	3.50	95.05	0.25	21.95	23.90	53.90
6	SF	0.3	36.75	9.25	52.20	1.80	92.05	3.45	4.50	0.00	3.95	5.90	**72.25**	17.90	48.20	9.45	41.25	1.10	0.00	0.00	2.30	97.70	0.00	0.10	25.25	74.65
(h=1,m=2)		0.5	36.50	7.30	54.50	1.70	92.60	2.55	4.85	0.00	3.75	4.15	**73.85**	18.25	48.90	6.75	43.25	1.10	0.00	0.00	1.85	98.15	0.00	0.15	24.75	75.10
7	RF	0.3	1.05	98.55	0.40	0.00	4.95	95.05	0.00	0.00	0.05	88.75	9.25	1.95	0.45	97.90	1.60	0.05	0.00	0.00	0.05	99.95	0.00	1.20	2.10	96.70
(h=1,m=1)		0.5	0.00	99.60	0.40	0.00	0.00	100.00	0.00	0.00	0.00	89.20	9.15	1.65	0.00	98.55	1.45	0.00	0.00	0.00	0.10	99.90	0.00	1.40	2.55	96.05
	SF	0.3	10.55	**86.75**	2.65	0.05	52.10	47.90	0.00	0.00	1.65	77.70	16.15	4.50	15.05	83.70	1.25	0.00	0.00	1.20	3.90	94.90	0.05	16.60	25.95	57.40
		0.5	6.45	**90.80**	2.65	0.10	41.50	58.50	0.00	0.00	0.85	78.85	15.75	4.55	9.60	89.05	1.35	0.00	0.00	1.50	3.35	95.15	0.05	18.25	25.65	56.05
8	RF	0.3	23.55	1.25	75.05	0.15	71.80	0.80	27.40	0.00	0.90	0.25	83.80	15.05	9.20	0.90	88.15	1.75	0.00	0.00	0.05	99.95	0.00	0.00	0.95	99.05
(h=2,m=1)		0.5	0.65	0.00	98.80	0.55	9.35	0.05	90.60	0.00	0.00	0.00	85.40	14.60	0.15	0.00	98.05	1.80	0.00	0.00	0.05	99.95	0.00	0.00	1.35	98.65
	SF	0.3	1.15	96.35	2.45	0.05	14.90	85.10	0.00	0.00	0.05	79.20	16.95	3.80	1.45	**96.80**	1.75	0.00	0.00	0.80	3.40	95.80	0.00	16.40	24.50	59.10
		0.5	0.95	96.25	2.75	0.05	13.10	86.90	0.00	0.00	0.10	80.15	16.25	3.50	1.50	**97.10**	1.40	0.00	0.00	0.90	4.15	94.95	0.00	17.45	24.60	57.95
9	SF	0.3	62.05	10.70	26.50	0.75	97.95	1.40	0.65	0.00	16.85	10.65	**59.10**	13.40	74.00	8.30	17.50	0.20	0.00	0.00	1.45	98.55	0.35	0.75	21.90	77.00
(h=2,m=2)		0.5	59.60	9.35	30.10	0.95	97.90	1.25	0.85	0.00	15.80	9.10	**62.40**	12.70	70.95	7.70	21.10	0.25	0.00	0.00	2.05	97.95	0.25	0.55	25.00	74.20

Notes: Rows report the percentage of estimated breaks for each information criterion out of 2000 iterations and across the 9 cases defined in Table 1. The reduced form breaks estimation is denoted RF and is given once before every structural form case (SF) that has the same reduced form. R^2 refers to the theoretical R^2 in the reduced form DGP. The IC that yields the highest rejection frequency for the true SF break is marked in bold.

Table 3: Unknown reduced form breaks, AR(1) errors, T=120, $\beta_2 = 0.1$

Case			BIC				SBBIC				HQ				SBHQ				AIC				SBAIC			
Case		R^2	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3
1	RF	0.3	99.90	0.10	0.00	0.00	100.00	0.00	0.00	0.00	92.80	6.10	0.85	0.25	99.50	0.50	0.00	0.00	0.00	0.00	0.10	99.90	1.05	1.00	2.60	95.35
(h=0,m=0)		0.5	99.90	0.10	0.00	0.00	100.00	0.00	0.00	0.00	92.80	6.10	0.85	0.25	99.50	0.50	0.00	0.00	0.00	0.00	0.10	99.90	1.05	1.00	2.60	95.35
	SF	0.3	58.75	18.90	14.15	8.20	**93.15**	5.80	1.00	0.05	17.35	14.55	22.55	45.55	67.95	17.20	10.00	4.85	0.05	0.00	0.10	99.85	0.90	1.10	4.30	93.70
		0.5	58.55	18.85	13.90	8.70	**93.10**	5.85	1.00	0.05	17.50	14.15	22.35	46.00	68.20	17.15	9.90	4.75	0.00	0.00	0.15	99.85	0.75	0.75	4.80	93.70
2	SF	0.3	1.45	57.60	24.85	16.10	12.55	**79.75**	7.10	0.60	0.25	19.40	22.35	58.00	2.30	64.80	22.30	10.60	0.00	0.00	0.15	99.85	0.00	0.65	2.80	96.55
(h=0,m=1)		0.5	1.40	56.95	25.50	16.15	12.10	**80.35**	6.90	0.65	0.25	19.60	22.70	57.45	2.10	65.00	22.55	10.35	0.00	0.00	0.05	99.95	0.00	0.70	2.35	96.95
3	SF	0.3	10.00	7.50	54.00	28.50	53.75	11.00	32.65	2.60	0.85	2.30	27.85	69.00	14.30	9.25	**54.70**	21.75	0.00	0.00	0.05	99.95	0.00	0.00	1.40	98.60
(h=0,m=2)		0.5	9.25	7.55	54.45	28.75	53.65	10.70	33.25	2.40	0.75	2.15	27.25	69.85	13.80	9.20	**54.95**	22.05	0.00	0.00	0.05	99.95	0.00	0.05	1.40	98.55
4	RF	0.3	0.25	99.45	0.30	0.00	2.45	97.55	0.00	0.00	0.00	87.75	9.85	2.40	0.05	98.65	1.25	0.05	0.00	0.00	0.00	100.00	0.00	0.90	2.25	96.85
(n=1,m=0)		0.5	0.00	99.70	0.30	0.00	0.05	99.95	0.00	0.00	0.00	88.55	9.15	2.30	0.00	98.85	1.10	0.05	0.00	0.00	0.05	99.95	0.00	0.85	2.45	96.70
	SF	0.3	60.45	19.30	12.70	7.55	**94.00**	5.00	0.95	0.05	17.85	15.30	22.60	44.25	68.25	18.05	9.50	4.20	0.00	0.00	0.05	99.95	0.75	1.10	4.40	93.75
		0.5	60.85	18.00	13.05	8.10	**94.10**	4.95	0.90	0.05	17.90	14.35	22.55	45.20	68.75	16.70	9.95	4.60	0.00	0.00	0.20	99.80	0.50	0.90	4.35	94.25
5	SF	0.3	10.90	51.30	23.20	14.60	47.85	47.05	4.85	0.25	1.10	20.95	22.05	55.90	14.85	**54.75**	20.75	9.65	0.00	0.00	0.00	100.00	0.05	0.65	3.20	96.10
(h=1,m=1)		0.5	5.65	55.45	23.95	14.95	31.55	**62.65**	5.60	0.20	0.75	20.80	21.75	56.70	8.10	60.70	21.00	10.20	0.00	0.00	0.00	100.00	0.05	0.40	2.65	96.90
6	SF	0.3	11.60	13.00	48.00	27.40	56.60	15.45	25.70	2.25	1.15	3.80	27.40	67.65	16.25	13.70	**50.15**	19.90	0.00	0.00	0.00	100.00	0.00	0.00	1.60	98.40
(h=1,m=2)		0.5	12.25	10.90	49.50	27.35	58.05	13.20	26.55	2.20	1.05	3.35	27.80	67.80	17.20	12.10	**50.20**	20.50	0.00	0.00	0.00	100.00	0.00	0.05	1.70	98.25
7	RF	0.3	2.30	97.20	0.50	0.00	9.40	90.60	0.00	0.00	0.05	87.60	10.25	2.10	1.00	97.75	1.25	0.00	0.00	0.00	0.00	100.00	0.00	0.90	2.30	96.80
(h=1,m=1)		0.5	0.00	99.75	0.25	0.00	0.05	99.95	0.00	0.00	0.00	88.35	10.00	1.65	0.00	98.65	1.30	0.05	0.00	0.05	0.05	99.90	0.00	0.95	2.35	96.70
	SF	0.3	7.65	51.85	25.30	15.20	35.95	**58.10**	5.40	0.55	1.00	17.90	24.40	56.70	10.70	56.85	23.25	9.20	0.00	0.00	0.00	100.00	0.00	0.45	2.75	96.80
		0.5	5.80	54.30	25.30	14.60	30.00	**63.80**	5.65	0.55	0.75	18.60	23.60	57.05	7.75	60.25	22.90	9.10	0.00	0.00	0.00	100.00	0.00	0.60	2.00	97.40
8	RF	0.3	29.95	2.35	67.50	0.20	75.25	1.65	23.10	0.00	2.25	0.50	80.60	16.65	13.80	2.25	82.35	1.60	0.00	0.00	0.05	99.95	0.00	0.00	0.80	99.20
(h=2,m=1)		0.5	2.20	0.15	97.30	0.35	15.55	0.40	84.05	0.00	0.00	0.00	83.55	16.45	0.75	0.15	97.15	1.95	0.00	0.00	0.05	99.95	0.00	0.00	1.35	98.65
	SF	0.3	1.70	57.85	25.80	14.65	14.50	**78.85**	6.10	0.55	0.30	17.50	24.60	57.60	2.50	63.95	23.60	9.95	0.00	0.00	0.05	99.95	0.00	0.65	2.60	96.75
		0.5	1.70	57.10	26.05	15.15	13.85	**79.20**	6.55	0.40	0.35	18.45	24.95	56.25	2.45	64.80	22.75	10.00	0.00	0.00	0.05	99.95	0.00	0.60	2.60	96.80
9	SF	0.3	22.70	15.80	**38.15**	23.35	71.40	14.05	12.85	1.70	3.30	5.55	23.75	67.40	29.35	17.40	36.05	17.20	0.00	0.00	0.00	100.00	0.05	0.00	2.05	97.90
(h=2,m=2)		0.5	22.75	14.35	**39.80**	23.10	71.80	13.20	13.35	1.65	3.35	4.75	25.35	66.55	28.60	15.85	39.00	16.55	0.00	0.00	0.00	100.00	0.05	0.10	1.80	98.05

Notes: Same as Table 2.

Table 4: Unknown reduced form breaks, i.i.d. errors, T=240, $\beta_2 = 0.1$

			BIC				SBBIC				HQ				SBHQ				AIC				SBAIC			
Case		R^2	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3
1	RF	0.3	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	97.30	2.20	0.45	0.05	99.95	0.05	0.00	0.00	0.00	0.05	0.35	99.60	2.40	2.75	7.40	87.45
(h=0,m=0)		0.5	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	97.30	2.20	0.45	0.05	99.95	0.05	0.00	0.00	0.00	0.05	0.35	99.60	2.40	2.75	7.40	87.45
	SF	0.3	99.30	0.70	0.00	0.00	**100.00**	0.00	0.00	0.00	88.70	8.30	2.45	0.55	99.30	0.70	0.00	0.00	1.25	1.35	6.45	90.95	35.70	10.95	19.90	33.45
		0.5	99.15	0.85	0.00	0.00	**100.00**	0.00	0.00	0.00	88.95	8.15	2.50	0.40	99.30	0.70	0.00	0.00	1.00	1.30	5.80	91.90	32.00	11.55	20.55	35.90
2	SF	0.3	0.00	98.50	1.45	0.05	0.05	**99.95**	0.00	0.00	0.00	85.15	12.80	2.05	0.00	98.90	1.05	0.05	0.00	0.80	4.75	94.45	0.00	21.80	25.60	52.60
(h=0,m=1)		0.5	0.00	98.45	1.50	0.05	0.00	**100.00**	0.00	0.00	0.00	85.10	12.65	2.25	0.00	98.90	1.05	0.05	0.00	0.85	3.80	95.35	0.00	20.90	25.25	53.85
3	SF	0.3	0.75	0.25	**97.80**	1.20	31.60	0.90	67.50	0.00	0.00	0.00	86.00	14.00	1.15	0.35	97.50	1.00	0.00	0.00	2.95	97.05	0.00	0.00	27.60	72.40
(h=0,m=2)		0.5	0.60	0.15	**98.10**	1.15	29.00	0.85	70.15	0.00	0.00	0.00	85.30	14.70	0.90	0.20	97.95	0.95	0.00	0.00	2.80	97.20	0.00	0.00	26.60	73.40
4	RF	0.3	0.00	99.90	0.10	0.00	0.00	100.00	0.00	0.00	0.00	95.95	3.60	0.45	0.00	99.55	0.45	0.00	0.00	0.10	0.35	99.55	0.00	2.70	6.95	90.35
(n=1,m=0)		0.5	0.00	99.90	0.10	0.00	0.00	100.00	0.00	0.00	0.00	96.30	3.30	0.40	0.00	99.60	0.40	0.00	0.00	0.10	0.40	99.50	0.00	3.00	7.20	89.80
	SF	0.3	99.35	0.65	0.00	0.00	**100.00**	0.00	0.00	0.00	89.70	7.60	2.25	0.45	99.40	0.60	0.00	0.00	0.55	1.45	6.65	91.35	25.50	14.25	24.60	35.65
		0.5	99.25	0.70	0.05	0.00	**100.00**	0.00	0.00	0.00	90.30	7.10	2.30	0.30	99.50	0.50	0.00	0.00	1.00	1.55	5.80	91.65	28.85	12.00	23.20	35.95
5	SF	0.3	0.70	98.00	1.20	0.10	14.85	85.15	0.00	0.00	0.00	84.60	13.10	2.30	0.90	**98.00**	1.00	0.10	0.00	1.05	3.70	95.25	0.00	20.75	25.50	53.75
(h=1,m=1)		0.5	0.00	98.85	1.05	0.10	2.40	97.60	0.00	0.00	0.00	85.00	13.00	2.00	0.00	**99.00**	0.90	0.10	0.00	1.00	4.05	94.95	0.00	21.60	24.10	54.30
6	SF	0.3	2.60	1.65	**94.70**	1.05	48.95	5.95	45.10	0.00	0.00	0.10	85.90	14.00	2.95	2.10	94.25	0.70	0.00	0.00	2.50	97.50	0.00	0.00	25.25	74.75
(h=1,m=2)		0.5	2.05	0.80	**96.15**	1.00	47.40	2.85	49.75	0.00	0.00	0.00	85.80	14.20	2.85	1.05	95.35	0.75	0.00	0.00	2.40	97.60	0.00	0.00	25.50	74.50
7	RF	0.3	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	96.30	3.25	0.45	0.00	99.65	0.35	0.00	0.00	0.10	0.40	99.50	0.00	3.55	6.90	89.55
(h=1,m=1)		0.5	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	96.55	3.10	0.35	0.00	99.75	0.25	0.00	0.00	0.15	0.40	99.45	0.00	3.80	7.55	88.65
	SF	0.3	0.10	98.55	1.35	0.00	5.80	94.20	0.00	0.00	0.00	85.50	11.75	2.75	0.15	**98.65**	1.20	0.00	0.00	0.95	3.55	95.50	0.00	17.65	24.25	58.10
		0.5	0.05	98.55	1.40	0.00	3.20	96.80	0.00	0.00	0.00	85.55	11.95	2.50	0.05	**98.75**	1.20	0.00	0.00	0.85	3.90	95.25	0.00	19.90	23.30	56.80
8	RF	0.3	0.10	0.05	99.65	0.20	4.75	0.00	95.25	0.00	0.00	0.00	94.90	5.10	0.00	0.00	99.35	0.65	0.00	0.00	0.15	99.85	0.00	0.00	3.65	96.35
(h=2,m=1)		0.5	0.00	0.00	99.85	0.15	0.00	0.00	100.00	0.00	0.00	0.00	95.40	4.60	0.00	0.00	99.35	0.65	0.00	0.00	0.20	99.80	0.00	0.00	4.60	95.40
	SF	0.3	0.00	98.55	1.35	0.10	0.25	**99.65**	0.10	0.00	0.00	85.30	12.25	2.45	0.00	98.80	1.15	0.05	0.00	0.75	3.90	95.35	0.00	16.20	26.30	57.50
		0.5	0.00	98.70	1.20	0.10	0.20	**99.75**	0.05	0.00	0.00	85.85	11.85	2.30	0.00	98.80	1.15	0.05	0.00	1.05	4.00	94.95	0.00	18.70	24.40	56.90
9	SF	0.3	19.80	8.15	71.25	0.80	85.80	3.70	10.50	0.00	0.95	1.75	**85.70**	11.60	23.00	8.50	67.85	0.65	0.00	0.00	1.90	98.10	0.00	0.00	25.40	74.60
(h=2,m=2)		0.5	17.00	6.15	76.20	0.65	83.00	3.35	13.65	0.00	0.75	1.30	**87.10**	10.85	20.35	6.30	72.85	0.50	0.00	0.00	2.35	97.65	0.00	0.00	28.45	71.55

Notes: Same as Table 2.

Table 5: Unknown reduced form breaks, AR(1) errors, T=240, $\beta_2 = 0.1$

Case		R^2	BIC 0	BIC 1	BIC 2	BIC ≥ 3	SBBIC 0	SBBIC 1	SBBIC 2	SBBIC ≥ 3	HQ 0	HQ 1	HQ 2	HQ ≥ 3	SBHQ 0	SBHQ 1	SBHQ 2	SBHQ ≥ 3	AIC 0	AIC 1	AIC 2	AIC ≥ 3	SBAIC 0	SBAIC 1	SBAIC 2	SBAIC ≥ 3
1	RF	0.3	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	97.05	2.45	0.50	0.00	99.75	0.25	0.00	0.00	0.05	0.15	0.80	99.00	2.35	3.00	8.45	86.20
(h=0,m=0)		0.5	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	97.05	2.45	0.50	0.00	99.75	0.25	0.00	0.00	0.05	0.15	0.80	99.00	2.35	3.00	8.45	86.20
	SF	0.3	66.05	18.40	10.70	4.85	**95.90**	3.60	0.50	0.00	17.60	15.20	24.65	42.55	69.20	17.55	9.70	3.55	0.00	0.00	0.10	99.90	0.65	0.30	3.10	95.95
		0.5	66.30	18.10	10.80	4.80	**95.65**	3.80	0.55	0.00	17.65	15.20	24.85	42.30	69.25	17.50	9.70	3.55	0.00	0.00	0.05	99.95	0.45	0.50	2.70	96.35
2	SF	0.3	0.05	64.65	22.80	12.50	1.20	**94.20**	4.30	0.30	0.00	18.45	23.30	58.25	0.05	68.10	21.45	10.40	0.00	0.05	0.00	99.95	0.00	0.25	1.90	97.85
(h=0,m=1)		0.5	0.05	64.55	22.90	12.50	1.10	**94.25**	4.35	0.30	0.00	18.90	22.80	58.30	0.05	67.80	21.75	10.40	0.00	0.05	0.00	99.95	0.00	0.30	1.50	98.20
3	SF	0.3	1.40	1.45	67.95	29.20	18.20	5.75	**73.00**	3.05	0.00	0.10	24.95	74.95	1.80	1.50	70.10	26.60	0.00	0.00	0.00	100.00	0.00	0.00	1.05	98.95
(h=0,m=2)		0.5	1.50	1.45	67.90	29.15	17.25	5.65	**73.95**	3.15	0.00	0.05	24.60	75.35	1.70	1.55	70.70	26.05	0.00	0.00	0.00	100.00	0.00	0.00	1.00	99.00
4	RF	0.3	0.00	99.95	0.05	0.00	0.00	100.00	0.00	0.00	0.00	95.75	3.90	0.35	0.00	99.40	0.60	0.00	0.00	0.00	0.40	99.60	0.00	2.60	5.65	91.75
(n=1,m=0)		0.5	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	95.90	3.70	0.40	0.00	99.50	0.50	0.00	0.00	0.05	0.40	99.55	0.00	3.00	6.35	90.65
	SF	0.3	68.25	16.05	10.80	4.90	**96.40**	3.05	0.50	0.05	19.15	14.40	24.25	42.20	71.40	15.85	9.05	3.70	0.00	0.00	0.00	100.00	0.25	0.50	3.15	96.10
		0.5	68.25	15.90	10.90	4.95	**96.40**	3.15	0.45	0.00	19.65	13.85	24.55	41.95	71.90	15.15	9.05	3.90	0.00	0.00	0.00	100.00	0.35	0.50	2.85	96.30
5	SF	0.3	1.65	63.40	22.40	12.55	16.95	**78.60**	4.25	0.20	0.15	18.35	23.75	57.75	2.00	66.15	21.45	10.40	0.00	0.00	0.00	100.00	0.05	0.15	2.00	97.80
(h=1,m=1)		0.5	0.55	64.45	22.20	12.80	6.35	**89.00**	4.45	0.20	0.05	18.75	23.20	58.00	0.70	67.35	21.10	10.85	0.00	0.00	0.00	100.00	0.00	0.05	1.90	98.05
6	SF	0.3	2.05	3.80	66.90	27.25	25.35	11.05	60.50	3.10	0.20	0.45	26.00	73.35	2.50	4.00	**69.15**	24.35	0.00	0.00	0.00	100.00	0.00	0.00	0.95	99.05
(h=1,m=2)		0.5	2.20	2.80	67.25	27.75	25.65	8.10	63.15	3.10	0.20	0.20	26.90	72.70	2.60	3.30	**68.95**	25.15	0.00	0.00	0.00	100.00	0.00	0.00	0.75	99.25
7	RF	0.3	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	96.15	3.30	0.55	0.00	99.55	0.45	0.00	0.00	0.10	0.45	99.45	0.00	2.85	5.75	91.40
(h=1,m=1)		0.5	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	96.45	3.00	0.55	0.00	99.65	0.35	0.00	0.00	0.15	0.65	99.20	0.00	3.50	6.50	90.00
	SF	0.3	0.95	63.95	23.70	11.40	10.15	**85.05**	4.45	0.35	0.05	20.30	24.90	54.75	1.30	66.45	22.95	9.30	0.00	0.00	0.05	99.95	0.00	0.30	2.15	97.55
		0.5	0.60	64.50	22.95	11.95	6.85	**88.85**	3.95	0.35	0.05	20.70	24.70	54.55	0.75	67.80	21.80	9.65	0.00	0.00	0.00	100.00	0.00	0.30	2.00	97.70
8	RF	0.3	0.45	0.05	99.40	0.10	8.45	0.15	91.40	0.00	0.00	0.00	95.05	4.95	0.00	0.00	99.25	0.75	0.00	0.00	0.15	99.85	0.00	0.00	3.25	96.75
(h=2,m=1)		0.5	0.00	0.00	99.95	0.05	0.00	0.00	100.00	0.00	0.00	0.00	95.20	4.80	0.00	0.00	99.35	0.65	0.00	0.00	0.30	99.70	0.00	0.00	4.45	95.55
	SF	0.3	0.10	64.90	24.35	10.65	1.85	**93.10**	4.75	0.30	0.00	18.45	25.70	55.85	0.10	68.25	22.60	9.05	0.00	0.00	0.00	100.00	0.00	0.15	1.45	98.40
		0.5	0.10	64.85	24.15	10.90	2.00	**93.05**	4.75	0.20	0.00	18.70	25.15	56.15	0.15	68.15	22.55	9.15	0.00	0.00	0.05	99.95	0.00	0.20	1.15	98.65
9	SF	0.3	9.10	10.05	57.10	23.75	51.30	15.40	32.35	0.95	0.50	1.60	26.15	71.75	10.80	10.80	**57.90**	20.50	0.00	0.00	0.00	100.00	0.00	0.05	0.60	99.35
(h=2,m=2)		0.5	8.40	8.75	58.85	24.00	50.55	13.50	34.50	1.45	0.55	1.30	26.55	71.60	9.85	9.15	**61.00**	20.00	0.00	0.00	0.00	100.00	0.00	0.05	0.85	99.10

Notes: Same as Table 2.

Table 6: Unknown reduced form breaks, i.i.d. errors, T=120, $\beta_2 = 1$

Case		R^2	BIC 0	BIC 1	BIC 2	BIC ≥ 3	SBBIC 0	SBBIC 1	SBBIC 2	SBBIC ≥ 3	HQ 0	HQ 1	HQ 2	HQ ≥ 3	SBHQ 0	SBHQ 1	SBHQ 2	SBHQ ≥ 3	AIC 0	AIC 1	AIC 2	AIC ≥ 3	SBAIC 0	SBAIC 1	SBAIC 2	SBAIC ≥ 3
1	RF	0.3	99.80	0.20	0.00	0.00	100.00	0.00	0.00	0.00	93.10	5.45	1.30	0.15	99.35	0.60	0.05	0.00	0.00	0.10	0.05	99.85	0.95	1.35	3.10	94.60
(h=0,m=0)		0.5	99.80	0.20	0.00	0.00	100.00	0.00	0.00	0.00	93.10	5.45	1.30	0.15	99.35	0.60	0.05	0.00	0.00	0.10	0.05	99.85	0.95	1.35	3.10	94.60
	SF	0.3	98.45	1.35	0.20	0.00	**100.00**	0.00	0.00	0.00	86.90	9.05	3.05	1.00	99.15	0.75	0.10	0.00	7.85	4.25	14.85	73.05	69.55	8.15	11.75	10.55
		0.5	98.50	1.30	0.20	0.00	**100.00**	0.00	0.00	0.00	87.15	8.95	2.90	1.00	99.10	0.80	0.10	0.00	7.75	4.05	14.50	73.70	68.80	8.25	12.15	10.80
2	SF	0.3	0.00	89.75	9.35	0.90	0.00	**99.40**	0.50	0.10	0.00	65.35	22.70	11.95	0.00	92.80	6.80	0.40	0.00	4.05	7.60	88.35	0.00	34.25	21.50	44.25
(h=0,m=1)		0.5	0.00	89.70	9.30	1.00	0.00	**99.35**	0.60	0.05	0.00	65.45	22.45	12.10	0.00	92.65	6.90	0.45	0.00	4.05	7.80	88.15	0.00	35.20	22.05	42.75
3	SF	0.3	0.25	0.30	94.00	5.45	23.75	1.65	74.50	0.10	0.00	0.00	73.60	26.40	0.55	0.50	**95.70**	3.25	0.00	0.00	7.75	92.25	0.00	0.00	42.05	57.95
(h=0,m=2)		0.5	0.00	0.00	94.55	5.45	3.45	0.65	95.75	0.15	0.00	0.00	73.40	26.60	0.05	0.05	**96.40**	3.50	0.00	0.00	7.40	92.60	0.00	0.00	41.75	58.25
4	RF	0.3	0.00	99.60	0.40	0.00	0.80	99.20	0.00	0.00	0.00	88.85	8.95	2.20	0.00	99.00	1.00	0.00	0.00	0.05	0.20	99.75	0.00	1.20	2.25	96.55
(n=1,m=0)		0.5	0.00	99.70	0.30	0.00	0.00	100.00	0.00	0.00	0.00	89.15	8.70	2.15	0.00	99.15	0.85	0.00	0.00	0.05	0.15	99.80	0.00	1.40	2.35	96.25
	SF	0.3	99.50	0.30	0.20	0.00	**100.00**	0.00	0.00	0.00	93.15	4.30	2.15	0.40	99.80	0.10	0.10	0.00	6.10	5.30	14.25	74.35	60.60	11.25	14.25	13.90
		0.5	99.65	0.25	0.10	0.00	**100.00**	0.00	0.00	0.00	93.50	4.00	1.95	0.55	99.75	0.15	0.10	0.00	7.70	4.45	12.90	74.95	65.55	9.40	12.70	12.35
5	SF	0.3	0.50	90.50	7.80	1.20	11.65	87.80	0.45	0.10	0.05	67.55	21.10	11.30	0.90	**93.20**	5.45	0.45	0.00	4.15	6.65	89.20	0.00	34.30	20.80	44.90
(h=1,m=1)		0.5	0.00	90.80	8.15	1.05	0.05	**99.35**	0.50	0.10	0.00	68.05	20.90	11.05	0.00	93.80	5.70	0.50	0.00	4.50	7.45	88.05	0.00	35.40	21.05	43.55
6	SF	0.3	3.25	9.70	**82.40**	4.65	50.20	17.10	32.70	0.00	0.20	1.95	72.25	25.60	6.60	12.05	78.50	2.85	0.00	0.00	6.80	93.20	0.05	0.00	39.65	60.30
(h=1,m=2)		0.5	0.20	0.15	93.85	5.80	14.15	3.10	82.65	0.10	0.00	0.00	73.35	26.65	0.40	0.35	**96.20**	3.05	0.00	0.00	7.55	92.45	0.00	0.00	40.55	59.45
7	RF	0.3	1.05	98.55	0.40	0.00	4.95	95.05	0.00	0.00	0.05	88.75	9.25	1.95	0.45	97.90	1.60	0.05	0.00	0.00	0.05	99.95	0.00	1.20	2.10	96.70
(h=1,m=1)		0.5	0.00	99.60	0.40	0.00	0.00	100.00	0.00	0.00	0.00	89.20	9.15	1.65	0.00	98.55	1.45	0.00	0.00	0.00	0.10	99.90	0.00	1.40	2.55	96.05
	SF	0.3	2.35	93.05	4.25	0.35	21.50	78.50	0.00	0.00	0.10	75.55	17.20	7.15	3.00	**94.40**	2.40	0.20	0.00	2.60	6.40	91.00	0.10	30.90	23.55	45.45
		0.5	0.00	96.30	3.25	0.45	0.20	**99.80**	0.00	0.00	0.00	76.80	14.85	8.35	0.00	97.65	2.10	0.25	0.00	3.00	6.55	90.45	0.00	34.35	21.95	43.70
8	RF	0.3	23.55	1.25	75.05	0.15	71.80	0.80	27.40	0.00	0.90	0.25	83.80	15.05	9.20	0.90	88.15	1.75	0.00	0.00	0.05	99.95	0.00	0.00	0.95	99.05
(h=2,m=1)		0.5	0.65	0.00	98.80	0.55	9.35	0.05	90.60	0.00	0.00	0.00	85.40	14.60	0.15	0.00	98.05	1.80	0.00	0.00	0.05	99.95	0.00	0.00	1.35	98.65
	SF	0.3	0.30	92.00	6.50	1.20	4.65	94.45	0.80	0.10	0.00	77.90	14.80	7.30	0.30	**96.25**	2.95	0.50	0.00	2.35	7.40	90.25	0.00	26.30	26.40	47.30
		0.5	0.00	95.10	4.25	0.65	0.35	**99.50**	0.10	0.05	0.00	79.55	13.20	7.25	0.00	97.30	2.40	0.30	0.00	3.00	7.40	89.60	0.00	30.20	23.45	46.35
9	SF	0.3	29.95	5.70	63.15	1.20	88.15	3.10	8.75	0.00	3.05	1.35	**82.20**	13.40	35.20	4.70	59.65	0.45	0.00	0.00	5.55	94.45	0.15	0.05	40.95	58.85
(h=2,m=2)		0.5	5.10	1.75	**91.70**	1.45	50.10	3.10	46.80	0.00	0.25	0.40	86.30	13.05	7.75	2.10	89.50	0.65	0.00	0.00	6.20	93.80	0.00	0.00	45.10	54.90

Notes: Same as Table 2.

Table 7: Unknown reduced form breaks, AR(1) errors, T=120, $\beta_2 = 1$

Case		R^2	BIC 0	BIC 1	BIC 2	BIC ≥ 3	SBBIC 0	SBBIC 1	SBBIC 2	SBBIC ≥ 3	HQ 0	HQ 1	HQ 2	HQ ≥ 3	SBHQ 0	SBHQ 1	SBHQ 2	SBHQ ≥ 3	AIC 0	AIC 1	AIC 2	AIC ≥ 3	SBAIC 0	SBAIC 1	SBAIC 2	SBAIC ≥ 3
1	RF	0.3	99.90	0.10	0.00	0.00	100.00	0.00	0.00	0.00	92.90	5.85	0.85	0.40	99.45	0.55	0.00	0.00	0.00	0.00	0.10	99.90	1.10	1.20	2.50	95.20
(h=0,m=0)		0.5	99.90	0.10	0.00	0.00	100.00	0.00	0.00	0.00	92.80	6.10	0.85	0.25	99.50	0.50	0.00	0.00	0.00	0.00	0.10	99.90	1.05	1.00	2.60	95.35
	SF	0.3	93.95	5.10	0.90	0.05	**99.85**	0.15	0.00	0.00	67.50	16.20	10.55	5.75	96.45	3.00	0.55	0.00	1.15	1.40	7.00	90.45	41.10	9.15	16.30	33.45
		0.5	85.05	11.20	3.20	0.55	**99.25**	0.55	0.20	0.00	47.00	18.40	18.50	16.10	89.10	8.65	1.95	0.30	0.20	0.50	2.50	96.80	22.80	7.05	15.15	55.00
2	SF	0.3	0.00	69.50	21.60	8.90	0.10	**94.15**	5.50	0.25	0.00	32.60	28.90	38.50	0.00	75.10	18.95	5.95	0.00	0.20	0.95	98.85	0.00	5.65	11.25	83.10
(h=0,m=1)		0.5	0.00	69.20	21.45	9.35	0.00	**94.10**	5.65	0.25	0.00	32.75	28.85	38.40	0.00	74.70	19.10	6.20	0.00	0.05	1.10	98.85	0.00	5.10	10.95	83.95
3	SF	0.3	0.60	1.35	76.25	21.80	20.80	5.20	71.45	2.55	0.05	0.00	44.70	55.25	1.40	1.90	**80.10**	16.60	0.00	0.00	0.40	99.60	0.00	0.00	9.95	90.05
(h=0,m=2)		0.5	0.10	0.20	76.20	23.50	6.30	1.70	**88.80**	3.20	0.00	0.05	43.50	56.45	0.15	0.30	81.55	18.00	0.00	0.00	0.60	99.40	0.00	0.00	10.00	90.00
4	RF	0.3	0.25	99.45	0.30	0.00	2.45	97.55	0.00	0.00	0.00	87.75	9.85	2.40	0.05	98.65	1.25	0.05	0.00	0.00	0.00	100.00	0.00	0.90	2.25	96.85
(n=1,m=0)		0.5	0.00	99.70	0.30	0.00	0.05	99.95	0.00	0.00	0.00	88.55	9.15	2.30	0.00	98.85	1.10	0.05	0.00	0.00	0.05	99.95	0.00	0.85	2.45	96.70
	SF	0.3	90.45	6.95	2.10	0.50	**99.65**	0.20	0.15	0.00	57.35	15.55	13.65	13.45	93.55	4.95	1.40	0.10	0.10	0.55	3.15	96.20	18.35	8.10	16.70	56.85
		0.5	91.20	6.20	2.25	0.35	**99.80**	0.10	0.10	0.00	58.50	13.50	14.35	13.65	94.10	4.45	1.40	0.05	0.05	0.55	2.05	97.35	18.40	7.35	16.20	58.05
5	SF	0.3	0.80	69.10	20.95	9.15	13.90	**81.20**	4.70	0.20	0.00	33.25	29.65	37.10	1.65	74.75	17.85	5.75	0.00	0.00	0.75	99.25	0.00	4.40	10.90	84.70
(h=1,m=1)		0.5	0.00	69.65	20.90	9.45	0.25	**93.90**	5.55	0.30	0.00	33.15	29.30	37.55	0.00	75.80	18.10	6.10	0.00	0.15	0.75	99.10	0.00	4.25	11.40	84.35
6	SF	0.3	4.00	9.15	66.85	20.00	42.20	19.15	37.20	1.45	0.15	2.00	42.35	55.50	6.45	11.10	**67.60**	14.85	0.00	0.00	0.60	99.40	0.00	0.00	8.85	91.15
(h=1,m=2)		0.5	0.45	0.85	75.10	23.60	15.50	5.30	76.90	2.30	0.00	0.05	42.20	57.75	0.85	1.45	**79.30**	18.40	0.00	0.00	0.75	99.25	0.00	0.00	9.00	91.00
7	RF	0.3	2.30	97.20	0.50	0.00	9.40	90.60	0.00	0.00	0.05	87.60	10.25	2.10	1.00	97.75	1.25	0.00	0.00	0.00	0.00	100.00	0.00	0.90	2.30	96.80
(h=1,m=1)		0.5	0.00	99.75	0.25	0.00	0.05	99.95	0.00	0.00	0.00	88.35	10.00	1.65	0.00	98.65	1.30	0.05	0.00	0.05	0.05	99.90	0.00	0.95	2.35	96.70
	SF	0.3	2.35	74.05	17.90	5.70	23.10	74.55	2.10	0.25	0.15	39.80	27.75	32.30	3.00	**79.30**	14.25	3.45	0.00	0.00	1.25	98.75	0.05	4.55	11.40	84.00
		0.5	0.00	76.00	17.25	6.75	0.40	**97.20**	2.20	0.20	0.00	40.10	24.80	35.10	0.00	83.25	13.05	3.70	0.00	0.05	0.70	99.25	0.00	4.75	9.80	85.45
8	RF	0.3	29.95	2.35	67.50	0.20	75.25	1.65	23.10	0.00	2.25	0.50	80.60	16.65	13.80	2.25	82.35	1.60	0.00	0.00	0.05	99.95	0.00	0.00	0.80	99.20
(h=2,m=1)		0.5	2.20	0.15	97.30	0.35	15.55	0.40	84.05	0.00	0.00	0.00	83.55	16.45	0.75	0.15	97.15	1.95	0.00	0.00	0.05	99.95	0.00	0.00	1.35	98.65
	SF	0.3	0.35	74.20	18.55	6.90	5.95	**90.00**	3.75	0.30	0.00	40.45	29.15	30.40	0.30	81.85	14.15	3.70	0.00	0.05	1.20	98.75	0.00	3.65	10.75	85.60
		0.5	0.00	75.35	18.00	6.65	0.85	**95.85**	3.20	0.10	0.00	42.90	27.20	29.90	0.00	81.95	14.35	3.70	0.00	0.10	0.70	99.20	0.00	3.30	11.10	85.60
9	SF	0.3	22.55	9.50	**56.65**	11.30	77.25	9.00	13.60	0.15	2.95	1.85	48.95	46.25	28.50	8.50	55.75	7.25	0.00	0.00	0.60	99.40	0.00	0.00	9.50	90.50
(h=2,m=2)		0.5	6.00	3.20	77.00	13.80	46.40	6.85	46.40	0.35	0.35	0.70	52.80	46.15	9.00	3.55	**78.80**	8.65	0.00	0.00	0.70	99.30	0.00	0.00	9.75	90.25

Notes: Same as Table 2.

Table 8: Unknown reduced form breaks, i.i.d. errors, T=240, $\beta_2 = 1$

Case		R^2	BIC				SBBIC				HQ				SBHQ				AIC				SBAIC			
			0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3
1	RF	0.3	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	97.30	2.20	0.45	0.05	99.95	0.05	0.00	0.00	0.00	0.05	0.35	99.60	2.40	2.75	7.40	87.45
(h=0,m=0)		0.5	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	97.30	2.20	0.45	0.05	99.95	0.05	0.00	0.00	0.00	0.05	0.35	99.60	2.40	2.75	7.40	87.45
	SF	0.3	99.10	0.85	0.05	0.00	**100.00**	0.00	0.00	0.00	89.25	8.35	2.00	0.40	99.30	0.70	0.00	0.00	10.30	4.00	14.50	71.20	82.45	4.40	7.60	5.55
		0.5	99.10	0.85	0.05	0.00	**100.00**	0.00	0.00	0.00	88.65	8.55	2.35	0.45	99.25	0.75	0.00	0.00	11.05	4.40	14.85	69.70	82.90	4.30	7.00	5.80
2	SF	0.3	0.00	94.45	5.00	0.55	0.00	**99.80**	0.20	0.00	0.00	71.90	19.65	8.45	0.00	95.35	4.15	0.50	0.00	4.30	7.65	88.05	0.00	37.00	21.05	41.95
(h=0,m=1)		0.5	0.00	94.45	4.95	0.60	0.00	**99.75**	0.25	0.00	0.00	71.80	19.45	8.75	0.00	95.25	4.30	0.45	0.00	4.15	8.40	87.45	0.00	37.55	20.70	41.75
3	SF	0.3	0.00	0.00	97.40	2.60	0.00	0.00	**99.85**	0.15	0.00	0.00	79.85	20.15	0.00	0.00	97.85	2.15	0.00	0.00	9.35	90.65	0.00	0.00	47.50	52.50
(h=0,m=2)		0.5	0.00	0.00	97.50	2.50	0.00	0.00	**99.85**	0.15	0.00	0.00	79.85	20.15	0.00	0.00	98.00	2.00	0.00	0.00	10.00	90.00	0.00	0.00	48.60	51.40
4	RF	0.3	0.00	99.90	0.10	0.00	0.00	100.00	0.00	0.00	0.00	95.95	3.60	0.45	0.00	99.55	0.45	0.00	0.00	0.10	0.35	99.55	0.00	2.70	6.95	90.35
(n=1,m=0)		0.5	0.00	99.90	0.10	0.00	0.00	100.00	0.00	0.00	0.00	96.30	3.30	0.40	0.00	99.60	0.40	0.00	0.00	0.10	0.40	99.50	0.00	3.00	7.20	89.80
	SF	0.3	99.65	0.35	0.00	0.00	**100.00**	0.00	0.00	0.00	95.20	3.30	1.25	0.25	99.70	0.30	0.00	0.00	7.75	4.00	15.45	72.80	76.15	6.65	9.20	8.00
		0.5	99.70	0.30	0.00	0.00	**100.00**	0.00	0.00	0.00	95.65	2.85	1.20	0.30	99.75	0.25	0.00	0.00	9.35	3.40	15.65	71.60	79.20	5.25	8.05	7.50
5	SF	0.3	0.00	94.65	4.80	0.55	0.05	**99.70**	0.25	0.00	0.00	71.70	19.25	9.05	0.00	95.15	4.40	0.45	0.00	4.05	7.70	88.25	0.00	36.85	18.95	44.20
(h=1,m=1)		0.5	0.00	94.20	5.20	0.60	0.00	**99.80**	0.20	0.00	0.00	72.55	18.45	9.00	0.00	95.00	4.50	0.50	0.00	5.15	8.05	86.80	0.00	37.45	19.05	43.50
6	SF	0.3	0.00	0.40	96.40	3.20	2.30	7.70	89.85	0.15	0.00	0.00	79.90	20.10	0.00	0.50	**96.80**	2.70	0.00	0.00	7.95	92.05	0.00	0.00	42.90	57.10
(h=1,m=2)		0.5	0.00	0.00	96.90	3.10	0.00	0.00	**99.85**	0.15	0.00	0.00	79.50	20.50	0.00	0.00	97.45	2.55	0.00	0.00	9.30	90.70	0.00	0.00	45.80	54.20
7	RF	0.3	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	96.30	3.25	0.45	0.00	99.65	0.35	0.00	0.00	0.10	0.40	99.50	0.00	3.55	6.90	89.55
(h=1,m=1)		0.5	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	96.55	3.10	0.35	0.00	99.75	0.25	0.00	0.00	0.15	0.40	99.45	0.00	3.80	7.55	88.65
	SF	0.3	0.00	97.85	2.00	0.15	0.45	**99.55**	0.00	0.00	0.00	82.65	11.80	5.55	0.00	98.30	1.55	0.15	0.00	3.25	7.05	89.70	0.00	33.35	21.35	45.30
		0.5	0.00	98.25	1.55	0.20	0.00	**100.00**	0.00	0.00	0.00	82.45	11.25	6.30	0.00	98.55	1.25	0.20	0.00	3.50	7.00	89.50	0.00	35.85	19.05	45.10
8	RF	0.3	0.10	0.05	99.65	0.20	4.75	0.00	95.25	0.00	0.00	0.00	94.90	5.10	0.00	0.00	99.35	0.65	0.00	0.00	0.15	99.85	0.00	0.00	3.65	96.35
(h=2,m=1)		0.5	0.00	0.00	99.85	0.15	0.00	0.00	100.00	0.00	0.00	0.00	95.40	4.60	0.00	0.00	99.35	0.65	0.00	0.00	0.20	99.80	0.00	0.00	4.60	95.40
	SF	0.3	0.00	97.40	2.25	0.35	0.00	**99.60**	0.40	0.00	0.00	79.95	13.70	6.35	0.00	98.25	1.55	0.20	0.00	2.55	7.45	90.00	0.00	28.95	26.25	44.80
		0.5	0.00	98.00	1.75	0.25	0.00	**99.90**	0.10	0.00	0.00	81.25	12.35	6.40	0.00	98.40	1.40	0.20	0.00	3.15	7.20	89.65	0.00	32.00	21.75	46.25
9	SF	0.3	0.90	0.25	**98.15**	0.70	33.00	1.85	65.10	0.05	0.00	0.00	90.05	9.95	1.30	0.30	97.90	0.50	0.00	0.00	6.70	93.30	0.00	0.00	47.05	52.95
(h=2,m=2)		0.5	0.00	0.05	**99.35**	0.60	1.90	0.25	97.85	0.00	0.00	0.00	91.35	8.65	0.00	0.05	99.60	0.35	0.00	0.00	8.95	91.05	0.00	0.00	50.70	49.30

Notes: Same as Table 2.

Table 9: Unknown reduced form breaks, AR(1) errors, T=240, $\beta_2 = 1$

Case		R^2	BIC				SBBIC				HQ				SBHQ				AIC				SBAIC			
			0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3	0	1	2	≥ 3
1	RF	0.3	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	97.05	2.45	0.50	0.00	99.75	0.25	0.00	0.00	0.05	0.15	0.80	99.00	2.35	3.00	8.45	86.20
(h=0,m=0)		0.5	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	97.05	2.45	0.50	0.00	99.75	0.25	0.00	0.00	0.05	0.15	0.80	99.00	2.35	3.00	8.45	86.20
	SF	0.3	89.40	7.90	2.20	0.50	**99.45**	0.50	0.05	0.00	49.65	19.70	17.05	13.60	90.90	7.25	1.50	0.35	0.25	0.10	2.15	97.50	34.10	3.90	12.15	49.85
		0.5	89.30	8.05	2.15	0.50	**99.45**	0.50	0.05	0.00	49.50	19.70	17.35	13.45	90.75	7.45	1.45	0.35	0.25	0.10	1.75	97.90	33.20	3.30	12.55	50.95
2	SF	0.3	0.00	73.80	18.25	7.95	0.00	**96.15**	3.45	0.40	0.00	33.70	29.50	36.80	0.00	76.80	16.95	6.25	0.00	0.15	0.75	99.10	0.00	4.45	10.10	85.45
(h=0,m=1)		0.5	0.00	74.25	18.45	7.30	0.00	**96.10**	3.50	0.40	0.00	33.70	30.00	36.30	0.00	77.05	16.95	6.00	0.00	0.15	0.80	99.05	0.00	4.45	9.10	86.45
3	SF	0.3	0.00	0.00	82.75	17.25	0.25	0.10	**98.05**	1.60	0.00	0.00	43.45	56.55	0.00	0.00	84.75	15.25	0.00	0.00	0.55	99.45	0.00	0.00	8.65	91.35
(h=0,m=2)		0.5	0.00	0.00	82.00	18.00	0.05	0.00	**98.20**	1.75	0.00	0.00	43.65	56.35	0.00	0.00	83.90	16.10	0.00	0.00	0.45	99.55	0.00	0.00	8.75	91.25
4	RF	0.3	0.00	99.95	0.05	0.00	0.00	100.00	0.00	0.00	0.00	95.75	3.90	0.35	0.00	99.40	0.60	0.00	0.00	0.00	0.40	99.60	0.00	2.60	5.65	91.75
(n=1,m=0)		0.5	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	95.90	3.70	0.40	0.00	99.50	0.50	0.00	0.00	0.05	0.40	99.55	0.00	3.00	6.35	90.65
	SF	0.3	94.55	3.85	1.20	0.40	**99.75**	0.25	0.00	0.00	59.85	13.25	14.70	12.20	95.45	3.40	0.90	0.25	0.00	0.15	2.30	97.55	27.50	5.30	13.80	53.40
		0.5	94.50	3.65	1.40	0.45	**99.85**	0.15	0.00	0.00	60.85	13.05	14.25	11.85	95.75	3.05	0.90	0.30	0.05	0.30	1.95	97.70	29.55	4.20	11.80	54.45
5	SF	0.3	0.00	73.60	18.60	7.80	0.25	**96.05**	3.30	0.40	0.00	34.45	28.55	37.00	0.00	76.75	16.90	6.35	0.00	0.05	0.45	99.50	0.00	4.20	9.05	86.75
(h=1,m=1)		0.5	0.00	74.25	18.30	7.45	0.00	**96.35**	3.30	0.35	0.00	34.60	29.40	36.00	0.00	76.95	16.70	6.35	0.00	0.15	0.65	99.20	0.00	4.55	8.95	86.50
6	SF	0.3	0.15	0.90	81.40	17.55	4.90	9.00	**84.40**	1.70	0.00	0.00	44.70	55.30	0.15	1.15	83.65	15.05	0.00	0.00	0.45	99.55	0.00	0.00	8.35	91.65
(h=1,m=2)		0.5	0.00	0.00	81.95	18.05	0.10	0.05	**97.80**	2.05	0.00	0.00	45.00	55.00	0.00	0.00	85.00	15.00	0.00	0.00	0.45	99.55	0.00	0.00	7.70	92.30
7	RF	0.3	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	96.15	3.30	0.55	0.00	99.55	0.45	0.00	0.00	0.10	0.45	99.45	0.00	2.85	5.75	91.40
(h=1,m=1)		0.5	0.00	100.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00	96.45	3.00	0.55	0.00	99.65	0.35	0.00	0.00	0.15	0.65	99.20	0.00	3.50	6.50	90.00
	SF	0.3	0.00	83.15	12.00	4.85	0.80	**97.35**	1.75	0.10	0.00	40.15	26.30	33.55	0.00	85.05	10.95	4.00	0.00	0.00	0.65	99.35	0.00	3.55	8.80	87.65
		0.5	0.00	83.95	10.65	5.40	0.00	**98.00**	1.90	0.10	0.00	41.65	22.65	35.70	0.00	85.95	10.05	4.00	0.00	0.10	0.60	99.30	0.00	4.15	8.85	87.00
8	RF	0.3	0.45	0.05	99.40	0.10	8.45	0.15	91.40	0.00	0.00	0.00	95.05	4.95	0.00	0.00	99.25	0.75	0.00	0.00	0.15	99.85	0.00	0.00	3.25	96.75
(h=2,m=1)		0.5	0.00	0.00	99.95	0.05	0.00	0.00	100.00	0.00	0.00	0.00	95.20	4.80	0.00	0.00	99.35	0.65	0.00	0.00	0.30	99.70	0.00	0.00	4.45	95.55
	SF	0.3	0.00	81.00	14.10	4.90	0.15	**97.25**	2.40	0.20	0.00	43.25	27.35	29.40	0.00	83.05	13.00	3.95	0.00	0.00	0.70	99.30	0.00	3.10	9.50	87.40
		0.5	0.00	81.95	12.35	5.70	0.00	**98.40**	1.45	0.15	0.00	44.55	24.20	31.25	0.00	84.35	11.25	4.40	0.00	0.05	0.60	99.35	0.00	3.40	9.45	87.15
9	SF	0.3	2.00	1.00	87.70	9.30	33.50	4.65	61.60	0.25	0.10	0.05	53.70	46.15	2.25	1.05	**89.00**	7.70	0.00	0.00	0.30	99.70	0.00	0.00	7.85	92.15
(h=2,m=2)		0.5	0.05	0.05	91.40	8.50	4.40	1.05	94.10	0.45	0.00	0.00	54.95	45.05	0.10	0.10	**92.90**	6.90	0.00	0.00	0.45	99.55	0.00	0.00	9.55	90.45

Notes: Same as Table 2.